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Spectral Analysis and Stability of the Moore-Gibson-Thompson-Fourier Model

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Abstract
We consider the linear evolution system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta \Delta^2 u_t + \gamma \Delta^2 u = -\eta \Delta \theta \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t \end{cases}$$

describing the dynamics of a thermoviscoelastic plate of MGT type with Fourier heat conduction. The focus is the analysis of the energy transfer between the two equations, particularly when the first one stands in the supercritical regime, and exhibits an antidissipative character. The principal actor becomes then the coupling constant η, ruling the competition between the Fourier damping and the MGT antidamping. Indeed, we will show that a sufficiently large η is always able to stabilize the system exponentially fast. One of the features of this model is the presence of the bilaplacian in the first equation. With respect to the analogous model with the Laplacian, this introduces some differences in the mathematical approach. From the one side, the energy estimate method does not seem to apply in a direct way, from the other side, there is a gain of regularity allowing to rely on analytic semigroup techniques.

Keyword MGT equation, Fourier law, analytic semigroup, spectrum, exponential stability

Mathematics Subject Classification 35B40 · 35Q74 · 74F05

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1 Introduction

1.1 The Model System

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Given the parameters $\alpha, \beta, \gamma, \kappa > 0$ and $\eta \neq 0$, we consider the linear evolution PDE system in the unknown variables $u, \theta : \Omega \times \mathbb{R}^+ \to \mathbb{R}$

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta\Delta^2 u_t + \gamma\Delta^2 u = -\eta\Delta\theta, \\ \theta_t - \kappa\Delta\theta = \eta\Delta u_{tt} + \alpha\eta\Delta u_t, \end{cases} \tag{1.1}$$

subject to the initial conditions

$$u(0) = u_0, \qquad u_t(0) = v_0, \qquad u_{tt}(0) = w_0, \qquad \theta(0) = \theta_0,$$

where $u_0, v_0, w_0, \theta_0 : \Omega \to \mathbb{R}$ are assigned initial data. Such a system has been introduced as a model for the vibrations of a thermoviscoelastic plate of Moore-Gibson-Thompson type in the recent paper [7], where we address the interested reader for a detailed derivation. Accordingly, the variable u stands for the vertical displacement from equilibrium, while θ represents the relative temperature, namely, the temperature variation from a fixed reference temperature. Concerning the structural parameters of the first equation of (1.1), by considering unit mass density, we have the relations

$$\alpha = 1/\tau, \qquad \beta = k/\tau, \qquad \gamma = k^*/\tau,$$

where τ, k, k^* represent the thermal relaxation, the thermal conductivity, and the conductivity rate of the material, respectively. The second equation, ruling the evolution of θ, is the classical heat equation, obtained by substituting the Fourier thermal law into the energy balance identity, where κ is (proportional to) the thermal conductivity. Finally, the coupling parameter η is responsible for the interplay between the two equations. The system is complemented with the boundary conditions

$$u(t)_{|\partial\Omega} = \Delta u(t)_{|\partial\Omega} = \theta(t)_{|\partial\Omega} = 0,$$

expressing the fact that the ends of the plate are hinged and the boundary is kept at equilibrium temperature for all times.

In absence of coupling, the first equation ruling the evolution of u typically appears in the literature in the abstract form

$$u_{ttt} + \alpha u_{tt} + \beta A u_t + \gamma A u = 0, \tag{1.2}$$

where A is a strictly positive selfadjoint operator acting on some Hilbert space H (see, e.g., [19, 23]). Equation (1.2) is known as the Moore-Gibson-Thompson (MGT) equation, named after the works [26, 29], although it has been originally introduced by Stokes in the mid-nineteenth century [28] for the concrete choice $A = -\Delta$. In particular, when $A = -\Delta$, the MGT equation (1.2) is used in the description of acoustic wave propagation in viscous thermally relaxing fluids. In this context, α, β, γ represent the natural damping coefficient, the sound diffusivity, and the square of the sound speed, respectively. Other physical applications arise in viscoelasticity, thermal conduction, lithotripsy and high intensity focused ultrasounds (see [8, 13, 15, 18]).

