J. Math. Pures Appl. 209 (2026) 103866

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

journal homepage: www.elsevier.com/locate/matpur

Weak stability by noise for approximations of doubly nonlinear evolution equations

Carlo Orrieri[a], Luca Scarpa[b,*], Ulisse Stefanelli[c,d]

[a] *Department of Mathematics, Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy*
[b] *Department of Mathematics, Politecnico di Milano, Via E. Bonardi 9, 20133 Milano, Italy*
[c] *Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*
[d] *Vienna Research Platform on Accelerating Photoreaction Discovery, & University of Vienna, Währinger Str. 17, 1090 Vienna, Austria*

A R T I C L E   I N F O

Article history:
Received 23 July 2025
Available online 5 February 2026

MSC:
35K55
35R60
60H15

Keywords:
Doubly nonlinear stochastic equations
Friedrichs solutions
Regularisation by noise
Kolmogorov equation

A B S T R A C T

Doubly nonlinear stochastic evolution equations are considered. Upon assuming the additive noise to be rough enough, we prove the existence of probabilistically weak solutions of Friedrichs type and study their uniqueness in law. This entails stability for approximations of stochastic doubly nonlinear equations in a weak probabilistic sense. Such effect is a genuinely stochastic, as doubly nonlinear equations are not even expected to exhibit uniqueness in the deterministic case.

R É S U M É

On considère des équations d'évolution stochastiques doublement non linéaires. En supposant que le bruit additif soit suffisamment rugueux, nous prouvons l'existence de solutions probabilistes faibles de type Friedrichs et étudions leur unicité en loi. Cela implique la stabilité pour les approximations des équations stochastiques doublement non linéaires dans un sens probabiliste faible. Un tel effet est véritablement stochastique, car les équations doublement non linéaires ne présentent même pas d'unicité dans le cas déterministe.

© 2026 The Author(s). Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

* Corresponding author.
E-mail addresses: carlo.orrieri@unipv.it (C. Orrieri), luca.scarpa@polimi.it (L. Scarpa), ulisse.stefanelli@univie.ac.at (U. Stefanelli).
URLs: http://www-dimat.unipv.it/orrieri (C. Orrieri), https://sites.google.com/view/lucascarpa (L. Scarpa), http://www.mat.univie.ac.at/~stefanelli (U. Stefanelli).

https://doi.org/10.1016/j.matpur.2026.103866

1. Introduction

We are interested in abstract doubly nonlinear evolution equations in the form

$$A(\partial_t u) + B(u) \ni F(u)\,, \quad u(0) = u_0\,, \tag{1.1}$$

where the nonlinearities A and B are maximal monotone operators on a Hilbert space H, $F : H \to H$ is a Lipschitz-continuous forcing, and u_0 is a given initial datum. Solutions $u : [0,T] \to H$ to (1.1) may describe the dynamics of a given system, up to the final reference time $T > 0$. The term B represents the (sub-)differential of the energy driving the system, whereas A corresponds to the dissipation. In typical examples, B is a nonlinear differential operator while A is a nonlinear map, associated to a possibly multivalued graph. Partial differential equations in doubly nonlinear form ubiquitously arise in applications to classical thermodynamics, e.g. the well known two-phase Stefan problem, to mechanics, e.g. the quasistatic limit of a damped oscillator, to electronics, e.g. in RC circuits, and viscoelastoplasticity: we refer to [19, Sec. 6] for a brief survey of such examples.

The mathematical literature on deterministic doubly nonlinear evolutions is extensive. The special case $A = \mathrm{id}$ corresponds to perturbed gradient flows, and is rather classical. The genuine doubly nonlinear case has been originally tackled in [6,7,12], where existence of solutions was investigated. Solutions to (1.1) are however not expected to be unique, see [2]. Some sufficient conditions for uniqueness can be found in [7,12].

As a consequence of nonuniqueness, the convergence of approximations of deterministic doubly nonlinear equations may be restricted to subsequences only. More precisely, consider a family of approximated problems

$$A(\partial_t u_\lambda) + B_\lambda(u_\lambda) = F(u_\lambda)\,, \quad u_\lambda(0) = u_0^\lambda\,, \tag{1.2}$$

where $\lambda > 0$ is a positive parameter, $(B_\lambda)_\lambda$ is a family of single-valued approximations of B, and $(u_0^\lambda)_\lambda$ is a family of regularisations of the initial datum u_0. As the limiting problem (1.2) for $\lambda \to 0$ may have multiple solutions, convergence of the full sequence $(u_\lambda)_\lambda$ cannot be directly guaranteed, as different subsequences may have different limits.

The purpose of this paper is to show that the addition of a suitable noise may restore stability for such approximations, in a weak probabilistic formulation. The stochastic counterpart of doubly nonlinear equations (1.1) was firstly introduced in [19] and reads

$$\begin{cases} \mathrm{d}u = (\partial_t u^d)\,\mathrm{d}t + G\,\mathrm{d}W\,, \\ A(\partial_t u^d) + B(u) \ni F(u)\,, \\ u(0) = u_0\,, \end{cases} \tag{1.3}$$

where W is a H-cylindrical Wiener process and G is a Hilbert–Schmidt covariance operator. Let us briefly comment on the formulation (1.3). First, notice that when the noise vanishes, i.e. $G = 0$, then the stochastic equation (1.3) comes down to the deterministic one (1.1). In (1.3) we are assuming solutions to be of the form

$$u(t) = u^d(t) + \int_0^t G\,\mathrm{d}W(s)\,, \quad t \in [0,T]\,,$$

where u^d is an absolutely continuous process. Since the noise gives no contribution on average, the variation rate of $t \mapsto \mathbb{E}[u(t)]$ is given exactly by $\mathbb{E}[\partial_t u^d]$, where $\mathbb{E}$ denotes expectation. From the modelling point of view, the process u^d accounts for averaged dissipation along the dynamics, while the stochastic integral renders random perturbations of the trajectories. The second equation in (1.3) has then to be interpreted as a stochastic generalisation of the usual energy-dissipation balance, where the energy acts on the whole process u and the (averaged) dissipation acts on u^d only. In [19], existence of solutions to (1.3) has been obtained in the probabilistically weak and analytically strong sense. Stochastic doubly nonlinear equations were also investigated in [4,16] in the context of the two-phase stochastic Stefan problem, in [18] in a more abstract framework, and in [20] in the rate-independent case.

The stochastic version of the approximated problem (1.2) reads

$$\begin{cases} \mathrm{d}u_\lambda = (\partial_t u_\lambda^d)\,\mathrm{d}t + G_\lambda\,\mathrm{d}W\,, \\ A(\partial_t u_\lambda^d) + B_\lambda(u_\lambda) = F(u_\lambda)\,, \\ u_\lambda(0) = u_0^\lambda\,, \end{cases} \tag{1.4}$$

where $(G_\lambda)_\lambda$ are possibly regularised operators approximating G. Under suitable assumptions on the approximating families $(B_\lambda)_\lambda$, $(u_0^\lambda)_\lambda$, and $(G_\lambda)_\lambda$, we prove that the addition of noise stabilises the dynamics of the approximated problem (1.4). More precisely, we are able to show that, given the approximating sequence, there exists a unique probability measure μ on the space of trajectories of (1.2) such that the probability distributions of any solution to (1.4) weakly converge to μ as $\lambda \to 0$.

Let us briefly highlight the challenges of the analysis, as well as the technical strategy for overcome them. Existence of solutions for the stochastic system (1.3) was proved in [19] in the analytically-strong sense under the assumption that the noise coefficient G is coloured enough, precisely that G is Hilbert–Schmidt from H to V, where V is the natural space associated to the weak formulation of $B : V \to V^*$. Let us recall that for doubly nonlinear problems, the analytically-strong formulation is usually the only one available: indeed, since generally the operator A is of order zero on H, if one gives sense to the nonlinearity $A(\partial_t u^d)$, then by comparison in (1.3) it holds that $B(u) \in H$ at least almost everywhere, so that the variable u is required to take values in the domain $D(B)$ of B. In particular, weaker notions of solutions (e.g. variational ones) are not directly available and existence of (analytically strong) solutions can be expected only under the assumption that G is Hilbert–Schmidt with value in V.

The main technical idea is to rely on techniques from the research stream on regularisation by noise, namely, to exploit the Kolmogorov equation associated to (1.3). Without any claim of completeness, we refer to [1,5,8–10,14,15] and the references therein for the recent literature on uniqueness by noise for infinite dimensional stochastic evolution equations.

Due to the unusual form of the doubly nonlinear equation, this calls for reformulating the stochastic system (1.3) in a more classical way, namely

$$\mathrm{d}u + B(u)\,\mathrm{d}t = [F(u) + K(u)]\,\mathrm{d}t + G\,\mathrm{d}W\,, \qquad u(0) = u_0\,,$$

where the operator $K : D(B) \to H$ is defined as

$$K(z) := A^{-1}(F(z) - B(z)) - (F(z) - B(z))\,, \quad z \in D(B)\,.$$

The Kolmogorov equation associated to (1.3) formally reads

$$\begin{aligned} \alpha\varphi(x) - \frac{1}{2}\,\mathrm{Tr}\left[GG^* D^2\varphi(x)\right] + (B(x), D\varphi(x))_H \\ = g(x) + (F(x) + K(x), D\varphi(x))_H\,, \quad x \in D(B)\,, \end{aligned}$$

where $\alpha > 0$ is fixed and $g : H \to \mathbb{R}$ is a given forcing term. We point out that the Kolmogorov equation is extremely pathological due to presence of both the perturbation term K, which is of the same order of the leading operator B, and the coloured coefficient $G \in \mathscr{L}^2(H,V)$. This feature makes the first-order perturbation term too singular and the second-order diffusion too degenerate, respectively. Consequently, we do not expect to solve the Kolmogorov equation, not even in some suitably mild fashion.

In order to overcome this obstruction, we introduce a weaker notion of solution to the limiting stochastic equation (1.3), allowing for rougher noise coefficients G being only Hilbert–Schmidt with values in H. A natural candidate is the concept of Friedrichs-weak solution for (1.3), which consists in defining solutions to (1.3) as limit of solutions to the approximated problem (1.4). In this way, the limiting noise coefficient G is allowed to be in $\mathscr{L}^2(H,H) \setminus \mathscr{L}^2(H,V)$, whereas the approximations $(G_\lambda)_\lambda$ may more regular (e.g., even in $\mathscr{L}^2(H,V)$). The drawback of such weak formulation is that for the limit problem (1.3) one can identify the nonlinearity $B(u)$ weakly in V^*, while the identification of the nonlinearity $A(\partial_t u^d)$ (or equivalently of $K(u)$) is relaxed. For technical details, we refer to Definition 2.3.

The introduction of such relaxed notion of solution allows also to treat the operator K in the Kolmogorov equation. The main idea is to consider a hybrid Kolmogorov equation of the form

$$\begin{aligned}\alpha\varphi_\lambda(x) - \frac{1}{2}\operatorname{Tr}\left[GG^*D^2\varphi_\lambda(x)\right] + (B_\lambda(x), D\varphi_\lambda(x))_H \\ = g(x) + (F(x) + K_\lambda(x), D\varphi_\lambda(x))_H\,, \quad x \in H\,,\end{aligned} \tag{1.5}$$

where $K_\lambda : H \to H$ is defined as

$$K_\lambda(z) := A^{-1}(F(z) - B_\lambda(z)) - (F(z) - B_\lambda(z))\,, \quad z \in H\,.$$

The term *hybrid* refers to the fact that this is not the natural Kolmogorov equation associated to (1.4), due to the presence of the limit coefficient G. Since G can be taken in $\mathscr{L}^2(H,H)$ and K_λ is no longer singular, existence and uniqueness of strong solutions to (1.5) can be obtained via fixed point arguments by exploiting strong Feller properties of the associated transition semigroup. Furthermore, by exploiting structural assumptions on the data A and g, we show that the family $(\varphi_\lambda)_\lambda$ is uniformly bounded in $C^1_b(H)$. By exploiting suitable infinite dimensional compactness arguments, this allows to pass to the limit in the Itô formula for $\varphi_\lambda(u_\lambda)$, hence to show uniqueness of the law of the possible limiting points of the sequence $(u_\lambda)_\lambda$. Eventually, such uniqueness-in-law result for Friedrichs-weak solutions to (1.3) can be reformulated as a weak stability result for approximations of the stochastic equation (1.3) (see Theorem 2.8 below).

