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Eigensolution sensitivity revisited

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Abstract
An explicit formulation is developed for the sensitivity of eigensolutions associated with matrix-valued operator pencils depending nonlinearly on a scalar parameter. The eigenproblem is expressed as a nonlinear algebraic system for the eigenvalue and eigenvector, including a normalization constraint. Differentiation of this system yields the sensitivity equations. Regularization of the augmented system followed by symbolic inversion and limiting yields compact, closed-form expressions for the derivatives, applicable also in the presence of algebraic multiplicity. The interpretation of the sensitivity problem relies on an inversion formula for the sum of singular complex matrices, extended through decomposition-based constructions to ensure numerical robustness. Applications to structural dynamics, rigid-body degeneracies, time-delay systems, and continuum eigenproblems illustrate the method's capability to handle ill-conditioning near eigenvalue coalescence and to provide stable sensitivity estimates for both self-adjoint and non-self-adjoint operators.

Keywords Singular value decomposition · Spectral decomposition · Eigenvalue multiplicity

1 Introduction

The sensitivity of eigensolutions to parameter variations is a classical subject that continues to play a central role in computational mechanics and dynamical systems. Small changes in design or operating parameters may produce large shifts in eigenvalues, especially when operators are non-self-adjoint or when eigenvalues are close to coalescence, rendering continuation and optimization procedures numerically delicate. For linear and Hermitian problems, standard perturbation theory provides simple and well-conditioned results. However, nonlinear or non-Hermitian operator pencils require more general formulations [1–3].

In many engineering contexts, the governing equations lead to eigenproblems of the form

$$\mathbf{Z}(s, p)\mathbf{z} = \mathbf{0}, \tag{1}$$

where the operator $\mathbf{Z} \in \mathbb{C}^{n\times n}$, linear in the solution $\mathbf{z} \in \mathbb{C}^n$ (often as a result of a linearization process about a reference configuration, see for example [4, 5]), depends nonlinearly on the spectral parameter $s \in \mathbb{C}$ and on one or multiple design or physical parameters $p \in \mathbb{R}$. In the classic eigenproblem form, $\mathbf{Z}(s, p) = s\mathbf{I} - \mathbf{A}(p)$. Typical instances of engineering interest include undamped and damped structural dynamics, possibly incorporating rigid-body motions, follower-force or preload problems, aeroelastic models involving transcendental dependence on the Laplace variable, constrained dynamics problems resulting in systems of differential-algebraic equations, time-delay systems, and continuum eigenproblems constrained by boundary conditions.

The computation of derivatives of quantities defined implicitly by nonlinear algebraic and eigenvalue problems has a long history. Early applications appeared in aeroelasticity, where continuation techniques were obtained by

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differentiating the flutter equations with respect to flight parameters and solving the resulting linearized systems (see for example Mantegazza and Cardani [6]). Related sensitivity formulations were later proposed for aeroelastic derivatives and nonlinear equilibrium equations (Bindolino and Mantegazza [7]).

The smooth dependence of invariant eigenspaces on perturbation parameters is a classical result of perturbation theory (Kato [8]). More recent works have exploited this framework to construct eigenvector-derivative formulations applicable to repeated eigenvalues and mode veering phenomena (e.g., Ruiz et al. [9]).

More general eigensensitivity formulations were subsequently developed in the matrix-analysis literature, including the work of Andrew et al. [10], who derived eigenvalue and eigenvector derivatives for analytic matrix-valued functions, discussed normalization strategies, bordered formulations, and multiple eigenvalues. Subsequent developments addressed higher-order derivatives [11] and analyticity properties of semisimple eigensolutions [12, 13]. A recent and comprehensive survey of theory and applications of eigensolution derivatives is provided in Lin et al. [14].

The mathematical difficulties intensify when the eigenvalue has algebraic multiplicity greater than one, or when the operator pencil becomes singular at the eigenvalue. In such situations, classical linearization or direct inversion is ill-conditioned, and explicit, computationally stable sensitivity expressions are required [9, 15, 16].

The present study addresses these issues by combining a nonlinear algebraic formulation of the eigenproblem, that naturally accommodates sensitivity analysis of general nonlinear eigenproblems, with a regularization-based derivation of eigensolution sensitivities. The eigenproblem is expressed as an augmented system for the eigenvalue and eigenvector, including a normalization constraint. Differentiation of this system yields a linear system for the sensitivities. Because the coefficient matrix of this system includes a singular block, a scalar regularization parameter is introduced. The resulting system is solved symbolically, and the limit for vanishing regularization is evaluated. This procedure produces explicit formulas for both eigenvalue and eigenvector derivatives under broad nondegeneracy conditions and clarifies the algebraic structure responsible for ill-conditioning near degeneracies.

A key element for the mathematical and physical interpretation of the derivation is the inversion of the sum of singular complex matrices that together span the whole space. Building on recent work on the inversion of the sum of singular and low-rank matrices [17], the analysis provides an expression based on the spectral decomposition alternative to the original one based on the singular-value decomposition. The resulting formulas are compact, numerically stable, and suitable for embedding in eigensolvers or continuation algorithms.

The paper is organized as follows. Section 2 develops the nonlinear algebraic formulation and derives the sensitivity equations. Section 3 presents the inversion formulas required to evaluate the limiting expressions. Section 4 illustrates the method on representative problems from structural dynamics, rigid-body mechanics, time-delay systems, and continuum eigenanalysis. Section 5 summarizes the main findings and discusses possible extensions.

2 Eigensolution and its sensitivity

Consider the eigenproblem of Eq. (1). Being $\mathbf{z} = \mathbf{z}_\ell \neq \mathbf{0}$ the ℓth eigenvector requires the singularity of matrix $\mathbf{Z}$ when $s = s_\ell$ is the ℓth eigenvalue of the eigenproblem of Eq. (1).

Problems of this type arise in several circumstances of engineering relevance; for example,

- in a typical structural dynamics problem,

$$\mathbf{Z} = s^2\mathbf{M} + \mathbf{K} \tag{2}$$

where $\mathbf{M} \in \mathrm{R}^{n\times n}$ and $\mathbf{K} \in \mathrm{R}^{n\times n}$ are the usual mass and stiffness matrices, symmetric and positive definite (the latter at most positive semidefinite in case rigid body motions are allowed). In that case, $s^2 \leq 0$ and thus $s \in \mathbb{I}$, i.e., it is purely imaginary;

- or

$$\mathbf{Z} = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{3}$$

  when the damping matrix $\mathbf{C} \in \mathrm{R}^{n\times n}$ is also present;
- or in static stability of structures subjected to a preload s (usually, $s \in \mathbb{R}$ plays the role of *critical load*),

$$\mathbf{Z} = \mathbf{K} + s\mathbf{K}_p \tag{4}$$

  where $\mathbf{K}_p \in \mathrm{R}^{n\times n}$ is the preload matrix;
- or in "classical" aeroelasticity problems,

$$\mathbf{Z} = s^2\mathbf{M} + \mathbf{K} - q\mathbf{H}_{am}(p) \tag{5}$$

  where $\mathbf{H}_{am}$ is the linear aeroelastic operator that describes generalized unsteady aerodynamic forces implicitly and transcendentally dependent on the non-dimensional Laplace variable[1] $p = sb/V_\infty$, b being a reference length[2], V_∞ a reference wind speed, and $q = \rho V_\infty^2/2$ the dynamic pressure, a function of the air density ρ (for an early example of eigensolution sensitivity analysis applied to flutter continuation, see [6]);
- or in constrained dynamics problems with kinematic constraints in form of algebraic equations, which result in Index 3 systems of differential-algebraic equations, yielding, after linearization about a reference configuration, a problem formally analogous to that of conventional structural dynamics but with a structurally singular mass matrix, as discussed in Sect. 4.6;
- or in quasi-linear problems with time delays, expressed in the Laplace domain as $\mathrm{e}^{\tau s}$, yielding

$$\mathbf{Z} = s^2\mathbf{M} + \mathbf{K} + \mathrm{e}^{\tau s}\mathbf{H} \tag{6}$$

- or in continuum dynamics problems where a wave function is determined depending on some parameter that is evaluated by enforcing appropriate boundary conditions, as will be discussed in Sect. 4.8.

In the following we will:

- formulate the eigenproblem as a nonlinear algebraic problem in the eigenvalue s and eigenvector $\mathbf{z}$ (Sect. 2.1);
- formulate the sensitivity of a generic eigensolution with respect to the parameter p (Sect. 2.2);
- formulate an explicit solution of the eigensolution sensitivity problem to highlight some important properties (Sect. 2.3, a contribution considered original);
- develop the case of eigenvalue multiplicity greater than one for both eigenproblem and eigensolution sensitivity, including the latter's explicit solution (Sect. 2.4 and 2.5, contributions considered original);
- present a general formulation and two numerical methods for the explicit solution of the sum of singular complex matrices (Sect. 3, a contribution considered at least partly original, as it adds a novel formulation based on the spectral decomposition to the singular value decomposition-based one proposed in [17]);
- apply the proposed methods to some of the above outlined contexts (Sect. 4).

[1] It is unfortunate that the traditional symbol for the non-dimensional (or reduced) Laplace variable, p, clashes with the one in use here for the parameter, but there is no risk of confusion or ambiguity.

[2] For the *typical section* problem, b is half the airfoil's chord.

2.1 Eigenproblem as nonlinear problem

The eigenproblem of Eq. (1) can be reformulated as a nonlinear algebraic problem in the eigenvector $\mathbf{z}$ and eigenvalue s by adding the eigenvector L^2, or Euclidean, weighted normalization equation

$$\frac{1}{2}\mathbf{z}^{\mathsf{H}}\mathbf{W}\mathbf{z} = \frac{1}{2} \tag{7}$$

where $(\cdot)^{\mathsf{H}}$ is the conjugate-transpose (or Hermitian transpose) operator and $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a symmetric, positive-definite normalization matrix (without excessive loss of generality, we can assume $\mathbf{W} = \mathbf{I}$), i.e.,

$$\mathbf{Z}\mathbf{z} = \mathbf{0} \tag{8a}$$

$$\frac{1}{2}\mathbf{z}^{\mathsf{H}}\mathbf{z} = \frac{1}{2} \tag{8b}$$

Other normalizations would be legit: for example, prescribed (e.g., unit) value of the largest absolute value component (L^∞ norm), although for practical implementation it requires prior knowledge of the index of such component.

Indeed, if we formulate the eigenproblem of Eqs. (8) as a real-valued problem, we obtain

$$\mathrm{Re}(\mathbf{Z})\,\mathrm{Re}(\mathbf{z}) - \mathrm{Im}(\mathbf{Z})\,\mathrm{Im}(\mathbf{z}) = \mathbf{0} \quad \rightarrow \text{ the real part of Eq. (8}a\text{)} \tag{9a}$$

$$\mathrm{Im}(\mathbf{Z})\,\mathrm{Re}(\mathbf{z}) + \mathrm{Re}(\mathbf{Z})\,\mathrm{Im}(\mathbf{z}) = \mathbf{0} \quad \rightarrow \text{ the imaginary part of Eq. (8}a\text{)} \tag{9b}$$

$$\mathrm{Re}(\mathbf{z})^{\mathsf{T}}\,\mathrm{Re}(\mathbf{z}) + \mathrm{Im}(\mathbf{z})^{\mathsf{T}}\,\mathrm{Im}(\mathbf{z}) = 1 \quad \rightarrow \text{ Eq. (8}b\text{) multiplied by 2} \tag{9c}$$

Considering that the unknowns are $\mathrm{Re}(\mathbf{z})$, $\mathrm{Im}(\mathbf{z})$, $\mathrm{Re}(s)$, and $\mathrm{Im}(s)$, i.e., $2n + 2$ terms, Eqs. (9) are short of one equation, resulting in an indetermination in the absolute phase of the elements of the eigenvector. In fact, normalizing it with unit L^2 norm leaves the eigenvector undetermined with respect to multiplication by a unit-norm complex number, $e^{j\theta}$, with $\theta \in \mathbb{R}$.

The problem of Eqs. (8) can be solved through an iterative Newton-like process. After linearizing about a tentative solution $\{\mathbf{z}_0; s_0\}$,

$$\mathbf{Z}\mathbf{z}_0 + \mathbf{Z}\Delta\mathbf{z} + \mathbf{Z}_{/s}\mathbf{z}_0\Delta s = \mathbf{0} \tag{10a}$$

$$\frac{1}{2}\left(\mathbf{z}_0^{\mathsf{H}}\mathbf{z}_0 + \mathbf{z}_0^{\mathsf{H}}\Delta\mathbf{z} + \mathbf{z}_0^{\mathsf{T}}\mathrm{conj}\,(\Delta\mathbf{z})\right) = \frac{1}{2} \tag{10b}$$

where matrix $\mathbf{Z}$ and its derivative with respect to s, $\mathbf{Z}_{/s} = \partial\mathbf{Z}/\partial s$ (in the following, we use $\clubsuit_{/\spadesuit}$ to indicate the partial derivative of $\clubsuit$ with respect to $\spadesuit$) are evaluated for $s = s_0$, the problem can be reorganized as

$$\begin{bmatrix} \mathbf{Z} & \mathbf{Z}_{/s}\mathbf{z}_0 \\ \mathbf{z}_0^{\mathsf{H}} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{z} \\ \Delta s \end{Bmatrix} = \begin{Bmatrix} -\mathbf{Z}\mathbf{z}_0 \\ (1 - \mathbf{z}_0^{\mathsf{H}}\mathbf{z}_0)/2 \end{Bmatrix} \tag{11}$$

where $\mathbf{z}_0^{\mathsf{T}}\mathrm{conj}\,(\Delta\mathbf{z})$ was arbitrarily replaced by $\mathbf{z}_0^{\mathsf{H}}\Delta\mathbf{z}$. The rationale of this modification and its justification is to get rid of $\mathrm{conj}\,(\Delta\mathbf{z})$ and keep only $\Delta\mathbf{z}$ as the unknown of the linearized problem. Its immediate consequence is that now the last block-row of Eq. (11) is no longer the linearization of Eq. (9c); it has an imaginary part, which is prescribed to vanish as well.

In fact, reformulating Eq. (11) as a real-valued problem, we obtain

$$\begin{bmatrix} \mathrm{Re}(\mathbf{Z}) & -\mathrm{Im}(\mathbf{Z}) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) & -\mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) \\ \mathrm{Im}(\mathbf{Z}) & \mathrm{Re}(\mathbf{Z}) & \mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) \\ \mathrm{Re}(\mathbf{z}_0)^{\mathsf{T}} & \mathrm{Im}(\mathbf{z}_0)^{\mathsf{T}} & 0 & 0 \\ -\mathrm{Im}(\mathbf{z}_0)^{\mathsf{T}} & \mathrm{Re}(\mathbf{z}_0)^{\mathsf{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathrm{Re}(\Delta\mathbf{z}) \\ \mathrm{Im}(\Delta\mathbf{z}) \\ \mathrm{Re}(\Delta s) \\ \mathrm{Im}(\Delta s) \end{Bmatrix} = \begin{Bmatrix} -\mathrm{Re}(\mathbf{Z}\mathbf{z}_0) \\ -\mathrm{Im}(\mathbf{Z}\mathbf{z}_0) \\ (1-\mathbf{z}_0^{\mathsf{H}}\mathbf{z}_0)/2 \\ 0 \end{Bmatrix} \tag{12}$$

i.e., an additional, unjustified scalar equation appears, which however is linearly independent from the others. Specifically, the corresponding matrix row, the last in Eq. (12), is of exactly unit L^2 norm and orthogonal to the second last row. It serves the purpose of making the problem well-posed, at the cost of breaking the superlinear convergence of the Newton–Raphson solution algorithm, but without otherwise affecting the outcome of the solution process.