From the mathematical viewpoint, system (1.1) falls into the class of hyperbolic-parabolic systems, whose prototype examples are the wave-heat and the plate-heat systems (see, e.g., [10, 16, 17] and references therein). Nonetheless, models of this kind usually consist of a conservative equation coupled with a dissipative one. On the contrary, system (1.1) exhibits a peculiar characteristic, due to the fact that the stability properties of the MGT equation are strongly influenced by the sign of the so-called stability number

$$\mu = \gamma - \alpha\beta.$$

Indeed, in the *subcritical case* $\mu < 0$ the MGT equation (1.2) is exponentially stable, in the *critical case* $\mu = 0$ it has a conservative-type dynamics, while in the *supercritical case* $\mu > 0$ there exist trajectories blowing up exponentially fast [11, 19, 23]. In particular, the MGT equation either decays exponentially to zero, or it does not decay at all, similarly to what happens in the finite-dimensional case. It is important to point out that these stability properties are independent of the particular choice of the operator A appearing in (1.2). More results concerning the well-posedness and the stability of the solutions to the MGT equation, both in the linear and the nonlinear setting, can be found for instance in [3–5, 20–22, 24].

In the light of the discussion above, a natural question to address is how the dissipation produced by the heat equation influences the asymptotic dynamics of the system. In the subcritical case, both equations in (1.1) are dissipative (actually, exponentially stable) and good stabilization properties are expected. This insight has been confirmed in [7], where it is shown that if $\mu < 0$ then the solution semigroup associated to (1.1) is exponentially stable and analytic as well. This means that the coupling enables not only a transfer of dissipation between the equations, but somehow also a transfer of regularity. The focus of the present paper is to understand what happens in the critical and supercritical regimes, that is, when $\mu \geq 0$. In this situation, system (1.1) consists of a conservative (if $\mu = 0$) or antidissipative (if $\mu > 0$) equation coupled with a dissipative one, making the picture intriguing and highly nontrivial.

1.2 The Results

First, we prove that system (1.1) generates a strongly continuous analytic semigroup $S(t) = e^{t\mathbb{A}}$ for every value of the stability number μ, and we provide a complete characterization of the spectrum $\sigma(\mathbb{A})$ of its infinitesimal generator $\mathbb{A}$, including a detailed analysis of the point spectrum. Then we show that, when $\mu \geq 0$, the exponential stability of $S(t)$ occurs if and only if the coupling parameter η satisfies the constraint

$$\eta^2 > x_\star, \tag{1.3}$$

where $x_\star$ is a certain constant depending only on $\alpha, \beta, \gamma, \kappa$ and Ω. In particular, $x_\star$ turns out to be zero when $\mu = 0$, meaning that in the critical regime $S(t)$ is always exponentially stable. Condition (1.3) also tells that a sufficiently large (in modulus) coupling parameter is able to stabilize the dynamics to zero exponentially fast, no matter how large the stability number might be. This phenomenon highlights that the antidissipation in the MGT equation is weaker, or more precisely of lower quality, than the dissipation in the heat equation. To some extent, being the MGT equation hyperbolic and the heat equation parabolic, this might not be that surprising. Nevertheless, the result is not obvious, for there are several examples of hyperbolic-parabolic systems where exponential stability does not occur due to an inefficient (or a too efficient) coupling mechanism (see, e.g., [2, 9, 16] and references therein). Our achievements reveal that there is a good communication between the two equations in (1.1), and the dissipation is shared is an effective way. Even more so, the coupling always plays a key role. Indeed, in the last part of the paper we will show that $x_\star \to \infty$ when $\kappa \to \infty$, meaning that the thermal dissipation alone is not able to drive the solutions to zero exponentially fast without a significant help from the coupling parameter.