We briefly summarise here the contents of the paper. Section 2 presents the mathematical setting and the main results. In Section 3, we present an example of deterministic doubly nonlinear PDE with multiple solutions, whose stochastic counterpart falls within our setting. In Section 4 we prove existence of Friedrichs-weak solutions for the equation (1.3). Section 5 contains the analysis of the Kolmogorov equation, while in Section 6, we prove the main results on uniqueness and stability by noise.

2. Main results

2.1. Notation and setting

For a given Banach space E, the symbols $\mathcal{B}(E)$ and $\mathscr{P}(E)$ denote the Borel σ-algebra of E and the space of probability measures on $\mathcal{B}(E)$, respectively. We indicate the space of bounded measurable real functions on E by $\mathscr{B}_b(E)$. Moreover, for $k \in \mathbb{N}$ and $s \in (0,1]$, the symbol $C^{k,s}_b(E)$ denotes the space of real-valued bounded Borel-measurable functions on E which are k-times Fréchet-differentiable with s-Hölder-continuous derivatives. In particular, we indicate by $C^{0,1}_b(E;E)$ the space of Lipschitz continuous functions from E to itself endowed with the norm

$$\|F\|_{C^{0,1}_b(E;E)} := \sup_{v\in E}\|F(v)\|_E + \sup_{u\neq v\in E}\frac{\|F(u)-F(v)\|_E}{\|u-v\|_E}.$$

Given two Hilbert spaces K_1 and K_2, we use the symbols $\mathscr{L}(K_1, K_2)$, $\mathscr{L}^1(K_1, K_2)$, and $\mathscr{L}^2(K_1, K_2)$ to indicate the spaces of bounded continuous linear operators, trace-class operators, and Hilbert–Schmidt operators from K_1 to K_2, respectively. We shall use the subscript '+' to indicate the respective subspaces of nonnegative operators.

For every probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a saturated and right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ (i.e. $(\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions), the progressive sigma algebra on $\Omega \times (0, +\infty)$ is denoted by $\mathcal{P}$. Given a final reference time $T > 0$, we use the classical symbols $L^s(\Omega; E)$ and $L^r(0, T; E)$ for the spaces of strongly measurable Bochner-integrable functions on Ω and $(0, T)$, respectively, for all $s, r \in [1, +\infty]$ and for every Banach space E. If $s, r \in [1, +\infty)$ we use $L^s_{\mathcal{P}}(\Omega; L^r(0, T; E))$ to indicate that measurability is intended with respect to $\mathcal{P}$. In the case that $s \in (1, +\infty)$, $r = +\infty$, and E is separable, we set

$$L^s_w(\Omega; L^\infty(0, T; E^*)) := \left\{v : \Omega \to L^\infty(0, T; E^*) \text{ weakly* measurable: } \mathbb{E}\,\|v\|^s_{L^\infty(0,T;E^*)} < \infty\right\},$$

and recall that by [13, Thm. 8.20.3] we have the identification

$$L^s_w(\Omega; L^\infty(0, T; E^*)) = \left(L^{\frac{s}{s-1}}(\Omega; L^1(0, T; E))\right)^* .$$

Throughout the paper, H is a fixed real separable Hilbert space, with scalar product and norm denoted by $(\cdot, \cdot)_H$ and $\|\cdot\|_H$, respectively. The following assumptions will be assumed throughout the paper.

A1: $A : H \to 2^H$ is a maximal monotone operator, and there exist constants $c_A, C_A > 0$ such that, for all $u \in H$ and $v \in A(u)$,

$$\|v\|_H \leq C_A(1 + \|u\|_H), \qquad (v, u)_H \geq c_A\,\|u\|^2_H - C_A .$$

Moreover, $A^{-1} \in C^{0,s_A}(H; H)$ for some $s_A \in (0, 1)$, and there is a constant $k_A > 0$ such that the operator $A^{-1} - k_A I$ (I is the identity operator) has a bounded range, i.e.,

$$\sup_{v\in H} \left\|A^{-1}(v) - k_A v\right\|_H =: C'_A < +\infty ,$$

for some $C'_A > 0$.

A2: L is a linear, symmetric, maximal monotone, unbounded operator on H with effective domain $D(L)$, such that $D(L) \hookrightarrow H$ compactly. For every $\sigma \in (0, 1)$ we classically define L^σ by spectral theory, we set

$$V_{2\sigma} := D(L^\sigma) , \quad V := V_1 = D(L^{1/2}) ,$$

and we assume, with no loss of generality, that $(Lu, u)_H = \|u\|^2_V$ for all $u \in V_2 = D(L)$. Moreover, $f : H \to H$ is maximal monotone and bounded, i.e., there exists a constant $C_f > 0$ such that, for all $u \in H$,

$$\|f(u)\|_H \leq C_f .$$

The nonlinear operator B (of semilinear type) is then defined as

$$B := L + f ,$$

which is maximal monotone on H with effective domain $D(B) = D(L) = V_2$.

A3: $F : H \to H$ is bounded Lipschitz-continuous, and we set $C_F := \|F\|_{C^{0,1}_b(H;H)}$.

A4: $G \in \mathscr{L}^2(H, H)$ commutes with L and $\ker G = \{0\}$.

A5: $u_0 \in H$.

We are interested in the doubly nonlinear problem

$$du = (\partial_t u^d)\,dt + G\,dW\,, \tag{2.1}$$

$$A(\partial_t u^d) + B(u) \ni F(u)\,, \tag{2.2}$$

$$u(0) = u_0\,. \tag{2.3}$$

We consider two notions of solution for (2.1)–(2.3), both in the probabilistically weak sense: analytically strong solution and analytically weak solution in the sense of Friedrichs.

Definition 2.1 *(Analytically strong solution).* If $u_0 \in V$ and $G \in \mathscr{L}^2(H,V)$, an *analytically strong solution* to (2.1)–(2.3) is a family $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u, u^d, v)$ where, for all $T > 0$,

- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- W is a cylindrical Wiener process on H,
- $u \in L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)) \cap L^2_w(\Omega; L^\infty(0,T;V)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0,T;V_2))$,
- $u^d \in L^2_{\mathcal{P}}(\Omega; H^1(0,T;H))$,
- $v \in L^2_{\mathcal{P}}(\Omega; L^2(0,T;H))$

and it holds that

$$u(t) = u_0 + \int_0^t \partial_t u^d(s)\,ds + \int_0^t G\,dW(s) \qquad \text{in } H\,, \quad \forall\, t \geq 0\,, \quad \mathbb{P}\text{-a.s.}\,, \tag{2.4}$$

$$v + B(u) = F(u) \qquad \text{in } H\,, \quad \text{a.e. in } \Omega \times (0,+\infty)\,, \tag{2.5}$$

$$v \in A(\partial_t u^d) \qquad \text{a.e. in } \Omega \times (0,+\infty)\,. \tag{2.6}$$

Existence of analytically strong solutions for doubly nonlinear equations (2.1)–(2.3) in the sense of Definition 2.1 has been proved in [19]. In particular, note that existence of analytically strong solutions cannot be expected solely under **A4**, as the condition $G \in \mathscr{L}^2(H,V)$ is needed. This motivates the introduction of a weaker notion of solution for (2.1)–(2.3) allowing for more general noise coefficients G satisfying only assumption **A4**.

Before moving to the definition of the weak solution in the sense of Friedrichs, we need an equivalent formulation of the concept of strong solution given by Definition 2.1. First of all, observe that equation (2.5) and inclusion (2.6) satisfied by analytically strong solutions can be written equivalently as

$$\partial_t u^d = A^{-1}(F(u) - B(u)) \quad \text{in } H\,, \quad \text{a.e. in } \Omega \times (0,+\infty)\,.$$

Consequently, when intended in the strong analytical sense of Definition 2.1, problem (2.1)–(2.3) can also be formulated as

$$du = A^{-1}(F(u) - B(u))\,dt + G\,dW\,, \qquad u(0) = u_0\,.$$

Now, taking assumption **A1** into account, let us introduce the operator

$$K : D(L) \to H\,, \qquad K(x) := A^{-1}(F(x) - B(x)) - k_A(F(x) - B(x))\,, \quad x \in V_2\,, \tag{2.7}$$

which has bounded range, namely,

$$\|K(x)\|_H \le C'_A \qquad \forall\, x \in V_2\,. \tag{2.8}$$

Taking these remarks into account, an equivalent formulation of problem (2.1)–(2.3) in the strong analytical sense is

$$\mathrm{d}u + k_A B(u)\,\mathrm{d}t = [k_A F(u) + K(u)]\,\mathrm{d}t + G\,\mathrm{d}W\,, \qquad u(0) = u_0\,. \tag{2.9}$$

Definition 2.2 *(Analytically strong solution: equivalent formulation).* If $u_0 \in V$ and $G \in \mathscr{L}^2(H,V)$, an analytically strong solution to (2.9) is a family $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u)$ where, for all $T > 0$,

- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- W is a cylindrical Wiener process on H,
- $u \in L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)) \cap L^2_w(\Omega; L^\infty(0,T;V)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0,T;V_2))$,

and it holds that, for every $t \ge 0$, $\mathbb{P}$-almost surely,

$$u(t) + k_A \int_0^t B(u(s))\,\mathrm{d}s = u_0 + \int_0^t [k_A F(u(s)) + K(u(s))]\,\mathrm{d}s + \int_0^t G\,\mathrm{d}W(s) \quad \text{in } H\,. \tag{2.10}$$

The equivalence between Definitions 2.1 and 2.2 is immediate by means of position (2.7) above.

The doubly nonlinear evolution (2.1)–(2.3) inherits a semilinear structure from the structural form of A and B. On the other hand, (2.9) cannot be viewed as a classical semilinear stochastic equation. Indeed, the nonlinear perturbation K has the same differential order of the linear part L (i.e., K is defined on $D(L)$), while in classical semilinear problems the linear component needs to be dominant with respect to the others.

Bearing these considerations in mind, we introduce a weaker notion of solution to (2.9). To this end, for every $\lambda > 0$ let $B_\lambda : H \to H$ denote the Yosida approximation of B: it is well known that B_λ is λ^{-1}-Lipschitz continuous (see for example [3] for classical results on monotone analysis). We define the operator

$$K_\lambda : H \to H\,, \qquad K_\lambda(x) := A^{-1}(F(x) - B_\lambda(x)) - k_A(F(x) - B_\lambda(x))\,, \quad x \in H\,. \tag{2.11}$$

Note that the Lipschitz-continuity of B_λ and **A1** ensure that $K_\lambda \in C^{0,s_A}_b(H;H)$ with

$$\|K_\lambda(x)\|_H \le C'_A \qquad \forall\, x \in H\,. \tag{2.12}$$

Definition 2.2 can be easily adapted to the case when K is replaced by K_λ, so that one can speak of analytically strong solutions for $\lambda > 0$, as well.

