



Correspondences acting on constant cycle curves on K3 surfaces

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Received: 7 April 2025 / Accepted: 17 October 2025
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Abstract

Constant cycle curves on a K3 surface X over \mathbb{C} are curves whose points all define the same class in the Chow group. In this paper we study correspondences $Z \subseteq X \times X$ over \mathbb{C} acting on the group $\text{ccc}(X)$ of cycles generated by irreducible constant cycle curves. We construct for any $n \geq 2$ and any very ample line bundle L a locus $Z_n(L) \subseteq X \times X$ which is expected to have dimension 2 and which yields a correspondence that acts on $\text{ccc}(X)$, when it has dimension 2. We provide examples of $Z_n(L)$ for low n and exhibit one correspondence different from $Z_n(L)$ acting on $\text{ccc}(X)$.

Keywords K3 surface · Chow group · Correspondences

Mathematics Subject Classification 14C15 · 14J28

1 Introduction

Constant cycle curves have been considered in [22] and formally introduced in [12] with the motivation of getting a more thorough picture of Chow groups and rational curves on K3 surfaces. On a K3 surface X defined over \mathbb{C} , they are curves whose points all define the same class in the Chow group of zero cycles of X . Rational curves are important examples of constant cycle curves. On the other hand, examples of non-rational constant cycle curves are constructed in [12]. Overall, constant cycle curves behave similarly to rational curves. They are rigid, as the existence of a family dominating X would force $CH_0(X)$ to be isomorphic to \mathbb{Z} , contradicting Mumford's theorem (see [16]). They are finite in linear systems, for bounded order, as proven in [12]. Their union is even dense with respect to the classical topology (see [12] or directly [22]). Yet the set of constant cycle curves on K3 surfaces is not well understood.

In this paper, we study correspondences $Z \subseteq X \times X$ over \mathbb{C} preserving the set of constant cycle curves. More precisely, we introduce the group $\text{ccc}(X) \subseteq Z_1(X)$ of cycles generated by irreducible constant cycle curves (see Definition 2.3). Then we study correspondences

This work was supported by the Alexander von Humboldt Foundation.

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$Z \subseteq X \times X$ acting on $\text{ccc}(X)$, i.e. inducing a homomorphism $Z_* : \text{ccc}(X) \rightarrow \text{ccc}(X)$ (see Definition 2.4).

Notice that the notion of constant cycle curve is not well defined at the level of Chow groups, since constant cycle curves are rigid. This means that not all curves that are rationally equivalent to a constant cycle curve are constant cycle curves. It is therefore essential to work at the level of cycles.

Natural examples to look at are 2-cycles $Z \subseteq X \times X$ such that for any point $(p, q) \in Z$, p is rationally equivalent to q in X . These correspondences have been studied before (see [6, 21]).

In the paper, we focus on examples of cycles defined by curves of genus $g \geq 3$. We construct for any $n \geq 2$ and any very ample line bundle L a cycle $Z_n(L) \subseteq X \times X$ that is expected to have dimension 2 and induces a correspondence that acts on $\text{ccc}(X)$, when of dimension 2. We then provide examples for low n having dimension 2.

Theorem 1.1 *Let X be a smooth projective K3 surface over \mathbb{C} and L a very ample line bundle. Then for any integer $n \geq 2$ the locus $Z_n(L) \subseteq X \times X$ defined as the closure of*

$$Z_n(L)^0 = \{(p, q) \in X \times X \mid \exists C \in |L| \text{ smooth s.t. } p, q \in C, p \neq q, n[p - q] = 0 \in JC\}$$

is a locus of expected dimension 2. Moreover, $Z_n(L)$ acts on $\text{ccc}(X)$ when it has dimension 2 (see Definition 2.4).

The construction of the correspondences $Z_n(L)$ generalizes ideas of [12, 22], that the locus of n -torsion points of the fibers of an elliptic fibration $X \rightarrow \mathbb{P}^1$ defines a constant cycle curve on X .

Different to [12], the loci we construct are expected to be of dimension 2 when not empty rather than 1. Notice that for any point $(p, q) \in Z_n(L)$, p is rationally equivalent to q and so all points $(p, q) \in Z_n(L)$ lie on a fibre of the difference map $X \times X \rightarrow CH_0(X)$. Mumford proved that the dimension of this fibre is at most 2. Nonetheless, whether the locus $Z_n(L)$ is not empty or of dimension 1 or 2 is not yet determined.

The proof of Theorem 1.1 is contained in Sect. 3, and follows from a series of more precise statements (see Theorems 3.2, 3.5, 3.7).

Determining whether $Z_n(L)$ is non-empty is not trivial, as the difference map $C \times C \rightarrow JC$ is not in general surjective for $g \geq 3$ (as it was for elliptic curves). Therefore, the existence of points $p, q \in X$ defining n -torsion points $p - q$ of JC is no longer guaranteed, for $g \geq 3$. Notice that when m divides n , $Z_m(L)$ defines a component of $Z_n(L)$, as m -torsion points are also n -torsion points. Therefore, the problem of non-emptiness for n is in this case reduced to that for m . However, the investigation of a component in $Z_n(L)$ cut out by n -torsion points that are not m -torsion points remains an interesting problem. The problem can be interpreted as a more refined problem of Brill–Noether theory. This because a point $(p, q) \in Z_n(L)$ is given by $p, q \in C$, for $C \in |L|$, with the property that $n[p - q] = 0 \in JC$. The latter condition defines a degree n map $C \rightarrow \mathbb{P}^1$ totally ramified at p and q , which is in particular a g_n^1 . Recall that a Brill–Noether general curve admits a g_n^1 if and only if the Brill–Noether number is non-negative, which happens precisely for $g \leq 2n - 2$ (see [1]). By [14], every smooth curve in the linear system $|L|$ is Brill–Noether general when $\text{Pic } X$ is generated by L . Therefore, for a very general K3 surface X , n must satisfy $g \leq 2n - 2$ for $Z_n(L)$ to be non-empty. The important observation is that these smooth curves do not necessarily admit a g_n^1 with the special ramification profile induced by points of $Z_n(L)$.

Conjecture 1 *Let (X, L) be a very general polarized K3 surface of genus $g \geq 3$. Then $Z_n(L)$ is not empty for any $g \leq 2n - 2$.*

The case of negative Brill–Noether number, i.e. $g \geq 2n - 2$, is also interesting on K3 surfaces of Picard rank at least two. Indeed, curves of every gonality in certain K3 surfaces of Picard rank 2 have been constructed in [13]. Again the examples do not satisfy the ramification profile induced by $Z_n(L)$ in general, but they might be worth further investigation.

In the paper, we confirm the above conjecture for $n = 3, 4$ in the following cases.

Theorem 1.2 *Let $X \subseteq \mathbb{P}^3$ be a smooth quartic not containing a line and L a hyperplane section. Then $Z_3(L)$ is non-empty of dimension 2.*

Theorem 1.3 *Let $X \subseteq \mathbb{P}^4$ be a general smooth complete intersection of a cubic and a quadric and L a hyperplane section. Then $Z_4(L)$ is non-empty of dimension 2.*

The proofs of these theorems are contained in Sects. 5 and 6 and rely on the study of two further correspondences J and T , related to the geometry of these concrete examples. Furthermore, since in genus $g = 2$ the line bundle L of Theorem 1.1 cannot be very ample and therefore the assumptions of the theorem are not satisfied, we devote Sect. 4 to carry out by hand the example for $n = 2$ and L the pullback of a line via the degree 2 cover from the K3 surface to \mathbb{P}^2 .

It is worth mentioning that the correspondences $Z_n(L)$ satisfy the so-called easy direction of the generalized Bloch conjecture (see Sect. 2.3 or directly [19]). Namely, they act by multiplication on the Chow group of 0-cycles, thus they act by multiplication on the transcendental cohomology.

Theorem 1.4 *In the setting of Theorem 1.1, $Z_n(L)_* : CH_0(X) \rightarrow CH_0(X)$ acts by multiplication, if $Z_n(L)$ has dimension 2. In particular, $Z_n(L)^*$ acts by multiplication on $H_{\text{tr}}^2(X, \mathbb{Q})$.*

We would like to spend a couple of words on the correspondences J and T involved in the proofs of Theorems 1.2, 1.3, because they are quite interesting on their own.

In fact, we can use J to prove the existence of at least one correspondence different from $Z_n(L)$ of Theorem 1.1 acting on the group $\text{ccc}(X)$. More importantly, $J \subseteq X \times X$ does not satisfy the property that for any $(p, q) \in J$, p is rationally equivalent to q , providing an example of a correspondence acting on constant cycle curves, which does not arise from the natural ones explained in the previous part of the introduction.

Theorem 1.5 *Let X be a smooth quartic surface not containing a line. Define $J \subseteq X \times X$ as the closure of the locus*

$$J^0 = \{(p, q) \in X \times X \mid p \neq q, X \cdot l_{p,q} = 3p + q \in Z_0(X)\}, \quad (1)$$

where $l_{p,q}$ denotes the line between p, q . Then J is a 2-cycle which yields a correspondence with generically finite projections and acting on the group $\text{ccc}(X)$.

As an application, this correspondence allows in some special cases to construct non-rational constant cycle curves starting from smooth conics, which are rational curves. The proof of the last theorem is contained in Sect. 5. We will actually see that $Z_n(L)$ and J share some interesting properties, concerning the study of constant cycle curves.