1.3 Methodology

In the very recent article [6], some of the authors of the present paper have obtained similar results for the MGT-Fourier system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u = -\eta \Delta \theta, \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t, \end{cases} \tag{1.4}$$

where the bilaplacian Δ^2 in the MGT equation is replaced by $-\Delta$. The techniques employed in [6] are based on the use of appropriate energy-like functionals able to enucleate the antidissipation produced by the MGT equation. Due to the higher regularity of the variable u, such techniques do not seem to apply to (1.1). In order to overcome this difficulty, we rely on the analyticity of $S(t)$ and the fact that an analytic semigroup is exponentially stable if (and only if) the spectrum of its infinitesimal generator is contained in the open left half plane

$$\mathbb{C}^- = \{z \in \mathbb{C} : \mathfrak{Re}\,(z) < 0\}.$$

Thanks to the aforementioned characterization of $\sigma(\mathbb{A})$, and making use of the Routh-Hurwitz stability criterion, we translate this abstract condition into a condition on the structural parameters of the system, which eventually leads to (1.3).

1.4 Plan of the Paper

In the next Sect. 2, after introducing the proper functional setting, we rewrite system (1.1) in an abstract form, and we view it as an ordinary differential equation on a suitable Hilbert space. In Sect. 3 we state the results on the generation of the analytic solution semigroup (Theorem 3.1), and its exponential decay (Theorem 3.3). The rest of the paper is devoted to the proof of Theorem 3.3. This requires a detailed knowledge of the spectrum of the infinitesimal generator of the semigroup, discussed in Sect. 4, and some stability criteria, presented in Sect. 5. The proof of the theorem is carried out in Sect. 6. Some remarks and comments are given in the final Sect. 7. In the Appendix we state and prove some operator theoretical results used in Sect. 4.

2 The Abstract Problem

2.1 Functional Setting

Let $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ be a real Hilbert space, and let

$$A : \mathfrak{D}(A) \subset H \to H$$

be a strictly positive selfadjoint unbounded linear operator, with inverse A^{-1} not necessarily compact. Recall that, in this case, the spectrum $\sigma(A)$ of A belongs to $\mathbb{R}^+$ and its minimum is strictly positive. For $r \in \mathbb{R}$, we define the hierarchy of continuously nested Hilbert spaces

(r will be always omitted whenever zero)

$$H^r = \mathfrak{D}(A^{r/2}), \qquad \langle u, v\rangle_r = \langle A^{r/2}u, A^{r/2}v\rangle, \qquad \|u\|_r = \|A^{r/2}u\|.$$

For $r > 0$ it is understood that H^{-r} denotes the completion of the domain, so that H^{-r} is the dual space of H^r. The phase space of our problem is the product Hilbert space

$$\mathcal{H} = H^2 \times H^2 \times H \times H,$$

endowed with the standard product norm

$$\|(u, v, w, \theta)\|^2_{\mathcal{H}} = \|u\|^2_2 + \|v\|^2_2 + \|w\|^2 + \|\theta\|^2.$$

2.2 The System

In greater generality, we will consider the abstract system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta A^2 u_t + \gamma A^2 u = \eta A\theta, \\ \theta_t + \kappa A\theta = -\eta A u_{tt} - \alpha\eta A u_t, \end{cases} \tag{2.1}$$

in the unknown variables $u, \theta : \mathbb{R}^+ \to H$.

Remark 2.1 System (1.1) turns out to be the concrete realization of (2.1), corresponding to the choice $H = L^2(\Omega)$ and

$$A = -\Delta \qquad \text{with domain} \qquad \mathfrak{D}(A) = H^2(\Omega) \cap H^1_0(\Omega).$$

Introducing the state vector $\boldsymbol{u} = (u, v, w, \theta) \in \mathcal{H}$, we view (2.1) as the ordinary differential equation in $\mathcal{H}$

$$\frac{d}{dt}\boldsymbol{u} = \mathbb{A}\boldsymbol{u},$$

where $\mathbb{A} : \mathfrak{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator acting as

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\alpha w - A^2(\beta v + \gamma u) + \eta A\theta \\ -\kappa A\theta - \eta A w - \alpha\eta A v \end{pmatrix}, \tag{2.2}$$

with (dense) domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ (u, v, w, \theta) \in \mathcal{H} \;\middle|\; \begin{matrix} w \in H^2 \\ \beta v + \gamma u \in H^4 \\ \theta \in H^2 \end{matrix} \right\}.$$

3 The Semigroup and its Exponential Stability

Our first result concerns with the generation of the solution semigroup for system (2.1).