An alternative approach, that preserves superlinear convergence and (arbitrarily) eliminates the aforementioned indetermination would be to replace the last block-row in Eq. (12) with a simple de-phasing equation that prescribes an arbitrary component of the eigenvector to be real-valued, namely

$$\mathbf{n}^{\mathsf{T}}\mathrm{Im}(\Delta\mathbf{z}) = 0 \tag{13}$$

with

$$\mathbf{n} = \left[0 \ldots 1 \ldots 0\right]^{\mathsf{T}} \tag{14}$$

i.e., a zero-valued vector except for the kth coefficient, which has unit value. The index of the component of $\mathbf{z}_0$ with largest absolute value can be used for k, to guarantee the well-posedness of the problem.

Such modification, however, would break the simple structure of Eq. (11), requiring the solution of

$$\begin{bmatrix} \mathrm{Re}(\mathbf{Z}) & -\mathrm{Im}(\mathbf{Z}) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) & -\mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) \\ \mathrm{Im}(\mathbf{Z}) & \mathrm{Re}(\mathbf{Z}) & \mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_0\right) \\ \mathrm{Re}(\mathbf{z}_0)^{\mathsf{T}} & \mathrm{Im}(\mathbf{z}_0)^{\mathsf{T}} & 0 & 0 \\ \mathbf{0}^{\mathsf{T}} & \mathbf{n}^{\mathsf{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathrm{Re}(\Delta\mathbf{z}) \\ \mathrm{Im}(\Delta\mathbf{z}) \\ \mathrm{Re}(\Delta s) \\ \mathrm{Im}(\Delta s) \end{Bmatrix} = \begin{Bmatrix} -\mathrm{Re}(\mathbf{Z}\mathbf{z}_0) \\ -\mathrm{Im}(\mathbf{Z}\mathbf{z}_0) \\ (1-\mathbf{z}_0^{\mathsf{H}}\mathbf{z}_0)/2 \\ 0 \end{Bmatrix} \tag{15}$$

instead.

Under broad assumptions, the linear problem of Eq. (11) can be solved, although losing superlinear convergence (more details in the following). The solution is then used to update the tentative solution, namely $\mathbf{z}_0 := \mathbf{z}_0 + \Delta\mathbf{z}$ and $s_0 := s_0 + \Delta s$, and the process is repeated until the norm of the right-hand side of Eq. (11) falls below the desired threshold, and $\{\mathbf{z}; s\} \to \{\mathbf{z}_\ell; s_\ell\}$ as long as $\{\mathbf{z}_0; s_0\}$ were within the basin of attraction of the solution. At this stage, matrix $\mathbf{Z}$ becomes singular; it is legitimate to expect that while approaching convergence, it quickly becomes ill-conditioned.

2.2 Sensitivity of eigensolution

Assuming that the problem of Eq. (1) depends on a set $\mathcal{P}$ of parameters $p \in \mathcal{P}$, the sensitivity of the ℓth eigensolution to a given parameter p is obtained by considering the derivative of the nonlinear problem with respect to the parameter itself, following the classical bordered-system approach of Andrew et al. [10]. The derivative of the eigenproblem of Eq. (8a) yields

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{Z}\mathbf{z}_\ell\right) = \mathbf{Z}\mathbf{z}_{\ell/p} + \left(\mathbf{Z}_{/s}s_{\ell/p} + \mathbf{Z}_{/p}\right)\mathbf{z}_\ell = \mathbf{0}. \tag{16}$$

Under broad regularity conditions, differentiation of matrix $\mathbf{Z}$ with respect to the parameter p, namely $\mathbf{Z}_{/p}$, can be computed analytically or numerically evaluated using finite differences, possibly using complex variable perturbation when the problem is, or can be reduced to, (combinations of) real-valued matrices, as proposed by Squire and Trapp [18]. Particular care must be taken, however, in case of nonlinear manifolds, to ensuring that the sensitivity of the local tangent matrix is computed accordingly, as highlighted later in the example of Sect. 4.5.2.

The derivative of the normalization equation, Eq. (8b), with respect to p would yield

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{2}\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z}_\ell - \frac{1}{2}\right) = \frac{1}{2}\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z}_{\ell/p} + \frac{1}{2}\mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}(\mathbf{z}_{\ell/p}) = 0. \tag{17}$$

As discussed earlier for Eq. (11), one may be tempted to get rid of the term containing $\mathrm{conj}(\mathbf{z}_{\ell/p})$ by only considering $\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z}_{\ell/p}$. However, this would lead to computing the sensitivity of the eigenvector inconsistently with respect to the selected normalization criterion.

In analogy with Eq. (15), the sensitivity problem consistent with L^2 normalization of the eigenvector would require the solution of the problem

$$\begin{bmatrix} \mathrm{Re}(\mathbf{Z}) & -\mathrm{Im}(\mathbf{Z}) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_\ell\right) & -\mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_\ell\right) \\ \mathrm{Im}(\mathbf{Z}) & \mathrm{Re}(\mathbf{Z}) & \mathrm{Im}\left(\mathbf{Z}_{/s}\mathbf{z}_\ell\right) & \mathrm{Re}\left(\mathbf{Z}_{/s}\mathbf{z}_\ell\right) \\ \mathrm{Re}(\mathbf{z}_\ell)^{\mathsf{T}} & \mathrm{Im}(\mathbf{z}_\ell)^{\mathsf{T}} & 0 & 0 \\ \mathbf{0}^{\mathsf{T}} & \mathbf{n}^{\mathsf{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathrm{Re}\left(\mathbf{z}_{\ell/p}\right) \\ \mathrm{Im}\left(\mathbf{z}_{\ell/p}\right) \\ \mathrm{Re}\left(s_{\ell/p}\right) \\ \mathrm{Im}\left(s_{\ell/p}\right) \end{Bmatrix} = \begin{Bmatrix} -\mathrm{Re}\left(\mathbf{Z}_{/p}\mathbf{z}_\ell\right) \\ -\mathrm{Im}\left(\mathbf{Z}_{/p}\mathbf{z}_\ell\right) \\ 0 \\ 0 \end{Bmatrix} \tag{18}$$

where vector $\mathbf{n}$ can be constructed as in Eq. (14) by considering the largest absolute value component, of index k, of the now known reference eigenvector, $\mathbf{z}_\ell$. We call this L^2-norm sensitivity, and indicate its result as $\mathbf{z}_{\ell/p_{\|\cdot\|_2}}$.

As an alternative, we can use the same largest absolute value component of $\mathbf{z}_\ell$ to devise a unit L^∞-norm normalization of the eigenvector, rescaling and de-phasing the eigenvector with respect of its kth component as $\mathbf{z}_\ell := \mathbf{z}_\ell/\mathbf{z}_\ell(k)$. Then, the normalization equation can be replaced by $\mathbf{n}^{\mathsf{T}}\mathbf{z}_{\ell/p} = 0$.

At this point, the differentiation with respect to the parameter p of the eigenproblem written as a nonlinear problem yields

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{Z}\mathbf{z}_\ell\right) = \mathbf{0} \tag{19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{n}^{\mathsf{T}}\mathbf{z}_\ell - 1\right) = 0 \tag{19b}$$

i.e.,

$$\mathbf{Z}\mathbf{z}_{\ell/p} + \left(\mathbf{Z}_{/s}s_{\ell/p} + \mathbf{Z}_{/p}\right)\mathbf{z}_\ell = \mathbf{0} \tag{20a}$$

$$\mathbf{n}^{\mathsf{T}}\mathbf{z}_{\ell/p} = 0 \tag{20b}$$

which in matrix form is

$$\begin{bmatrix} \mathbf{Z} & \mathbf{Z}_{/s}\mathbf{z}_\ell \\ \mathbf{n}^{\mathsf{T}} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{z}_{\ell/p} \\ s_{\ell/p} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{Z}_{/p}\mathbf{z}_\ell \\ 0 \end{Bmatrix} \tag{21}$$

preserving the simple structure of Eq. (11). We call this L^∞-norm sensitivity, and indicate its result as $\mathbf{z}_{\ell/p_{\|\cdot\|_\infty}}$.

The two definitions produce different results that are however equivalent in the sense that

$$\left\|\mathbf{z}_{\ell_{\|\cdot\|_2}} + \mathbf{z}_{\ell_{\|\cdot\|_2}/p}\Delta p\right\|_{\clubsuit} = \left\|\mathbf{z}_{\ell_{\|\cdot\|_\infty}} + \mathbf{z}_{\ell_{\|\cdot\|_\infty}/p}\Delta p\right\|_{\clubsuit} + o(\Delta p) \tag{22}$$

where $\clubsuit = \{2, \infty\}$, i.e., the two norms yield equivalent first-order approximations of the perturbed eigenvector. It is worth stressing that the two norms (actually, any normalization criterion, including that of Eq. (17), although not representative of any meaningful normalization criterion) yield identical eigenvalue sensitivity, as they only differ in the normalization of the eigenvector. This is demonstrated in Sect. 3.4.

One may also observe that the L^2-norm sensitivity differs from the L^∞-norm by an unknown vector, $\mathbf{v}$, i.e., $\mathbf{z}_{\ell_{\|\cdot\|_2}/p} := \mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{v}$. At the same time, it must satisfy Eqs. (16) and (17), i.e.,

$$\mathbf{Z}\left(\cancel{\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p}} + \mathbf{v}\right) + \cancel{\left(\mathbf{Z}_{/s}s_{\ell/p} + \mathbf{Z}_{/p}\right)\mathbf{z}_\ell} = \mathbf{0} \tag{23a}$$

$$\mathbf{z}_\ell^{\mathsf{H}}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{v}\right) + \mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{v}\right) = 0, \tag{23b}$$

which implies $\mathbf{Z}\mathbf{v} = \mathbf{0}$, i.e., $\mathbf{v} \equiv \mathbf{z}\alpha$, where α is a complex scalar, and thus

$$\mathbf{z}_\ell^{\mathsf{H}}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{z}\alpha\right) + \mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{z}\alpha\right) = 0. \tag{24}$$

Considering that $\mathrm{conj}(\mathbf{z}\alpha) = \mathrm{conj}(\mathbf{z})\,\mathrm{conj}(\alpha)$ and $\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z} = \mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}\,(\mathbf{z}) = 1$, one obtains

$$\alpha + \mathrm{conj}(\alpha) = 2\mathrm{Re}(\alpha) = -\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} - \mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p}\right) \tag{25}$$

and thus the L^2-norm sensitivity can be recovered as

$$\mathbf{z}_{\ell_{\|\cdot\|_2}/p} = \mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} - \mathbf{z}\frac{\mathbf{z}_\ell^{\mathsf{H}}\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p} + \mathbf{z}_\ell^{\mathsf{T}}\mathrm{conj}\left(\mathbf{z}_{\ell_{\|\cdot\|_\infty}/p}\right)}{2}, \tag{26}$$

where the imaginary part of α is arbitrarily set to zero.

Under broad assumptions, the problem of Eq. (21) can usually be solved numerically without particular issues. In fact, as will be analyzed further in the following, despite matrix $\mathbf{Z}$ being structurally singular when $s = s_\ell$, such that vector $\mathbf{z}_\ell$ represents its nullspace, the expanded problem of Eq. (21) is well-posed thanks to the structural properties of matrix $\mathbf{Z}_{/s}$, such that the combination of the rank-deficient matrix $\mathbf{Z}$ and of vectors $\mathbf{Z}_{/s}\mathbf{z}$ and $\mathbf{n}^{\mathsf{T}}$ restore the full column- and row-rank of the problem.

2.3 Explicit solution of sensitivity problem

Obtaining an explicit solution of Eq. (21) is not trivial. It is worth pointing out that it is not even necessary, since the solution can be obtained numerically solving the problem of Eq. (21). However, such explicit solution is sought here to better understand the nature and properties of the problem.

Owing to the zero in the bottom-right corner of the matrix in the said equation, it would be tempting to formally write

$$\mathbf{z}_{\ell/p} = -\mathbf{Z}^{-1}\left(\mathbf{Z}_{/p}\mathbf{z}_\ell + \mathbf{Z}_{/s}\mathbf{z}_\ell s_{\ell/p}\right) \tag{27}$$

which would yield

$$s_{\ell/p} = -\left(\mathbf{n}^{\mathsf{T}}\mathbf{Z}^{-1}\mathbf{Z}_{/s}\mathbf{z}_\ell\right)^{-1}\mathbf{n}^{\mathsf{T}}\mathbf{Z}^{-1}\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{28a}$$

$$\mathbf{z}_{\ell/p} = -\mathbf{Z}^{-1}\left(\mathbf{I} - \mathbf{Z}_{/s}\mathbf{z}_\ell\left(\mathbf{n}^{\mathsf{H}}\mathbf{Z}^{-1}\mathbf{Z}_{/s}\mathbf{z}_\ell\right)^{-1}\mathbf{n}^{\mathsf{H}}\mathbf{Z}^{-1}\right)\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{28b}$$

However, this approach is not feasible because of the singularity of matrix $\mathbf{Z}$ when $s = s_\ell$. An alternative, robust formulation can be obtained by:

a. "regularizing" the problem by replacing the bottom-right zero in the matrix of Eq. (21) with a coefficient $\gamma \neq 0$;
b. solving the problem in symbolic form;
c. considering the limit of the solution for $\gamma \to 0$.

a. *Modified problem.* The modified problem is

$$\begin{bmatrix} \mathbf{Z} & \mathbf{Z}_{/s}\mathbf{z}_\ell \\ \mathbf{n}^{\mathsf{H}} & \gamma \end{bmatrix} \begin{Bmatrix} \mathbf{z}_{\ell/p} \\ s_{\ell/p} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{Z}_{/p}\mathbf{z}_\ell \\ 0 \end{Bmatrix} \tag{29}$$

b. *Explicit solution.* Its solution is obtained by first solving the second block equation for $s_{\ell/p}$, thanks to the added scalar γ,

$$s_{\ell/p} = -\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\mathbf{z}_{\ell/p} \tag{30}$$

and replacing it in the first block equation, yielding

$$\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)\mathbf{z}_{\ell/p} = -\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{31}$$

and thus

$$\mathbf{z}_{\ell/p} = -\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)^{-1} \mathbf{Z}_{/p}\mathbf{z}_\ell \tag{32a}$$

$$s_{\ell/p} = \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)^{-1} \mathbf{Z}_{/p}\mathbf{z}_\ell \tag{32b}$$

How to obtain an explicit inversion of matrix $\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell\mathbf{n}^{\mathsf{H}}/\gamma$ is discussed in Sect. 3.
c. *Limit.* The limit of Eqs. (32) for $\gamma \to 0$ is challenging, because γ appears at the denominator of multiple expressions. This aspect of the problem will be discussed in Sect. 3.4, where the explicit inversion of the matrix based on the algorithm illustrated in Sect. 3 and some of its properties will be exploited.

2.4 Multiplicity

The solution of Sect. 2.1 and the eigensolution sensitivity formulated in Sect. 2.2 become ill-posed when the eigenvalue has algebraic multiplicity greater than one. If the geometrical multiplicity of the problem is exactly equal to one, the nonlinear eigenproblem can be reformulated as follows.

Define a set of m vectors,

$$\mathbf{z} = \left[\mathbf{z}_1, \ \ldots, \ \mathbf{z}_m\right] \tag{33}$$

where $\mathbf{z} \in \mathbb{C}^{n\times m}$, $1 \leq m \leq n$. Its number of columns must be $m = M$, where M is the (expected) multiplicity of the eigenvalue. For $m = 1$ the original method is obtained.