This fact is no longer true for the correspondence T studied in Sect. 6, whose definition is a natural generalization of J to the case of a K3 surface realized as a complete intersection in \mathbb{P}^4 . Namely, we look for triples of points (p, q, r) on the surface whose intersection with the plane generated by these points is $4p + q + r$. Already as a locus $T \subseteq X \times X \times X$, T is not fitting our previous setting.

To conclude, notice that there is one further explicit presentation of a K3 surface in \mathbb{P}^5 as a complete intersection of 3 quadrics. One would wonder whether we can generalize the

constructions of J and T to this case, by looking at 3-planes intersecting the K3 surface in a point of multiplicity 5. This however would need a new argument.

2 Preliminaries

2.1 Correspondences on smooth projective surfaces

Let X be a smooth projective surface over \mathbb{C} , $Z_i(X)$ the group of cycles of dimension i and $CH_i(X)$ the Chow group. Let $Z \subseteq X \times X$ be an irreducible subvariety and $p_{i,X} : X \times X \rightarrow X$ the projection on the i -th factor. We denote by $p_{i,Z} : Z \rightarrow X$ the restrictions of $p_{i,X}$ to Z and call them projections from Z .

For any 2-cycle $Z \in Z_2(X \times X)$, Fulton’s refined intersection (see [10, Chapter 8]) gives a homomorphism

$$Z_* : CH_i(X) \rightarrow CH_i(X), \quad Z_*[D] = p_{2,X*}(Z \cdot p_{1,X}^*([D])), \tag{2}$$

which is compatible with the schematic inverse image in the following sense. Let $D \in Z_1(X)$ be reduced irreducible, then

$$Z_*[D] \in CH_1(p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|))). \tag{3}$$

More precisely, on any $D \in Z_1(X)$ reduced irreducible, the pullback $p_{1,X}^*(D)$ is well defined without any assumption of flatness and gives a class $p_{1,X}^*(D) \in CH_*(p_{1,X}^{-1}(|D|))$. The intersection product $Z \cdot p_{1,X}^*(D)$ given by the Fulton’s refined intersection is obtained by taking the diagonal $\delta_{X \times X}$ and applying the refined Gysin homomorphism $\delta_{X \times X}^!$ to the class $Z \times p_{1,X}^*(D)$ in $CH_*(X \times X \times X \times X)$. We obtain a cycle,

$$Z \cdot p_{1,X}^*(D) = \delta_{X \times X}^!(Z \times p_{1,X}^*(D)) \in CH_*(|Z| \cap p_{1,X}^{-1}(|D|)) \tag{4}$$

with support in $|Z| \cap p_{1,X}^{-1}(|D|)$. By push forward we have a cycle $p_{2,X*}(Z \cdot p_{1,X}^*(D))$ with support in $p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|))$ as wanted. The definition is extended by linearity.

Overall, this refinement gives us a class with a precise support $p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|))$, for a given precise support $|D|$. Furthermore, it allows to define pullbacks of non-flat projections. We use this refinement to lift Z_* at the level of cycles under certain assumptions.

Lemma 2.1 *Let X be a smooth projective surface and $Z \in Z_2(X \times X)$ be an irreducible reduced subvariety. Assume that one of the following holds:*

- (1) $p_{2,X}(Z) \subseteq X$ has dimension ≤ 1 ;
- (2) $p_{1,X}$ is generically finite.

Then there exists a homomorphism $Z_* : Z_1(X) \rightarrow Z_1(X)$ making the diagram

$$\begin{array}{ccc} Z_* : Z_1(X) & \longrightarrow & Z_1(X) \\ \downarrow & & \downarrow \\ Z_* : CH_1(X) & \longrightarrow & CH_1(X) \end{array} \tag{5}$$

commutative.

Proof All we have to check is that to any irreducible curve of $C \in Z_1(X)$, we can associate a cycle $Z_*C \in Z_1(X)$ such that $[Z_*C] = Z_*[C] \in CH_1(X)$. Namely, a cycle Z_*C making Diagram (5) commutative. Then the statement follows by linearity on the intersection product.

Let $C \in Z_1(X)$ be an irreducible reduced curve. The refined intersection of Fulton gives on Chow groups a class $Z_*[C] \in CH_1(p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|)))$.

Assume that any irreducible component of the support $p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|))$ has dimension ≤ 1 . Then we can define a cycle Z_*C as it follows. If the support is either empty or zero dimensional, we set $Z_*C = 0$, which is compatible with the definition of Chow groups. Otherwise the support is a curve, after eventually removing isolated points, which do not contribute to define $Z_*[C]$. Since the Chow group of a curve is generated by its irreducible components, we can recover from its generators a cycle C' in $Z_1(X)$. Setting $Z_*C = C'$ we obtain a cycle, which by definition makes the diagram commutative.

Therefore, we just have to check that the support $p_{2,X}(|Z| \cap p_{1,X}^{-1}(|D|))$ has dimension ≤ 1 . This is obvious if $p_{2,X}(Z) \subseteq X$ has dimension ≤ 1 . Otherwise, $p_{1,X}$ is generically finite by assumption. Thus $|Z| \cap p_{1,X}^{-1}(|D|)$ is already a curve and we conclude. \square

2.2 Constant cycle curves and correspondences on K3 surfaces

Constant cycle curves have been formally introduced in [12] and they can be presented using a few different equivalent definitions. In the paper, we work over \mathbb{C} and we will always use the following definition.

Definition 2.2 Let X be a smooth projective K3 surface. A curve $C \subseteq X$ is a constant cycle curve if all points $p \in C$ define the same class $[p] \in CH_0(X)$.

It is easy to see (see [12]) that on any K3 surface X any point of a constant cycle curve defines the distinguished class of Beauville–Voisin c_X (see [5]) and so the class of a point of a rational curve.

In the paper, we want to study the constant cycle curves under the action of suitable correspondences.

Definition 2.3 We define the group $ccc(X)$ generated by constant cycle curves as the free abelian subgroup

$$ccc(X) \subseteq Z_1(X)$$

of the group of 1-cycles generated by irreducible constant cycle curves. We say that $C \in ccc(X)$ is a constant cycle cycle.

Definition 2.4 We say that a correspondence $Z \in Z_2(X \times X)$ acts on constant cycle curves (or on $ccc(X)$) if the natural homomorphism $Z_* : CH_1(X) \rightarrow CH_1(X)$ induces a homomorphism

$$Z_* : ccc(X) \rightarrow ccc(X).$$

In other words, if for any irreducible constant cycle curve C , Z_*C is a well defined 1-cycle that can be written as a linear combination of irreducible constant cycle curves.

In particular, we want to understand how many of these correspondences we can find on a K3 surface and whether we can use them to construct non-rational constant cycle curves acting on rational curves. As said in Conjecture 1 in the introduction, we expect to find many of them.

2.3 Generalized Bloch conjecture

The Generalized Bloch conjecture generalizes the classical Bloch conjecture for surfaces (see e.g. [19, Conjecture 11.2]) to the case of 2-dimensional correspondences in the product of two possibly different smooth projective surfaces.

In the paper, we restrict our attention to the case of the product $X \times X$ of two copies of a smooth projective surface X . Let $CH_0(C)_{\text{hom}} \subseteq CH_0(X)$ denote the subgroup of 0-cycles homologically equivalent to 0, modulo rational equivalence, and let $\text{alb}_X : CH_0(X)_{\text{hom}} \rightarrow \text{Alb}(X)$ be the Albanese map restricted to $CH_0(C)_{\text{hom}}$. Define a filtration on $CH_0(X)$ as

$$\begin{aligned} F^0 CH_0(X) &= CH_0(X), & F^1 CH_0(X) &= CH_0(X)_{\text{hom}}, \\ F^2 CH_0(X) &= \ker \text{alb}_X \subseteq F^1 CH_0(X). \end{aligned} \tag{6}$$

Recall that since X is projective, there are well defined homomorphisms

$$Z^* : CH_0(X) \rightarrow CH_0(X), \quad Z^*([Y]) = [\pi_{1,X*}(Z \cdot \pi_{2,X}^*(Y))] \tag{7}$$

and $[Z]^* : H^2(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$.

In this setting the Generalized Bloch conjecture (see [19, conjecture 11.19]) is rephrased as it follows.

Conjecture 2.5 (Generalized Bloch conjecture) *For any 2-cycle $Z \subseteq X \times X$ such that $[Z]^* : H^2(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$ vanishes on $H^0(X, \Omega_X^2)$, the morphism $Z_* : CH_0(X) \rightarrow CH_0(X)$ vanishes on $F^2 CH_0(X)$.*

The reader can find the definition of $[Z]^*$ in [19, Chapter 11] and of Z_* in Sect. 2.1.

The inverse of the Generalized Bloch–Beilinson conjecture is easier and already established ([19, proposition 11.18]). We recall here the statement in view also of [19, remark 11.20].

Lemma 2.6 *Let Z be a 2 cycle on X and assume that Z_* vanishes on $F^2 CH_0(X)$. Then $[Z]^* : H^2(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$ vanishes on $H^0(X, \Omega_X^2) \subseteq H^2(X, \mathbb{C})$ and furthermore Z^* vanishes on $H^2(X, \mathbb{Q})_{\text{tr}}$, namely on the smallest Hodge substructure containing $H^0(X, \Omega_X^2)$.*

As an application, consider a correspondence Z and an integer d . Take the correspondence $Z'' = Z - d\Delta_X$, where $\Delta_X \subseteq X \times X$ denotes the diagonal. Then Z'' satisfies the assumptions of Lemma 2.6 if and only if Z_* behaves as a multiple of the diagonal. We will see examples in the next sections.