Definition 2.3 *(Friedrichs-weak solution).* A *Friedrichs-weak solution* to (2.1)–(2.3) is a family $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P}, W, u, y, \Lambda)$, where, for all $T > 0$,

- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- W is a cylindrical Wiener process on H,
- $u \in L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0,T;V))$,
- $y \in L^\infty_{\mathcal{P}}(\Omega \times (0,T); H)$,
- $\Lambda = (\lambda_n)_n \subset (0, +\infty)$,

such that there exist a sequence of data $(u_0^n, G_n)_n \subset H \times \mathscr{L}^2(H,H)$, and a sequence of analytically strong solutions $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u_n)$ to the problem

$$\mathrm{d}u_n + k_A B_{\lambda_n}(u_n)\,\mathrm{d}t = [k_A F(u_n) + K_{\lambda_n}(u_n)]\,\mathrm{d}t + G_n\,\mathrm{d}W\,, \qquad u_n(0) = u_0^n\,, \tag{2.13}$$

satisfying, as $n \to \infty$, for all $T > 0$,

$$\begin{aligned}
&u_0^n \to u_0 \quad \text{in } H\,,\\
&G_n \to G \quad \text{in } \mathscr{L}^2(H,H)\,,\\
&\lambda_n \searrow 0 \quad \text{in } \mathbb{R}\,,\\
&u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^2_w(\Omega; L^\infty(0,T;H)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0,T;V))\,,\\
&u_n \to u \quad \text{in } L^2(0,T;H)\,, \quad \mathbb{P}\text{-a.s.}\,,\\
&K_{\lambda_n}(u_n) \overset{*}{\rightharpoonup} y \quad \text{in } L^\infty_{\mathcal{P}}(\Omega \times (0,T); H)\,.
\end{aligned}$$

Remark 2.4. The concept of Friedrichs-weak solution in Definition 2.3 is related to the concept of closure of analytically strong solutions. Let us stress that in Definition 2.3 the initial datum u_0 and the operator G are allowed to satisfy assumptions **A4**–**A5** only. In particular, neither u_0 is required to be in V nor G is required to be in $\mathscr{L}^2(H,V)$. On the other hand, one has much more freedom on the approximating sequences. These can indeed be taken either more regular (e.g. in V and $\mathscr{L}^2(H,V)$), or simply set as $G_n = G$ for every n if one is interested in the asymptotic behaviour of analytically weak solutions instead. Moreover, note that even if $(u_0^n)_n$ and $(G_n)_n$ are only in H and $\mathscr{L}^2(H,H)$, respectively, the existence of analytically strong solution for $n > 0$ is guaranteed since $D(B_{\lambda_n}) = H$ (see [19]).

Remark 2.5. Let us point out that if $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u, y, \Lambda)$ is a Friedrichs-weak solution in the sense of Definition 2.3 above, it holds that

$$u(t) + k_A \int_0^t B(u(s))\,\mathrm{d}s = u_0 + \int_0^t [k_A F(u(s)) + y(s)]\,\mathrm{d}s + \int_0^t G\,\mathrm{d}W(s) \quad \text{in } V^* \tag{2.14}$$

for every $t \geq 0$, $\mathbb{P}$-almost surely. It is important to note however that one *cannot* infer that $y = K(u)$, due to the lack of regularity of the solution u and the fact that K is defined on $D(L)$. This weak notion of solution still features the identification of the nonlinearity B. In contrast, the limit of K is not identified and the condition $y = K(u)$ needs to be interpreted in a very weak sense via the limiting procedure highlighted above. Equivalently, this corresponds to relaxing the identification of the nonlinearity A.

Our first result is an existence proof for Friedrichs-weak solutions.

Theorem 2.6 *(Existence of Friedrichs-weak solutions). Assume **A1**–**A5**. Then, problem* (2.1)–(2.3) *admits at least a Friedrichs-weak solution in the sense of* Definition 2.3 *satisfying $u_0^n \in V$ and $G_n \in \mathscr{L}^2(H,V)$ for all n.*

Our main result concerns the uniqueness in distribution for Friedrichs-weak solutions. For this purpose, we need the following additional structural assumption.

A6: there exists $\delta \in (0, \frac{1}{2})$ such that $G(H) = V_{2\delta}$ and

$$s_A + \frac{2}{1+2\delta} > 2\,.$$

Note that the condition $\delta < \frac{1}{2}$ in **A6** implies that the range of G cannot be V, and one has $V \hookrightarrow G(H)$ compactly, hence $G \notin \mathscr{L}^2(H,V)$. On the other hand, the requirement $\delta > 0$ is redundant as it is trivially implied by the Hilbert–Schmidt conditions on G. All in all, it holds that $V \hookrightarrow G(H) \hookrightarrow H$ compactly. Moreover, we stress that for all $\delta \in (0, \frac{1}{2})$, assumption **A6** is satisfied with $s_A \in (\frac{4\delta}{1+2\delta}, 1)$. We have the following.

Theorem 2.7 *(Uniqueness in law). Assume **A1**–**A6**, and let*

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u_1, y_1, \Lambda_1) \qquad \text{and} \qquad (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u_2, y_2, \Lambda_2)$$

be two Friedrichs-weak solutions to problem (2.1)–(2.3) with respect to the same initial datum u_0 and defined on the same stochastic basis. If $\Lambda_1 = \Lambda_2$, then u_1 and u_2 have the same law on $C^0([0,+\infty);H)$, i.e.

$$\mathbb{P}\left(u_1 \in E\right) = \mathbb{P}\left(u_2 \in E\right) \quad \forall\, E \in \mathcal{B}(C^0([0,+\infty);H))\,.$$

As a corollary of Theorem 2.7 we are able to establish a stability-by-noise result. Indeed, let $\Lambda = (\lambda_n)_n$ be an infinitesimal sequence of positive real numbers, and let $(u_0^n, G_n)_n \subset H \times \mathscr{L}^2(H,H)$ such that $u_0^n \to u_0$ in H and $G_n \to G$ in $\mathscr{L}^2(H,H)$. Given the sequences $(B_{\lambda_n}, K_{\lambda_n})_n$ of approximated operators, problem (2.13) cannot be assumed to show the uniqueness in law property, even if the approximating operators $(K_{\lambda_n})_n$ are Hölder continuous on H. The reason for this is that G_n may be in $\mathscr{L}^2(H,V)$ (hence the Ornstein–Uhlenbeck semigroup associated to L and G_n may not be *strong Feller*). Let us set then

$$\mathcal{S}_n := \{\text{analytically strong solutions of (2.13) in the sense of Definition 2.2}\}$$

and let $\mathscr{P}_n$ be their respective distributions on $C^0([0,+\infty);H)$, namely

$$\begin{aligned}\mathscr{P}_n := \big\{\mu \in \mathscr{P}(C^0([0,+\infty);H)) :\ \exists\, (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, u) \in \mathcal{S}_n : \\ \mu(E) = \mathbb{P}(u \in E) \quad \forall\, E \in \mathcal{B}(C^0([0,+\infty);H))\big\}\,.\end{aligned}$$

This means that for every $n \in \mathbb{N}$, $\mathscr{P}_n$ represents the set of distributions on $C^0([0,+\infty);H)$ of all the analytically strong solutions of (2.13). The weak stability-by-noise result is the following.

Theorem 2.8 *(Weak stability). Assume **A1**–**A6** and let*

$$(\lambda_n, u_0^n, G_n)_n \subset (0,+\infty) \times H \times \mathscr{L}^2(H,H)$$

be such that, as $n \to \infty$, $\lambda_n \searrow 0$, $u_0^n \to u_0$ in H, and $G_n \to G$ in $\mathscr{L}^2(H,H)$. Then, there exists a unique probability $\mu \in \mathscr{P}(L^2_{loc}(0,+\infty;H))$ such that for every sequence $(\mu_n)_n \subset \mathscr{P}(C^0([0,+\infty);H))$ satisfying

$$\mu_n \in \mathscr{P}_n \quad \forall\, n \in \mathbb{N}\,,$$

as $n \to \infty$, it holds that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $L^2_{loc}(0,+\infty;H)$, i.e.,

$$\lim_{n\to\infty} \int_{L^2_{loc}(0,+\infty;H)} \varphi \,\mathrm{d}\mu_n = \int_{L^2_{loc}(0,+\infty;H)} \varphi \,\mathrm{d}\mu \qquad \forall\, \varphi \in C^0_b\left(L^2_{loc}(0,+\infty;H)\right)\,.$$

Moreover, μ is concentrated on $C^0([0,+\infty);H) \cap L^2_{loc}(0,+\infty;V)$.

3. An explicit example

3.1. Nonuniqueness for a deterministic doubly nonlinear equation

The uniqueness-in-law result of Theorem 2.7 is of a genuinely stochastic nature and has no deterministic counterpart. In fact, deterministic doubly nonlinear evolution equations may fail to have unique solutions [6,7]. In order to illustrate this fact, we follow Akagi [2] and present in this section the concrete example of the deterministic parabolic PDE

$$|\partial_t u|^{p-2}\partial_t u - \Delta u = \lambda u \quad \text{in } (0,T)\times D\,. \tag{3.1}$$

Here, $D \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) is a nonempty, open, and bounded domain with smooth boundary ∂D and $p > 2$. Note that equation (3.1) features a single nonlinearity acting on the time derivative, which does not satisfy our assumption **A1** in general: this issue is overcome at the end of the section where a modified version of (3.1) is presented. We complement equation (3.1) by homogeneous Dirichlet boundary conditions and by a null initial condition, namely,

$$u = 0 \quad \text{on } (0,T)\times\partial D\,, \tag{3.2}$$

$$u(\cdot,0) = 0 \quad \text{on } D\,. \tag{3.3}$$

Assuming that $\lambda > \lambda_1$ where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D with homogeneous Dirichlet conditions, Theorem 2.1 in [2] proves that equation (3.1) together with conditions (3.2)–(3.3) has infinitely many strong solutions.

Indeed, one looks for solutions to problem (3.1)–(3.3) having the form $u(t,x) = \theta(t)v(x)$. Here, one lets $v = 0$ on ∂D and $\theta(0) = 0$, so that conditions (3.2) and (3.3) are respectively satisfied. Equation (3.1) then reads

$$|\dot\theta|^{p-2}\dot\theta|v|^{p-2}v - \theta\Delta v = \theta\lambda v \quad \text{in } (0,T)\times D\,.$$

One finds a solution to the latter by splitting the variables and solving the decoupled system

$$\dot\theta = |\theta|^{p'-2}\theta \qquad \text{in } (0,T)\,, \tag{3.4}$$

$$-\Delta v + |v|^{p-2}v = \lambda v \qquad \text{in } D\,, \tag{3.5}$$

where $1/p' + 1/p = 1$. For all $t_* \in [0,T]$, the functions $\theta_*(t) = c_p((t-t_*)^+)^{1/(2-p')}$ with $c_p = (2-p')^{1/(2-p')} > 0$ solve (3.4) with $\theta_*(0) = 0$. For later purposes, we remark that for all such functions θ_* one has $\max_{[0,T]}|\theta_*| \le c_p T^{1/(2-p')} =: M_1$.

As far as equation (3.5) is concerned, one considers the minimization of the functional

$$I(v) = \int_D \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{p}|v|^p - \frac{\lambda}{2}|v|^2\right)\mathrm{d}x$$

on $H^1_0(D)$. As $p > 2$, global minimizers v_* can be proved to exist and to be strong solutions to (3.5) with homogeneous Dirichlet boundary conditions. On the other hand, setting $v_1 \not\equiv 0$ to be the eigenfunction related to the first eigenvalue λ_1 of the Laplacian with homogeneous Dirichlet boundary conditions and using the fact that $\|\nabla v_1\|^2_{L^2(D;\mathbb{R}^d)} = \lambda_1\|v_1\|^2_{L^2(D)}$, for all $s > 0$ one has that

$$I(sv_1) = \int_D \left(s^2 \frac{\lambda_1 - \lambda}{2} |v_1|^2 + \frac{s^p}{p} |v_1|^p \right) \mathrm{d}x \,.$$

As $\lambda_1 - \lambda < 0$, one has that $I(sv_1) < 0$ for s small enough. In particular, $I(v_*) < 0$ on the global minimizer, which implies that $v_* \neq 0$. This proves that (3.5) has at least two distinct nontrivial solutions, namely, v_* and $-v_*$. All in all, we have checked that (3.1)–(3.3) has at least the solutions

$$u_*(t,x) = \pm v_*(x)(2-p')^{1/(2-p')}((t-t_*)^+)^{1/(2-p')} \quad \forall (x,t) \in \Omega \times [0,T], \tag{3.6}$$

for any given $t_* \in [0,T]$. Note that $u_* \equiv 0$ for $t_* = T$.