The nonlinear problem becomes

$$\mathbf{Z}(s)\mathbf{z} = \mathbf{0} \tag{34a}$$

$$\frac{1}{2}\mathbf{z}^{\mathsf{H}}\mathbf{z} = \frac{1}{2}\mathbf{I} \tag{34b}$$

Its linearization yields

$$\begin{bmatrix} \mathbf{Z}(s) & \mathbf{Z}_{/s}\mathbf{z} \\ \mathbf{z}^{\mathsf{H}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{s} \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}(s)\mathbf{z} \\ \frac{1}{2}\left(\mathbf{I} - \mathbf{z}^{\mathsf{H}}\mathbf{z}\right) \end{bmatrix} \tag{35}$$

where $\Delta\mathbf{s} \in \mathbb{C}^{m\times m}$ is a matrix of increments for the estimated eigenvalues. At each iteration of the solution process, the tentative eigenvalue s is updated as s += $\Delta\mathbf{s}(1, 1)$, arbitrarily choosing the first of the diagonal elements of the tentative eigenvalue increments matrix (the choice of any other diagonal element converges to the same result), whereas the tentative eigenvectors matrix is consistently updated as $\mathbf{z} + = \Delta\mathbf{z}$. In this manner, the columns of $\mathbf{z}$ are expected to converge to the eigenvectors corresponding to the desired eigenvalue whatever the multiplicity M. Otherwise, if $M > 1$, more eigenvectors will converge to the corresponding eigensolution associated with the same eigenvalue, which is used to update $\mathbf{Z}(s)$ and its gradient, $\mathbf{Z}_{/s}$, while remaining mutually orthogonal thanks to Eq. (34b).

The sensitivity problem requires further manipulation. Consider

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{Z}(s_\ell)\mathbf{z}_\ell\right) = \mathbf{0}, \tag{36}$$

where

$$\mathbf{z}_\ell = [\mathbf{z}_{\ell_1}, \ldots, \mathbf{z}_{\ell_m}] \tag{37}$$

is the matrix $\in \mathbb{C}^{n\times m}$ obtained by aligning all distinct eigenvectors that correspond to the eigenvalue s_ℓ of multiplicity m, i.e.,

$$\mathbf{Z}\mathbf{z}_{\ell/p} + \mathbf{Z}_{/s}\mathbf{z}_\ell\mathbf{s}_{\ell/p} + \mathbf{Z}_{/p}\mathbf{z}_\ell = \mathbf{0}, \tag{38}$$

along with

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{n}^{\mathsf{T}}\mathbf{z}_\ell - \mathbf{I}\right) = \mathbf{0}, \tag{39}$$

where matrix $\mathbf{n} \in \mathbb{R}^{n\times m}$ is now constituted by putting side by side vectors defined according to Eq. (14). Their unit-value element is associated with the largest absolute value element of the corresponding eigenvector and, in case of multiple elements of the same eigenvector of identical largest absolute value, the index of the unit-value element of each Eq. (14) vector must be chosen to avoid identical vectors.

The corresponding linear problem in matrix form is

$$\begin{bmatrix} \mathbf{Z}(s_\ell) & \mathbf{Z}_{/s}\mathbf{z}_\ell \\ \mathbf{n}^{\mathsf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\ell/p} \\ \mathbf{s}_{\ell/p} \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}_{/p}\mathbf{z}_\ell \\ \mathbf{0} \end{bmatrix}. \tag{40}$$

The solution, in terms of eigenvalue sensitivity $\mathbf{s}_{\ell/p} \in \mathbb{C}^{m\times m}$, is a square matrix, whose elements express how each direction, represented by the corresponding column of $\mathbf{z}_\ell$, contributes to the sensitivity of each instance of the eigenvalue with multiplicity greater than one. The spectral decomposition of the eigenvalue sensitivity matrix yields

$$\mathbf{s}_{\ell/p} = \mathbf{X}\Lambda\mathbf{X}^{-1}, \tag{41}$$

where Λ is a diagonal matrix whose diagonal elements $s_{\ell/p}$ represent the M sensitivities of the eigenvalue; the non-zero ones are associated with the eigensolutions that change according to their dependence on p,

$$\hat{\mathbf{s}}_{/p} = \mathrm{diag}\,(\Lambda)\,. \tag{42}$$

The sensitivity eigenvector matrix $\mathbf{X}$ describes the directions, in the space of the eigenvectors, where the sensitivity produces the variations of the eigenvectors that are associated with the eigenvalue variations $\hat{\mathbf{s}}_{/p}$. As such, the corrected sensitivity of the eigenvectors is

$$\hat{\mathbf{z}}_{\ell/p} = \mathbf{z}_{\ell/p}\mathbf{X} \tag{43}$$

It is worth noticing that the same recombination can also be applied to the eigenvectors originally determined by the algorithm, as they are defined in excess of a recombination that preserves their orthogonality and the spanned subspace. As such, it may be convenient to reformulate them as

$$\hat{\mathbf{z}}_{\ell} = \mathbf{z}_{\ell}\mathbf{X} \tag{44}$$

The importance—and convenience—of this result will be illustrated in the examples in Sect. 4.

2.5 Explicit sensitivity for multiplicity greater than one

Obtaining an explicit solution of Eq. (40) is even less trivial because matrix $\mathbf{Z}$ for $s = s_\ell$ is m times singular, as its rank reduces to $n - m$.

The approach outlined in Sect. 2.3 can be modified by:

a. "regularizing" the problem by replacing the bottom-right zero submatrix in the matrix of Eq. (40) with a coefficient $\gamma \neq 0$ that multiplies an identity matrix of appropriate size;
b. solving the problem in symbolic form;
c. considering the limit for $\gamma \to 0$.

a. *Modified problem.* The modified problem is

$$\begin{bmatrix} \mathbf{Z} & \mathbf{Z}_{/s}\mathbf{z}_\ell \\ \mathbf{n}^{\mathsf{H}} & \gamma\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{z}_{\ell/p} \\ \mathbf{s}_{\ell/p} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{Z}_{/p}\mathbf{z}_\ell \\ \mathbf{0} \end{Bmatrix} \tag{45}$$

b. *Explicit solution.* Its solution is obtained by first solving the second block equation for $\mathbf{s}_{\ell/p}$, thanks to the added scalar γ,

$$\mathbf{s}_{\ell/p} = -\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\mathbf{z}_{\ell/p} \tag{46}$$

and replacing it in the first block equation, yielding

$$\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)\mathbf{z}_{\ell/p} = -\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{47}$$

and thus

$$\mathbf{z}_{\ell/p} = -\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)^{-1}\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{48a}$$

$$\mathbf{s}_{\ell/p} = \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_{\ell}\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)^{-1}\mathbf{Z}_{/p}\mathbf{z}_{\ell} \tag{48b}$$

How to obtain an explicit inversion of matrix $\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_{\ell}\mathbf{n}^{\mathsf{H}}/\gamma$ is discussed in Sect. 3.
c. *Limit.* The same considerations made in Sect. 2.3 also apply in this case.

3 Sum of singular matrices inversion revisited

The inversion of the non-singular sum of real singular matrices, as formulated in [17], was revisited and extended in Appendix A of [19], in a context analogous to that of Sect. 2.3 applied to eigenproblems within the Transfer Matrix Method. Such extension is recalled here because it is central to the considerations that will be developed further in Sect. 4.

The matrix inversion of Eq. (32a) can be expressed through the algorithm originally proposed in [17]. Consider the problem

$$\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{e}\mathbf{D}\mathbf{f}^{\mathsf{H}} \tag{49}$$

where, in the context of Sect. 2.3 and 2.5, and significantly of Eqs. (32a) and (48a),

$$\mathbf{A} \in \mathbb{C}^{n\times n} := \mathbf{Z} \in \mathbb{C}^{n\times n} \tag{50a}$$

$$\mathbf{e} \in \mathbb{C}^{n\times k} := -\mathbf{Z}_{/s}\mathbf{z}_{\ell} \in \mathbb{C}^{n\times 1}\ (\text{Eq. } (32a)) \text{ viz. } \mathbb{C}^{n\times m}\ (\text{Eq. } (48a)) \tag{50b}$$

$$\mathbf{D} \in \mathbb{C}^{k\times k} := \frac{1}{\gamma} \in \mathbb{R}\ (\text{Eq. } (32a)) \text{ viz. } \frac{1}{\gamma}\mathbf{I} \in \mathbb{R}^{m\times m}(48a)) \tag{50c}$$

$$\mathbf{f} \in \mathbb{C}^{n\times k} := \mathbf{n} \in \mathbb{C}^{n\times 1}\ (\text{Eq. } (32a)) \text{ viz. } \mathbb{C}^{n\times m}(48a)) \tag{50d}$$

with $n > r \geq 1$ and $k = n - r$ (in the present case, $r = n - 1$ viz. $r = n - m$, and $k = 1$ viz. $k = m$). It is worth noting that in [17] matrix $\mathbf{D}$ must be invertible; in the present context, $\mathbf{D} \to \infty\mathbf{I}$; as such, it can be considered invertible, with $\mathbf{D}^{-1} = \gamma\mathbf{I} \to \mathbf{0}$.

3.1 Singular value decomposition

Consider the complex singular value decomposition (SVD) of matrix $\mathbf{A} \equiv \mathbf{Z}$; then

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathsf{H}} = \begin{bmatrix}\mathbf{U}_r & \mathbf{U}_k\end{bmatrix}\begin{bmatrix}\Sigma_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_r^{\mathsf{H}}\\ \mathbf{V}_k^{\mathsf{H}}\end{bmatrix} = \mathbf{U}_r\Sigma_r\mathbf{V}_r^{\mathsf{H}} \tag{51}$$

with $\mathbf{U}_r, \mathbf{V}_r \in \mathbb{C}^{n\times r}$ and $\mathbf{U}_k, \mathbf{V}_k \in \mathbb{C}^{n\times k}$, and $\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{I}$, $\mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}$, which implies $\mathbf{U}_r^{\mathsf{H}}\mathbf{U}_r = \mathbf{I}$, $\mathbf{U}_k^{\mathsf{H}}\mathbf{U}_k = \mathbf{I}$, and $\mathbf{U}_r^{\mathsf{H}}\mathbf{U}_k = \mathbf{0}$, and $\mathbf{V}_r^{\mathsf{H}}\mathbf{V}_r = \mathbf{I}$, $\mathbf{V}_k^{\mathsf{H}}\mathbf{V}_k = \mathbf{I}$, and $\mathbf{V}_r^{\mathsf{H}}\mathbf{V}_k = \mathbf{0}$, and finally $\Sigma_r \in \mathbb{R}^{r\times r}$ diagonal, with strictly greater than zero diagonal elements.

We can rewrite matrix $\tilde{\mathbf{A}}$ as

$$\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{e}\mathbf{D}\mathbf{f}^{\mathsf{H}} = \mathbf{U}_r\Sigma_r\mathbf{V}_r^{\mathsf{H}} + \mathbf{e}\mathbf{D}\mathbf{f}^{\mathsf{H}} = \begin{bmatrix}\mathbf{U}_r & \mathbf{e}\end{bmatrix}\begin{bmatrix}\Sigma_r & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{V}_r^{\mathsf{H}}\\ \mathbf{f}^{\mathsf{H}}\end{bmatrix} \tag{52}$$

Under the assumption that the columns of $\mathbf{A}$ and $\mathbf{e}$ span $\mathbb{C}^n$, matrix $[\mathbf{U}_r,\ \mathbf{e}]$ is invertible; in fact,

$$\begin{bmatrix}\mathbf{U}_r^{\mathsf{H}}\\ \mathbf{U}_k^{\mathsf{H}}\end{bmatrix}\begin{bmatrix}\mathbf{U}_r\ \mathbf{e}\end{bmatrix}=\begin{bmatrix}\mathbf{U}_r^{\mathsf{H}}\mathbf{U}_r\ \mathbf{U}_r^{\mathsf{H}}\mathbf{e}\\ \cancel{\mathbf{U}_k^{\mathsf{H}}\mathbf{U}_r}\ \mathbf{U}_k^{\mathsf{H}}\mathbf{e}\end{bmatrix}=\begin{bmatrix}\mathbf{I}\ \mathbf{U}_r^{\mathsf{H}}\mathbf{e}\\ \mathbf{0}\ \mathbf{U}_k^{\mathsf{H}}\mathbf{e}\end{bmatrix} \tag{53}$$

i.e.,

$$\begin{bmatrix}\mathbf{U}_r\ \mathbf{e}\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{I}\ \mathbf{U}_r^{\mathsf{H}}\mathbf{e}\\ \mathbf{0}\ \mathbf{U}_k^{\mathsf{H}}\mathbf{e}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{U}_r^{\mathsf{H}}\\ \mathbf{U}_k^{\mathsf{H}}\end{bmatrix}=\begin{bmatrix}\mathbf{U}_r^{\mathsf{H}}-\mathbf{U}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{U}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{U}_k^{\mathsf{H}}\\ \left(\mathbf{U}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{U}_k^{\mathsf{H}}\end{bmatrix} \tag{54}$$

Similarly, under the assumption that the columns of $\mathbf{A}^{\mathsf{H}}$ and $\mathbf{f}$ span $\mathbb{C}^n$, matrix $[\mathbf{V}_r,\ \mathbf{f}]^{\mathsf{H}}$ is invertible; in fact,

$$\begin{bmatrix}\mathbf{V}_r\ \mathbf{f}\end{bmatrix}^{-\mathsf{H}}=\begin{bmatrix}\mathbf{V}_r^{\mathsf{H}}\\ \mathbf{f}^{\mathsf{H}}\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{V}_r-\mathbf{V}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{V}_k\right)^{-1}\mathbf{f}^{\mathsf{H}}\mathbf{V}_r\ \mathbf{V}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{V}_k\right)^{-1}\end{bmatrix} \tag{55}$$

The inverse of matrix $\tilde{\mathbf{A}}$ is thus

$$\tilde{\mathbf{A}}^{-1}=\begin{bmatrix}\mathbf{V}_r^{\mathsf{H}}\\ \mathbf{f}^{\mathsf{H}}\end{bmatrix}^{-1}\begin{bmatrix}\Sigma_r\ \mathbf{0}\\ \mathbf{0}\ \mathbf{D}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{U}_r\ \mathbf{e}\end{bmatrix}^{-1}=\mathbf{G}+\mathbf{x}\mathbf{D}^{-1}\mathbf{y}^{\mathsf{H}} \tag{56}$$

with

$$\mathbf{G}=\left(\mathbf{V}_r-\mathbf{V}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{V}_k\right)^{-1}\mathbf{f}^{\mathsf{H}}\mathbf{V}_r\right)\Sigma_r^{-1}\left(\mathbf{U}_r^{\mathsf{H}}-\mathbf{U}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{U}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{U}_k^{\mathsf{H}}\right) \tag{57a}$$

$$\mathbf{x}=\mathbf{V}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{V}_k\right)^{-1} \tag{57b}$$

$$\mathbf{y}=\mathbf{U}_k\left(\mathbf{e}^{\mathsf{H}}\mathbf{U}_k\right)^{-1} \tag{57c}$$

3.2 Spectral decomposition

Alternatively, consider the spectral decomposition of the complex matrix $\mathbf{A}$ resulting from its eigenanalysis, namely

$$\mathbf{A}=\mathbf{X}\Lambda\mathbf{X}^{-1}=\mathbf{X}\Lambda\mathbf{Y}^{\mathsf{H}} \tag{58}$$

where $\mathbf{X}$ and $\mathbf{Y}=\mathbf{X}^{-\mathsf{H}}$ respectively are the right and left eigenvector matrices and Λ is the diagonal matrix of the eigenvalues of $\mathbf{A}$, under the assumption that $\mathbf{A}$ is diagonalizable. Matrix $\mathbf{A}$ is assumed to be singular; thus, $k=n-r$ eigenvalues are zero. Assuming the r non-zero eigenvalues are listed first, the spectral decomposition can be rewritten as

$$\mathbf{A}=\begin{bmatrix}\mathbf{X}_r\ \mathbf{X}_k\end{bmatrix}\begin{bmatrix}\Lambda_r\ \mathbf{0}\\ \mathbf{0}\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{Y}_r^{\mathsf{H}}\\ \mathbf{Y}_k^{\mathsf{H}}\end{bmatrix}=\mathbf{X}_r\Lambda_r\mathbf{Y}_r^{\mathsf{H}} \tag{59}$$

with $\mathbf{X}_r,\mathbf{Y}_r\in\mathbb{C}^{n\times r}$ and $\mathbf{X}_k,\mathbf{Y}_k\in\mathbb{C}^{k\times k}$, and $\mathbf{Y}^{\mathsf{H}}\mathbf{X}=\mathbf{I}$, which implies $\mathbf{Y}_r^{\mathsf{H}}\mathbf{X}_r=\mathbf{I}$, $\mathbf{Y}_k^{\mathsf{H}}\mathbf{X}_k=\mathbf{I}$, $\mathbf{Y}_r^{\mathsf{H}}\mathbf{X}_k=\mathbf{0}$, and $\mathbf{Y}_k^{\mathsf{H}}\mathbf{X}_r=\mathbf{0}$, and finally $\Lambda_r\in\mathbb{C}^{r\times r}$ diagonal, with non-zero diagonal elements.