3 Correspondences from n -torsion points of linear systems on $K3$ surfaces

In this section we introduce for any integer $n \geq 2$ a locus $Z_n(L) \subseteq X \times X$ parametrizing n -torsion points of curves C in the linear system $|L|$ on a $K3$ surface X . We prove that the expected dimension of such a locus is 2. When the locus $Z_n(L)$ has dimension 2, we show that it acts on the group $\text{ccc}(X)$ generated by constant cycle curves (Definition 2.4) and that it satisfies the so-called easy direction of the generalized Bloch conjecture (Sect. 2.3).

3.1 The loci $Z'_n(L)$ and $Z_n(L)$

Let X be a K3 surface over \mathbb{C} and L be a very ample line bundle. In particular, $L^2 > 2$ and any smooth irreducible curve $C \in |L|$ has genus $g = \frac{L^2}{2} + 1 > 2$.

Let $\mathcal{C} \rightarrow |L|$ be the universal family over the linear system $|L|$. The fiber product

$$\mathcal{C} \times_{|L|} \mathcal{C} \rightarrow |L|$$

parametrizes triples (C, p, q) such that $C \in |L|$, $p, q \in C$. We denote by $\mathcal{C}^0 \rightarrow |L|^0$ and $\mathcal{C}^0 \times_{|L|^0} \mathcal{C}^0 \rightarrow |L|^0$ the restrictions to the locus of smooth irreducible curves $|L|^0$ in $|L|$.

Definition 3.1 We define the locus $Z'_n(L)$ of n -torsion points of a linear system $|L|$ as closure in $\mathcal{C} \times_{|L|} \mathcal{C}$ of the locus

$$Z_n^0(L) := \{(C, p, q) \in \mathcal{C}^0 \times_{|L|^0} \mathcal{C}^0 \mid p \neq q, n[p - q] = 0 \in JC\} \subseteq \mathcal{C} \times_{|L|} \mathcal{C}. \tag{8}$$

These loci have a natural interpretation in terms of coverings of curves. Let $\mathcal{M}_{g,2}$ be the moduli space of genus g smooth projective curves with 2 marked points and let $\overline{\mathcal{M}}_{g,2}$ be the Deligne Mumford compactification, i.e. the moduli space of stable curves with 2 marked points. We define the locus $\mathcal{H}_n \subseteq \overline{\mathcal{M}}_{g,2}$ as the closure of the locus

$$\mathcal{H}_n^0 := \{(C, p, q) \in \mathcal{M}_{g,2} \mid \exists \pi : C \rightarrow \mathbb{P}^1 \text{ cover, } \deg \pi = n, R(\pi) \supseteq (n - 1)(p - q)\}, \tag{9}$$

where $R(\pi)$ denotes the ramification divisor. In words, \mathcal{H}_n^0 is the locus of two pointed smooth curves admitting a degree n cover totally ramified at the two marked points.

By definition, a point $(C, p, q) \in Z_n^0(L)$ satisfies the condition $n[p - q] = 0 \in JC$. This condition defines a cover $\pi : C \rightarrow \mathbb{P}^1$ of degree n , ramified at p and q with total ramification. Thus, a point $(C, p, q) \in \mathcal{H}_n^0$.

Consider now the natural modular rational map

$$\phi : \mathcal{C} \times_{|L|} \mathcal{C} \dashrightarrow \overline{\mathcal{M}}_{g,2}, \tag{10}$$

defined by sending a triple (C, p, q) to the stable model of C with two marked points p, q . We use it to study the image of the loci $Z'_n(L)$ in $\overline{\mathcal{M}}_{g,2}$ and more precisely in the loci \mathcal{H}_n .

Theorem 3.2 *Let X be a K3 surface and L a very ample line bundle. Then the expected dimension of $Z'_n(L)$ is 2.*

Proof Recall that points of $\mathcal{C} \times_{|L|} \mathcal{C}$ are just triples (C, p, q) , where $C \in |L|$ and $p, q \in C$. The rational map $\phi : \mathcal{C} \times_{|L|} \mathcal{C} \dashrightarrow \overline{\mathcal{M}}_{g,2}$ sends a point $(C, p, q) \in \mathcal{C} \times_{|L|} \mathcal{C}$ to $(C, p, q) \in \overline{\mathcal{M}}_{g,2}$. We want to prove that this rational map is generically finite. Suppose that $\phi(C, p, q) = \phi(C', p', q')$. Then C and C' are isomorphic, and there exists an automorphism f of C such that $f(p) = p'$ and $f(q) = q'$. Since the group of automorphisms of a curve of genus $g \geq 2$ is finite, we only have finitely many choices of p', q' for given p, q . Therefore, to prove that this map is generically finite, it suffices to prove that generically we have only finitely many choices of $C \in |L|$ in the fiber of $C \in \overline{\mathcal{M}}_g$. In other words, it suffices to prove that the rational map $|L| \dashrightarrow \overline{\mathcal{M}}_g$ is generically finite. This fact follows by [4], since we work under the assumption that L is very ample.

As $\mathcal{C} \times_{|L|} \mathcal{C} \dashrightarrow \overline{\mathcal{M}}_{g,2}$ is generically finite, we can compute the expected dimension of $Z_n(L)$ in $\overline{\mathcal{M}}_{g,2}$.

As observed above, a point $(C, p, q) \in Z_n^0(L)$ defines a point in \mathcal{H}_n^0 , as defined in (9). Namely, the condition $n[p - q] = 0 \in JC$ defines a cover $\pi : C \rightarrow \mathbb{P}^1$ of degree n , totally ramified at p and q . Thus, the expected dimension of $Z_n'(L)$ is given by

$$\dim \mathcal{H}_n + \dim(C \times_{|L|} C) - \dim \overline{\mathcal{M}}_{g,2}, \tag{11}$$

since all these varieties have only quotient singularities. Notice that \mathcal{H}_n has dimension $2g - 1$. Indeed, the ramification divisor $R(\pi)$ depends in general on $r = 2g - 2 + 2n$ parameters. Then we must subtract the contribution $2(n - 1)$ given by the two (fixed) marked points p, q , and take into account the action of the space of automorphisms of \mathbb{P}^1 that fix the two marked points p, q , which is of dimension 1.

The dimension of $C \times_{|L|} C$ is $g + 2$. Indeed, X is a K3 surface, $|L|$ has dimension g . Since the dimension of $\overline{\mathcal{M}}_{g,2}$ is $3g - 1$, putting all together we conclude that the expected dimension of $Z_n'(L)$ is 2. \square

Remark 3.3 As a bibliographic information, the behavior of $|L| \dashrightarrow \mathcal{M}_g$ for L very ample was already known before [4] in many cases, established for instance by [2, 3, 8, 9, 15, 24].

We now define a cycle $Z_n(L) \subseteq X \times X$ starting from $Z_n'(L)$. Consider the forgetful map $\psi : C \times_{|L|} C \rightarrow X \times X$, sending (C, p, q) to (p, q) .

Definition 3.4 We define $Z_n(L) \subseteq X \times X$ as $Z_n(L) := \psi(Z_n'(L))$. Moreover, we denote by $p_{i,Z_n(L)} : Z_n(L) \rightarrow X$ the projection over the i -th factor, for $i = 1, 2$.

Explicitly, by definition $Z_n(L)$ is realized as the closure of the locus

$$Z_n^0(L) = \{(p, q) \in X \times X \mid \exists C \in |L| \text{ smooth s.t. } p, q \in C, p \neq q, n[p - q] = 0 \in JC\}, \tag{12}$$

as in the statement of Theorem 1.1.

Notice that a priori $Z_n(L)$ and $Z_n'(L)$ can have different dimension if the map ψ is not generically finite over $Z_n'(L)$. Let $|L_{p,q}|$ the linear system of $C \in |L|$ passing through p, q . For general p, q , the rational map $|L_{p,q}| \dashrightarrow \mathcal{M}_g$ is generically finite. In the next theorem we will see how this leads to conclude that $Z_n(L)$ and $Z_n'(L)$ share the same expected dimension.

Theorem 3.5 *The expected dimensions of $Z_n'(L)$ and $Z_n(L)$ coincide.*

Proof Suppose that $Z_n(L)$ has expected dimension smaller than $Z_n'(L)$. This means that the general fiber of the restriction $\psi : Z_n'(L) \rightarrow Z_n(L)$ to $Z_n'(L)$ of $\psi : C \times_{|L|} C \rightarrow X \times X$ has positive dimension. Namely for a general $(p, q) \in Z_n(L)$, the linear system $|C' - p - q|$ for $C' \in |L|$ and $p, q \in C'$ contains a positive dimensional subvariety Y_{pq} of curves with the property that $n[p - q] = 0 \in JC_t$ for any $C_t \in Y \subseteq |C' - p - q|$.