Before moving on, in order to provide a nonuniqueness deterministic counterexample which still fits to our assumptions, let us remark that any solution $v_* \in H^1_0(D)$ of (3.5) is bounded in $C^0(\overline{D})$ in terms of data. Given any $q > 1$, multiply equation (3.5) by $|v_*|^{q-2}v_*$ and integrate in space and by parts. By using the fact that

$$-\int_D \Delta v_* \, |v_*|^{q-2} v_* \, \mathrm{d}x = (q-1) \int_D |v_*|^{q-2} |\nabla v_*|^2 \, \mathrm{d}x \geq 0$$

one obtains

$$\int_D |v_*|^{p+q-2} \, \mathrm{d}x \leq \lambda \int_D |v_*|^q \, \mathrm{d}x,$$

which implies that v_* is bounded in $L^{p+q-2}(D)$ in terms of data. This can indeed be made rigorous by a bootstrap argument on q. In particular, one has that v_* is bounded in $L^r(D)$ in terms of data for any $r > 1$. Choose now $r > d(p-1)/2$. By a comparison in (3.5) one has that $-\Delta v_*$ is bounded in $L^{r/(p-1)}(D)$ in terms of data, as well. By elliptic regularity and the fact that $r/(p-1) > d/2$ one gets that v_* is bounded in $C^0(\overline{D})$ in terms of data. More precisely, there exists $M_2 = M_2(D,p,\lambda)$ such that $\|v\|_{C^0(\overline{D})} \leq M_2$.

At this point, we are ready to present a deterministic counterexample to uniqueness, which is fitting within our setting. Consider the equation

$$\alpha(\partial_t u) - \Delta u = \ell(u) \quad \text{in } (0,T) \times D \,, \tag{3.7}$$

where the linearly bounded monotone mapping $\alpha \in C^1(\mathbb{R})$ is defined by

$$\alpha(x) := \begin{cases} |x|^{p-2}x & \text{for } |x| \leq M \,, \\ (p-1)M^{p-2}x - (p-2)M^{p-1}\dfrac{x}{|x|} & \text{for } |x| > M \,, \end{cases} \tag{3.8}$$

and $\ell \in C^{0,1}(\mathbb{R})$ is given by

$$\ell(x) := \begin{cases} \lambda x & \text{for } |x| \leq M \,, \\ \lambda M x/|x| & \text{for } |x| > M \,, \end{cases} \tag{3.9}$$

for $M := M_1 M_2$. The same argument leading to (3.6) ensures that equation (3.7) with conditions (3.2)–(3.3) has infinite many solutions. In fact, for all u_* from (3.6) one has $|u_*| = |\theta_*||v_*| \leq M_1 M_2 = M$, so that $\alpha(\partial_t u_*) = |\partial_t u_*|^{p-2}\partial_t u_*$ and $\ell(u_*) = \lambda u_*$. We hence have that equations (3.1) and (3.7) coincide and u_* solves (3.7), as well. Note that the truncations allow the operators α and ℓ to satisfy **A1**–**A3**.

3.2. Restoring uniqueness by noise

Before closing this section, let us consider the stochastic counterpart of the deterministic equation (3.7). In the stochastic case, equation (3.7) together with the boundary and initial conditions (3.2)–(3.3) can be variationally formulated as (2.1)–(2.3) by letting $H = L^2(D)$, $A(v)(x) = \alpha(v(x))$ where α is given in (3.8), $B = -\Delta$ with $D(B) = V_2 = H^2(D) \cap H^1_0(D)$, $f = 0$, and $F(v) = \ell(v)$ where ℓ is given in (3.9). By letting $k_A := M^{2-p}/(p-1)$ we have

$$A^{-1}(v)(x) = \begin{cases} |v(x)|^{p'-2}v(x) & \text{if } |v(x)| \le M^{p-1}\,, \\ k_A\left(v(x) + (p-2)M^{p-1}\dfrac{v(x)}{|v(x)|}\right) & \text{if } |v(x)| > M^{p-1}\,. \end{cases}$$

One easily checks that $A^{-1} \in C^{0,p'-1}(H;H)$ (recall that $p > 2 > p' > 1$) and

$$\begin{aligned} &\sup_{v\in H} \|A^{-1}(v) - k_A v\|_H^2 \\ &= \sup_{v\in H}\left(\int\limits_{\{|v|\le M^{p-1}\}} ||v|^{p'-2}v - k_A v|^2 \,\mathrm{d}x + \int\limits_{\{|v|>M^{p-1}\}} |k_A(p-2)M^{p-1}|^2 \,\mathrm{d}x \right) \\ &\le |D|(M + k_A M^{p-1})^2 + |D|(k_A(p-2)M^{p-1})^2 =: (C'_A)^2 < +\infty. \end{aligned}$$

At the same time one has that $F \in C^{0,1}_b(H,H)$ with

$$\begin{aligned} \|F\|^2_{C^{0,1}_b(H;H)} &\le 2\sup_{v\in H}\int_D |\ell(v)|^2 \mathrm{d}x + 2\sup_{u\ne v\in H}\left(\int_D |\ell(u)-\ell(v)|^2\mathrm{d}x\right)\left(\int_D |u-v|^2\mathrm{d}x\right)^{-1} \\ &\le 2|D|M^2 + 2\lambda^2 =: C_F^2 < +\infty\,. \end{aligned}$$

Letting G fulfill **A4**, we have that condition in **A6** holds for any $\delta \in (0, 1/3)$ by taking $p > 2$ large enough. Given $u_0 = 0$, we have proved that Friedrichs-weak solution to (2.1)–(2.3) exist and are unique in law in the sense of Theorem 2.7. On the other hand, taking $G = 0$, the deterministic equation (3.7) with $u_0 = 0$ has infinitely many solutions.

4. Existence of Friedrichs-weak solutions

This section is devoted to the proof of Theorem 2.6 concerning the existence of Friedrichs-weak solutions.

First of all, since $V \hookrightarrow H$ densely, there exists a sequence $(u_0^n)_n \subset V$ such that $\|u_0^n - u_0\|_H \to 0$ as $n \to \infty$. Moreover, let us fix a sequence $\Lambda = (\lambda_n)_n$ such that $\lambda_n \searrow 0$ as $n \to \infty$. Let us construct a sequence $(G_n)_n \subset \mathscr{L}^2(H,V)$ such that $\|G_n - G\|_{\mathscr{L}^2(H,H)} \to 0$ as $n \to \infty$. To this end, for every $x \in H$ and $n \in \mathbb{N}$ we define $G_n x := x_n \in H$ as the unique solution to the singular perturbation problem

$$x_n + \frac{1}{n}Lx_n = Gx\,.$$

In a more compact form, this means that we define $G_n := (I + \frac{1}{n}L)^{-1}G$, so that clearly $G_n \in \mathscr{L}^2(H,V)$ for every $n \in \mathbb{N}$. Let us show the convergence as $n \to \infty$. Testing the equation above by x_n yields directly

$$\frac{1}{2}\|G_n x\|_H^2 + \frac{1}{n}\|G_n x\|_V^2 \le \frac{1}{2}\|Gx\|_H^2 \quad \forall\, x \in H\,,$$

so that

$$\limsup_{n\to\infty} \|G_n x\|_H^2 \leq \|Gx\|_H^2 \quad \forall\, x \in H\,,$$

hence $G_n x \to Gx$ in H for every $x \in H$. Moreover, given a complete orthonormal system $(e_j)_j$ of H, choosing $x = e_j$, and summing over $j \in \mathbb{N}$, we also obtain

$$\frac{1}{2}\|G_n\|_{\mathscr{L}^2(H,H)}^2 + \frac{1}{n}\|G_n\|_{\mathscr{L}^2(H,V)}^2 \leq \frac{1}{2}\|G\|_{\mathscr{L}^2(H,H)}^2\,.$$

This implies analogously that $G_n \to G$ in $\mathscr{L}^2(H,H)$ as $n \to \infty$.

Since K_{λ_n} and F are Lipschitz-continuous on H for every $n \in \mathbb{N}$, there exists an analytically strong solution $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u_n)$ in the sense of Definition 2.2 for the problem with K replaced by K_{λ_n}, i.e.,

$$u_n \in L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)) \cap L^2_w(\Omega; L^\infty(0,T;V)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0,T;V_2))$$

and

$$u_n(t) + k_A \int_0^t B(u_n(s))\,\mathrm{d}s = u_0^n + \int_0^t [k_A F(u_n(s)) + K_{\lambda_n}(u_n(s))]\,\mathrm{d}s + \int_0^t G_n\,\mathrm{d}W(s) \quad \text{in } H\,,$$

for every $t \geq 0$, $\mathbb{P}$-almost surely.

Now, the Itô formula for the square of the H-norm yields

$$\begin{aligned}
&\frac{1}{2}\|u_n(t)\|_H^2 + k_A \int_0^t \|u_n(s)\|_V^2\,\mathrm{d}s \\
&= \frac{1}{2}\|u_0^n\|_H^2 + \int_0^t (k_A F(u_n(s)) + K_{\lambda_n}(u_n(s)) - k_A f(u_n(s)), u_n(s))_H\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|G_n(s)\|_{\mathscr{L}^2(H,H)}^2\,\mathrm{d}s + \int_0^t (u_n(s), G_n \mathrm{d}W(s))_H\,.
\end{aligned}$$

By **A2**–**A3**, (2.12), and the properties of $(u_0^n, G_n)_n$ we deduce that

$$\frac{1}{2}\|u_n(t)\|_H^2 + k_A \int_0^t \|u_n(s)\|_V^2\,\mathrm{d}s \leq C\left(1 + \int_0^t \|u_n(s)\|_H^2\,\mathrm{d}s\right) + \int_0^t (u_n(s), G_n \mathrm{d}W(s))_H$$

for some constant $C > 0$ independent of n. A standard application of the Burkholder–Davis–Gundy inequality together with the Gronwall lemma implies that

$$\|u_n\|_{L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)\cap L^2(0,T;V))} \leq C\,.$$

Moreover, we recall also that by (2.12) and **A2** we have

$$\|K_{\lambda_n}(u_n)\|_{L^\infty_{\mathcal{P}}(\Omega\times(0,T);H)} + \|f(u_n)\|_{L^\infty_{\mathcal{P}}(\Omega\times(0,T);H)} \leq C\,,$$

while from the convergence $G_n \to G$ in $\mathscr{L}^2(H,H)$ it follows that

$$\int_0^\cdot G_n \,\mathrm{d}W(s) \to \int_0^\cdot G \,\mathrm{d}W(s) \quad \text{in } L^2_{\mathcal{P}}(\Omega; C^0([0,T];H)) .$$

Since $C^0([0,T];H) \cap L^2(0,T;V)$ is compactly embedded in $L^2(0,T;H)$ as effect of the Aubin–Lions Lemma, by identifying W with a constant sequence of random variables in $C^0([0,T];\tilde{H})$, where $\tilde{H}$ is a Hilbert–Schmidt extension of H, by the Prokhorov and Skorokhod theorem there exists a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ and a sequence $(\Gamma_n)_n$ such that $\Gamma_n : \hat{\Omega} \to \Omega$ satisfies $(\Gamma_n)_\# \hat{\mathbb{P}} = \mathbb{P}$ for every $n \in \mathbb{N}$ and

$$\begin{aligned}
\hat{u}_n := u_n \circ \Gamma_n &\to \hat{u} \quad \text{in } L^2(0,T;H) , \quad \hat{\mathbb{P}}\text{-a.s.} ,\\
\hat{u}_n &\overset{*}{\rightharpoonup} \hat{u} \quad \text{in } L^2_w(\hat{\Omega}; L^\infty(0,T;H)) \cap L^2(\hat{\Omega}; L^2(0,T;V)) ,\\
K_{\lambda_n}(\hat{u}_n) &\overset{*}{\rightharpoonup} \hat{y} \quad \text{in } L^\infty(\hat{\Omega} \times (0,T); H) ,\\
\hat{W}_n := W \circ \Gamma_n &\to \hat{W} \quad \text{in } C^0([0,T];\tilde{H}) , \quad \hat{\mathbb{P}}\text{-a.s.} ,\\
\hat{I}_n := \left(\int_0^\cdot G_n \,\mathrm{d}W(s) \right) \circ \Gamma_n &\to \hat{I} \quad \text{in } C^0([0,T];H) , \quad \hat{\mathbb{P}}\text{-a.s.} ,
\end{aligned}$$

for some processes

$$\begin{aligned}
\hat{u} &\in L^2_w(\hat{\Omega}; L^\infty(0,T;H)) \cap L^2(\hat{\Omega}; L^2(0,T;V)) ,\\
\hat{y} &\in L^\infty(\hat{\Omega} \times (0,T); H) ,\\
\hat{W} &\in L^2(\hat{\Omega}; C^0([0,T];\tilde{H})) ,\\
\hat{I} &\in L^2(\hat{\Omega}; C^0([0,T];H)) .
\end{aligned}$$