We can rewrite matrix $\tilde{\mathbf{A}}$ as

$$\tilde{\mathbf{A}}=\mathbf{A}+\mathbf{e}\mathbf{D}\mathbf{f}^{\mathsf{H}}=\mathbf{X}_r\Lambda_r\mathbf{Y}_r^{\mathsf{H}}+\mathbf{e}\mathbf{D}\mathbf{f}^{\mathsf{H}}=\begin{bmatrix}\mathbf{X}_r\ \mathbf{e}\end{bmatrix}\begin{bmatrix}\Lambda_r\ \mathbf{0}\\ \mathbf{0}\ \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{Y}_r^{\mathsf{H}}\\ \mathbf{f}^{\mathsf{H}}\end{bmatrix} \tag{60}$$

Under the assumption that the columns of $\mathbf{A}$ and $\mathbf{e}$ span $\mathbb{C}^n$, matrix $[\mathbf{X}_r,\ \mathbf{e}]$ is invertible; in fact,

$$\begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}} \\ \mathbf{Y}_k^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_r\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}}\mathbf{X}_r & \mathbf{Y}_r^{\mathsf{H}}\mathbf{e} \\ \cancel{\mathbf{Y}_k^{\mathsf{H}}\mathbf{X}_r} & \mathbf{Y}_k^{\mathsf{H}}\mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{Y}_r^{\mathsf{H}}\mathbf{e} \\ \mathbf{0} & \mathbf{Y}_k^{\mathsf{H}}\mathbf{e} \end{bmatrix} \tag{61}$$

i.e.,

$$\begin{bmatrix} \mathbf{X}_r\ \mathbf{e} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{Y}_r^{\mathsf{H}}\mathbf{e} \\ \mathbf{0} & \mathbf{Y}_k^{\mathsf{H}}\mathbf{e} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}} \\ \mathbf{Y}_k^{\mathsf{H}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}} - \mathbf{Y}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}} \\ \left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}} \end{bmatrix} \tag{62}$$

Similarly, under the assumption that the columns of $\mathbf{A}^{\mathsf{H}}$ and $\mathbf{f}$ span $\mathbb{C}^n$, matrix $[\mathbf{Y}_r,\ \mathbf{f}]^{\mathsf{H}}$ is invertible; in fact,

$$\begin{bmatrix} \mathbf{Y}_r\ \mathbf{f} \end{bmatrix}^{-\mathsf{H}} = \begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}} \\ \mathbf{f}^{\mathsf{H}} \end{bmatrix}^{-1} = \left[\mathbf{X}_r - \mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1}\mathbf{f}^{\mathsf{H}}\mathbf{X}_r\ \mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1} \right] \tag{63}$$

The inverse of matrix $\tilde{\mathbf{A}}$ is thus

$$\tilde{\mathbf{A}}^{-1} = \begin{bmatrix} \mathbf{Y}_r^{\mathsf{H}} \\ \mathbf{f}^{\mathsf{H}} \end{bmatrix}^{-1} \begin{bmatrix} \Lambda_r & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_r\ \mathbf{e} \end{bmatrix}^{-1} = \mathbf{G} + \mathbf{x}\mathbf{D}^{-1}\mathbf{y}^{\mathsf{H}} \tag{64}$$

with

$$\mathbf{G} = \left(\mathbf{X}_r - \mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1}\mathbf{f}^{\mathsf{H}}\mathbf{X}_r\right)\Lambda_r^{-1}\left(\mathbf{Y}_r^{\mathsf{H}} - \mathbf{Y}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}}\right) \tag{65a}$$

$$\mathbf{x} = \mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1} \tag{65b}$$

$$\mathbf{y} = \mathbf{Y}_k\left(\mathbf{e}^{\mathsf{H}}\mathbf{Y}_k\right)^{-1} \tag{65c}$$

The elements of the inverse of matrix $\tilde{\mathbf{A}}$ from this formulation are identical to those of Eqs. (57).

3.3 Useful properties

Along the lines of Corollary 1 in [17], it is worth noticing that

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{66a}$$

$$\mathbf{y}^{\mathsf{H}}\mathbf{A} = \mathbf{0} \tag{66b}$$

$$\mathbf{G}\mathbf{e} = \mathbf{0} \tag{66c}$$

$$\mathbf{f}^{\mathsf{H}}\mathbf{G} = \mathbf{0} \tag{66d}$$

$$\mathbf{f}^{\mathsf{H}}\mathbf{x} = \mathbf{I} \tag{66e}$$

$$\mathbf{y}^{\mathsf{H}}\mathbf{e} = \mathbf{I} \tag{66f}$$

The proofs are trivial; they can be found in [19]. The property of Eq. (66d) is used in the next Section to resolve the limit for $\gamma \to 0$ in Eq. (32b). Its proof, and that of Eq. (66e), are reported there for completeness.

3.4 Explicit solution

The application of the proposed sum of singular matrices inversion requires matrices $\mathbf{A}$ and $\mathbf{e}$ to fully describe the space $\mathbb{C}^n$. In the present context, with $\mathbf{A} := \mathbf{Z}$ and $\mathbf{e} := \mathbf{Z}_{/s}\mathbf{z}_\ell$, this is guaranteed as long as

$$\mathbf{n}^{\mathsf{H}}\mathbf{Z}_{/s}\mathbf{z}_\ell \neq 0 \tag{67}$$

i.e., the vector $\mathbf{e} = -\mathbf{Z}_{/s}\mathbf{z}_\ell$ is not orthogonal to $\mathbf{z}_\ell$. In fact, matrix $\mathbf{Z}$ is singular, with $\mathbf{z}_\ell$ describing its nullspace. Under the assumption of Eq. (67), the entire space $\mathbb{C}^n$ is is spanned. As a consequence, recalling Eq. (50c),

$$\left(\mathbf{Z} - \mathbf{Z}_{/s}\mathbf{z}_\ell \frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\right)^{-1} = \mathbf{G} + \mathbf{x}\mathbf{D}^{-1}\mathbf{y}^{\mathsf{H}} = \mathbf{G} + \mathbf{x}\gamma\mathbf{y}^{\mathsf{H}} \tag{68}$$

with $\mathbf{G}$, $\mathbf{x}$, and $\mathbf{y}$ defined according to either Eqs. (57) or (65), and $\mathbf{D}^{-1} = \gamma\mathbf{I}$; thus, Eq. (32a) becomes

$$\mathbf{z}_{\ell/p} = \lim_{\gamma\to 0}\left(-\left(\mathbf{G} + \mathbf{x}\gamma\mathbf{y}^{\mathsf{H}}\right)\mathbf{Z}_{/p}\mathbf{z}_\ell\right) = -\mathbf{G}\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{69}$$

Furthermore,

$$s_{\ell/p} = \lim_{\gamma\to 0}\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\left(\mathbf{G} + \mathbf{x}\gamma\mathbf{y}^{\mathsf{H}}\right)\mathbf{Z}_{/p}\mathbf{z}_\ell = \lim_{\gamma\to 0}\left(\frac{1}{\gamma}\mathbf{n}^{\mathsf{H}}\mathbf{G} + \mathbf{n}^{\mathsf{H}}\mathbf{x}\mathbf{y}^{\mathsf{H}}\right)\mathbf{Z}_{/p}\mathbf{z}_\ell = \mathbf{y}^{\mathsf{H}}\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{70}$$

since $\mathbf{n}^{\mathsf{H}}\mathbf{G} \equiv \mathbf{0}$ thanks to the property of Eq. (66d), and $\mathbf{n}^{\mathsf{H}}\mathbf{x} \equiv \mathbf{I}$ thanks to the property of Eq. (66e), considering Eq. (50d), i.e., $\mathbf{f} := \mathbf{z}_\ell$.

Proof considering for example the spectral decomposition of Sect. 3.2, the property of Eq. (66d) can be proved by noticing that

$$\begin{aligned}\mathbf{n}^{\mathsf{H}}\mathbf{G} &= \mathbf{f}^{\mathsf{H}}\left(\mathbf{X}_r - \mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1}\mathbf{f}^{\mathsf{H}}\mathbf{X}_r\right)\Lambda_r^{-1}\left(\mathbf{Y}_r^{\mathsf{H}} - \mathbf{Y}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}}\right)\\ &= \cancel{\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_r - \underbrace{\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1}}_{\mathbf{I}}\mathbf{f}^{\mathsf{H}}\mathbf{X}_r\right)}\Lambda_r^{-1}\left(\mathbf{Y}_r^{\mathsf{H}} - \mathbf{Y}_r^{\mathsf{H}}\mathbf{e}\left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}}\right) = \mathbf{0}\end{aligned} \tag{71}$$

The property of Eq. (66e) can be proved by considering

$$\mathbf{n}^{\mathsf{H}}\mathbf{x} = \mathbf{f}^{\mathsf{H}}\mathbf{X}_k\left(\mathbf{f}^{\mathsf{H}}\mathbf{X}_k\right)^{-1} = \mathbf{I} \tag{72}$$

The proof based on the complex SVD of Sect. 3.1 follows analogously. □

Finally, it is worth noting that the sensitivity of the eigenvalue, $s_{\ell/p}$, does not depend on the normalization of the eigenvector.

Proof Eq. (70) involves only a subspace of either the SVD ($\mathbf{U}_k$) or the spectral decomposition ($\mathbf{Y}_k$) of matrix $\mathbf{Z}$, together with the vectors/submatrices $\mathbf{Z}_{/s}\mathbf{z}_\ell$ (with $\mathbf{e}$ defined in Eq. (50b)) and $\mathbf{Z}_{/p}\mathbf{z}_\ell$, and therefore does not depend on the normalization vector $\mathbf{n}$.

Furthermore, considering, for instance, the spectral decomposition discussed in Sect. 3.2, with $\mathbf{y}$ given by Eq. (65c), the case of unit multiplicity results in

$$s_{\ell/p} = \mathbf{y}^{\mathsf{H}}\mathbf{Z}_{/p}\mathbf{z}_\ell = \left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{e}\right)^{-1}\mathbf{Y}_k^{\mathsf{H}}\mathbf{Z}_{/p}\mathbf{z}_\ell = -\left(\mathbf{Y}_k^{\mathsf{H}}\mathbf{Z}_{/s}\mathbf{z}_\ell\right)^{-1}\mathbf{Y}_k^{\mathsf{H}}\mathbf{Z}_{/p}\mathbf{z}_\ell \tag{73}$$

Multiplication of the eigenvector $\mathbf{z}_\ell$ by an arbitrary complex scalar leaves these expressions unchanged. The proof based on the complex SVD of Sect. 3.1 follows analogously. □

As corollary, replacing the sensitivity of the L^2-norm normalization equation of the eigenvector with $\mathbf{z}_\ell^{\mathsf{T}}\mathbf{z}_{\ell/p} = 0$ would yield an eigenvector sensitivity $\mathbf{z}_{\ell/p}$ inconsistent with the L^2-norm normalization, but the resulting sensitivity of the eigenvalue, $s_{\ell/p}$, would still be correct.

4 Applications and discussion

The practical behavior of the proposed formulation is illustrated in a set of representative examples drawn from structural dynamics, rigid-body mechanics, time-delay systems, and one-dimensional continuum eigenproblems. These cases were selected to highlight how algebraic multiplicity, operator nonlinearity, and non-self-adjoint effects influence eigensolution sensitivity and numerical conditioning. The results demonstrate how the explicit sensitivity expressions clarify the dependence of the spectrum on physical parameters and provide numerically stable estimates even in the presence of near-degenerate or coalescent eigenvalues.

4.1 Structural dynamics

In the structural dynamics case, matrix $\mathbf{Z}$ is

$$\mathbf{Z} = s_\ell^2\mathbf{M} + \mathbf{K} \tag{74}$$

with $\mathbf{M}, \mathbf{K} \in \mathbb{R}^{n\times n} > 0$; thus, $s = \mathrm{j}\omega \in \mathbb{I}$, consequently with $\omega \in \mathbb{R}$, and $\mathbf{z}_\ell \in \mathbb{R}^n$. In this specific case, the L^2 norm of the eigenvector can be used also for the sensitivity problem, since the eigenvector is real.

The sensitivity of $\mathbf{Z}$ with respect to the eigenvalue is thus

$$\mathbf{Z}_{/s} = 2s_\ell\mathbf{M} \tag{75}$$

and the requirement of Eq. (67), using the L^2 norm and thus $\mathbf{z}_\ell^{\mathsf{T}}$ instead of $\mathbf{n}^{\mathsf{T}}$, yields

$$2s_\ell\mathbf{z}_\ell^{\mathsf{T}}\mathbf{M}\mathbf{z}_\ell = 2s_\ell m_\ell \neq 0 \qquad \text{iff } s_\ell \neq 0 \tag{76}$$

where $m_\ell = \mathbf{z}_\ell^{\mathsf{T}}\mathbf{M}\mathbf{z}_\ell > 0$ is the modal mass of the ℓth mode, positive thanks to the intrinsic positive definiteness of mass matrices. The condition $s_\ell \neq 0$ means that the formulation does not work for rigid-body modes, discussed in the next Section.

If the L^∞ norm is used, the condition is still satisfied as long as the unit-valued component of $\mathbf{n}$ corresponds to a non-zero component of $\mathbf{M}\mathbf{z}_\ell$.

Although in this case the square of the eigenvalue, namely s^2, may be directly computed (this is further discussed in Sect. 4.3), the approach using s is necessary when damping is present. In that case, matrix $\mathbf{Z}$ becomes

$$\mathbf{Z} = s_\ell^2\mathbf{M} + s_\ell\mathbf{C} + \mathbf{K} \tag{77}$$

 https://doi.org/10.1007/s11012-026-02190-8

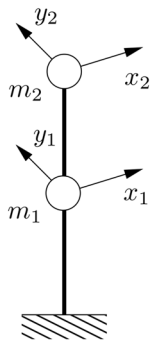


Fig. 1 Sketck of 4-Dof problem

Its sensitivity with respect to the eigenvalue is thus

$$\mathbf{Z}_{/s} = 2s_\ell \mathbf{M} + \mathbf{C} \tag{78}$$

and the requirement of Eq. (67), using the L^∞-norm sensitivity, yields

$$2s_\ell \mathbf{n}^\mathsf{T}\mathbf{M}\mathbf{z}_\ell + \mathbf{n}^\mathsf{T}\mathbf{C}\mathbf{z}_\ell \neq 0 \tag{79}$$

If Rayleigh damping is used, i.e., $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$, then again the eigenvectors are real and their L^2 norm can be used, yielding $\mathbf{z}_\ell^\mathsf{T}\mathbf{Z}_{/s}\mathbf{z}_\ell = m_\ell\left(2s_\ell + \alpha + \omega_\ell^2\beta\right)$, where $\omega_\ell = \sqrt{\mathbf{z}_\ell^\mathsf{T}\mathbf{K}\mathbf{z}_\ell/(\mathbf{z}_\ell^\mathsf{T}\mathbf{M}\mathbf{z}_\ell)}$.

4.2 Structural dynamics: multiplicity

Consider a problem made of two masses connected by a spring, with one of the masses grounded by another spring, namely

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{80}$$

Consider now an identical problem, with identical data, fully decoupled from the original one. As sketched in Fig. 1, this may be considered representative, for example, of the transverse vibrations of a discretized beam with identical bending stiffness properties in two orthogonal directions (e.g., with circular or square section). The equations of motion are

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \\ 0 & 0 & k_1+k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{81}$$

Clearly, the eigenvalues of the problem have multiplicity 2, but the corresponding eigenvectors are distinct.