Theorem 3.1 *The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous analytic semigroup $S(t) : \mathcal{H} \to \mathcal{H}$.*

The proof is based on the following well-known perturbation lemma (see, e.g., [25]).

Lemma 3.2 *Let $\mathbb{A}$ be the infinitesimal generator of an analytic semigroup. If $\mathbb{B}$ is a bounded linear operator, then $\mathbb{A}+\mathbb{B}$ is the infinitesimal generator of an analytic semigroup.*

Proof of Theorem 3.1 The goal is to see (2.1) as a suitable perturbed system. To this end, we make use of a "pumping" technique firstly devised in [11]. Choosing $m \geq 0$ large enough that

$$\mu_m = \gamma - (\alpha + m)\beta < 0,$$

and calling $\alpha_m = \alpha + m$, we rewrite (2.1) as

$$\begin{cases} u_{ttt} + \alpha_m u_{tt} + \beta A^2 u_t + \gamma A^2 u = \eta A\theta + m u_{tt}, \\ \theta_t + \kappa A\theta = -\eta A u_{tt} - \alpha_m \eta A u_t + m\eta A u_t, \end{cases}$$

or, equivalently, as

$$\frac{d}{dt}\boldsymbol{u} = (\mathbb{A}_m + \mathbb{B})\boldsymbol{u}.$$

Here $\mathbb{A}_m : \mathfrak{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ is the operator defined as in (2.2) with α replaced by α_m, while $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ is the operator acting as

$$\mathbb{B}\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ mw \\ m\eta A v \end{pmatrix}.$$

Since $\mu_m < 0$, from the results of [7] we know that $\mathbb{A}_m$ is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$. As for $\mathbb{B}$, it is straightforward to see that for every $\boldsymbol{u} \in \mathcal{H}$

$$\|\mathbb{B}\boldsymbol{u}\|_{\mathcal{H}}^2 = m^2\|w\|^2 + m^2\eta^2\|v\|_2^2 \leq m^2(1+\eta^2)\|\boldsymbol{u}\|_{\mathcal{H}}^2.$$

Thus $\mathbb{B}$ is a bounded operator, and the conclusion follows from Lemma 3.2. □

We now turn to the asymptotic properties of the semigroup. Recall that $S(t)$ is said to be *exponentially stable* if

$$\|S(t)\|_{L(\mathcal{H})} \leq M e^{-\nu t},$$

for some constants $\nu > 0$ and $M \geq 1$, where the norm is meant in the space of bounded linear operators on $\mathcal{H}$. In the subcritical case $\mu < 0$, it is known from [7] that $S(t)$ is exponentially stable for every choice of the coupling parameter $\eta \neq 0$. The next theorem deals with the more delicate situation where the MGT equation lives in its critical or supercritical regime. Here, the size of the coupling parameter becomes crucial.

Theorem 3.3 *Let $\mu \geq 0$. Then there exists a number $x_\star \geq 0$ independent of η, called stability threshold, such that the analytic semigroup $S(t)$ is exponentially stable if and only if*

$$\eta^2 > x_\star.$$

Moreover, $x_\star = 0$ if and only if $\mu = 0$.

The remaining of the paper is devoted to the proof of Theorem 3.3, which requires some tools from the stability theory of abstract semigroups, based in turn on the knowledge of the spectrum of the infinitesimal generator.

4 Spectral Analysis

The object of this section is the description of the spectrum $\sigma(\mathbb{A})$ of the (complexification of the) closed operator $\mathbb{A}$. Besides having an interest by itself, this will be the main tool for the stability analysis of the semigroup $S(t)$.