We now prove that Y has expected dimension 0. Recall that $|L_{pq}|$ is the linear system $|C' - p - q|$, let $C \rightarrow |L_{pq}|$ be the universal family over $|L_{pq}|$ and consider the fiber product $C \times_{|L_{pq}|} C \rightarrow |L_{pq}|$. We have a rational map

$$\psi_{pq} : C \times_{|L_{pq}|} C \dashrightarrow \overline{\mathcal{M}}_{g,2}$$

defined by mapping (C', p, q) to the curve C' with marked points p, q , analogously to the map (10) defined for L . To compute the expected dimension of Y_{pq} we can thus repeat the same computation used in the proof of Theorem 3.2 to compute the expected dimension of $Z_n'(L)$. We can assume that ψ_{pq} is generically quasi-finite for a general (p, q) , otherwise ψ

would not be generically quasi-finite, contrary to our assumptions in Theorem 3.2. Then, the expected dimension of Y_{pq} is thus given by

$$\dim \mathcal{H}_n + \dim(C \times_{|L_{pq}|} C) - \dim \overline{\mathcal{M}}_{g,2}, \tag{13}$$

since all these varieties have only quotients singularities. Now as in Theorem 3.2, \mathcal{H}_n has dimension $2g - 1$ and the dimension of $\overline{\mathcal{M}}_{g,2}$ is $3g - 1$. But now the dimension of $C \times_{|L_{pq}|} C$ is g , since L is very ample and so $\dim |L_{pq}| = \dim |L| - 2 = g - 2$. We conclude that the expected dimension of Y_{pq} is 0. This prove that the expected dimension of $Z_n(L)$ is not smaller and so equal to the expected dimension of $Z'_n(L)$. \square

This concludes the proof of the first part of Theorem 1.1.

In the paper, we look for examples acting on the group $\text{ccc}(X)$ generated by constant cycle curves (Definition 2.4). For this, it is very important to find non-empty $Z_n(L)$ with projections satisfying the assumptions of Lemma 2.1. This properties will be discussed in the next subsections.

3.2 Properties of the correspondence $Z_n(L)$

We will now see how the correspondences $Z_n(L)$ have significant applications to the study of $CH_0(X)$ and $Z_1(X)$ and in particular to the study of the group $\text{ccc}(X)$ generated by constant cycle curves. In this section, we assume that $Z_n(L)$ has the expected dimension, i.e. it is a 2 cycle.

The following lemma is the key special property satisfied by $Z_n(L)$, from which all other properties follow.

Lemma 3.6 *For any point $(p, q) \in Z_n(L)$, p is rationally equivalent to q on X .*

Proof It is enough to prove the statement for $Z_n^0(L)$. Then it holds on the closure $Z_n(L)$ of $Z_n^0(L)$ by specialization to points of the closure. Indeed, for any point (p, q) in the closure there is a dvr R together with a map $\text{Spec } R \rightarrow X \times X$ mapping the generic point η of $\text{Spec } R$ to the interior $(p_\eta, q_\eta) \in Z_n^0(L)$ and the closed point $\text{Spec } k$ to (p, q) . The family $X \times_{\text{Spec } R} \text{Spec } R \rightarrow \text{Spec } R$ induces a specialization map $CH_0(X \times_{\text{Spec } R} \eta/\eta) \rightarrow CH_0(X \times_{\text{Spec } R} \text{Spec } k/\text{Spec } k)$. Assume the lemma holds on the interior. Then $p_\eta - q_\eta$ is a zero cycle rationally equivalent to 0 on $X \times_{\text{Spec } R} \eta/\eta$. By using the constructed specialization map $p - q$ is a cycle rationally equivalent to 0 on $X \times_{\text{Spec } R} \text{Spec } k/\text{Spec } k$ and so on X . For more details see for instance [10, 20.3].

To prove the statement for $Z_n^0(L)$, let $(p, q) \in Z_n^0(L)$. By Definition 3.4, there is $C \subseteq X$ such that $n[p - q] = 0 \in JC$. Take the image of $n[p - q] = 0$ via $JC \rightarrow CH_0(X)$. Since $CH_0(X)$ is torsion free, we must have that $[p - q] = 0 \in CH_0(X)$, which means that p and q are rationally equivalent on X . \square

As an application, we show that the correspondences $Z_n(L)$ act on the group $\text{ccc}(X)$ generated by constant cycle curves.

Since $Z_n(X)$ is a 2-cycle which is symmetric with respect to the involution of $X \times X$ interchanging the two factors, the projections $\pi_{i, Z_n(L)} : Z_n(L) \rightarrow X$, for $i = 1, 2$ are either both generically finite or they map onto the same curve C .

Thus, by Lemma 2.1 the homomorphism $Z_n(L)_* : CH_1(X) \rightarrow CH_1(X)$ lifts to a homomorphism of cycles

$$Z_n(L)_* : Z_1(X) \rightarrow Z_1(X) \tag{14}$$

and we can ask whether it preserves the group $\text{ccc}(X)$.

Theorem 3.7 *Assume that $Z_n(L)$ has dimension 2. Then $Z_n(L)$ acts on the group $\text{ccc}(X)$ generated by constant cycle curves (Definition 2.4). Namely, $Z_n(L)_* : CH_1(X) \rightarrow CH_1(X)$ induces a homomorphism*

$$Z_n(L)_* : \text{ccc}(X) \rightarrow \text{ccc}(X).$$

Proof Assume that $Z_n(L)$ has dimension 2. As observed above, the projections satisfy Lemma 2.1, and so we have a well defined homomorphism $Z_n(L)_* : Z_1(X) \rightarrow Z_1(X)$.

Thus, all we have to prove is that for any irreducible constant cycle curve $C \in \text{ccc}(X)$, $C' = Z_n(L)_*C \in \text{ccc}(X)$, i.e. it is a linear combination of irreducible constant cycle curves. Then the statement follows by linearity of $Z_n(L)$.

Assume that C' is not 0, otherwise there is nothing to prove. By definition of $Z_n(L)_*$, $C' = \sum_i \alpha_i C'_i$, where any C'_i is a reduced irreducible curve and α_i is the multiplicity along C'_i . To prove that $C' \in \text{ccc}(X)$, it is enough to show that C'_i is a constant cycle curve, for any i . For this, we need to show that two general points $q, q' \in C'_i$ are rationally equivalent. By Lemma 2.1, $C' = Z_n(L)_*C \in Z_1(p_{2,Z_n(L)}(|Z_n(L)| \cap p_{1,Z_n(L)}^{-1}(|C|)))$ and so $q, q' \in p_{2,Z_n(L)}(|Z_n(L)| \cap p_{1,Z_n(L)}^{-1}(|C|))$. Then there exist $p, p' \in C$ (not necessarily distinct) such that $(p, q), (p', q') \in |Z_n(L)| \cap p_{1,Z_n(L)}^{-1}(C)$. Now since $p, p' \in C$ and C is a constant cycle curve, p and p' are rationally equivalent on X . By Lemma 3.6 we conclude that q and q' are rationally equivalent on X . \square

Theorem 3.7 proves the second part of Theorem 1.1.

A last application concerns the relation with the generalized Bloch conjecture (see Subsection 2.3). More precisely we can find a non-zero integer d such that $Z_n(L) - d\Delta_X$, where $\Delta_X \subseteq X \times X$ is the diagonal, satisfies the so called easy-direction of the generalized Bloch conjecture (see Lemma 2.6, or directly [19, proposition 11.18]). In other words $Z_n(L)$ behaves as a multiple of the diagonal on zero cycles.

Theorem 3.8 *Assume $Z_n(L)$ has dimension 2. Then $Z_n(L)_*$ acts by multiplication on $CH_0(X)$. In particular, $[Z_n(L)]^*$ acts by multiplication on $H^0(X, \Omega_X^2)$ and $(Z_n(L))^*$ acts by multiplication on $H^2(X, \mathbb{Q})_{\text{tr}}$.*

Proof Let $d = \deg \pi_{2,*}\psi_{*Z_n}$. We claim that for any $p \in X$, $Z_n(L)_*([p]) = d[p]$. Indeed, under our assumptions $(Z_n(L))_*([p]) = [q_1] + [q_2] + \dots + [q_d]$, i.e. sum of d points not necessarily distinct. By Lemma 3.6, p and q_i are rationally equivalent. This concludes the proof of the claim.

Define $\Gamma = Z_n(L) - d\Delta_X$, where Δ_X denotes the diagonal of $X \times X$. We conclude from above that $\Gamma_* = 0$ on $CH_0(X)$. Now notice that X is K3 surface and so $\text{Alb}(X)$ is 0, the map alb_X is 0 and $F^2CH_0(X) = CH_0(X)_{\text{hom}}$ (as introduced in Subsection 2.3). We can then apply [19, proposition 11.18] and [19, remark 11.20] (see directly Subsection 2.3, Lemma 2.6) and conclude that $[\Gamma]^* = 0$ on $H^0(X, \Omega_X^2)$, which proves the statement. \square

4 Correspondences on K3 surfaces of degree 2

In this section we construct a natural locus $Z'_2(L)$ as in Definition 3.1, or more precisely $Z_2(L)$ as in Definition 3.4, but for a line bundle L of genus 2, which is not very ample. This provides an example with non-generically finite projections and we will see how this reflects

on the study of constant cycle curves. This will hopefully clarify why we are particularly interested in having finite projections.