In particular, these convergences imply also that

$$L\hat{u}_n \rightharpoonup L\hat{u} \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;V^*)) ,$$

and, by the strong-weak closure of maximal monotone operator f and by the Lipschitz-continuity of F,

$$\begin{aligned}
f(\hat{u}_n) &\overset{*}{\rightharpoonup} f(\hat{u}) \quad \text{in } L^\infty(\hat{\Omega} \times (0,T); H) ,\\
F(\hat{u}_n) &\overset{*}{\rightharpoonup} F(\hat{u}) \quad \text{in } L^2(0,T;H) , \quad \hat{\mathbb{P}}\text{-a.s.}
\end{aligned}$$

Define now the filtrations $(\hat{\mathscr{F}}_{n,t})_t$ and $(\hat{\mathscr{F}}_t)_t$ by

$$\hat{\mathscr{F}}_{n,t} := \sigma \left\{ \hat{u}_n(s), \hat{I}_n(s), \hat{W}_n(s) : s \in [0,t] \right\} , \quad \hat{\mathscr{F}}_t := \sigma \left\{ \hat{u}(s), \hat{y}(s), \hat{I}(s), \hat{W}(s) : s \in [0,t] \right\} .$$

It is a standard matter (see [17] for details) to check that $\hat{W}_n$ and $\hat{W}$ are H-cylindrical Wiener processes with respect to $(\hat{\mathscr{F}}_{n,t})_t$ and $(\hat{\mathscr{F}}_t)_t$, respectively. Moreover, by using classical martingale arguments, it also holds that $\hat{I}_n = \int_0^\cdot G_n \,\mathrm{d}\hat{W}_n(s)$ and $\hat{I} = \int_0^\cdot G \,\mathrm{d}\hat{W}(s)$. We now note that

$$\hat{u}_n(t) + k_A \int_0^t B(\hat{u}_n(s)) \,\mathrm{d}s = u_0^n + \int_0^t [k_A F(\hat{u}_n(s)) + K_{\lambda_n}(\hat{u}_n(s))] \,\mathrm{d}s + \int_0^t G_n \,\mathrm{d}\hat{W}_n(s) \quad \text{in } H ,$$

for every $t \geq 0$, $\hat{\mathbb{P}}$-almost surely. By using the convergences above and classical arguments we infer then that $\hat{u} \in L^2(\hat{\Omega}; C^0([0,T]; H))$ with

$$\hat{u}(t) + k_A \int_0^t B(\hat{u}(s))\,\mathrm{d}s = u_0 + \int_0^t [k_A F(\hat{u}(s)) + \hat{y}(s)]\,\mathrm{d}s + \int_0^t G\,\mathrm{d}\hat{W}(s) \quad \text{in } H\,,$$

for every $t \geq 0$, $\hat{\mathbb{P}}$-almost surely. This concludes the proof of Theorem 2.6.

5. The Kolmogorov equation

Let us focus now on the Kolmogorov equation associated to the stochastic evolution equation (2.1)–(2.3), as this will be crucial in proving uniqueness.

The equivalent reformulation (2.9) of the doubly nonlinear problem (2.1)–(2.3) suggests the form of the associated Kolmogorov equation, namely

$$\begin{aligned} &\alpha\varphi(x) - \frac{1}{2}\operatorname{Tr}\left[QD^2\varphi(x)\right] + k_A\left(Lx, D\varphi(x)\right)_H \\ &\quad = g(x) + \left(k_A F(x) + K(x) - k_A f(x), D\varphi(x)\right)_H\,, \quad x \in V_2\,, \end{aligned} \tag{5.1}$$

where $\alpha > 0$ is fixed and g is a suitable forcing term. Clearly, due to the singularity of the perturbation K, in general one cannot expect to prove existence of mild solutions to (5.1) via classical arguments, i.e., by directly exploiting the possible regularising properties of the Ornstein–Uhlenbeck semigroup. The idea is then to characterise the solution of (5.1) in a different way, still relying on approximating arguments à la Friedrichs.

From now on, for brevity of notation we will assume with no loss of generality that

$$k_A = 1\,.$$

Furthermore, we recall that by assumptions **A2**–**A3** one has

$$\|F(x)\|_H \leq C_F\,, \quad \|f(x)\|_H \leq C_f \qquad \forall\, x \in H\,. \tag{5.2}$$

This will be essential in the analysis of the Kolmogorov equation.

5.1. The Ornstein–Uhlenbeck semigroup

We collect here some preliminary results on the Ornstein–Uhlenbeck semigroup associated to (5.1). First, note that by assumption **A2** we have that $-L$ generates a strongly continuous semigroup $(e^{-tL})_{t\geq 0}$ of contractions on H. Introducing then the operator

$$Q_t := \int_0^t e^{-2sL} Q\,\mathrm{d}s\,, \quad t \geq 0\,, \tag{5.3}$$

with $Q = G^*G$, it is clear that $Q_t \in \mathscr{L}^1(H,H)$ for every $t \geq 0$ since $G \in \mathscr{L}^2(H,H)$ by assumption **A4**. With this notation, we define the Ornstein–Uhlenbeck semigroup as

$$(R_t\varphi)(x) := \int_H \varphi(e^{-tL}x + y) N_{Q_t}(\mathrm{d}y)\,, \quad x \in H\,, \quad \varphi \in \mathscr{B}_b(H)\,, \quad t \geq 0\,, \tag{5.4}$$

where N_{Q_t} denotes a Gaussian measure on H, centred at 0, and with covariance operator Q_t. Note that an easy computation shows that

$$Q_t = \frac{1}{2} L^{-1} \left(I - e^{-2tL}\right) Q \quad \forall\, t \geq 0\,.$$

Lemma 5.1 *(Strong Feller property). Assume* ***A2****,* ***A4****, and* ***A6****. Then, the semigroup R is strong Feller. More specifically, there exists a constant $C_R > 0$ such that, for every $t > 0$ and $\varphi \in \mathscr{B}_b(H)$, it holds that $R_t\varphi \in C_b^\infty(H)$ and*

$$\sup_{x\in H} |(R_t\varphi)(x)| \leq \sup_{x\in H} |\varphi(x)|\,, \tag{5.5}$$

$$\sup_{x\in H} \|D(R_t\varphi)(x)\|_H \leq \frac{C_R}{t^{\frac{1}{2}+\delta}} \sup_{x\in H} |\varphi(x)|\,. \tag{5.6}$$

Moreover, for every $\varphi \in C_b^1(H)$, it holds that

$$\sup_{x\in H} \|D^2(R_t\varphi)(x)\|_{\mathscr{L}(H,H)} \leq \frac{C_R}{t^{\frac{1}{2}+\delta}} \sup_{x\in H} \|D\varphi(x)\|_H\,. \tag{5.7}$$

Furthermore, for every $\varepsilon \in (0, \frac{1}{2} - \delta)$ and $\eta \in (0, \frac{1-2\delta}{1+2\delta})$, there exist constants $C_{R,\varepsilon}, C_{R,\eta} > 0$ such that, for every $t > 0$ and $\varphi \in \mathscr{B}_b(H)$, it holds that $D(R_t\varphi)(H) \subset V_{2\varepsilon}$ and

$$\sup_{x\in H} \|L^\varepsilon D(R_t\varphi)(x)\|_H \leq \frac{C_{R,\varepsilon}}{t^{\frac{1}{2}+\delta+\varepsilon}} \sup_{x\in H} |\varphi(x)|\,, \tag{5.8}$$

$$\|R_t\varphi\|_{C^{1,\eta}(H)} \leq \frac{C_{R,\eta}}{t^{(1+\eta)(\frac{1}{2}+\delta)}} \sup_{x\in H} |\varphi(x)| \tag{5.9}$$

The proof of Lemma 5.1 follows immediately from [5, Lem. 4.1–4.2].

Remark 5.2. Note that since by assumption **A6** one has $\delta \in (0, \frac{1}{2})$, it follows that the exponent $\frac{1}{2} + \delta$ also belongs to $(0, 1)$. Moreover, the conditions on $\varepsilon \in (0, \frac{1}{2} - \delta)$ and $\eta \in (0, \frac{1-2\delta}{1+2\delta})$ ensure that also $\frac{1}{2} + \delta + \varepsilon \in (0, 1)$ and $(1 + \eta)(\frac{1}{2} + \delta) \in (0, 1)$, as one can easily check.

5.2. The regularised Kolmogorov equation

The main technical challenge in the study of the Kolmogorov equation (5.1) is that of taming the singularity of K. We proceed by approximation and, for all $\lambda > 0$, argue with $K_\lambda \in C_b^{0,s_A}(H; H)$, which satisfies the uniform bound (2.12), and with the Yosida approximation $f_\lambda : H \to H$ of f.

Given a fixed $\alpha > 0$ and a forcing $g \in C_b^0(H)$ (some additional qualification for α and g will be introduced later), the regularised Kolmogorov equation reads, for $\lambda > 0$,

$$\begin{aligned} &\alpha\varphi_\lambda(x) - \frac{1}{2} \operatorname{Tr}\left[QD^2\varphi_\lambda(x)\right] + (Lx, D\varphi_\lambda(x))_H \\ &\quad = g(x) + (F(x) + K_\lambda(x) - f_\lambda(x), D\varphi_\lambda(x))_H\,, \quad x \in V_2\,. \end{aligned} \tag{5.10}$$

Solutions to the regularised Kolmogorov equation (5.10) can be intended either in the mild sense, by exploiting the representation formula for the resolvent of the semigroup R, or in the strong classical sense, as specified in the following.

Definition 5.3 *(Mild solution).* A function $\psi \in C_b^1(H)$ is said to be a *mild solution* to (5.10) if

$$\psi(x) = \int_0^{+\infty} e^{-\alpha t} R_t \left[g + (F + K_\lambda - f_\lambda, D\psi)_H\right](x)\, \mathrm{d}t \qquad \forall\, x \in H\,.$$

Definition 5.4 *(Classical solution).* A function $\psi \in C_b^2(H)$ is said to be a *classical solution* to (5.10) if it is a mild solution and it satisfies, for every $x \in V_2$,

$$\alpha\psi(x) - \frac{1}{2}\operatorname{Tr}\left[QD^2\psi(x)\right] + (Lx, D\psi(x))_H = g(x) + (F(x) + K_\lambda(x) - f_\lambda(x), D\psi(x))_H\,.$$

The following result ensures that the regularised Kolmogorov equation (5.10) is well posed in both the mild and strong sense, and provides estimates on the solution φ_λ which are uniform with respect to the parameter λ.