4.1 The Spectrum

We provide the complete characterization of $\sigma(\mathbb{A})$, in dependence of the structural parameters of the model. To this end, for any $z \in \mathbb{C}$, we define the polynomial in the variable $\lambda \in \sigma(A)$, hence $\lambda > 0$, as

$$p_z(\lambda) = \kappa(\beta z + \gamma)\lambda^3 + z(\beta z + \gamma + \eta^2(z + \alpha))\lambda^2 + \kappa z^2(z + \alpha)\lambda + z^3(z + \alpha).$$

The following theorem holds.

Theorem 4.1 *The spectrum of* $\mathbb{A}$ *is the set*

$$\sigma(\mathbb{A}) = \bigcup_{\lambda \in \sigma(A)} \big\{z \in \mathbb{C} : \; p_z(\lambda) = 0\big\} \cup \big\{ -\tfrac{\gamma}{\beta}\big\}.$$

Proof Let $z \in \mathbb{C}$ be fixed. For an arbitrarily given $\hat{\boldsymbol{u}} = (\hat{u}, \hat{v}, \hat{w}, \hat{\theta}) \in \mathcal{H}$, we look for the unique solution $\boldsymbol{u} = (u, v, w, \theta) \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation

$$z\boldsymbol{u} - \mathbb{A}\boldsymbol{u} = \hat{\boldsymbol{u}}.$$

In components,

$$\begin{cases} zu - v = \hat{u}, \\ zv - w = \hat{v}, \\ zw + \alpha w + A^2(\beta v + \gamma u) - \eta A\theta = \hat{w}, \\ z\theta + \kappa A\theta + \eta Aw + \alpha\eta Av = \hat{\theta}. \end{cases} \tag{4.1}$$

The first two equations yield

$$v = zu - \hat{u},$$
$$w = z^2 u - z\hat{u} - \hat{v}.$$

Then, recalling that A is invertible and $\eta \neq 0$, from the third equation we deduce that

$$\theta = \frac{1}{\eta}\big[(z^3 + \alpha z^2)A^{-1}u + (\beta z + \gamma)Au + \phi\big],$$

where

$$\phi = -\big[(z^2 + \alpha z)A^{-1}\hat{u} + (z + \alpha)A^{-1}\hat{v} + A^{-1}\hat{w} + \beta A\hat{u}\big] \in H.$$

Plugging everything into the last equation of (4.1), and applying ηA to both sides, we finally obtain

$$p_z(A)u = \psi, \tag{4.2}$$

having set

$$\psi = \eta^2(z + \alpha)A^2\hat{u} + \eta^2 A^2\hat{v} + \eta A\hat{\theta} - zA\phi - \kappa A^2\phi. \tag{4.3}$$

Note that, for every possible choice of the structural parameters,

$$\psi \in H^{-4},$$

and in general this is the best regularity that ψ can attain. We now claim that if (4.2) has a unique solution $u \in H^2$, then we have the unique solution $\boldsymbol{u} \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation, and we conclude that z belongs to the resolvent set of $\mathbb{A}$. Indeed, once we find such a $u \in H^2$, then v and w belong to H^2, and from the fourth equation of (4.1) we learn that $\theta \in H^2$ as well. From the third equation we conclude that $\beta v + \gamma u \in H^4$.

At this point, we apply the abstract results stated in the final Appendix, allowing us to conclude that (4.2) has a unique solution $u \in H^2$ for every ψ given by (4.3) if and only if

$$p_z(\lambda) \neq 0, \quad \text{for all} \quad \lambda \in \sigma(A) \qquad \text{and} \qquad \beta z + \gamma \neq 0.$$