Theorem 4.1 *Let $\pi : X \rightarrow \mathbb{P}^2$ be a double cover of \mathbb{P}^2 ramified over a sextic D . Consider the linear system $|L|$ defined by $L = \mathcal{O}_X(C)$, where $C = \pi^*l$ is the pullback of a line l of \mathbb{P}^2 . Then $Z_2(L)$ is not empty of dimension two, isomorphic to $D \times D$ up to base change and the projections $\pi_{i, Z_2(L)} : Z_2(L) \rightarrow X$ map onto D with all fibers isomorphic to D .*

Proof Recall that a point $(C', p, q) \in Z'_2(L)$, with $C' \in |L|$ and $p, q \in C'$, satisfies $2[p - q] = 0 \in JC'$. On the one hand, $2[p - q] = 0 \in JC'$ for $C' \in |L|$ induces a degree 2 covering totally ramified at p and q , i.e. a g_2^1 . On the other hand, by definition $C' = \pi^*l'$ for some line $l' \subseteq \mathbb{P}^2$, so the restriction of $\pi : X \rightarrow \mathbb{P}^2$ to $C' \rightarrow l'$ is a degree 2 cover ramified in 6 points of the sextic D , defining a g_2^1 on C' . These two degree 2 coverings must coincide up to an isomorphism preserving the ramification locus, because the g_2^1 of a hyperelliptic curve is unique. Thus p, q lie on D . Conversely, any pair of points p, q on D satisfy $2[p - q] = 0 \in JC'$ for $C' = \pi^*l'$ and l' the line of \mathbb{P}^2 through the two unique images p', q' on D of the points p, q , since D is the ramification locus of π and π is totally ramified on D . Since p' and q' are linearly equivalent on the sextic by definition, their pullbacks $2p, 2q$ are linearly equivalent on C' , which means that $2[p - q] = 0 \in JC'$. It follows that $Z'_2(L)$ is the locus of (D, p, q) such that $p, q \in D$ and so it is isomorphic to a isotrivial family constructed from $D \times D$ by using the monodromy action. Since D does not vary, the same holds for $Z_2(L)$. Now the projections have clearly image D and all fibers D . \square

It has been proven by [12] that the sextic $D \subseteq \mathbb{P}^2$ defines a constant cycle curve in both X and \mathbb{P}^2 . We can now use Z_2 to prove this fact again.

Theorem 4.2 *The ramification locus D of a K3 $\pi : X \rightarrow \mathbb{P}^2$ is a constant cycle curve in both X and \mathbb{P}^2 .*

Proof The proof is a corollary of Lemma 3.6, which doesn't require the assumption on L to be very ample. For any $p \in D$, $Z_2(L)_*(p) \simeq D$ by Theorem 4.1 and we conclude by Lemma 3.6 that all points in D are linearly equivalent, which means that D is a constant cycle curve on X . By looking at the map $CH^1(X) \rightarrow CH^1(\mathbb{P}^2)$ induced by π we conclude that D is a constant cycle curve also on \mathbb{P}^2 . \square

Remark 4.3 It is clear that the only constant cycle curves that can be investigated by $Z_n(L)$ with projections mapping to a curve are the ramification locus and the fibers. Thus $Z_2(L)$ does not provide any new information to the study of constant cycle curves on this K3 surface.

5 Correspondences on K3 surfaces of degree 4

In this section we construct and study a correspondence J on a smooth quartic surface X over \mathbb{C} , by detecting lines intersecting X in at most two distinct points, one of them with multiplicity at least 3. We use J to prove non-emptiness of $Z_3(H)$ associated to the linear system $|H|$ of a hyperplane section (as in Definitions 3.1, 3.4). We will see that there is a very nice geometric relation between J and $Z_3(H)$ given by g_1^3 s defined by points of J and 3-torsion points.

5.1 Construction and properties of the correspondence J

Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface over \mathbb{C} . For any pair of distinct points p, q of X we denote by $l_{p,q}$ the line in \mathbb{P}^3 through p and q and by $\Delta_X \subseteq X \times X$ the diagonal. Since X is a smooth quartic surface, it contains at most finitely many lines (see e.g. [11]), and outside of them the intersection $C \cdot l_{p,q}$ defines a 0-cycle of degree 4 in X , i.e. an element in $Z_0(X)$.

Theorem 5.1 *Let $X \subseteq \mathbb{P}^3$ be a smooth quartic surface over \mathbb{C} . Then the closure $J \subseteq X \times X$ of*

$$J^0 = \{(p, q) \in X \times X \setminus \Delta_X \mid X \cdot l_{p,q} = 3p + q \in Z_0(X)\} \tag{15}$$

is a 2 dimensional cycle of $X \times X$ yielding a correspondence whose first projection $\pi_{1,J} : J \rightarrow X$ is generically finite of degree 2. Moreover, if we assume that X does not contain a line, the second projection $\pi_{2,J} : J \rightarrow X$ has degree 68.

Remark 5.2 Later in the paper we will provide an alternative proof to the finiteness of the second projection which holds without the assumption that X does not contain a line. The computation of the degree is not recovered by this alternative proof.

Proof We first prove that $\pi_{1,J}$ is generically finite dominant of degree 2. It follows from this that J is 2 dimensional.

Notice that as observed above X contains at most finitely many lines (see e.g. [11]), and therefore it is enough to work with J^0 . Let $p \in X$ be a general point. The fiber $\pi_{1,J}^{-1}(p)$ of p is parameterized by all lines in \mathbb{P}^3 intersecting p with multiplicity at least 3. Any of these lines will in fact intersect X in exactly one more point q (not necessarily distinct from p), defining a point (p, q) of the fiber. Now notice that any line intersecting p with multiplicity at least 2 must lie in the tangent space T_pX . Therefore we have reduced the problem to counting lines in T_pX intersecting p with multiplicity 3.

Consider the quartic plane curve $Q_p = T_{p,X} \cap X \subseteq T_{p,X}$ cut out by the tangent space $T_{p,X}$. By definition the curve Q_p is singular at p . We know that any line in T_pX through p intersects p with multiplicity at least 2. Among them, the lines determining the tangent cone of Q_p at p are exactly those intersecting p with multiplicity at least 3. In conclusion, we need to determine the number of lines generating the tangent cone, counted without multiplicity. A double line will in fact determine only one point on the fiber $\pi_{1,J}^{-1}(p)$.

Now for a general p , Q_p is irreducible and p is a node of Q_p . Thus the tangent cone is generated by two distinct lines l_1, l_2 , each of them intersecting X in exactly one more point different from p . We obtain two points q_1, q_2 that must be distinct by construction and so the fiber $\pi_{1,J}^{-1}(p)$ is given by two points $(p, q_1), (p, q_2)$.

This proves that $\pi_{1,J}$ is generically finite of degree 2.

We now prove that the second projection $\pi_{2,J}$ is generically finite of degree 68. Let q be a point of X and consider the projection $\pi_q : X \setminus \{q\} \rightarrow \mathbb{P}^2$ from the point q , which is a finite rational map on X . The blow up $\epsilon : X' \rightarrow X$ of X at q resolves the indeterminacy of π_q and gives a morphism $\pi'_q : X' \rightarrow \mathbb{P}^2$, mapping the exceptional divisor E to a line L in \mathbb{P}^2 . Recall that the projection is defined for a choice of a suitable hyperplane H not containing q (which is then abstractly identified with \mathbb{P}^2) by taking for any point $p \neq q$ of X the line $l_{p,q}$ through p, q and sending p to the intersection point of $l_{p,q} \cdot H$. Since X is a quartic surface, the intersection of X with any line through q is four points counted with multiplicity, one of each is q itself. Thus the morphism π'_q has degree 3 outside the exceptional divisor. The points of total ramification of π'_q are exactly the points $p \in X$ such that $l_{pq} \cdot X = 3p + q$. Therefore,

the sublocus of points of total ramification in the ramification locus R of π'_q coincides with the fiber $\pi_{2,J}^{-1}(q)$ of $\pi_{2,J}$ over q . We now compute such a locus.

Denote by B the branch locus of π'_q . Since the degree of the morphism π'_q outside the exceptional divisor is 3, the points of ramification are either simple, i.e. exactly two branches come together, or of total ramification, i.e. all branches come together. In particular, either the general point of ramification is simple or all points of ramification have total ramification and the cover is Galois. By [25, Theorem 1] the set of points q corresponding to morphisms π'_q that come from Galois covers is finite and by [25, Corollary 2.2] described by the points q such that $X \cap T_q X$ is given by 4 distinct lines. Since by assumption X does not contain lines, such a set is empty. Therefore, for all points $q \in X$, the morphism π'_q is such that the general point of R has simple ramification. Because π'_q has degree 3 outside the exceptional divisor, we can describe $\pi_q'^{-1}(B)$ as the union of two curves. One, which is the ramification locus R and whose general point has multiplicity 2 at the intersection with each line through q . The other, denoted by A , is cut out by the last intersection point of each line with X' . The intersection $R \cdot A$ is a zero cycle whose points are exactly the points of R having non-simple ramification, i.e. having more than two branches coming together. These points are exactly the points $p \in X$ such that $q + 3p = l_{p,q} \cdot X$. Therefore, for a general q , they are the points in the fiber of the second projection at q . We then conclude that the degree of the second projection $\pi_{2,J}$ is exactly the degree of the cycle $A \cdot R$.