Consider $m_1 = m_2 = 1$ kg and $k_1 = k_2 = 1$ N/m. The eigenvalues of the problem are $s_{1\pm,2\pm} = \pm$j0.618033988749895 rad/s and $s_{3\pm,4\pm} = \pm$j1.618033988749895 rad/s, i.e., all with algebraic multiplicity equal to 2. The corresponding eigenvectors are

$$\mathbf{z}_{1,2} = \begin{bmatrix} 0.525731112119133 & 0 \\ 0.850650808352040 & 0 \\ 0 & 0.525731112119133 \\ 0 & 0.850650808352040 \end{bmatrix} \tag{82a}$$

Table 1 Condition numbers of 4-DoF problem not considering multiplicity ($m = 1$)

Iteration	cond($\mathbf{Z}$)	cond(Eq. (11))
1	4.8452e+00	6.9323e+00
2	2.0556e+01	3.3437e+01
3	3.8957e+02	5.7111e+02
4	3.6445e+04	5.2749e+04
5	4.7766e+08	6.9127e+08
6	5.6474e+15	8.0550e+15

$$\mathbf{z}_{3,4} = \begin{bmatrix} 0.850650808352040 & 0 \\ -0.525731112119133 & 0 \\ 0 & 0.850650808352040 \\ 0 & -0.525731112119133 \end{bmatrix} \tag{82b}$$

Their partition to match their non-zero elements with the related independent portions of coordinates is completely arbitrary, as any recombination that preserves their orthogonality would be acceptable. A large number of digits is used for ease of verification.

4.2.1 Neglecting multiplicity

The problem is first analyzed using the proposed method for $m = 1$; the eigensolution is initialized randomly with

$$s_0 = +\mathrm{j}1.788730244623709 \text{ rad/s} \tag{83a}$$

$$\mathbf{z}_0 = \begin{Bmatrix} 0.184477659263231 \\ -0.660569274228337 \\ 0.727569854980361 \\ 0.016072750295559 \end{Bmatrix} \tag{83b}$$

where the eigenvector has unit norm.

With a tolerance on the norm of the residual of 1e-12, the method converges in 6 iterations to the solution $s_\ell = +\mathrm{j}1.618033988749895$ rad/s, namely the larger solution with positive imaginary part, which is also the closest to the initial value. The resulting eigenvector is

$$\mathbf{z}_\ell = \begin{Bmatrix} 0.688862126751785 \\ -0.425740207895141 \\ -0.499074912289715 \\ 0.308445258727417 \end{Bmatrix} \tag{84}$$

This vector is orthogonal to $\mathbf{z}_{1,2}$; it represents a combination of eigenvectors $\mathbf{z}_{3,4}$. It is easy to check that

$$\mathbf{z}_\ell = -0.809805997935055\,\mathbf{z}_3 + 0.586697746466108\,\mathbf{z}_4 \tag{85}$$

So we obtained the correct eigenvalue and a single eigenvector that is a linear combination of the correct ones. However, during the iterative process, the condition numbers of matrix $\mathbf{Z}$ and of the matrix of Eq. (11) have drastically deteriorated, as shown in Table 1, pointing out the expected near-singularity of matrix $\mathbf{Z}$ but also the ill-posedness of the nonlinear eigenvalue problem when an eigenvalue multiplicity is greater than one.

An attempt to solve the sensitivity problem would fail because, considering the condition number of the matrix of Eq. (11), the problem is clearly ill-posed.

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Table 2 Condition numbers of 4-DoF problem with $m = 2$

Iteration	cond($\mathbf{Z}$)	cond(Eq. (35))
1	4.8452e+00	7.8518e+00
2	4.8244e+00	8.0018e+00
3	1.6569e+01	6.3252e+00
4	1.6600e+02	4.0068e+00
5	3.7524e+03	3.3090e+00
6	6.8780e+05	3.2369e+00
7	1.1185e+10	3.2361e+00
8	4.8465e+16	3.2361e+00

4.2.2 Considering multiplicity

Then the problem is analyzed using the proposed method for $m = 2$; the eigensolution is initialized randomly with

$$s_0 = +\mathrm{j}1.788730244623709 \text{ rad/s} \tag{86a}$$

$$\mathbf{z}_0 = \begin{bmatrix} 1.310080341410262 & -1.080661851211695 \\ -0.327097515918198 & 0.407730865297905 \\ -0.672993163854711 & 0.379344537288795 \\ 0.149327499476731 & -0.330361045798659 \end{bmatrix} \tag{86b}$$

i.e., the same initial value of the previous case for the eigenvalue and the first eigenvector, and a second randomly generated eigenvector of unit norm and orthogonal to the previous one.

The same solution, $s_\ell = \mathrm{j}1.618033988749895$ rad/s, is obtained in 8 iterations. The resulting eigenvectors are

$$\mathbf{z}_\ell = \begin{bmatrix} 0.434396926347324 & 0.731372755939115 \\ -0.268472065091131 & -0.452013221616055 \\ -0.731372755939115 & 0.434396926347324 \\ 0.452013221616055 & -0.268472065091131 \end{bmatrix} \tag{87}$$

These vectors are orthogonal to $\mathbf{z}_{1,2}$; they are a combination of eigenvectors $\mathbf{z}_{3,4}$. So we obtained the correct eigenvalue and two eigenvectors that are linear combinations of the correct ones. During the iterative process, the L^2-norm condition number of matrix $\mathbf{Z}$ has drastically deteriorated, but that of the matrix of Eq. (35) remained quite limited, as shown in Table 2, pointing out the well-posedness of the proposed approach to eigenanalysis in case of eigenvalue multiplicity greater than one.

4.2.3 Sensitivity to mass perturbation

Consider now a sensitivity analysis; for example, addressing the mass associated with coordinate y_1, i.e., $p := m_1$ the right-hand side of the problem of Eq. (40) becomes

$$\begin{Bmatrix} -s_\ell^2 \mathbf{M}_{/p} \mathbf{z}_\ell \\ \mathbf{0} \end{Bmatrix} \tag{88}$$

with

$$\mathbf{M}_{/p} = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,0 \end{bmatrix} \tag{89}$$

Considering the L^2-norm sensitivity, the solution of the sensitivity problem is

$$\mathbf{z}_{\ell/p} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.3668\text{e-}01 & -1.4057\text{e-}01 \\ 3.8295\text{e-}01 & -2.2745\text{e-}01 \end{bmatrix} \text{kg}^{-1} \tag{90a}$$

$$\mathbf{s}_{\ell/p} = \begin{bmatrix} -\text{j}4.3275\text{e-}01 & \text{j}2.5703\text{e-}01 \\ \text{j}2.5703\text{e-}01 & -\text{j}1.5266\text{e-}02 \end{bmatrix} \text{rad} \cdot \text{s}^{-1} \cdot \text{kg}^{-1} \tag{90b}$$

The same solution can be obtained by the explicit formulas of Eqs. (69) and (70).

The spectral decomposition of matrix $\mathbf{s}_{\ell/p}$ in Eq. (90b) yields

$$\mathbf{X} = \begin{bmatrix} 8.5978\text{e-}01 & 5.1066\text{e-}01 \\ -5.1066\text{e-}01 & 8.5978\text{e-}01 \end{bmatrix} \tag{91a}$$

$$\Lambda = \begin{bmatrix} -\text{j}5.8541\text{e-}01 & 0 \\ 0 & 0 \end{bmatrix} \text{rad}\cdot\text{s}^{-1}\cdot\text{kg}^{-1} \tag{91b}$$

i.e.,

$$s_{\ell/p} = \Lambda(1,1) = -\text{j}5.8541\text{e-}01\ \text{rad}\cdot\text{s}^{-1}\cdot\text{kg}^{-1} \tag{92}$$

The eigenvalue's imaginary part reduces, something intuitively correct for a positive mass change.

Matrix $\mathbf{X}$ can be used to re-sort the eigenvector sensitivity,

$$\mathbf{z}_{\ell/p}\mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.7528\text{e-}01 & 0 \\ 4.4541\text{e-}01 & 0 \end{bmatrix} \text{kg}^{-1} \tag{93}$$

i.e., only the first eigenvector is affected by the parameter, indeed only impacting the components of the eigenvector associated with coordinates y_1 and y_2, those that interact with the portion of model affected by the parameter perturbation.

The original eigenvectors, re-sorted using the same matrix, are

$$\mathbf{z}_{\ell}\mathbf{X} = \begin{bmatrix} 0 & 8.5065\text{e-}01 \\ 0 & -5.2573\text{e-}01 \\ -8.5065\text{e-}01 & 0 \\ 5.2573\text{e-}01 & 0 \end{bmatrix} \tag{94}$$

i.e., they are decoupled in such a manner that the first only involves coordinates y_1 and y_2, whereas the second only involves coordinates x_1 and x_2.

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Considering instead the L^∞-norm sensitivity, the solution is again

$$s_{\ell/p} = \Lambda(1,1) = -\text{j5.8541e-01 rad}\cdot\text{s}^{-1}\cdot\text{kg}^{-1} \tag{95}$$

and

$$\mathbf{z}_{\ell/p}\mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -6.1554\text{e-}01 & 0 \end{bmatrix} \text{kg}^{-1} \tag{96}$$

the re-sorted eigenvectors being

$$\mathbf{z}_{\ell}\mathbf{X} = \begin{bmatrix} 0 & 8.5065\text{e-}01 \\ 0 & -5.2573\text{e-}01 \\ 8.5065\text{e-}01 & 0 \\ -5.2573\text{e-}01 & 0 \end{bmatrix} \tag{97}$$

The eigenvectors and their sensitivity have been rescaled to have unit norm and thus remain comparable with those obtained using unit L^2-norm normalization. The eigenvalue sensitivity, Eq. (95), is unaltered. In the eigenvector sensitivity of Eq. (96) only the first eigenvector, affecting coordinates y_1 and y_2, only the component associated with y_2 is non-zero, as the L^∞-norm normalization expressly requires the largest component, that associated with y_1, to remain constant. The sign of the first eigenvector changes with respect to Eq. (94) because the normalization process made the largest absolute value component positive. The other eigenvector is unaltered.

4.2.4 Sensitivity to stiffness perturbation

Addressing the stiffness k_2 associated with coordinates x_1 and x_2, the right-hand side of the problem of Eq. (40) becomes

$$\begin{Bmatrix} -\mathbf{K}_{/p}\mathbf{z}_{\ell} \\ \mathbf{0} \end{Bmatrix} \tag{98}$$

with

$$\mathbf{K}_{/p} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{99}$$

Considering the L^2-norm sensitivity, the solution of the sensitivity problem is

$$\mathbf{z}_{\ell/p} = \begin{bmatrix} -5.3694\text{e-}02 & -9.0403\text{e-}02 \\ -8.6879\text{e-}02 & -1.4627\text{e-}01 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{m}\cdot\ \text{N}^{-1} \tag{100a}$$

$$\mathbf{s}_{\ell/p} = \begin{bmatrix} \text{j1.5266e-01} & \text{j2.5703e-01} \\ \text{j2.5703e-01} & \text{j4.3275e-02} \end{bmatrix} \text{rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \tag{100b}$$

The spectral decomposition of matrix $\mathbf{s}_{\ell/p}$ yields

$$\mathbf{X} = \begin{bmatrix} 5.1066\text{e-}01 & 8.5978\text{e-}01 \\ 8.5978\text{e-}01 & -5.1066\text{e-}01 \end{bmatrix} \tag{101a}$$

$$\Lambda = \begin{bmatrix} \text{j}5.8541\text{e-}01 & 0 \\ 0 & 0 \end{bmatrix} \text{ rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \tag{101b}$$

i.e.,

$$s_{\ell/p} = \Lambda(1,1) = \text{j}5.8541\text{e-}01 \text{ rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \tag{102}$$

The eigenvalue's imaginary part increases, something intuitively correct for a positive stiffness change.

Matrix $\mathbf{X}$ can be used to re-sort the eigenvector sensitivity,

$$\mathbf{z}_{\ell/p}\mathbf{X} = \begin{bmatrix} 0 & -1.0515\text{e-}01 \\ 0 & -1.7013\text{e-}01 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ m}\cdot\text{N}^{-1} \tag{103}$$

i.e., only the second eigenvector is affected by the parameter, indeed only impacting the components of the eigenvector associated with coordinates y_1 and y_2, which interact with the perturbed parameter.

The original eigenvectors, re-sorted using the same matrix, are

$$\mathbf{z}_{\ell}\mathbf{X} = \begin{bmatrix} 0 & 8.5065\text{e-}01 \\ 0 & -5.2573\text{e-}01 \\ -8.5065\text{e-}01 & 0 \\ 5.2573\text{e-}01 & 0 \end{bmatrix} \tag{104}$$

i.e., they are again decoupled in such a manner that the first only involves coordinates y_1 and y_2, whereas the second only involves coordinates x_1 and x_2.

One may also use the re-sorted eigenvectors of Eq. (94), resulting from the sensitivity to a variation of mass m_1 affecting coordinate y_1 to re-evaluate the sensitivity problem, i.e., redefining $\mathbf{z}_\ell := \mathbf{z}_\ell\mathbf{X}$ and using it in Eq. (40). The resulting solution corresponds directly to that of Eqs. (102) and (103)

Consider now the same problem when the stiffness k_2 associated with coordinates y_1 and y_2 is slightly perturbed with respect to the nominal value by 0.01%, i.e., $k_{2_y} = 1.0001$. The largest eigenvalue eigensolution of subsystem y is now $s_{\ell_y} = +\text{j}1.618092528986939$ rad/s, with

$$\mathbf{z}_{\ell_y} = \begin{Bmatrix} 0 \\ 0 \\ 0.850640294400804 \\ -0.525748123669228 \end{Bmatrix}, \tag{105}$$

while that of subsystem x is unaffected; thus, both eigenvalues have multiplicity $m = 1$.

Iterating the sensitivity analysis and the corresponding eigenanalysis, setting at each iteration the parameter increment equal to $(s_{\ell_x} - s_{\ell_y})/s_{\ell_y/p}$, with $p := k_{2_y}$, the results of Table 3 are obtained, namely the convergence of k_{2_y} to k_{2_x} and of s_{ℓ_y} to s_{ℓ_x}, while the condition number of the matrix in Eq. (11) grows unbounded as the two eigenvalues become closer and closer, resulting in the need to switch to Eqs. (35) and (40) with $m = 2$ to handle the increased multiplicity of the problem.

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Table 3 Continuation of stiffness parameter in 4-DoF problem until eigenvalue coalescence

Iter	s_{ℓ_y}, rad/s	$s_{\ell_y} - s_{\ell_x}$, rad/s	$s_{\ell_y/p}$, rad·s^{-1}·m·N^{-1}	Δp, N/m	cond(Eq. (11))
1	j1.618092528986939	j5.8540e-05	j0.585394544335300	−1.0000134e-04	1.7083e+04
2	j1.618033987967262	−j7.8263e-10	j0.585410196834226	1.3368957e-09	1.2777e+09
3	j1.618033988749894	j6.6613e-16	j0.585410196624969	7.5859493e-16	1.4729e+15

4.2.5 Sensitivity to damping perturbation

Assuming a damping matrix

$$\mathbf{C} = \begin{bmatrix} c_1 + c_{2_x} & -c_{2_x} & 0 & 0 \\ -c_{2_x} & c_{2_x} & 0 & 0 \\ 0 & 0 & c_1 + c_{2_y} & -c_{2_y} \\ 0 & 0 & -c_{2_y} & c_{2_y} \end{bmatrix} \tag{106}$$

and addressing the damping associated with coordinates x_1 and x_2, i.e., $p := c_{2_x}$, the right-hand side of the problem of Eq. (40) becomes

$$\begin{Bmatrix} -s\mathbf{C}_{/p}\mathbf{z}_\ell \\ \mathbf{0} \end{Bmatrix} \tag{107}$$

with

$$\mathbf{C}_{/p} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{108}$$

The L$^\infty$-norm sensitivity, directly using the re-sorted eigenvectors of Eq. (94), is

$$\mathbf{z}_{\ell/p} = \begin{bmatrix} 0 & 0 \\ 0 & -\text{j3.8042e-01} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ m·N}^{-1} \tag{109a}$$

$$\mathbf{s}_{\ell/p} = \begin{bmatrix} 0 & 0 \\ 0 & -9.4821\text{e-}01 \end{bmatrix} \text{ rad·s}^{-2}\text{·m·N}^{-1} \tag{109b}$$

i.e., matrix $\mathbf{s}_{\ell/p}$ is already in diagonal form, yielding

$$s_{\ell/p} = \Lambda(2,2) = -9.4821\text{e-}01 \text{ rad·s}^{-2}\text{·m·N}^{-1} \tag{110}$$

The eigenvalue's real part decreases, yielding asymptotic stability with a perturbed eigenvector that becomes imaginary, something intuitively correct for a positive damping change. Moreover, the eigenvector perturbation is imaginary, making the perturbed eigenvector complex; again, something expected in a problem with structural damping.