Indeed:
• If $p_z(\lambda) \neq 0$ and $\beta z + \gamma \neq 0$, then the existence of the unique solution is guaranteed by Theorem A.3 with $r = 2$ and $s = -4$.
• If $p_z(\lambda) = 0$ for some $\lambda \in \sigma(A)$, then we apply Theorem A.1 with $r = 2$ and $s = -2$. This is possible since the set of vectors ψ of the form (4.3) covers the whole space H^{-2}, and this can be easily seen by choosing $\hat{\boldsymbol{u}} = (0, 0, 0, \hat{\theta})$.
• If $\beta z + \gamma = 0$, the polynomial $p_z(\lambda)$ becomes of order two at most. We may also assume that it has no roots in $\sigma(A)$, otherwise we fall into the previous case. The conclusion follows now from Theorem A.2 with $r = 2$, $s = -4$ and $\varepsilon = 2$. This is possible since we can always choose $\hat{\boldsymbol{u}}$ in such a way that ψ belongs to H^{-4} but not to H^{-2}.

In summary, we proved that $z \in \sigma(\mathbb{A})$ if and only if one of the following situations (or both) occur:

– $p_z(\lambda) = 0$ for some $\lambda \in \sigma(A)$;
– $\beta z + \gamma = 0$.

This finishes the proof. □

4.2 The Eigenvalues

We now deepen our analysis, establishing a necessary and sufficient condition to be an eigenvalue.

Theorem 4.2 *Let $z \in \mathbb{C}$ be given.*

(i) If $\mu \neq 0$, then z is an eigenvalue of $\mathbb{A}$ if and only if the equation $p_z(\lambda) = 0$ is satisfied for some eigenvalue λ of A.
(ii) If $\mu = 0$, then point (i) *continues to hold except for the special value $z = -\frac{\gamma}{\beta}$, which is always an eigenvalue of $\mathbb{A}$.*

Remark 4.3 A couple of comments before going to the proof. Since A is selfadjoint, we know that $\sigma(A)$ is the disjoint union of the continuous spectrum and the point spectrum (the eigenvalues). In particular, if A has compact inverse, then the spectrum is purely punctual. In that case, Theorem 4.2 merely says that the whole spectrum of $\mathbb{A}$, except possibly the point $-\frac{\gamma}{\beta}$, is made by eigenvalues. Observe also that, in principle, for a given z there may exist multiple values λ such that $p_z(\lambda) = 0$. Hence, in order for z to be an eigenvalue of $\mathbb{A}$, at least one of those λ, but not necessarily all of them, must be an eigenvalue of A.

Proof A complex number z is an eigenvalue of $\mathbb{A}$ if and only if there is a nontrivial solution to the equation

$$\mathbb{A}\boldsymbol{u} = z\boldsymbol{u}.$$

As we saw in the proof of Theorem 4.1, this happens if and only if

$$p_z(A)u = 0,$$

for some $u \neq 0$. In other words,

$$\text{z is an eigenvalue of} \mathbb{A} \quad \Leftrightarrow \quad 0 \text{ is an eigenvalue of} p_z(A).$$

Assume first that λ is an eigenvalue of A such that $p_z(\lambda) = 0$. Then 0 is clearly an eigenvalue of $p_z(A)$. Conversely, let 0 be an eigenvalue of $p_z(A)$. In which case, we claim that the equation $p_z(\lambda) = 0$ has at least a solution in $\sigma(A)$. Indeed, if not, $p_z(\lambda) \neq 0$ on the closed set $\sigma(A)$. Since $|p_z(\lambda)| \to \infty$ as $\lambda \to \infty$, we meet the hypotheses of Theorem A.3 in the Appendix (with $r = s = 2$), and we deduce that the equation $p_z(A)u = 0$ has only the trivial solution, against the fact that 0 is an eigenvalue of $p_z(A)$. Therefore, unless p_z is identically zero, we fall exactly within the hypotheses of the abstract Theorem A.4 with $\zeta = 0$ stated in the Appendix. This allows us to conclude that at least one of the roots of $p_z(\lambda)$ is an eigenvalue of A. In order to complete the proof, we note that when $z \neq -\frac{\gamma}{\beta}$ then p_z cannot be the null polynomial. On the other hand, when $z = -\frac{\gamma}{\beta}$, we have

$$p_z(\lambda) = \frac{\gamma\mu}{\beta^4}\big[\beta^2\eta^2\lambda^2 - \beta\gamma\kappa\lambda + \gamma^2\big],$$

telling that $p_z \equiv 0$ whenever $\mu = 0$. In that case, the value $z = -\frac{\gamma}{\beta}$ turns out to be always an eigenvalue of $\mathbb{A}$, even if $\sigma(A)$ is purely continuous. □