Proposition 5.5 *(Well-posedness of the Kolmogorov equation). Assume **A1–A6**, let $g \in C_b^{0,s_A}(H)$, $\varepsilon \in (0, \frac{1}{2} - \delta)$, and $\eta \in (0, \frac{1-2\delta}{1+2\delta})$. Then, there exists $\alpha_0 > 0$ such that, for every $\alpha > \alpha_0$, the following holds:*

(i) *for every $\lambda > 0$, equation (5.10) admits a unique mild solution $\varphi_\lambda \in C_b^1(H)$;*

(ii) *there exists a constant $C_1 > 0$, independent of λ, such that*

$$\|\varphi_\lambda\|_{C_b^1(H)} \leq C_1 \qquad \forall\, \lambda > 0\,; \tag{5.11}$$

(iii) *there exists a constant $C_\varepsilon > 0$, independent of λ, such that $D\varphi_\lambda(H) \subset V_{2\varepsilon}$ and*

$$\|L^\varepsilon D\varphi_\lambda\|_{C_b^0(H;H)} \leq C_\varepsilon \qquad \forall\, \lambda > 0\,; \tag{5.12}$$

(iv) *there exists a constant $C_\eta > 0$, independent of λ, such that*

$$\|\varphi_\lambda\|_{C_b^{1,\eta}(H)} \leq C_\eta \qquad \forall\, \lambda > 0\,; \tag{5.13}$$

(v) *for every $\lambda > 0$, the unique mild solution satisfies $\varphi_\lambda \in C_b^2(H)$ and is also a classical solution of equation (5.10);*

(vi) *there exists a constant $C_2 > 0$, independent of λ, such that*

$$\|\varphi_\lambda\|_{C_b^2(H)} \leq C_2 \qquad \forall\, \lambda > 0\,. \tag{5.14}$$

Proof of Proposition 5.5. (i). We exploit a fixed point argument. Let $S_\alpha : C_b^1(H) \to C_b^0(H)$ be defined as

$$S_\alpha v(x) := \int_0^{\infty} e^{-\alpha t} R_t[g + (F + K_\lambda - f_\lambda, Dv)_H](x)\, \mathrm{d}t\,, \quad v \in C_b^1(H)\,.$$

Note that

$$\begin{aligned} &e^{-\alpha t}\|DR_t[g + (F + K_\lambda - f_\lambda, Dv)_H](x)\|_H \\ &\overset{(5.6)}{\leq} C_R \frac{e^{-\alpha t}}{t^{\frac{1}{2}+\delta}} \sup_{y\in H} |g(y) + (F(y) + K_\lambda(y) - f_\lambda(y), Dv(y))_H| \\ &\leq C_R \left(\|g\|_{C_b^0(H)} + (C_F + C_A' + C_f)\, \|Dv\|_{C_b^0(H;H)}\right) \frac{e^{-\alpha t}}{t^{\frac{1}{2}+\delta}}\,. \end{aligned}$$

Hence, thanks to the Dominated Convergence Theorem and to a differentiation under the integral sign, we get that S_α is well-defined as a map from $C_b^1(H)$ into itself, and it holds that

$$D(S_\alpha v)(x) = \int_0^\infty e^{-\alpha t} DR_t[g + (F + K_\lambda - f_\lambda, Dv)_H](x)\,\mathrm{d}t\,, \quad v \in C_b^1(H)\,.$$

Furthermore, for every $v_1, v_2 \in C_b^1(H)$, by arguing on the difference of the respective equations it is immediate to see that

$$|S_\alpha v_1(x) - S_\alpha v_2(x)| \le \frac{1}{\alpha}(C_F + C_A' + C_f)\,\|Dv_1 - Dv_2\|_{C_b^0(H;H)}$$

and

$$\begin{aligned}\|DS_\alpha v_1(x) - DS_\alpha v_2(x)\|_H &\le C_R(C_F + C_A' + C_f)\,\|Dv_1 - Dv_2\|_{C_b^0(H;H)} \int_0^{+\infty} e^{-\alpha t}\frac{1}{t^{\frac12+\delta}}\,\mathrm{d}t \\ &= \frac{C_R}{\alpha^{\frac12-\delta}}\Gamma\left(\frac12 - \delta\right)(C_F + C_A' + C_f)\,\|Dv_1 - Dv_2\|_{C_b^0(H;H)}\end{aligned}$$

where Γ is the Riemann Γ-function. It follows that S_α is a contraction on $C_b^1(H)$ provided that α is chosen sufficiently large, e.g., for $\alpha > \alpha_0$ for some α_0 only depending on $C_F, C_A', C_f, \delta, C_R$. Existence and uniqueness of a mild solution is then a consequence of the Banach Fixed Point Theorem.
(ii). This is an immediate consequence of the computations in point (i). Indeed, since for $\alpha > \alpha_0$ one has in particular that $\frac{C_R}{\alpha^{\frac12-\delta}}\Gamma\left(\frac12-\delta\right)(C_F + C_A' + C_f) < 1$ and the already performed computations yield

$$\left(1 - \frac{C_R}{\alpha^{\frac12-\delta}}\Gamma\left(\frac12-\delta\right)(C_F + C_A' + C_f)\right)\|D\varphi_\lambda\|_{C_b^0(H;H)} \le \frac{C_R}{\alpha^{\frac12-\delta}}\Gamma\left(\frac12-\delta\right)\|g\|_{C_b^0(H)}\,,$$

hence (5.11) follows.
(iii). By (5.8) and the Dominated Convergence Theorem, we have that

$$\begin{aligned}\|L^\varepsilon D\varphi_\lambda\|_{C_b^0(H;H)} &\overset{(5.8)}{\le} \int_0^\infty e^{-\alpha t}\frac{C_{R,\varepsilon}}{t^{\frac12+\delta+\varepsilon}}\sup_{x\in H}\left(|g(x)| + (C_F + C_A' + C_f)\,\|D\varphi_\lambda(x)\|_H\right)\mathrm{d}t \\ &\le \frac{C_{R,\varepsilon}}{\alpha^{\frac12-\delta-\varepsilon}}\Gamma\left(\frac12-\delta-\varepsilon\right)\left(\|g\|_{C_b^0(H)} + (C_F + C_A' + C_f)\,\|D\varphi_\lambda\|_{C_b^0(H;H)}\right)\,.\end{aligned}$$

Hence, by possibly choosing a larger α_0 such that for all $\alpha > \alpha_0$ it also holds that

$$\frac{C_{R,\varepsilon}}{\alpha^{\frac12-\delta-\varepsilon}}\Gamma\left(\frac12-\delta-\varepsilon\right)(C_F + C_A' + C_f) < 1\,,$$

estimate (5.12) follows.
(iv). For every $\eta \in (0, (1-2\delta)/(1+2\delta))$, by (5.9) one has, for all $t > 0$,

$$\begin{aligned}&e^{-\alpha t}\,\|DR_t[g + (F + K_\lambda - f_\lambda, D\varphi_\lambda)_H]\|_{C_b^{0,\eta}(H)} \\ &\quad\overset{(5.9)}{\le} \frac{C_{R,\eta}}{t^{(1+\eta)(\frac12+\delta)}}e^{-\alpha t}\left(\|g\|_{C_b^0(H)} + (C_F + C_A' + C_f)\,\|D\varphi_\lambda\|_{C_b^0(H;H)}\right)\,,\end{aligned}$$

where $(1+\eta)(\frac{1}{2}+\delta) \in (0,1)$. Using the Dominated Convergence Theorem and estimate (5.11) one has that $\varphi_\lambda \in C_b^{1,\eta}(H)$ and

$$\|\varphi_\lambda\|_{C_b^{1,\eta}(H)} \le C_{R,\eta} \left(\|g\|_{C_b^0(H)} + C_1(C_F + C'_A + C_f) \right) \int_0^\infty \frac{1}{t^{(1+\eta)(\frac{1}{2}+\delta)}} e^{-\alpha t} \, dt \,,$$

so that also the estimate (5.13) is proved.
(v). We use the Schauder estimates as in [11, Prop. 6.4.2]. Let $\eta \in (0, (1-2\delta)/(1+2\delta))$ and set $\eta_0 := \min\{s_A, \eta\}$. Then, recalling that $g \in C_b^{0,s_A}(H)$, we note that

$$x \mapsto g(x) + (F(x) + K_\lambda(x) - f_\lambda(x), D\varphi_\lambda(x))_H$$

is η_0-Hölder continuous and bounded in H. It follows then by [11, Prop. 6.4.2] that

$$\varphi_\lambda \in C_b^{\eta_0 + \frac{2}{1+2\delta}}(H) \,.$$

This implies then that

$$D\varphi_\lambda \in C_b^{0,\eta_0+\frac{1-2\delta}{1+2\delta}}(H;H) \,.$$

By iterating the argument, this implies that

$$x \mapsto g(x) + (F(x) + K_\lambda(x) - f_\lambda(x), D\varphi_\lambda(x))_H$$

is $\min\{s_A, \eta_0 + \frac{1-2\delta}{1+2\delta}\}$-Hölder continuous and bounded in H. Again by [11, Prop. 6.4.2] we infer that

$$\varphi_\lambda \in C_b^{\min\{s_A+\frac{2}{1+2\delta}, \eta_0+\frac{3-2\delta}{1+2\delta}\}}(H) \,.$$

This implies then that

$$D\varphi_\lambda \in C_b^{0,\min\{s_A+\frac{1-2\delta}{1+2\delta}, \eta_0+2\frac{1-2\delta}{1+2\delta}\}}(H;H) \,.$$

By iterating the argument, there exists k sufficiently large such that $\eta_0 + 2\frac{1-2\delta}{1+2\delta} > s_A$: hence, one deduces that

$$\varphi_\lambda \in C_b^{s_A+\frac{2}{1+2\delta}}(H) \,.$$

Since

$$s_A + \frac{2}{1+2\delta} > 2$$

by assumption **A6**, this implies in particular that $\varphi_\lambda \in C_b^2(H)$. Moreover, the same result [11, Prop. 6.4.2] also ensures that φ_λ is a strong solution to the regularised Kolmogorov equation in the sense of Definition 5.4.
(vi). Again by [11, Prop. 6.4.2], since

$$x \mapsto g(x) + (F(x) + K_\lambda(x) - f_\lambda(x), D\varphi_\lambda(x))_H$$

is bounded in $C_b^{0,s_A}(H)$ uniformly in λ, one has that

$$\|\varphi_\lambda\|_{C_b^2(H)} \leq C\alpha^{2\delta - s_A(\frac{1}{2}+\delta)} =: C_2$$

for a positive constant C independent of λ, so that (5.14) follows. $\square$

5.3. Passage to the limit in the Kolmogorov equation

We are now ready to address the asymptotic behaviour of the solutions $(\varphi_\lambda)_{\lambda>0}$ to the Kolmogorov equation (5.10) as $\lambda \to 0$. We show that the sequence of approximated solutions $(\varphi_\lambda)_{\lambda>0}$ admits a limit φ, which may be interpreted as a Friedrichs-type solution of the limiting Kolmogorov equation (5.1). This is rigorously stated in the following result.

Proposition 5.6 *(Convergence of φ_λ). Assume **A1**–**A6**, let $g \in C_b^{0,s_A}(H)$, let $\alpha > \alpha_0$ be fixed, and let $(\varphi_\lambda)_{\lambda>0}$ be the family of solutions to (5.10) as given in* Proposition 5.5*. Then, for every sequence $\Lambda = (\lambda_n)_n$ with $\lambda_n \searrow 0$ as $n \to \infty$, there exists a unique*

$$\varphi \in C_b^{1,\frac{1-2\delta}{1+2\delta}-}(H) \qquad \textit{with} \qquad D\varphi \in C_b^0(H; D(L^{(\frac{1}{2}-\delta)-})\,,$$

depending only on A, B, F, G, and Λ, and a subsequence $(\varphi_{\lambda_{n_k}})_{k\in\mathbb{N}}$ satisfying, as $k \to \infty$,

$$\lim_{k\to\infty} |\varphi_{\lambda_{n_k}}(x) - \varphi(x)| = 0 \qquad \forall\, x \in H\,,$$
$$\lim_{k\to\infty} \|D\varphi_{\lambda_{n_k}}(x) - D\varphi(x)\|_H = 0 \qquad \forall\, x \in H\,.$$

Proof of Proposition 5.6. STEP 1. For every $r > 0$, set $B_r^V := \{x \in V : \|x\|_V \leq r\}$ to be the V-closed ball of radius r. Let $\Lambda = (\lambda_n)_n$ be an arbitrary sequence satisfying $\lambda_n \searrow 0$ as $n \to \infty$. Let also $j \in \mathbb{N}_+$ be fixed and let us consider $(\varphi_{\lambda_n})_n$ restricted to B_j^V: since $V \hookrightarrow H$ compactly, the ball B_j^V endowed with the H-distance is a compact metric space. Thanks to (5.11) it readily follows that the sequence $\varphi_{\lambda_n} : B_j^V \to \mathbb{R}$ are uniformly equicontinuous and bounded: hence, by the Ascoli–Arzelà Theorem there exists a subsequence $(\varphi_{\lambda^j_{n_k}})_k$ and $\varphi_j : B_j^V \to \mathbb{R}$, continuous with respect to the topology of H, such that

$$\lim_{k\to\infty} \sup_{x\in B_j^V} |\varphi_{\lambda^j_{n_k}}(x) - \varphi_j(x)| = 0\,.$$