Assuming now a damping matrix

$$\mathbf{C} = \begin{bmatrix} c_1 + (c_2 + p) & -(c_2 + p) & 0 & 0 \\ -(c_2 + p) & (c_2 + p) & 0 & 0 \\ 0 & 0 & c_1 + (c_2 - p) & -(c_2 - p) \\ 0 & 0 & -(c_2 - p) & (c_2 - p) \end{bmatrix} \tag{111}$$

such that the parameter affects both portions of the system but in opposite manner, the sensitivity of the matrix is

$$\mathbf{C}_{/p} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{112}$$

The L^{∞}-norm sensitivity of the eigensolution is

$$\mathbf{z}_{\ell/p} = \begin{bmatrix} 0 & 0 \\ \text{j1.9427e-01} & \text{-j3.2708e-01} \\ 0 & 0 \\ \text{j3.2708e-01} & \text{j1.9427e-01} \end{bmatrix} \text{m}\cdot\text{N}^{-1} \tag{113a}$$

$$\mathbf{s}_{\ell/p} = \begin{bmatrix} \text{4.5319e-01} & \text{8.3177e-01} \\ \text{8.3177e-01} & \text{-4.5319e-01} \end{bmatrix} \text{rad}\cdot\text{s}^{-2}\cdot\text{m}\cdot\text{N}^{-1} \tag{113b}$$

The spectral decomposition of matrix $\mathbf{s}_{\ell/p}$ yields

$$\mathbf{X} = \begin{bmatrix} \text{8.5978e-01} & \text{-5.1066e-01} \\ \text{5.1066e-01} & \text{8.5978e-01} \end{bmatrix} \tag{114a}$$

$$\Lambda = \begin{bmatrix} \text{j9.4721e-01} & 0 \\ 0 & \text{-j9.4721e-01} \end{bmatrix} \text{rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \tag{114b}$$

i.e.,

$$s_{\ell_1/p} = \Lambda(1,1) = \text{9.4721e-01 rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \qquad \mathbf{z}_{\ell_1/p} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \text{j3.8042e-01} \end{Bmatrix} \text{m}\cdot\text{N}^{-1} \tag{115a}$$

$$s_{\ell_2/p} = \Lambda(2,2) = \text{-9.4721e-01 rad}\cdot\text{s}^{-1}\cdot\text{m}\cdot\text{N}^{-1} \qquad \mathbf{z}_{\ell_2/p} = \begin{Bmatrix} 0 \\ \text{-j3.8042e-01} \\ 0 \\ 0 \end{Bmatrix} \text{m}\cdot\text{N}^{-1} \tag{115b}$$

The eigenvalues' real part either increases or decreases, something intuitively correct for a negative vs. positive damping change. The re-sorted eigenvectors show that the increase in real part, yielding instability, is associated with the portion of problem described by the y_1 and y_2 coordinates, those incurring in a reduction of the damping. The opposite occurs for the other portion of the system, affected by a damping increase. The eigenvector perturbation is again imaginary, making the perturbed eigenvector complex, as expected in a problem where structural damping is either added or subtracted.

4.3 Structural dynamics with rigid body motion

When the ℓth eigensolution corresponds to a rigid-body motion, $s_\ell \equiv 0$ by definition. As such, according to Eq. (76) the requirement of Eq. (67) cannot be satisfied.

In principle, this should be a moot point, because a rigid-body solution by definition should not be sensitive to parameters that characterize any of the mass or stiffness matrices, unless it results from the degeneration of some parameter-dependent term of the stiffness matrix.

At any rate, the problem can be circumvented by defining a new parameter, e.g., $\lambda = s^2$, to replace the pair of eigenvalues that correspond to the rigid body motion, such that

$$\mathbf{Z} = \lambda\mathbf{M} + \mathbf{K} \tag{116}$$

and

$$\mathbf{Z}_{/\lambda} = \mathbf{M} \tag{117}$$

such that

$$\mathbf{z}_\ell^{\mathsf{T}}\mathbf{Z}_{/\lambda}\mathbf{z}_\ell = \mathbf{z}_\ell^{\mathsf{T}}\mathbf{M}\mathbf{z}_\ell = m_\ell > 0 \tag{118}$$

4.4 Structural dynamics with rigid body motion: multiplicity

Practical structural problems often present multiple rigid-body modes, by definition with coincident eigenvalues since they are all equal to zero, and thus with algebraic multiplicity greater than one, although with geometrical multiplicity equal to one, since the associated eigenvectors are independent.

Consider the same problem of Sect. 4.2 omitting the two instances of spring k_1 that in the previous example connected the respective instances of mass m_1 to the ground. The eigensolution of the problem is now $s_{1\pm,2\pm} = 0$ rad/s, with multiplicity 4, and $s_{3\pm,4\pm} = \pm\mathrm{j}\sqrt{2}$ rad/s, each with multiplicity equal to 2. The corresponding eigenvectors are

$$\mathbf{z}_{1,2} = \begin{bmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix} \qquad \mathbf{z}_{3,4} = \begin{bmatrix} \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 \end{bmatrix} \tag{119}$$

While the non-zero eigenvalue solution, $s_{3\pm,4\pm}$, can be found like in Sect. 4.2, the zero-valued eigenvalue solution, $s_{1\pm,2\pm}$, requires the use of the square of the eigenvalue, instead.

Using s as the (tentative) eigenvalue, the nonlinear algebraic method with $m = 1$ consistently fails; with $m = 2$ it succeeds, although requiring a large number of iterations, since the condition number of the matrix of Eq. (11) rapidly degenerates.

Using $\lambda = s^2$ as the (tentative) square of the eigenvalue, instead, the method quickly converges to the desired solution both with $m = 1$ and $m = 2$.

4.4.1 Neglecting multiplicity

With $m = 1$, initializing the eigenproblem with

$$s_0 = \mathrm{j}0.2 \text{ rad/s} \tag{120a}$$

Table 4 Condition numbers of 4-DoF rigid-body problem not considering multiplicity ($m = 1$)

Iteration	cond($\mathbf{Z}$)	cond(Eq. (11))
1	4.9000e+01	5.0340e+01
2	5.3400e+02	5.3400e+02
3	1.2192e+04	1.2192e+04
4	1.1438e+07	1.1438e+07
5	2.0080e+13	2.0119e+13

$$\mathbf{z}_0 = \begin{Bmatrix} 0.392635592314701 \\ 0.697390157985294 \\ -0.264035948012697 \\ -0.538302217485650 \end{Bmatrix} \tag{120b}$$

and a tolerance on the norm of the residual of 1e-12, the method converges in 5 iterations to the solution $\lambda_\ell =$ 9.9531e-14 rad/s. The resulting eigenvector is

$$\mathbf{z}_\ell = \begin{Bmatrix} 0.569469622414564 \\ 0.569469622414564 \\ -0.419171026130357 \\ -0.419171026130357 \end{Bmatrix} \tag{121}$$

i.e., a linear combination of $\mathbf{z}_{1,2}$. The condition numbers of matrix $\mathbf{Z}$ and of the matrix of Eq. (11) are shown in Table 4.

4.4.2 Considering multiplicity

With $m = 2$, and the same initial solution of the previous case, plus the initial guess of another eigenvector, namely

$$s_0 = \text{j}0.2 \text{ rad/s} \tag{122a}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0.392635592314701 & 0.684416022535477 \\ 0.697390157985294 & 0.127437788107550 \\ -0.264035948012697 & 0.453447186822459 \\ -0.538302217485650 & 0.556524902427349 \end{bmatrix} \tag{122b}$$

and a tolerance on the norm of the residual of 1e-12, the method converges in 6 iterations to the solution $\lambda_\ell =$ 1.4166e-16 rad/s. The resulting eigenvectors are

$$\mathbf{z}_\ell = \begin{bmatrix} 0.560334012182057 & 0.431307077141053 \\ 0.560334012182057 & 0.431307077141053 \\ -0.431307077141053 & 0.560334012182057 \\ -0.431307077141053 & 0.560334012182057 \end{bmatrix} \tag{123}$$

i.e., a linear combination of $\mathbf{z}_{1,2}$. The condition numbers of matrix $\mathbf{Z}$ and of the matrix of Eq. (35) are shown in Table 5. Again, while cond($\mathbf{Z}$) quickly deteriorates as its rank tends to 2, the matrix of Eq. (35) remains well conditioned.

Table 5 Condition numbers of 4-DoF rigid-body problem considering multiplicity ($m = 2$)

Iteration	cond($\mathbf{Z}$)	cond(Eq. (35))
1	4.9000e+01	2.6343e+00
2	5.1321e+02	1.9889e+00
3	1.2442e+04	2.0002e+00
4	4.9031e+06	2.0000e+00
5	4.7333e+11	2.0000e+00
6	1.0934e+16	2.0000e+00

4.4.3 Sensitivity to stiffness perturbation

At this stage, the sensitivity analysis is moot, since changes in mass properties would be multiplied by zero (the eigenvalue), while changes in stiffness properties that do not result in strain energy for the rigid body motion would be ineffective.

To this end, consider the sensitivity to the spring that grounds the coordinate x_1, k_1, which is zero in the reference configuration. Introducing such spring would eliminate one of the rigid-body modes. The right-hand side of the problem of Eq. (40) has the same form of Eq. (98), now with

$$\mathbf{K}_{/p} = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \tag{124}$$

The L^2-norm sensitivity is

$$\mathbf{z}_{\ell/p} = \begin{bmatrix} -1.4008\text{e-}01 & -1.0783\text{e-}01 \\ 1.4008\text{e-}01 & 1.0783\text{e-}01 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{m}\cdot\text{N}^{-1} \tag{125a}$$

$$\lambda_{\ell/p} = \begin{bmatrix} -3.1397\text{e-}01 & -2.4168\text{e-}01 \\ -2.4168\text{e-}01 & -1.8603\text{e-}02 \end{bmatrix} \text{rad}^2\cdot\text{s}^{-2}\cdot\text{m}^2\cdot\text{N}^{-2} \tag{125b}$$

The spectral decomposition of matrix $\lambda_{\ell/p}$ yields

$$\mathbf{X} = \begin{bmatrix} 7.9243\text{e-}01 & -6.0996\text{e}01 \\ 6.0996\text{e-}01 & 7.9243\text{e-}01 \end{bmatrix} \tag{126a}$$

$$\Lambda = \begin{bmatrix} -5.0000\text{e-}01 & 0 \\ 0 & 0 \end{bmatrix} \text{rad}^2\cdot\text{s}^{-2}\cdot\text{m}^2\cdot\text{N}^{-2} \tag{126b}$$

i.e.,

$$\lambda_{\ell/p} = \Lambda(1,1) = -5.0000\text{e-}01 \tag{127}$$

The eigenvalue reduces (i.e., becomes negative), which means that $s = \pm\sqrt{\lambda}$ becomes imaginary, something intuitively expected for a positive stiffness change.

Matrix $\mathbf{X}$ can be used to re-sort the eigenvector sensitivity,

$$\mathbf{z}_{\ell/p}\mathbf{X} = \begin{bmatrix} -1.7678\text{e-}01 & 0 \\ 1.7678\text{e-}01 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{m}\cdot\text{N}^{-1} \tag{128}$$

i.e., only the first eigenvector is affected by the parameter, indeed only impacting the components of the eigenvector associated with coordinates x_1 and x_2, which interact with the perturbed parameter.

The original eigenvectors, re-sorted using the same matrix, are

$$\mathbf{z}_{\ell}\mathbf{X} = \begin{bmatrix} 7.0711\text{e-}01 & 0 \\ 7.0711\text{e-}01 & 0 \\ 0 & 7.0711\text{e-}01 \\ 0 & 7.0711\text{e-}01 \end{bmatrix} \tag{129}$$

i.e., they are again decoupled in such a manner that the first only affects coordinates y_1 and y_2, whereas the second only affects coordinates x_1 and x_2. They correspond exactly to $\mathbf{z}_{1,2}$ of Eq. (119), since $\sqrt{2}/2 = 7.0711\text{e-}01$.

4.5 Dynamics of preloaded systems

Consider the linear eigenproblem

$$(\lambda\mathbf{M} + \mathbf{K} + p\mathbf{H})\,\mathbf{z} = \mathbf{0}, \tag{130}$$

where $\lambda = s^2$, in analogy with the case with rigid-body motion of Sect. 4.4. Considering its sensitivity with L^∞-norm, the problem becomes

$$\begin{bmatrix} \lambda_\ell\mathbf{M} + \mathbf{K} + p_\ell\mathbf{H} & \mathbf{M}\mathbf{z}_\ell \\ \mathbf{n}^{\text{H}} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{z}_{\ell/p} \\ \lambda_{\ell/p} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{H}\mathbf{z}_\ell \\ 0 \end{Bmatrix}, \tag{131}$$

where p_ℓ is the value of the parameter p that yields the eigensolution $\{\mathbf{z}_\ell, s_\ell\}$.

Its solution can be used to locate the value of p that leads to the vanishing of the eigenvalue, namely $p \setminus s_\ell \equiv 0$, i.e.,

$$\lambda_\ell + \lambda_{\ell/p}\Delta p = 0, \tag{132}$$

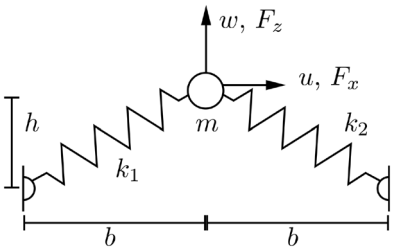
which yields

$$\Delta p = -\frac{\lambda_\ell}{\lambda_{\ell/p}}, \qquad p_\ell \mathrel{+}= \Delta p, \tag{133}$$

iterating until convergence, under the assumption that $|\lambda_{\ell/p}| > 0$. At convergence, $\lambda_\ell \to 0$, and the two corresponding values of $s_\ell = \pm\sqrt{\lambda_\ell}$ from imaginary conjugate tend to zero; for a further increment of p_ℓ, that makes λ_ℓ positive, they become real opposite.

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Fig. 2 Sketch of von Mises Truss



4.5.1 Simply supported beam with compressive preload

Consider for example the buckling of a simply supported rod of length ℓ, bending stiffness EJ, and mass per unit span m. The exact solution using the Euler-Bernoully model, is

$$s_w = \pm \mathrm{j} \left(\frac{\pi w}{\ell}\right)^2 \sqrt{\frac{EJ}{m}} \sqrt{1 + \frac{p_w}{EJ\left(\frac{\pi w}{\ell}\right)^2}}, \qquad w \in \mathbb{N}^+, \tag{134}$$

with a mode shape

$$v(x) = \sin\left(\pi w \frac{x}{\ell}\right), \tag{135}$$

where the critical value is obtained for $w = 1$,

$$p_{\text{crit}} = -EJ\left(\frac{\pi}{\ell}\right)^2. \tag{136}$$

Consider a single finite element based on third-order Hermitian shape functions which, after applying the simply-supported boundary conditions, yields the discrete problem

$$\frac{m\ell^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{Bmatrix} + \left(\frac{EJ}{\ell}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} + p\frac{\ell}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{137}$$

where φ_1 and φ_2 are the transverse rotations at the two ends. The two eigensolutions are

$$\lambda_1 = -\frac{120EJ + 10p\ell^2}{m\ell^4} \qquad \mathbf{z}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{138a}$$

$$\lambda_2 = -\frac{2520EJ + 42p\ell^2}{m\ell^4} \qquad \mathbf{z}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{138b}$$

For $p \equiv 0$, $\lambda_1 = 1.232 s_1^2$, i.e., a 23.2% error, consistent with the coarseness of the discretization.