Observe that, when $\mu \neq 0$, for the special value $z = -\frac{\gamma}{\beta}$ the equation $p_z(\lambda) = 0$ has two (possibly complex) solutions, namely,

$$\lambda^{\pm} = \frac{\gamma}{2\beta\eta^2}\Big[\kappa \pm \sqrt{\kappa^2 - 4\eta^2}\Big].$$

Accordingly, the conclusion of Theorem 4.2 for the particular case $z = -\frac{\gamma}{\beta}$ can be rephrased as follows:

(i) If $\mu = 0$ then $-\frac{\gamma}{\beta}$ is an eigenvalue of $\mathbb{A}$.
(ii) If $\mu \neq 0$ then $-\frac{\gamma}{\beta}$ is an eigenvalue of $\mathbb{A}$ if and only if at least one between λ^+ and λ^- is an eigenvalue of A. This can never occur when $\kappa < 2|\eta|$.

Remark 4.4 Although it goes beyond our scopes, we mention that with a more skillful exploitation of the functional calculus of A it is actually possible to show that if $z \in \sigma(\mathbb{A})$ then either z is an eigenvalue of $\mathbb{A}$, or z belongs to the continuous spectrum of $\mathbb{A}$. In other words, the residual spectrum of $\mathbb{A}$ is empty.

5 Stability Criteria

5.1 Exponential Stability of Analytic Semigroups

Along this section, let $S(t)$ be a generic analytic semigroup acting on a Hilbert (or Banach) space $\mathcal{H}$, with infinitesimal generator $\mathbb{A}$.

Definition 5.1 The *growth bound* of $S(t)$ is defined as

$$\omega_\star = \inf\big\{\omega \in \mathbb{R} : \ \|S(t)\|_{L(\mathcal{H})} \le Me^{\omega t} \ \text{ for some } M = M(\omega) \ge 1\big\}.$$

Accordingly, $S(t)$ is exponentially stable if and only if $\omega_\star < 0$. Although for general semigroups computing $\omega_\star$ might not be an easy task, in the analytic case the *spectrum determined growth condition* holds (see [14]), telling that

$$\omega_\star = \sigma_\star,$$

where $\sigma_\star$ is the *spectral bound* of $\mathbb{A}$, defined as

$$\sigma_\star = \sup\big\{\mathfrak{Re}\,(z) : \ z \in \sigma(\mathbb{A})\big\}.$$

Accordingly, the knowledge of the spectrum of $\mathbb{A}$ completely determines the decay properties of $S(t)$, whereas in general one only has the inequality $\omega_\star \ge \sigma_\star$. Even more, the following theorem holds.

Theorem 5.2 *The analytic semigroup $S(t)$ is exponentially stable (namely, $\sigma_\star < 0$) if and only if $\sigma(\mathbb{A}) \subset \mathbb{C}^-$.*

Actually, the conclusions of the theorem remain valid for the wider class of eventually norm continuous semigroups, containing in particular differentiable semigroups, compact semigroups and, closer to our interests, analytic semigroups. Although Theorem 5.2 is a well-known result, that can be found for instance in [1], for the reader's convenience we report the short proof.