We now compute the degree of $A \cdot R$. Let us first compute the ramification divisor R . Since X is a K3, the canonical divisor K_X is trivial and so the canonical divisor of the blow up $\epsilon : X' \rightarrow X$ at q with exceptional divisor E is just $K_{X'} = K_X + E = E$. Let L be the line such that $\pi'_q(E) = L$ and let $H' = \pi_q'^*L$. If H denotes a hyperplane of X , and $\epsilon : X' \rightarrow X$ is the blow up as above, then $\epsilon^*H = H' + E$. We have $\epsilon^*H \cdot E = 0$, that is $(H' + E) \cdot E = 0$, $E^2 = -1$, then $H' \cdot E = 1$ and $\epsilon^*H \cdot \epsilon^*H = (H' + E) \cdot (H' + E) = 4$. We conclude from this that $H'^2 = 3$. By the Hurwitz formula, $K_{X'} = \pi_q'^*K_{\mathbb{P}^2} + R$. Since $K_{X'} = E$ and $\pi_q'^*\mathcal{O}_{\mathbb{P}^2}(1) = H'$, we conclude that $R = 3H' + E$. We now compute A . Since $\pi_q'^*H' = 3L$, $\pi_q'^*E = L$, we obtain $\pi_q'^*R = 10L$ and so $B = 10L$. But now, $A + R = \pi_q'^*B = 10H'$ and since $R = 3H' + E$ we conclude that $A = 7H' - E$. To conclude, $A \cdot R = (7H' - E) \cdot (3H' + E) = 68$.

The above argument shows that for all $q \in X$ such that $\pi_{2,J}^{-1}(q) \subseteq J^0$, the projection $\pi_{2,J}$ is finite of degree 68. □

Remark 5.3 A classification theorem by [23, Lemma 1.1.6], describes the singularity of p as a point of the quartic curve cut out by its tangent space on X , for any point p of X . More precisely, p is either a node, a cusp or a tacnode (according to the definition given in [23] which is a bit unconventional) or a point of multiplicity 3. In all these cases the tangent cone is indeed generated by either one or two distinguished lines (see equations in [23, Figure 1.1.1]). It seems that with a little attention the closure of J could be fully explicitly understood in terms of this result, as well as the finiteness of the first projection.

A very important point in the study of the correspondence J is that it satisfies a "special property" analogous to the that of $Z_n(L)$ (Lemma 3.6).

Lemma 5.4 *Let $(p, q), (p', q') \in J$. Then p is rationally equivalent to p' on X if and only if q is rationally equivalent to q' on X .*

Proof As in Lemma 3.6, if we prove the statement for $(p, q), (p', q') \in J^0$, then we conclude for all points of J by specializing to points of the closure.

Let $(p, q), (p', q') \in J^0$. Since the Grassmannian variety of lines in \mathbb{P}^3 is rational, $X \cdot l_{p,q}$ is rationally equivalent to $X \cdot l_{p',q'}$. In other words, $3p + q$ is rationally equivalent to $3p' + q'$.

Assume that p is rationally equivalent to p' , then the same holds for $3p$ and $3p'$. From above, $3p + q$ is rationally equivalent to $3p' + q'$. Thus, by difference, q and q' are rationally equivalent.

Conversely, let q be rationally equivalent to q' . Since from above $3p + q$ is rationally equivalent to $3p' + q'$, by difference the same holds for $3p$ and $3p'$. This means that $p - p'$ is a 3 torsion cycle of X . Now since $CH_0(X)$ is torsion free (see [17] or also [19, theorem 14.14]), p and p' must be rationally equivalent on X . □

Remark 5.5 Alternatively, to prove the previous lemma one can compute the class of $l_{p,q} \cap X$, for a general pair of distinct $p, q \in X$. Since the line $l_{p,q}$ is intersection of a general hyperplane through p and a general hyperplane section through q , this class must be a multiple of the Beauville–Voisin class c_X (see [5]), and since X is a quartic surface, the it must be $[l_{p,q} \cdot X] = 4c_X$. Thus, in particular, any two lines through a pair of points of X are rationally equivalent.

Using the previous lemma we can provide a proof to the generic finiteness of the second projection $\pi_{2,J}$ of Theorem 5.1, which does not compute the degree but it does not require the assumption that X does not contain any line.

Alternative proof to the generic finiteness of $\pi_{2,J}$, which holds for any smooth quartic surface X . Assume by contradiction that $\pi_{2,J}$ is not generically finite and so the image $\pi_{2,J}(J)$ is a curve C since it is not constant. Using that $\pi_{1,J}$ is finite, we can show that there exists a curve $C'' \subseteq X$ such that $CH_0(C'') \rightarrow CH_0(X)$ is surjective. This leads to a contradiction because the Chow group of a curve is finitely generated but by Mumford’s Theorem (see [16]) the Chow group of X is not.

In other words, we prove that there exists a curve $C'' \subseteq X$ such that for any point $x \in X$ there is a point $p \in C$ such that x and p are rationally equivalent.

Since $\pi_{2,J} : J \rightarrow C$ is surjective by assumption and J is two dimensional, there exists a curve $C' \subseteq J$ such that the restriction $\pi_{2,J} : C' \rightarrow C$ is still surjective. We claim that its image with respect to the first projection $\pi_{1,J}(C')$ is a curve C'' with the property above.

Let $s \in X$ be a general point, then since $\pi_{1,J}$ is dominant there exists $q \in X$ such that $(s, q) \in J$ and $\pi_{1,J}((s, q)) = s$. By construction $\pi_{2,J}((s, q)) = q \in C$ and so using that $\pi_{2,J} : C' \rightarrow C$ is surjective there exists also $p' \in X$ such that $(p', q) \in C' \subseteq J$. Since $(p', q) \in C'$, $\pi_{1,J}((p', q)) = p' \in \pi_{1,J}(C') = C''$ and so for a general $s \in X$ we have found a rationally equivalent point $p' \in C''$ as claimed. □

As an application we recover a relation with the generalized Bloch conjecture (see Subsection 2.3). More precisely, we show that $J - 6\Delta_X$, where $\Delta_X \subseteq X \times X$ is the diagonal, satisfies the inverse direction of the generalized Bloch conjecture (see Lemma 2.6, or directly [19, proposition 11.18]).

Theorem 5.6 *Let $\Delta_X \subseteq X \times X$ denote the diagonal. The relation $J_* - 6\Delta_{X*} = 0$ holds on $CH_0(X)_{\text{hom}}$, and so $J^* - 6\Delta_X^* = 0$ on $H^2(X, \mathbb{Q})_{\text{tr}}$.*

Proof For a general $p \in X$, $J_*([p]) = [q_1] + [q_2]$ by definition. By Lemma 5.4, q_1 and q_2 are rationally equivalent and we obtain $J_*([p]) = 2[q_1]$. Now notice that the class $[X \cdot l_{p,q_1}]$ is the class of the intersection of X with the line l_{p,q_1} , which is equivalently realized as the intersection of any hyperplane through p and a hyperplane through q_1 . As explained in Remark 5.5, this class must be $4c_X$, where c_X is the Beauville–Voisin class. Equivalently, it must be $4[t]$, where t is a point of a rational curve. Consequently, we can write $[3p + q_1] =$

$[X \cdot l_{pq_1}] = 4[t]$. Then $J_*([p]) = 2[q] = 2[4t - 3p] = 8[t] + 6[p]$. In particular, on $CH_0(X)_{\text{hom}}$ we have $J_* = 6\Delta_{X^*}$. Now notice that X is a K3 surface and so $\text{Alb}(X)$ is 0, the map alb_X is 0 and $F^2CH_0(X) = CH_0(X)_{\text{hom}}$ (as introduced in Subsection 2.3). We can then apply [19, proposition 11.18] and [19, remark 11.20] (see directly Subsection 2.3, Lemma 2.6) to conclude the second part of the statement. \square

Corollary 5.7 *For the generic smooth quartic surface $[J] = -4[x \times X] + 62[X \times x] + 6[\Delta_X] + d[h \cdot h^{-1}]$, for $x \in X$ and h a curve in $X \times X$ that descends to $\mathcal{O}_X(1)$.*

Proof For a general quartic surface $[J] = a[x \times X] + b[X \times x] + c[\Delta_X] + d[h \cdot h^{-1}]$. From Theorem 5.6, $c = 6$. Now $\text{deg } \pi_{1,J} = 2$ and $\text{deg } \pi_{2,J} = 68$ so $a + c = 2$ and $b + c = 68$ and we conclude. \square

5.2 Non-emptiness of $Z_3(H)$ defined by a hyperplane section

We relate J to $Z_3(H)$, where H is a hyperplane section of the smooth quartic $X \subseteq \mathbb{P}^3$.

Let X be a smooth quartic surface that does not contain a line. Let $H \subseteq X$ be a hyperplane section of X and let $\mathcal{C} \rightarrow |H|$ be the universal curve of the linear system $|H|$, whose fiber over a general member $C \in |H|$ is the quartic curve C itself. Recall that $Z'_3(H) \subseteq \mathcal{C} \times_{|H|} \mathcal{C}$ is the sublocus of 3-torsion points of $|H|$ as defined in Sect. 3.

In these notations,

$$Z'_3(H) = \overline{\{(p, q, C) \in \mathcal{C} \times_{|H|} \mathcal{C} \mid p \neq q, 3[p - q] = 0 \in JC\}} \subseteq \mathcal{C} \times_{|H|} \mathcal{C}, \tag{16}$$

$$\begin{aligned} Z_3(H) &= \overline{\{(p, q) \in X \times X \mid \exists C \in |H| \text{ smooth s.t. } p, q \in C, p \neq q, 3[p - q] = 0 \in JC\}} \\ &\subseteq X \times X \end{aligned} \tag{17}$$

and they are related by the forgetful map $Z'_3(H) \rightarrow Z_3(H)$.