Clearly, it follows that $\varphi_{j+1} = \varphi_j$ on B_j^V for every $j \in \mathbb{N}_+$, so that the family $(\varphi_j)_{j>0}$ uniquely determines a function $\varphi : V \to \mathbb{R}$. Moreover, via a standard diagonal argument one can work with the same subsequence $(\varphi_{\lambda_{n_k}})_k$ for every $j \in \mathbb{N}$. Specifically, we have

$$\lim_{k\to\infty} \sup_{x\in B_j^V} |\varphi_{\lambda_{n_k}}(x) - \varphi(x)| = 0 \quad \forall\, j \in \mathbb{N}_+\,. \tag{5.15}$$

In particular, this implies the pointwise convergence

$$\lim_{k\to\infty} \varphi_{\lambda_{n_k}}(x) = \varphi(x) \quad \forall\, x \in V\,.$$

Let us prove that φ uniquely extends to a function in $C_b^{0,1}(H)$. To this end, thanks to (5.11) we note that for every $x_1, x_2 \in V$ we have

$$\begin{aligned}
&|\varphi(x_1) - \varphi(x_2)| \\
&\leq |\varphi(x_1) - \varphi_{\lambda_{n_k}}(x_1)| + |\varphi_{\lambda_{n_k}}(x_1) - \varphi_{\lambda_{n_k}}(x_2)| + |\varphi_{\lambda_{n_k}}(x_2) - \varphi(x_2)| \\
&\leq |\varphi(x_1) - \varphi_{\lambda_{n_k}}(x_1)| + C_1\,\|x_1 - x_2\|_H + |\varphi_{\lambda_{n_k}}(x_2) - \varphi(x_2)|
\end{aligned}$$

for every $k \in \mathbb{N}$, so that letting $k \to \infty$ yields

$$|\varphi(x_1) - \varphi(x_2)| \leq C_1 \|x_1 - x_2\|_H \quad \forall\, x_1, x_2 \in V\,.$$

By density of V in H this shows indeed that $\varphi \in C_b^{0,1}(H)$, where we have used the same symbol for the extension. Moreover, for every $x \in H$, let $(x_m)_{m\in\mathbb{N}} \subset V$ be such that $x_m \to x$ in H as $m \to \infty$. Then, again by (5.11) and the Lipschitz-continuity of $\varphi_{\lambda_{n_k}}$ and φ we have

$$\begin{aligned}
&|\varphi_{\lambda_{n_k}}(x) - \varphi(x)| \\
&\leq |\varphi_{\lambda_{n_k}}(x) - \varphi_{\lambda_{n_k}}(x_m)| + |\varphi_{\lambda_{n_k}}(x_m) - \varphi(x_m)| + |\varphi(x_m) - \varphi(x)| \\
&\leq 2C_1 \|x_m - x\|_H + |\varphi_{\lambda_{n_k}}(x_m) - \varphi(x_m)|\,,
\end{aligned}$$

and it is a standard matter to see that this yields

$$\lim_{k\to\infty} \varphi_{\lambda_{n_k}}(x) = \varphi(x) \quad \forall\, x \in H\,.$$

Step 2. Let us consider now the sequence of derivatives $D\varphi_{\lambda_n} : B_j^V \to H$, for $j \in \mathbb{N}_+$ fixed. Thanks to (5.13) it follows that $(D\varphi_{\lambda_n})_n$ is uniformly equicontinuous, while (5.12) implies that for every $x \in B_r^V$ the set $(D\varphi_{\lambda_n}(x))_n$ is relatively compact in H. Again by the Ascoli-Arzelà Theorem, for every $j \in \mathbb{N}$ there exists $\varphi'_j : B_j^V \to H$, continuous with respect to the topology of H on B_j^V, and a subsequence $(\lambda^j_{n_k})_k$ such that

$$\lim_{k\to\infty} \sup_{x\in B_j^V} \|D\varphi_{\lambda^j_{n_k}}(x) - \varphi'_j(x)\|_H = 0\,.$$

As before, we note that the sequence $(\varphi'_j)_j$ uniquely determines a function $\varphi' : V \to H$ such that, by possibly using a diagonal argument,

$$\lim_{k\to\infty} \sup_{x\in B_{r_j}^V} |D\varphi_{\lambda_{n_k}}(x) - \varphi'(x)| = 0 \quad \forall\, j \in \mathbb{N}_+\,. \tag{5.16}$$

In particular, the pointwise convergence

$$\lim_{k\to\infty} \|D\varphi_{\lambda_{n_k}}(x) - \varphi'(x)\|_H = 0 \quad \forall\, x \in V$$

follows. Let us prove that φ' uniquely extends to a function in $C_b^{0,\eta}(H; H)$, where $\eta \in (0, \frac{1-2\delta}{1+2\delta})$ is fixed. To this end, thanks to (5.13) we note that for every $x_1, x_2 \in V$ we have

$$\begin{aligned}
&\|\varphi'(x_1) - \varphi'(x_2)\|_H \\
&\leq \|\varphi'(x_1) - D\varphi_{\lambda_{n_k}}(x_1)\|_H + \|D\varphi_{\lambda_{n_k}}(x_1) - D\varphi_{\lambda_{n_k}}(x_2)\|_H + \|D\varphi_{\lambda_{n_k}}(x_2) - \varphi'(x_2)| \\
&\leq \|\varphi'(x_1) - D\varphi_{\lambda_{n_k}}(x_1)\|_H + C_\eta \|x_1 - x_2\|_H^\eta + \|D\varphi_{\lambda_{n_k}}(x_2) - \varphi'(x_2)\|_H
\end{aligned}$$

for every $k \in \mathbb{N}$, so that letting $k \to \infty$ yields

$$\|\varphi'(x_1) - \varphi'(x_2)\|_H \leq C_\eta \|x_1 - x_2\|_H^\eta \quad \forall\, x_1, x_2 \in V\,.$$

By density of V in H this shows indeed that $\varphi' \in C_b^{0,\eta}(H; H)$, where again we have used the same symbol for the extension. Furthermore, for every $x \in H$, let $(x_m)_{m\in\mathbb{N}} \subset V$ be such that $x_m \to x$ in H as $m \to \infty$. Then, again by (5.13) and the Hölder-continuity of $D\varphi_{\lambda_{n_k}}$ and φ' we have

$$
\begin{aligned}
&\|D\varphi_{\lambda_{n_k}}(x) - \varphi'(x)\|_H \\
&\le \|D\varphi_{\lambda_{n_k}}(x) - D\varphi_{\lambda_{n_k}}(x_m)\|_H + \|D\varphi_{\lambda_{n_k}}(x_m) - \varphi'(x_m)\|_H + \|\varphi'(x_m) - \varphi'(x)\|_H \\
&\le 2C_\eta \|x_m - x\|_H^\eta + \|D\varphi_{\lambda_{n_k}}(x_m) - \varphi'(x_m)\|_H \,,
\end{aligned}
$$

so that we deduce the convergence

$$
\lim_{k\to\infty} \|D\varphi_{\lambda_{n_k}}(x) - \varphi'(x)\|_H = 0 \quad \forall\, x \in H \,.
$$

Step 3. Let us show that $\varphi \in C^{1,\eta}(H)$ and $D\varphi = \varphi'$. By the convergences (5.15)–(5.16) proved above one has that

$$
(\varphi_{\lambda_{n_k}})_k \quad \text{is Cauchy in } C_b^1(B_j^V) \quad \forall\, j \in \mathbb{N}_+ \,.
$$

It follows in particular that φ satisfies $\varphi_{|B_j^V} \in C^1(B_j^V)$ and $D(\varphi_{|B_j^V}) = \varphi'_{|B_j^V}$ for every j. In particular, this implies that $\varphi \in C^1(V)$ and $D\varphi(x) = \varphi'(x)$ in V^* for every $x \in V$. Furthermore, let $x, h \in H$ be fixed, and let $(x_m)_m, (h_m)_m \subset V$ such that $x_m \to x$ and $h_m \to h$ in H as $m \to \infty$. Since $\varphi \in C^1(V)$ with $D\varphi = (\varphi')_{|V} : V \to V^*$ and $\varphi' \in C^{0,\eta}(H; H)$, one has that

$$
\varphi(x_m + h_m) - \varphi(x_m) = \int_0^1 \langle D\varphi(x_m + \sigma h_m), h_m\rangle_{V^*,V} \,\mathrm{d}\sigma = \int_0^1 (\varphi'(x_m + \sigma h_m), h_m)_H \,\mathrm{d}\sigma \,.
$$

Letting $m \to \infty$ yields, by Dominated Convergence Theorem,

$$
\varphi(x + h) - \varphi(x) = \int_0^1 (\varphi'(x + \sigma h), h)_H \,\mathrm{d}\sigma \qquad \forall\, x, h \in H \,,
$$

showing that $\varphi : H \to \mathbb{R}$ is Gâteaux differentiable with $D\varphi = \varphi' : H \to H$. Since $\varphi' \in C_b^{0,\eta}(H; H)$, this implies then that $\varphi \in C_b^{1,\eta}(H)$, as required. □

6. Uniqueness in law and weak stability

6.1. Proof of Theorem 2.7

This section is devoted to the proof of our main result concerning uniqueness in law, namely, Theorem 2.7. To this end, let $u_0 \in H$ be fixed, and let

$$
(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u_1, y_1, \Lambda) \qquad \text{and} \qquad (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u_2, y_2, \Lambda)
$$

be two Friedrichs-weak solutions to problem (2.1)–(2.3) with respect to the same initial datum u_0, defined on the same stochastic basis, and with the same approximating sequence $\Lambda = (\lambda_n)$. Let $(u_{0,i}^n, G_n^i)_n \subset H \times \mathscr{L}^2(H, H)$ and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u_n^i)$, for $i = 1, 2$, be some sequences of data and analytically strong solutions that approximate u_1 and u_2, respectively, in the sense of Definition 2.3: in particular, one has for $i = 1, 2$ and for all $T > 0$ that

$$
u_{0,i}^n \to u_0 \quad \text{in } H \,, \tag{6.1}
$$

$$
G_n^i \to G \quad \text{in } \mathscr{L}^2(H, H) \,, \tag{6.2}
$$

$$u_n^i \overset{*}{\rightharpoonup} u_i \quad \text{in } L^2_w(\Omega; L^\infty(0,T;H)) \cap L^2_{\mathscr{P}}(\Omega; L^2(0,T;V)), \tag{6.3}$$

$$u_n^i \to u_i \quad \text{in } L^2(0,T;H), \quad \mathbb{P}\text{-a.s.}, \tag{6.4}$$

$$K_{\lambda_n}(u_n^i) \overset{*}{\rightharpoonup} y_i \quad \text{in } L^\infty_{\mathscr{P}}(\Omega \times (0,T); H). \tag{6.5}$$

Let $(\varphi_\lambda)_\lambda$ be the sequence of solutions to the regularised Kolmogorov equation (5.10) as given in Proposition 5.5, associated to the same sequence Λ. Then, Proposition 5.6 ensures that there exists subsequence $(\lambda_{n_k})_k$ and

$$\varphi \in C_b^{1,\frac{1-2\delta}{1+2\delta}-}(H) \qquad \text{with} \qquad D\varphi \in C_b^0(H; D(L^{(\frac{1}{2}-\delta)-}),$$

depending only on A, B, F, G, and Λ (but not on u_1 and u_2), such that, as $k \to \infty$,

$$\lim_{k\to\infty} |\varphi_{\lambda_{n_k}}(x) - \varphi(x)| = 0 \qquad \forall\, x \in H, \tag{6.6}$$

$$\lim_{k\to\infty} \|D\varphi_{\lambda_{n_k}}(x) - D\varphi(x)\|_H = 0 \qquad \forall\, x \in H. \tag{6.7}$$