The sensitivity problem, for $p = 0$, yields $\lambda_{1/p} = -10/(m\ell^2)$ and $\mathbf{z}_{1/p} = \{0;\ 0\}$. For $p = -\lambda_1/\lambda_{1/p} = 120EJ/(m\ell^4)/(-10/(m\ell^2)) = -12EJ/\ell^2$, substituted in Eq. (138a), directly yields $\lambda_1 \equiv 0$.

4.5.2 Von Mises Truss

Consider now the von Mises truss sketched in Fig. 2, which shows a nonlinear dependence on the system's configuration with no explicit dependence on a loading parameter.

A point mass of mass m is connected to the ground by two linear springs of stiffness k_1 and k_2. The distances of the point mass from the ground points are $L_1 = \sqrt{(b+u)^2 + (h+w)^2}$ and $L_2 = \sqrt{(b-u)^2 + (h+w)^2}$,

Table 6 Data of von Mises Truss

Semi-width	b	$\sqrt{3}/2$	m
Height	h	1/2	m
Mass	m	1	kg
Stiffness of left spring	k_1	1	N/m
Stiffness of right spring	k_2	1	N/m

respectively. The springs are unloaded when u and w, the horizontal and vertical components of the mass displacement, are zero. The mass is loaded by a static force of components F_x and F_z. The numerical values of the parameters are reported in Table 6.

The problem takes the form

$$\mathbf{M}\ddot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, p), \tag{139}$$

where for simplicity the explicit dependence on the parameter p is limited to $\mathbf{f}$. Its linearization about a steady configuration, $\mathbf{z}_{\text{ref}}$, that results from

$$\mathbf{0} = \mathbf{f}(\mathbf{z}_{\text{ref}}, p), \tag{140}$$

yields the homogeneous equation

$$\mathbf{M}\Delta\ddot{\mathbf{z}} - \mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z} = \mathbf{0}, \tag{141}$$

and the eigenproblem

$$\left(\lambda\mathbf{M} - \mathbf{f}_{/\mathbf{z}}\right)\Delta\mathbf{z} = \mathbf{0}. \tag{142}$$

In addition to $\mathrm{d}(\lambda\mathbf{M}\Delta\mathbf{z})/\mathrm{d}p = \lambda_{/p}\mathbf{M}\Delta\mathbf{z}_\ell + \lambda_\ell\mathbf{M}\Delta\mathbf{z}_{/p}$, its sensitivity to p requires $\mathrm{d}(\mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z})/\mathrm{d}p$, which in turn can be obtained by considering

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z}\right) = \mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z}_{/p} + \left(\mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z}_\ell\right)_{/\mathbf{z}_{\text{ref}}}\mathbf{z}_{\text{ref}/p} + \mathbf{f}_{/\mathbf{z}p}\Delta\mathbf{z}_\ell, \tag{143}$$

where the matrix–vector multiplication differentiation with respect to another vector is intended as

$$(\mathbf{A}\mathbf{b})_{/\mathbf{c}} := \left[\mathbf{A}_{/c_1}\mathbf{b}\ \mathbf{A}_{/c_2}\mathbf{b}\ \ldots\ \mathbf{A}_{/c_n}\mathbf{b}\right], \tag{144}$$

with $\mathbf{A} := \mathbf{f}_{/\mathbf{z}}$, $\mathbf{b} := \Delta\mathbf{z}_\ell$, and $\mathbf{c} := \mathbf{z}_{\text{ref}}$. The sensitivity of the reference solution, $\mathbf{z}_{\text{ref}/p}$, is obtained by solving

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbf{f}(\mathbf{z}_{\text{ref}}, p) = \mathbf{f}_{/\mathbf{z}_{\text{ref}}}\mathbf{z}_{\text{ref}/p} + \mathbf{f}_{/p} = \mathbf{0}, \tag{145}$$

provided matrix $\mathbf{f}_{/\mathbf{z}_{\text{ref}}}$ is non-singular; when it becomes singular (i.e., when $\lambda_\ell = 0$), the problem can be solved using algorithms like Riks' pseudo-arc length [20].

The sensitivity problem is thus

$$\begin{bmatrix} \lambda_\ell\mathbf{M} - \mathbf{f}_{/\mathbf{z}} & \mathbf{M}\Delta\mathbf{z}_\ell \\ \mathbf{n}^{\mathsf{H}} & 0 \end{bmatrix}\begin{Bmatrix} \Delta\mathbf{z}_{/p} \\ \lambda_{/p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{/\mathbf{z}p}\Delta\mathbf{z}_\ell - \left(\mathbf{f}_{/\mathbf{z}}\Delta\mathbf{z}\right)_{/\mathbf{z}_{\text{ref}}}\mathbf{f}_{/\mathbf{z}_{\text{ref}}}^{-1}\mathbf{f}_{/p} \\ 0 \end{Bmatrix} \tag{146}$$

The stiffness of the springs, k_1 and k_2, and the components of the static load, F_x and F_y, are independently considered as parameters. The problem is formulated with the help of Matlab's symbolic engine and solved numerically. The static solutions have been actually obtained by prescribing the vertical component of the displacement, w, and determining the corresponding vertical component of the force, F_z. These solutions, and the corresponding eigensolutions, have been independently verified using the free general-purpose multibody solver MBDyn[3] [21]. Specifically, the corresponding eigenanalyses have been verified using the procedure illustrated in [22].

Figure 3 presents the equilibrium solutions in terms of the displacement components u and w (Fig. 3a) and corresponding vertical force F_z (Fig. 3b), all parameterized by the lateral load F_x, which breaks the symmetry of the problem. Figure 3b shows that, for zero vertical force, three equilibrium configurations exist: two stable configurations ($w = 0$ and $w = -1$ for $F_x = 0$, which move away from each other as F_x increases) and one unstable configuration ($w = -0.5$, independent of F_x). Figure 3c reports the corresponding eigenvalue λ, is negative for the stable equilibria and positive for the unstable one. The sensitivities of λ with respect to the spring stiffness are shown in Fig. 3d and e, whereas those with respect to the lateral and vertical load components are reported in Fig. 3f and g. The configurations for which $\lambda = 0$ coincide with limit points of the F_z curves in Fig. 3b; at these points the tangent stiffness matrix, $-\mathbf{f}_{/\mathbf{z}}$, becomes singular, causing the equilibrium sensitivity to become unbounded. Consequently, the eigenvalue sensitivities evaluated along the equilibrium manifold exhibit the vertical asymptotes observed in Fig. 3d–g.

It should be emphasized that the computed sensitivities correspond to perturbations restricted to the equilibrium manifold. Consequently, they differ from sensitivities obtained by perturbing the eigenproblem while keeping the reference equilibrium configuration fixed.

4.6 Constrained dynamics

Consider the two constrained masses problem proposed in Sect. 4.1 of [22], here sketched in Fig. 4.

Mass m_1 is connected to the ground by a spring of stiffness k and a damper of characteristic c. Another mass, m_2, is rigidly connected to mass m_1 by a kinematic constraint, such that the constraint

$$x_2 - x_1 = 0 \tag{147}$$

is enforced. The corresponding constrained dynamics equations are

$$m_1\ddot{x}_1 + c\dot{x}_1 + kx_1 - \lambda = 0 \tag{148a}$$

$$m_2\ddot{x}_2 + \lambda = 0 \tag{148b}$$

$$x_2 - x_1 = 0 \tag{148c}$$

where λ is the Lagrange multiplier associated with the constraint of Eq. (147). This is perhaps the simplest multi-degree of freedom problem with kinematic constraints that can be formulated. In the Laplace domain, it becomes

$$\left(s^2\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix} + s\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix} + \begin{bmatrix}\mathbf{K} & \mathbf{G}^{\mathsf{T}}\\ \mathbf{G} & 0\end{bmatrix}\right)\begin{Bmatrix}\mathbf{x}\\ \lambda\end{Bmatrix} = \begin{Bmatrix}\mathbf{0}\\ 0\end{Bmatrix} \tag{149}$$

with

$$\mathbf{M} = \begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix} \tag{150}$$

[3] https://www.mbdyn.org/, last accessed October 2026.

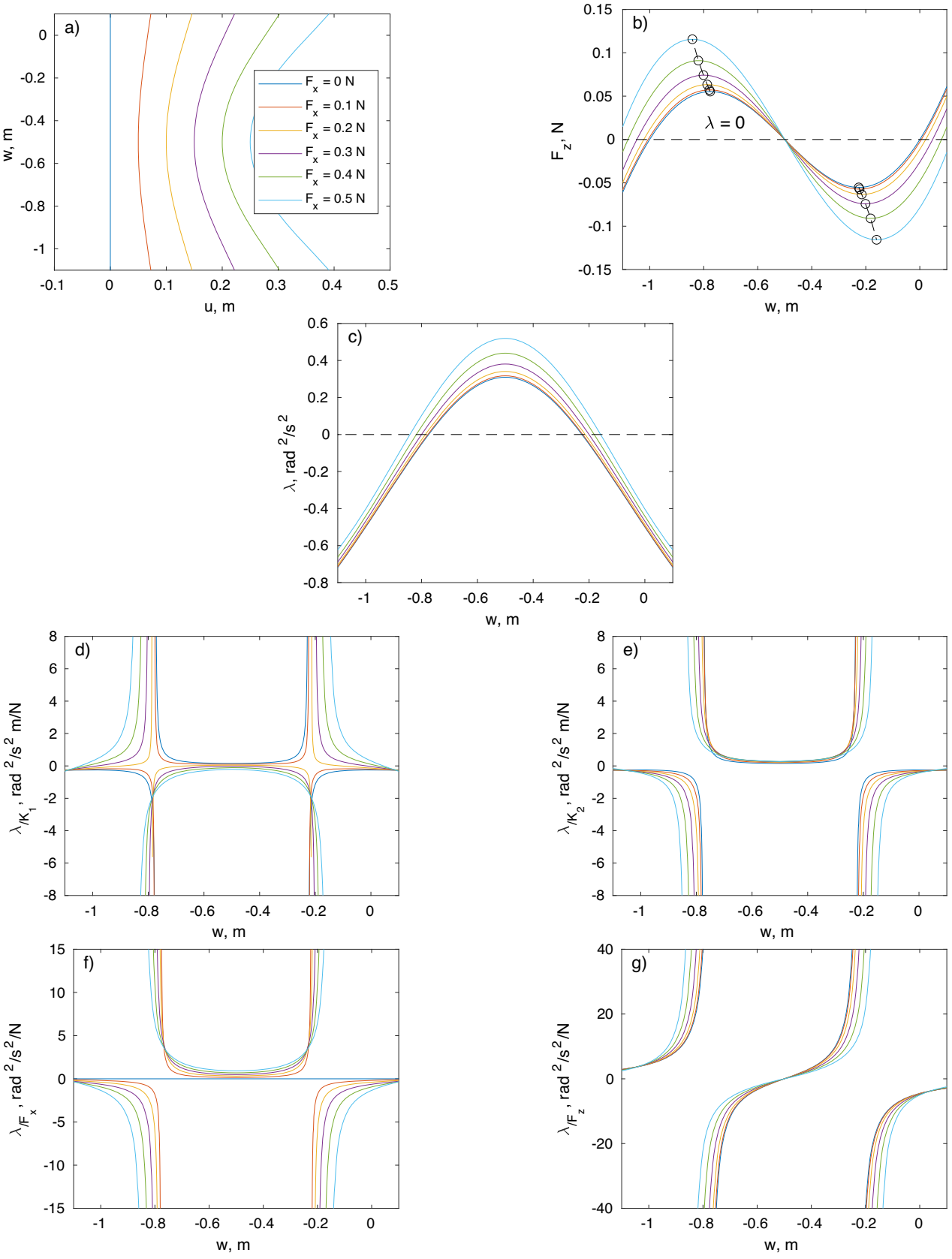


Fig. 3 Static solution (**a**, **b**), eigensolution (**c**), and sensitivity (**d**–**g**) of von Mises Truss

 https://doi.org/10.1007/s11012-026-02190-8

Fig. 4 Constrained dynamics: two constrained masses problem (adapted from [22])

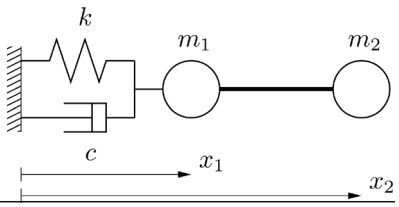


Table 7 Two masses problem data

m_1	2	kg
m_2	8	kg
k	1000	$\mathrm{N{\cdot}m^{-1}}$
c	10	$\mathrm{N{\cdot}s{\cdot}m^{-1}}$

$$\mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \tag{151}$$

$$\mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \tag{152}$$

$$\mathbf{G} = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad \text{the Jacobian matrix of the constraint} \tag{153}$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{154}$$

where it is worth noticing that the generalized mass matrix of Eq. (149) is structurally singular.

The problem is equivalent to a mass-damper-spring system whose equation of motion is

$$(m_1 + m_2)\,\ddot{x} + c\dot{x} + kx = 0, \tag{155}$$

so the expected eigenvalues are

$$s_{1,2} = -\frac{c}{2(m_1 + m_2)} \pm \sqrt{\left(\frac{c}{2(m_1 + m_2)}\right)^2 - \frac{k}{m_1 + m_2}} \tag{156}$$

and the components of the eigenvector associated with x_1 and x_2 are expected to be identical. With the data from Table 7, this corresponds to

$$s_{1,2} = -5.0000\text{e-}01 \pm \mathrm{j}9.9875\text{e+}00 \text{ rad/s} \tag{157}$$

The problem, analyzed with the proposed method initialized with

$$s_0 = \mathrm{j}10 \text{ rad/s} \qquad \mathbf{z}_0 = \begin{bmatrix} 0.605676867847688 \\ 0.605676867847688 \\ -0.516053353354505 \end{bmatrix} \tag{158}$$

yields

$$s_1 = (-5.0000\text{e-}01 + \mathrm{j}9.9875\text{e+}00) \text{ rad/s} \qquad \mathbf{z} = \begin{bmatrix} 1.2437\text{e-}03 - \mathrm{j}1.2484\text{e-}04 \\ 1.2437\text{e-}03 - \mathrm{j}1.2484\text{e-}04 \\ 1.0000\text{e+}00 + \mathrm{j}0.0000\text{e+}00 \end{bmatrix} \tag{159}$$

Table 8 Condition numbers of the two-masses problem

Iteration	cond($\mathbf{Z}$)	cond(Eq. (11))
1	5.2000e+06	5.4461e+03
2	5.1797e+06	1.3362e+06
3	1.3884e+07	2.0942e+06
4	8.1794e+07	2.5112e+06
5	3.9614e+09	2.5627e+06
6	1.7946e+13	2.5632e+06
7	2.2162e+20	2.5632e+06

where it is worth noticing that the constraint reaction must balance the inertia force acting on mass m_2, thus

$$\lambda = -m_2 s^2 x_2 = (1.0000\text{e+}00 + \text{j}0.0000\text{e+}00)\ \text{N} \tag{160}$$

i.e., the third component of the eigenvector when the second component is used for x_2.

Table 8 reports the condition number of the two-masses problem, which converged in seven iterations with a tolerance of 1e-12.