As stressed in Sect. 3, it is not clear in general whether $Z_3(H)$ is non-empty. We now show that $Z'_3(H)$ and $Z_3(H)$ are non-empty of dimension 2. We use J , or more precisely the fiber product $J \times_{\pi_{2,J}(J)} J$.

Notice that a point in this fiber product is given by two points $(p, q), (p', q') \in J$ such that $q = q'$. We will thus write a point of $J \times_{\pi_{2,J}(J)} J$ as a tuple $((p, q), (p', q))$.

Theorem 5.8 *There is a finite dominant rational map $J \times_{\pi_{2,J}(J)} J \dashrightarrow Z'_3(H)$. In particular, both $Z'_3(H)$ and $Z_3(H)$ have dimension 2.*

Proof Recall that $J^0 \subset J$ denotes the interior of J (see Theorem 5.1). Let $\Delta \subseteq J \times_{\pi_{2,J}(J)} J$ denote the locus of points $((p, q), (p', q))$ such that $p = p'$. Let $((p, q), (p', q))$ be a point of $J^0 \times_{\pi_{2,J}(J)^0} J^0 \setminus \Delta$. By definition of J (Theorem 5.1), $3p + q = X \cdot l_{p,q}$ and $3p' + q = X \cdot l_{p',q}$.

Consider the plane $H^q_{p,p'} \subseteq \mathbb{P}^3$ through p, p', q . Then $C = H^q_{p,p'} \cap X$ is a quartic containing the points p, p' . Furthermore, for a general p , the quartic C is smooth in the linear system $|H|$. Indeed, a quartic is singular if it is cut out by the tangent space at a certain point of X . If we consider the lines through p, q in the construction, for any p we have at most two choices of q , so they are parametrized by p up to a finite choice for q . We can fix a choice of q for any p so that we obtain a pencil of lines parametrized by p . We look at them as a line in $\mathbb{P}^3 \vee$ and intersect it with the dual variety defined by the Gauss map. The dual variety parametrizes tangent spaces and has degree $4(3)^2$. This number is lower than

the degree of the second projection, which is 68 by Theorem 5.1. This means that we can choose at least some p' such that the plane through p, q, p' is not a tangent space.

Now by definition of $J, 3p + q = X \cdot l_{p,q}$ and $3p' + q = X \cdot l_{p',q}$. Furthermore $3p + q$ and $3p' + q$ are rationally equivalent, since the Grassmannian of lines of \mathbb{P}^3 is rational. Consequently, $3p$ and $3p'$ are rationally equivalent on X . Since the Grassmannian of lines in \mathbb{P}^2 is also rational, by looking at the curve C and the lines $l_{p,q}, l_{p',q}$ in the hyperplane $H_{p,p'}^q \simeq \mathbb{P}^2$ we get $[3p - 3p'] = 0 \in JC$ and so $3p$ and $3p'$ are also rationally equivalent on the smooth quartic C , as we wanted.

This defines a map

$$U \setminus \Delta \rightarrow Z'_3(H),$$

from an open dense subset $U \subseteq J^0 \times_{\pi_{2,J^0}(J^0)} J^0$. The map sends a point $((p, q), (p', q)) \in U$ to the point $(p, p', C) \in Z'_3(H)$, where C defined as above. Namely, as the intersection $X \cap H_{p,p'}$ of X with the hyperplane $H_{p,p'}^q \subseteq \mathbb{P}^3$ generated by p, p', q .

We now prove that such a map is surjective on $Z_3^0(H)$. Let $(p, p', C) \in Z_3^0(H)$, then the condition $3[p - p'] = 0 \in JC$ defines a g_1^3 on C . Now it is well known that all the g_1^3 of quartic curves are obtained as projections from a point of the quartic (see [1]). Namely, there exists $q \in C$ such that the g_1^3 is defined by the projection $\pi_q : C \setminus \{q\} \rightarrow \mathbb{P}^1$ from q , which extends over q . By definition of π_q, p and p' are points of total ramification of the degree 3 map $\pi_q : C \setminus \{q\} \rightarrow \mathbb{P}^1$. Since the projection from a point q is defined by intersecting with lines through q , we obtain $X \cdot l_{p,q} = 3p + q$ and $X \cdot l_{p',q} = 3p' + q$. In particular, $((p, q), (p', q))$ defines a point of $J^0 \times_{\pi_{2,J^0}(J^0)} J^0$ as wanted.

We now prove that the rational map is finite. Fix a general point $(p, p', C') \in Z'_3(H)$. Then the fibers $\pi_{1,J}^{-1}(p), \pi_{1,J}^{-1}(p')$ of p, p' via the first projection of J are finite. Thus we have only finitely many q in the fiber over (p, p', C') .

Recall that J has dimension 2 and $\pi_{2,J}$ is generically finite. Therefore the fiber product $J^0 \times_{\pi_{2,J^0}(J^0)} J^0$ as well as $U \setminus \Delta$ are 2-dimensional. Thus we conclude that also $Z_3^0(H)$ is not empty of dimension 2, as it is dominated by a variety of dimension 2 via a generically finite map.

A similar argument shows that $Z_3(H)$ is also not empty of dimension 2. It is enough to replace the map $U \setminus \Delta \rightarrow Z'_3(H)$ with the analogous map $U \setminus \Delta \rightarrow Z_3(H)$, sending $((p, q), (p', q')) \in U \setminus \Delta$ to $(p, p') \in Z_3(H)$. □

5.3 The correspondence J and constant cycle curves

In this section, we study the correspondence J in relation to constant cycle curves. First of all, notice that J has generically finite projections by Theorem 5.1. Thus, by Lemma 2.1 there is a well-defined homomorphism at the level of cycles

$$J_* : Z_1(X) \rightarrow Z_1(X). \tag{18}$$

We can ask whether this homomorphism preserves the subgroup $\text{ccc}(X)$ defined by constant cycle curves (Definition 2.3).

Proposition 5.9 *J acts on the group $\text{ccc}(X)$ generated by constant cycle curves (Definition 2.4). In other words, J_* as in (18) induces a homomorphism*

$$J_* : \text{ccc}(X) \rightarrow \text{ccc}(X).$$

Proof We proceed as in Lemma 3.6. More precisely, we prove that for any $C \in \text{ccc}(X)$ irreducible constant cycle curve, $C' = J_*C$ is a linear combination of constant cycle curves if not 0. From this it follows by linearity that J acts on $\text{ccc}(X)$.

Assume C' is not 0. By definition, C' is of the form $C' = \sum_i \alpha_i C'_i$, where any C'_i is a irreducible reduced curve and α_i is the multiplicity along C'_i . We must show that every C_i is a constant cycle curve.

For this, we need to show that two distinct points $q, q' \in C'_i$ are rationally equivalent on X (Definition 2.2). By Lemma 2.1, $C' = J_*C \in Z_1(p_{2,J}(|J| \cap p_{1,J}^{-1}(C)))$ and so $q, q' \in p_{2,J}(|J| \cap p_{1,J}^{-1}(C))$. Thus there exist $p, p' \in C$ (not necessarily distinct) such that $(p, q), (p', q') \in |J| \cap p_{1,J}^{-1}(C)$. Since $p, p' \in C$ and C is a constant cycle curve, p and p' are rationally equivalent. By Lemma 5.4 we conclude that q and q' are rationally equivalent. \square

As an application, we can construct constant cycle curves by applying J_* to some known constant cycle curve, for instance a rational curve, and try to understand when the result is a new constant cycle curve. The first important step in this direction is to detect when the curve J_*C does not coincide with the curve C .

Proposition 5.10 *Let $C \subseteq X$ be a curve such that $J_*C = C$. Then C is either a planar curve of degree 4 or has degree greater than 4.*

Proof Let $x \in \mathbb{P}^3$ be a general point. By assumption $C = J_*C$, so there exist two points $p, q \in C$ such that $(p, q) \in J$ and p, q both differ from x . Denote with H the plane spanned by x, p, q . We have $C \cdot H = 3p + q + R$, with R a divisor of C .

If C is a planar curve in a quartic surface X , then R must be empty and C has degree 4. Otherwise C is not a planar. Consequently, for a general $x \in C$, there exists a non-empty R such that $C \cdot H = 3p + q + x + R$. In particular, the degree of C is greater than 4. \square

In particular, since conics are rational curves, Propositions 5.9 and 5.10 give rise to new constant cycle curves in the following.

Corollary 5.11 *Assume that X contains a smooth conic C , then J_*C is a constant cycle curve different from C .*

In particular, the corollary suggests how to construct non-rational constant cycle curves starting from conics in special position with respect to the branch divisor.

Remark 5.12 It is not so difficult to see that the branch curve of the projection $\pi_{1,J}$ coincides with the parabolic curve defined in [23]. It is conjectured in [23, page 4] that this curve should be a constant cycle curve. It would be interesting to determine the validity of the conjecture by using J and the properties stated above.

6 Correspondences on K3 surfaces of degree 6

In this section we introduce a correspondence T on $X \times X \times X$, i.e. 2-cycle of $X \times X \times X$ of dimension 2, for a smooth complete intersection $X \subseteq \mathbb{P}^4$ of a quadric and a cubic. The correspondence is defined by planes whose intersection with X contains a point of multiplicity at least 4. This would be an attempt to generalize what we have done with lines in the correspondence J of $X \times X$. We will actually see that we cannot immediately conclude that T acts on the group $\text{ccc}(X)$ generated by constant cycle curves from X . For instance, it

is defined on $X \times X \times X$. Nonetheless, we can prove a result similar to Lemma 5.4, from which we can deduce non-emptiness of the locus $Z_4(H)$, associated to the linear system of a hyperplane section (as in Definitions 3.1, 3.4).