Now, let $i \in \{1,2\}$ be fixed. Since $\varphi_{\lambda_{n_k}} \in C_b^2(H)$, Itô formula for $\varphi_{\lambda_{n_k}}(u^i_{n_k})$ yields

$$\begin{aligned}
&\mathbb{E}\,\varphi_{\lambda_{n_k}}(u^i_{n_k}(t)) - \frac{1}{2}\mathbb{E}\int_0^t \operatorname{Tr}\left[G_n^i(G_n^i)^* D^2\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right] \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t \left(Lu^i_{n_k}(s), D\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right)_H \mathrm{d}s \\
&= \varphi_{\lambda_{n_k}}(u^n_{0,i}) + \mathbb{E}\int_0^t \left(F(u^i_{n_k}(s)) + K_{\lambda_{n_k}}(u^i_{n_k}(s)) - f(u^i_{n_k}(s)), D\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right)_H \mathrm{d}s\,.
\end{aligned}$$

Recalling that $\varphi_{\lambda_{n_k}}$ solves (5.10), we deduce that

$$\begin{aligned}
&\mathbb{E}\,\varphi_{\lambda_{n_k}}(u^i_{n_k}(t)) - \alpha\,\mathbb{E}\int_0^t \varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\,\mathrm{d}s \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^t \operatorname{Tr}\left[\left(Q - G^i_{n_k}(G^i_{n_k})^*\right) D^2\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right] \mathrm{d}s + \mathbb{E}\int_0^t g(u^i_{n_k}(s))\,\mathrm{d}s \\
&= \varphi_{\lambda_{n_k}}(u^{n_k}_{0,i}) + \mathbb{E}\int_0^t \left(f_{\lambda_{n_k}}(u^i_{n_k}(s)) - f(u^i_{n_k}(s)), D\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right)_H \mathrm{d}s
\end{aligned}$$

for every $t \in [0,T]$. Now, since $f : H \to H$ is maximal monotone, single-valued, and of bounded range with $D(f) = H$, by the Dominated Convergence Theorem as $k \to +\infty$ one has that

$$f_{\lambda_{n_k}}(u^i_{n_k}) - f(u^i_{n_k}) \to 0 \quad \text{in } L^1_{\mathscr{P}}(\Omega; L^2(0,T;H)),$$

which yields in turn, thanks to (5.11), that

$$\mathbb{E}\int_0^t \left(f_{\lambda_{n_k}}(u^i_{n_k}(s)) - f(u^i_{n_k}(s)), D\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right)_H \mathrm{d}s \\ \le C_1 \left\| f_{\lambda_{n_k}}(u^i_{n_k}) - f(u^i_{n_k})\right\|_{L^1_{\mathscr{P}}(\Omega;L^2(0,T;H))} \to 0\,.$$

Moreover, given a fixed $\eta \in (0, \frac{1-2\delta}{1+2\delta})$, thanks to (5.13), (6.1), and (6.6), we have

$$\left|\varphi_{\lambda_{n_k}}(u^{n_k}_{0,i}) - \varphi(u_0)\right| \le \left|\varphi_{\lambda_{n_k}}(u^{n_k}_{0,i}) - \varphi_{\lambda_{n_k}}(u_0)\right| + \left|\varphi_{\lambda_{n_k}}(u_0) - \varphi(u_0)\right| \\ \le C\left\|u^{n_k}_{0,i} - u_0\right\|^\eta_H + \left|\varphi_{\lambda_{n_k}}(u_0) - \varphi(u_0)\right| \to 0\,.$$

Similarly, exploiting again (5.13) we have

$$\left|\varphi_{\lambda_{n_k}}(u^i_{n_k}(t)) - \varphi(u_i(t))\right| \le \left|\varphi_{\lambda_{n_k}}(u^i_{n_k}(t)) - \varphi_{\lambda_{n_k}}(u_i(t))\right| + \left|\varphi_{\lambda_{n_k}}(u_i(t)) - \varphi(u_i(t))\right| \\ \le C_\eta \left\|u^i_{n_k}(t) - u_i(t)\right\|^\eta_H + \left|\varphi_{\lambda_{n_k}}(u_i(t)) - \varphi(u_i(t))\right| \quad \forall t \in [0,T]\,,$$

which yields, thanks to (6.4), (6.6), and the Dominated Convergence Theorem, that

$$\varphi_{\lambda_{n_k}}(u^i_{n_k}(t)) \to \varphi(u_i(t)) \quad \text{in } L^1(\Omega) \quad \forall\, t \in [0,T]\,.$$

Eventually, using (5.14) we get for every $s > 0$ that

$$\begin{aligned}
&\mathrm{Tr}\left[\left(Q - G^i_{n_k}(G^i_{n_k})^*\right) D^2\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right] \\
&\overset{(5.14)}{\le} C_2 \left\|GG^* - G^i_{n_k}(G^i_{n_k})^*\right\|_{\mathscr{L}^1(H,H)} \\
&\le C_2 \left[\|G\|_{\mathscr{L}^2(H,H)} \left\|G^* - (G^i_{n_k})^*\right\|_{\mathscr{L}^2(H,H)} + \left\|G - G^i_{n_k}\right\|_{\mathscr{L}^2(H,H)} \left\|(G^i_{n_k})^*\right\|_{\mathscr{L}^2(H,H)}\right] \\
&\le C_2 \left\|G - G^i_{n_k}\right\|_{\mathscr{L}^2(H,H)} \left(\|G\|_{\mathscr{L}^2(H,H)} + \left\|G^i_{n_k}\right\|_{\mathscr{L}^2(H,H)}\right)\,.
\end{aligned}$$

Hence, by (6.2) we obtain

$$\left|\mathbb{E}\int_0^t \mathrm{Tr}\left[\left(Q - G^i_{n_k}(G^i_{n_k})^*\right) D^2\varphi_{\lambda_{n_k}}(u^i_{n_k}(s))\right] \mathrm{d}s\right| \le C_2 t \left\|G - G^i_{n_k}\right\|_{\mathscr{L}^2(H,H)} \to 0\,.$$

Putting everything together and letting $k \to +\infty$, for $i = 1, 2$ we infer by the arbitrariness of $T > 0$ that

$$\mathbb{E}\,\varphi(u_i(t)) - \alpha\,\mathbb{E}\int_0^t \varphi(u_i(s))\,\mathrm{d}s + \mathbb{E}\int_0^t g(u_i(s))\,\mathrm{d}s = \varphi(u_0) \quad \forall\, t \ge 0$$

An elementary computation yields then

$$e^{-\alpha t}\,\mathbb{E}\,\varphi(u_i(t)) + \mathbb{E}\int_0^t e^{-\alpha s} g(u_i(s))\,\mathrm{d}s = \varphi(u_0) \quad \forall\, t \ge 0\,,$$

so that letting $t \to +\infty$ yields

$$\varphi(u_0) = \mathbb{E}\int_0^{+\infty} e^{-\alpha t} g(u_1(s))\,\mathrm{d}s = \mathbb{E}\int_0^{+\infty} e^{-\alpha t} g(u_2(s))\,\mathrm{d}s\,. \tag{6.8}$$

Since g is arbitrary in $C_b^{0,s_A}(H)$ and $u_1, u_2 \in L^2_{\mathscr{P}}(\Omega; C^0([0,+\infty); H))$, this implies that u_1 and u_2 have the same law on $C^0([0,+\infty); H)$, as desired. For additional details, the Reader is referred to [5, sec. 4.1].

6.2. Proof of Theorem 2.8

The proof is a direct consequence of the arguments presented in the previous section. Indeed, given a sequence $(\mu_n)_n$ as in Theorem 2.8, one has that μ_n is the distribution of an analytically strong solution u_n of (2.13). Since $(u_0^n)_n$ are bounded in H and $(G_n)_n$ are bounded in $\mathscr{L}^2(H, H)$, the same arguments of Section 4 ensure that $(u_n)_n$ is bounded in $L^1_{\mathcal{P}}(\Omega; C^0([0,T]; H) \cap L^2(0,T; V))$, hence that $(\mu_n)_n$ is tight in $L^2(0,T; H)$. We infer that, possibly extracting a subsequence, one has that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $L^2(0,T; H)$. Moreover, the proof of Theorem 2.7 also ensures that such μ is unique and coincides with the law of u, where u is a Friedrichs-weak solution in the sense of Definition 2.3: hence, the convergence $\mu_n \overset{*}{\rightharpoonup} \mu$ hols for the entire sequence $(\mu_n)_n$, and this concludes the proof.

Declaration of competing interest

The authors have no conflicts of interest to declare.

Acknowledgement

This research was funded in whole or in part by the Austrian Science Fund (FWF) projects 10.55776/F65, 10.55776/I5149, 10.55776/P32788, as well as by the OeAD-WTZ project CZ 09/2023. C.O. and L.S. are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), Istituto Nazionale di Alta Matematica (INdAM). The present research is also part of the activities of "Dipartimento di Eccellenza 2023-2027" of Politecnico di Milano. For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

Data availability

No new data were created or analysed in this study. Data sharing is not applicable.

References

[1] D. Addona, D.A. Bignamini, Pathwise uniqueness for stochastic heat and damped equations with Hölder continuous drift, Preprint arXiv, https://arxiv.org/abs/2308.05415, 2023.
[2] G. Akagi, On some doubly nonlinear parabolic equations, in: Current Advances in Nonlinear Analysis and Related Topics, in: GAKUTO Internat. Ser. Math. Sci. Appl., vol. 32, Gakkōtosho, Tokyo, 2010, pp. 239–254.
[3] V. Barbu, Nonlinear Differential Equations of Monotone Types in Banach Spaces, Springer Monographs in Mathematics, Springer, New York, 2010.
[4] V. Barbu, G. Da Prato, The two phase stochastic Stefan problem, Probab. Theory Relat. Fields 124 (4) (2002) 544–560.
[5] F. Bertacco, C. Orrieri, L. Scarpa, Weak uniqueness by noise for singular stochastic PDEs, Trans. Am. Math. Soc. 378 (11) (2025) 7977–8023.
[6] P. Colli, On some doubly nonlinear evolution equations in Banach spaces, Jpn. J. Ind. Appl. Math. 9 (2) (1992) 181–203.
[7] P. Colli, A. Visintin, On a class of doubly nonlinear evolution equations, Commun. Partial Differ. Equ. 15 (5) (1990) 737–756.
[8] G. Da Prato, F. Flandoli, Pathwise uniqueness for a class of SDE in Hilbert spaces and applications, J. Funct. Anal. 259 (1) (2010) 243–267.

[9] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, Ann. Probab. 41 (5) (2013) 3306–3344.
[10] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations with unbounded measurable drift term, J. Theor. Probab. 28 (4) (2015) 1571–1600.
[11] G. Da Prato, J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.
[12] E. DiBenedetto, R.E. Showalter, Implicit degenerate evolution equations and applications, SIAM J. Math. Anal. 12 (5) (1981) 731–751.
[13] R.E. Edwards, Functional Analysis. Theory and Applications, Holt, Rinehart and Winston, New York-Toronto-London, 1965.
[14] E. Priola, An optimal regularity result for Kolmogorov equations and weak uniqueness for some critical SPDEs, Ann. Probab. 49 (3) (2021) 1310–1346.
[15] E. Priola, Erratum to "An optimal regularity result for Kolmogorov equations and weak uniqueness for some critical SPDEs", Ann. Probab. 51 (6) (2023) 2387–2395.
[16] N. Sapountzoglou, P. Wittbold, A. Zimmermann, On a doubly nonlinear PDE with stochastic perturbation, Stoch. Partial Differ. Equ., Anal. Computat. 7 (2) (2019) 297–330.
[17] L. Scarpa, The stochastic Cahn-Hilliard equation with degenerate mobility and logarithmic potential, Nonlinearity 34 (6) (2021) 3813–3857.
[18] L. Scarpa, U. Stefanelli, Doubly nonlinear stochastic evolution equations, Math. Models Methods Appl. Sci. 30 (5) (2020) 991–1031.
[19] L. Scarpa, U. Stefanelli, Doubly nonlinear stochastic evolution equations II, Stoch. Partial Differ. Equ., Anal. Computat. 11 (1) (2023) 307–347.
[20] L. Scarpa, U. Stefanelli, Rate-independent stochastic evolution equations: parametrized solutions, J. Funct. Anal. 285 (10) (2023) 110102, 48.