4.7 Time delay

Consider the problem of Eq. (6), with

$$\mathbf{Z} = s^2\mathbf{M} + s\mathbf{C}\mathbf{K} + \mathrm{e}^{\tau s}\mathbf{H} \tag{161}$$

where a damping contribution has been added, and

$$\mathbf{Z}_{/s} = 2s\mathbf{M} + \mathbf{C} + \tau\mathrm{e}^{\tau s}\mathbf{H} \tag{162}$$

This problem cannot be solved using conventional methods (e.g., using Matlab's `eig()` function with a first-order differential realization of the problem), since it is not polynomial with respect to the eigenvalue s.

Considering for example the time delay as the parameter, i.e., $p := \tau$, yields

$$\mathbf{Z}_{/p} = s_\ell \mathrm{e}^{\tau s_\ell}\mathbf{H} \tag{163}$$

Consider a problem with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{kg} \tag{164a}$$

$$\mathbf{C} = \pi \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{N}\cdot\text{s}\cdot\text{m}^{-1} \tag{164b}$$

$$\mathbf{K} = (2\pi)^2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{N}\cdot\text{m}^{-1} \tag{164c}$$

$$\mathbf{H} = h \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{N}\cdot\text{m}^{-1} \tag{164d}$$

where a non-conservative force acting on the second equation depends on the delayed coordinate of the first equation.

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With $h = 0$, the eigensolutions are

$$s_{1\pm} = -5.9999\text{e-}01 \pm \text{j}3.8366\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{1\pm} = \left\{ \begin{array}{c} 5.2573\text{e-}01 \\ 8.5065\text{e-}01 \end{array} \right\} \tag{165a}$$
$$s_{2\pm} = -4.1124\text{e+}00 \pm \text{j}9.2975\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{2\pm} = \left\{ \begin{array}{c} 8.5065\text{e-}01 \\ -5.2573\text{e-}01 \end{array} \right\} \tag{165b}$$

with real-valued eigenvectors, since the damping matrix is proportional to the stiffness matrix.

With $h = (2\pi)^2$ and $\tau = 0$ s, the proposed method yields

$$s_{1\pm} = -3.2490\text{e-}01 \pm \text{j}5.9902\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{1\pm} = \left\{ \begin{array}{c} 6.6451\text{e-}01 \pm \text{j}0.0000\text{e+}00 \\ 7.4612\text{e-}01 \mp \text{j}4.1565\text{e-}02 \end{array} \right\} \tag{166a}$$
$$s_{2\pm} = -4.3875\text{e+}00 \pm \text{j}8.2076\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{2\pm} = \left\{ \begin{array}{c} 9.6995\text{e-}01 \pm \text{j}0.0000\text{e+}00 \\ -2.3774\text{e-}01 \pm \text{j}5.1717\text{e-}02 \end{array} \right\} \tag{166b}$$

The solution of the nonlinear problem is coincident with the one resulting from Matlab's `eig()` function using a first-order differential realization of the problem.

With $h = (2\pi)^2$ and $\tau = 0.1$ s, the proposed method yields

$$s_{1\pm} = -1.5699\text{e+}00 \pm \text{j}5.6269\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{1\pm} = \left\{ \begin{array}{c} 6.6348\text{e-}01 \pm \text{j}3.6961\text{e-}02 \\ 7.4728\text{e-}01 \mp \text{j}0.0000\text{e+}00 \end{array} \right\} \tag{167a}$$
$$s_{2\pm} = -3.4399\text{e+}00 \pm \text{j}8.8136\text{e+}00 \quad \text{rad/s}$$
$$\mathbf{z}_{2\pm} = \left\{ \begin{array}{c} 9.6995\text{e-}01 \pm \text{j}0.0000\text{e+}00 \\ -2.3774\text{e-}01 \pm \text{j}5.1717\text{e-}02 \end{array} \right\} \tag{167b}$$

The L^∞-norm sensitivity to the time delay yields

$$s_{1\pm/\tau} = -7.6735\text{e+}00 \mp \text{j}9.7337\text{e+}00 \quad \text{rad/s}\cdot\text{s}^{-1}$$
$$\mathbf{z}_{1\pm/\tau} = \left\{ \begin{array}{c} -1.1760\text{e+}00 \pm \text{j}1.8145\text{e+}00 \\ 0.0000\text{e+}00 \mp \text{j}0.0000\text{e+}00 \end{array} \right\} \quad \text{s}^{-1} \tag{168a}$$
$$s_{2\pm/\tau} = 2.3067\text{e+}00 \pm \text{j}1.0685\text{e+}01 \quad \text{rad/s}\cdot\text{s}^{-1}$$
$$\mathbf{z}_{2\pm/\tau} = \left\{ \begin{array}{c} 0.0000\text{e+}00 \pm \text{j}0.0000\text{e+}00 \\ -2.5466\text{e+}00 \pm \text{j}3.8844\text{e+}00 \end{array} \right\} \quad \text{s}^{-1} \tag{168b}$$

The sensitivity solution can be used to predict the initial guess for the nonlinear problem at different values of τ, starting from $\tau = 0$, where a conventional eigensolution can be computed, namely

$$\mathbf{z}_{0_{i+1}} = \mathbf{z}_{\ell_i} + \mathbf{z}_{\ell/\tau}\Delta\tau \tag{169a}$$
$$s_{0_{i+1}} = s_{\ell_i} + s_{\ell/\tau}\Delta\tau \tag{169b}$$

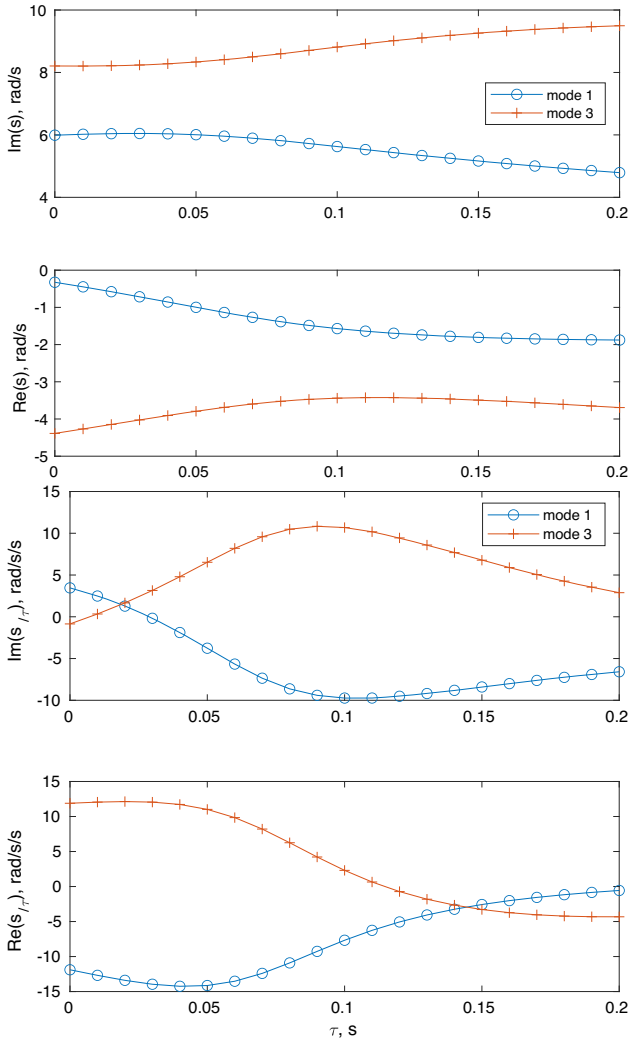


Fig. 5 Time-delay problem eigenvalues

Fig. 6 Sensitivity of time-delay problem eigenvalues to the time delay, τ

Fig. 5 shows the evolution of the two eigenvalues for increasing values of the time delay τ. With a tolerance on the residual of 1e-12, convergence requires at most four iterations, most commonly three.

Figure 6 shows the evolution of the two eigenvalues' sensitivity for increasing values of the time delay τ.

4.8 Continuum eigenanalysis

The eigenanalysis of one-dimensional continua is formulated in terms of a wave function, extracted from the indefinite equilibrium equation, which contains partial space and time derivatives of the continuum field (e.g., displacements and rotations in structural dynamics, temperature in thermal problems, etc.). The general solution of the indefinite equilibrium is then formulated in the form of the linear combination of a set of complex exponential functions, whose space and time exponents are defined in certain proportionality relationships that contain the parameters that characterize the indefinite equilibrium. Finally, appropriate boundary conditions are enforced at the ends of the one-dimensional continuum, resulting in a linear, homogeneous problem in the multipliers of the complex exponential functions, with the exponents playing the role of the eigenvalues.

Consider for example the heat flow problem along a one-dimensional continuum,

$$A\rho C_v \dot{T} - AKT_{/xx} = 0 \tag{170}$$

where A is the section area, ρ the density, C_v the thermal capacity, K the conductivity, and $T(x, t)$ is the temperature field, subjected to an appropriate set of boundary conditions (one for each end, either prescribed, i.e., Dirichlet,

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or natural, i.e., Neumann), namely either of

$$T(0,t) = T_0(t) \quad \text{or} \quad -Q_0(t) + AKT_{/x}(0,t) = 0 \tag{171a}$$
$$T(\ell,t) = T_\ell(t) \quad \text{or} \quad -Q_\ell(t) - AKT_{/x}(\ell,t) = 0 \tag{171b}$$

where the heat flows Q_0 and Q_ℓ at both ends are assumed positive when exiting the domain.

Through variable separation, the solution takes the form

$$T(x,t) = z\mathrm{e}^{\beta x + st} \tag{172}$$

resulting in

$$\left(sA\rho C_v - \beta^2 AK\right) T = 0 \tag{173}$$

which yields the condition

$$\beta^2 = s\frac{\rho C_v}{K} = -\alpha^2 \tag{174}$$

where we directly introduced $\alpha = \mathrm{j}\beta$ because we expect s to be negative. The general solution can be represented as

$$T(x,t) = \mathrm{e}^{st}\sum \phi_i(x) z_i = \left[\cos(\alpha x)\ \sin(\alpha x)\right]\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}\mathrm{e}^{st} = \boldsymbol{\phi}(\alpha, x)\mathbf{z}\mathrm{e}^{st} \tag{175}$$

Enforcing the boundary conditions yields

$$\Phi(\alpha)\mathbf{z}\mathrm{e}^{st} = \mathbf{0} \tag{176}$$

where Φ is a matrix constructed by applying the two boundary conditions, with the related differentiations of the functions in $\boldsymbol{\phi}$, and their evaluations at the beam ends.

Consider for example convective heat flow at one end, i.e., $Q_\ell(t) = h\left(T(0,t) - T_e\right)$, with $T_e = 0$, and prescribed temperature at the other end,

$$-hT(0,t) + AKT_{/x}(0,t) = 0 \tag{177a}$$
$$T(\ell,t) = 0 \tag{177b}$$

The problem becomes

$$\begin{bmatrix} -h & AK\alpha \\ \cos(\alpha\ell) & \sin(\alpha\ell) \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{178}$$

An analytical solution can be easily found by computing the determinant[4] of Φ,

$$\det(\Phi) = -h\sin(\alpha\ell) - AK\alpha\cos(\alpha\ell) \tag{179}$$

[4] The exact same determinant would be obtained by exchanging the boundary conditions; however, in that case the eigenvector would be trivial. The present form is used to preserve the generality of the example.

Table 9 Heat flow problem data

Quantity	Symbol	Value	Units
Section area	A	1	m^2
Length	ℓ	1	m
Density	ρ	1	$kg{\cdot}m^{-3}$
Thermal capacitance	C_v	1	$J{\cdot}kg^{-1}{\cdot}K^{-1}$
Thermal conductivity	K	1	$W{\cdot}m^{-1}{\cdot}K^{-1}$
Thermal convectivity	h	1	$W{\cdot}K^{-1}$

Table 10 Heat flow problem eigensolutions

$\alpha\ell$, no convection	$\alpha\ell$, infinite convection	Initial value, $\alpha_0\ell$	$\alpha\ell$	z_1	z_2
$\pi/2$	π	$\pi/2$	0.6458π	8.9696e−01	4.4212e−01
$3\pi/2$	2π	$3\pi/2$	1.5639π	9.7991e−01	1.9944e−01
$5\pi/2$	3π	$5\pi/2$	2.5397π	9.9224e−01	1.2436e−01
$7\pi/2$	4π	$7\pi/2$	3.5286π	9.9596e−01	8.9843e−02
$9\pi/2$	5π	$9\pi/2$	4.5224π	9.9753e−01	7.0212e−02

which vanishes for

$$\tan(\alpha\ell) = -\frac{AK}{h\ell}\alpha\ell \tag{180}$$

whose infinite solutions lie at the crossing of the line through the origin with negative slope $AK/(h\ell)$ and the tangent of $\alpha\ell$, with the limit cases

$$\alpha = \frac{1}{\ell}\left(\frac{\pi}{2} + w\pi\right) \qquad h \to 0 \text{ (no convection)} \tag{181}$$

$$\alpha = \frac{1}{\ell}w\pi \qquad h \to +\infty \text{ (infinite convection)} \tag{182}$$

with $w \in \mathbb{Z}$. Solutions for $0 \leq h \leq +\infty$ lie in between, and tend to the no-convection case for increasing w.

Using the proposed approach, with α playing the role of the eigenvalue, and considering

$$\mathbf{Z}_{/\alpha} = \begin{bmatrix} 0 & AK \\ -\ell\sin(\alpha\ell) & \ell\cos(\alpha\ell) \end{bmatrix} \tag{183}$$

with the data provided in Table 9, the solutions of Table 10 is obtained. The values of α converge towards the no convection case; the eigenvectors converge towards the solution $\{1; 0\}$, i.e., only the cosine.

The actual eigenvalue can be obtained from Eq. (174) as

$$s = -\frac{K}{\rho C_v}\alpha^2 \tag{184}$$

By computing the partial derivative of Eq. (184), the sensitivity of s thus becomes

$$s_{/p} = -\frac{K}{\rho C_v}2\alpha\alpha_{/p} \tag{185}$$

Table 11 Heat flow problem sensitivity

$\alpha\ell$	s, rad/s	$s_{/h}$, rad·s^{-1}·W^{-1}·K	$z_{1/h}$, W^{-1}·K	$z_{2/h}$, W^{-1}·K
0.6458π	−4.1159e+00	−1.3460e+00	−1.4666e−01	2.9754e−01
1.5639π	−2.4139e+01	−1.8470e+00	−3.7488e−02	1.8418e−01
2.5397π	−6.3659e+01	−1.9391e+00	−1.5112e−02	1.2057e−01
3.5286π	−1.2289e+02	−1.9680e+00	−7.9747e−03	8.8404e−02
4.5224π	−2.0185e+02	−1.9804e+00	−4.8934e−03	6.9523e−02

Considering for example the sensitivity to the convectivity, h, and thus

$$\mathbf{Z}_{/h} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{186}$$

for the eigensolution of Table 10 the results of Table 11 are obtained.

5 Conclusion

A general framework has been presented for the computation of eigensolution sensitivities in nonlinear operator pencils. The formulation expresses the eigenproblem as a nonlinear algebraic system augmented by a normalization condition and derives sensitivity equations by differentiation. Regularization of the resulting singular system permits explicit closed-form solutions for eigenvalue and eigenvector derivatives, valid even in the presence of algebraic multiplicity. The approach also clarifies the algebraic origin of ill-conditioning near coalescent eigenvalues and provides a systematic means of regularizing sensitivity computations.

An inversion formula for the sum of singular matrices has been applied to complex-valued matrices through decomposition-based constructions, including an original formulation based on spectral decomposition, ensuring numerical stability and analytical transparency. The examples demonstrate that the method can be applied to a variety of engineering problems, including undamped and damped structural systems, rigid-body degeneracies, nonlinear problems, differential-algebraic problems resulting from constrained dynamics, time-delay operators, and continuum eigenvalue formulations. The proposed framework offers a practical and theoretically consistent foundation for the analysis of parameter-dependent eigenproblems. It may serve as a basis for future developments in continuation, uncertainty quantification, and design sensitivity analysis.

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Data availability No datasets were produced or used in this study. All the data required to reproduce the results are provided in the manuscript.

Declarations

Ethical approval Not applicable.

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