6.1 Construction of the correspondence T

Let $X \subseteq \mathbb{P}^4$ be a smooth complete intersection of a quadric and a cubic defined over \mathbb{C} . For any triple of distinct and non-collinear points p, q, r , let $H_{p,q,r}$ be the plane in \mathbb{P}^4 generated by them. Denote by $\Delta_T \subseteq X \times X \times X$ the subset of triples (p, q, r) that do not define a line.

Theorem 6.1 *Let $X \subseteq \mathbb{P}^4$ be a general smooth complete intersection of a quadric and a cubic over \mathbb{C} . Then the closure $T \subseteq X \times X \times X$ of*

$$T^0 = \{(p, q, r) \in X \times X \times X \setminus \Delta_T \mid X \cdot H_{q,r}^p = 4p + q + r \in Z_0(X)\} \tag{19}$$

is a 2 dimensional cycle of $X \times X \times X$ yielding a correspondence from X to $X \times X$ with generically finite projections. Furthermore, the first projection $\pi_{1,T} : T \rightarrow X$ has degree 2.

Proof Notice that if $\pi_{1,T}$ is generically finite and dominant of degree 2, then T must have dimension 2.

We prove that $\pi_{1,T}$ is generically finite and dominant of degree 2. For this, it is enough to show that for a general $p \in X$, there exist exactly two points q, r of X such that $X \cdot H_{q,r}^p = 4p + q + r$. Then the two points (q, r) and (r, q) define the fiber $\pi_{1,T}^{-1}(p)$.

Notice that the only plane that can satisfy the property defining T is the tangent space $T_p X$. This can be checked with a local computation.

Recall that X is a complete intersection of a quadric A and a cubic B . For a general point p of X , let $A_p = T_{p,X} \cap A$ and $B_p = T_{p,X} \cap B$ be respectively the quadric and the cubic plane curves cut out by the tangent space $T_{p,X}$ on A and B . By definition they are both singular at p and for a general choice of p both the quadric and the cubic are nodal. In other words, p has multiplicity 2 on both the quadric and the surface. All together there are exactly two further points q, r in the intersection between the cubic and the quadric. They will be distinct for a general choice of p . This proves that $T_p X \cdot X = 4p + q + r$ and so the fiber of $\pi_{1,T}$ at p is given by the two points (q, r) and (r, q) .

Let us prove that the projection $\pi_{23,T}$ is finite onto its image. Let q, r be general points of X and let l_{qr} be the line through those points. Consider the projection $\pi_{qr} : X \setminus \{l_{qr}\} \rightarrow \mathbb{P}^2$ from the line l_{qr} . Since $l_{qr} \cap X = \{q, r\}$ for $q, r \in X$ general, we deduce that the map π_{qr} is a degree 4 rational map. The ramification divisor is a curve and for a general projection the general ramification point is simple. Thus the points of total ramification define a finite (possibly empty) subset of points of the branch divisor. These are exactly the points of the fiber of $(q, r) \in X \times X$, which is thus finite and non-empty since T is not empty. This proves that $\pi_{23,T}$ is generically finite. □

The natural generalization of Lemma 5.4 given for J and of Lemma 3.6 given for $Z_n(L)$ is the following.

Lemma 6.2 *Let $(p, q, r), (p', q', r') \in T^0$. Then p is rationally equivalent to p' on X if and only if $q + r$ is rationally equivalent to $q' + r'$ on $X \times X$.*

Proof Let $(p, q, r), (p', q', r') \in T^0$. Notice that by definition of T^0 , $4p + q + r = X \cdot T_p X$ and $4p' + q + r = X \cdot T_{p'} X$ and $4p + q + r$ and $4p' + q + r$ are rationally equivalent on X .

In fact, $T_p X$ and $T_{p'} X$ intersects in the line through r and q and define a pencil which is a \mathbb{P}^3 intersecting X on a curve C . The pencil restricted to C contains both divisors $4p + q + r$ and $4p' + q + r$.

Now let us assume that p is rationally equivalent to p' . Then the same holds for $4p$ and $4p'$. Moreover, from above $4p + q + r$ is rationally equivalent to $4p' + q' + r'$ and by difference, we obtain the same for $q + r$ and $q' + r'$.

Conversely, let $q + r$ be rationally equivalent to $q' + r'$. Using that $4p + q + r$ is rationally equivalent to $4p' + q' + r'$ we obtain the same for $4p$ and $4p'$ by difference. This means that $p - p'$ is a 4 torsion cycle of X . Since $CH_0(X)$ is torsion free (see for instance [19, theorem 14.14]) then p and p' must be rationally equivalent. □

6.2 Non-emptiness of $Z_4(H)$ defined by a hyperplane section

The correspondence T is strongly related to the existence of torsion points of order 4 on the general member of the linear system given by a hyperplane section on X .

Let $H \subseteq X$ be a hyperplane section of X and let $C \rightarrow |H|$ be the universal curve of the linear system $|H|$. As defined in Sect. 3,

$$Z'_4(H) = \overline{\{(p, q, C') \in C \times_{|H|} C \mid p \neq q, 4[p - q] = 0 \in JC'\}} \subseteq C \times_{|H|} C, \tag{20}$$

$$Z_4(H) = \overline{\{(p, q) \in X \times X \mid \exists C' \in |H| \text{ smooth s.t. } p, q \in C', p \neq q, 4[p - q] = 0 \in JC'\}} \subseteq X \times X \tag{21}$$

and they are related by the forgetful map $Z'_4(H) \rightarrow Z_4(H)$.

We show that $Z'_4(H)$ and $Z_4(H)$ are not empty. We use T , or more precisely, the fiber product $T \times_{\pi_{23}(T)} T$. Notice that a point in this fiber product is given by two points $(p, q, r), (p', q', r') \in T$ such that $q + r = q' + r'$ so we will write a point of $T \times_{\pi_{23}(T)} T$ just as a tuple $((p, q, r), (p', q, r))$.

Theorem 6.3 *There is a finite rational map $T \times_{\pi_{23}(T)} T \dashrightarrow Z'_4(H)$. In particular, $Z'_4(H)$ and $Z_4(H)$ are non-empty of dimension 2.*

Proof Recall that $T^0 \subset T$ denotes the interior of T (see Theorem 5.1). Let $\Delta_T \subseteq T \times_{\pi_{23}(T)} T$ denote the locus of points $((p', q, r), (p, q, r))$ such that $p = p'$. Let $((p, q, r), (p', q, r))$ be a point of $T^0 \times_{\pi_{23}(T^0)} T^0 \setminus \Delta_T$. Then by definition of T^0 , $4p + q + r = X \cdot T_p X$ and $4p' + q + r = X \cdot T_{p'} X$. Consider the hyperplane $H_{p,p'}^{q,r} \subseteq \mathbb{P}^4$ generated by $T_p X$ and $T_{p'} X$. Notice that $T_p X \cap T_{p'} X$ is the line generated by q, r . Consider the curve $C = H_{p,p'}^{q,r} \cap X$. For X general, C is smooth. Since such curve contains $4p + 4p' + q + r$, we conclude that $4p + q + r$ and $4p' + q + r$ are rationally equivalent on C .

By difference, $4p$ and $4p'$ are rationally equivalent on C , which means $[4p - 4q] = 0 \in JC$.

This defines a map $U \rightarrow Z'_4(H)$, from an open dense subset $U \subseteq T^0 \times_{\pi_{23}, T^0}(T^0) \setminus T^0$. The map sends a point $((p, q, r), (p', q, r)) \in U$ to the point $(p, p', C) \in Z'_4(H)$, where C defined as above.

We now prove that, up to shrinking U , the rational map is finite. For this, its enough to notice that a point $(p, p', C) \in Z'_4(H)$ in the image is of the form $(\pi_1(p', q, r), \pi_1(p', q, r), C = (T_p X, T_{p'} X) \cap X)$ and by Theorem 6.1 the fiber $\pi_{23}((r, s))$ is generically finite.

We now prove that the rational map is surjective. Let $(p, p', C) \in Z_4^0(H)$, then the condition $4[p - p'] = 0 \in JC$ define a g_1^4 . All g_1^4 in a complete intersection in \mathbb{P}^4 are given by projections from lines defined by the two residual points q, r . More precisely, for $p, p' \in C$ satisfying $4[p - p'] = 0 \in JC$, let H_p, H'_p be the hyperplanes through p, p' (respectively) and containing the line through q, r . Then $H_p \cdot C = 4p + q + r$ and $H'_p \cdot C = 4p' + q + r$. Thus, we find a point $((p, q, r)(p', q, r)) \in T^0 \times_{\pi_{23, T^0}(T^0)} T^0$. We conclude from above that $Z_4^0(H)$, and so also $Z_4(H)$, are non-empty of dimension 2. \square

Acknowledgements I am grateful to Stefan Schreieder, who introduced me to the topic and stimulated the preparation of the paper with many insightful conversations. I would also like to thank Gian Pietro Pirola, Claire Voisin and Daniel Huybrechts, for their comments on the paper and conversations we shared on the topic, and the anonymous referee for their very helpful comments.

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Data availability No data was collected for the purposes of the paper.

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