Proof If $\sigma_\star < 0$ it follows immediately that $\sigma(\mathbb{A}) \subset \mathbb{C}^-$. Assuming instead $\sigma(\mathbb{A}) \subset \mathbb{C}^-$, we need to show that $\sigma_\star < 0$. Clearly, $\sigma_\star \le 0$. If $\sigma_\star = 0$, there exists $z_n \in \sigma(\mathbb{A})$ such that $\mathfrak{Re}(z_n) \to 0$. For analytic (and more generally for eventually norm continuous) semigroups, the (closed) set

$$K = \{z \in \sigma(\mathbb{A}) : \mathfrak{Re}(z) \ge -1\}$$

is bounded, hence compact (see, e.g., [14]). Then, up to a subsequence, $z_n \to z$ for some $z \in K$, implying that $\mathfrak{Re}(z) = 0$. This is impossible since $\sigma(\mathbb{A}) \cap i\mathbb{R} = \emptyset$. □

Theorem 5.2 will be the key abstract result for proving our main Theorem 3.3.

5.2 The Routh-Hurwitz Criterion

Actually, another tool will be needed, namely, a stability criterion for fourth order polynomials apt to detect the sign of the real parts of its roots. Consider a monic fourth order complex polynomial

$$p(z) = z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0,$$

with strictly positive coefficients $a_j > 0$. The polynomial has four roots z_k, appearing in conjugate pairs. In particular, $z_k \neq 0$ being $a_0 > 0$. The well-known Routh-Hurwitz criterion establishes a necessary and sufficient condition on the coefficients of p in order for the real part of each root to be strictly negative. This is the content of the following result, suitably formulated for our scopes.

Theorem 5.3 *Defining the discriminant of* p *as*

$$\Delta = a_1 a_2 a_3 - a_0 a_3^2 - a_1^2,$$

the following hold:

- $\Delta > 0$ *if and only if* $\mathfrak{Re}\,(z_k) < 0$ *for all* k.
- $\Delta = 0$ *if and only if two of the roots are of the form* $\pm ir$ *with* $r > 0$. *In which case, the two remaining roots have strictly negative real part.*
- $\Delta < 0$ *if and only if* $\mathfrak{Re}\,(z_k) > 0$ *for some* k.

Proof The first point is exactly the Routh-Hurwitz criterion. We address the reader to [30, Theorem 2.4] for an elementary proof. To prove the second point, consider

$$p(ir) = r^4 - ia_3 r^3 - a_2 r^2 + ia_1 r + a_0.$$

It is apparent that ir, with $r \neq 0$, is a root if and only if

$$a_1 - a_3 r^2 = 0 \qquad \text{and} \qquad r^4 - a_2 r^2 + a_0 = 0,$$

which is satisfied if and only if

$$r^2 = \frac{a_1}{a_3} \qquad \text{and} \qquad \Delta = 0.$$

Besides, the two remaining roots have negative real part, since p decomposes as

$$p(z) = (z^2 + r^2)(z^2 + a_3 z + a_0 r^{-2}),$$

and the coefficients of the second quadratic equation are both strictly positive. The last point is an immediate consequence of the previous ones. □

6 Proof of Theorem 3.3

Due to the abstract Theorem 5.2, we only need to prove the existence of $x_\star \geq 0$ independent of η such that

$$\sigma(\mathbb{A}) \subset \mathbb{C}^- \quad \Leftrightarrow \quad \eta^2 > x_\star,$$

with $x_\star = 0$ if and only if $\mu = 0$. To this end, for every fixed $\lambda \in \sigma(A)$, we consider the fourth order polynomial

$$p_\lambda(z) = z^4 + (\alpha + \kappa\lambda)z^3 + (\beta\lambda^2 + \alpha\kappa\lambda + \eta^2\lambda^2)z^2 + (\gamma\lambda^2 + \beta\kappa\lambda^3 + \eta^2\lambda^2\alpha)z + \gamma\kappa\lambda^3,$$

which is nothing but the polynomial $p_z(\lambda)$ already encountered in Sect. 4, but this time viewed as a function of the variable z. Thanks to Theorem 4.1, the condition $\sigma(\mathbb{A}) \subset \mathbb{C}^-$ is equivalent to the fact that, for every $\lambda \in \sigma(A)$, all the roots of p_λ have negative real part.

For any $\lambda \in \sigma(A)$, hence $\lambda > 0$, the coefficients of p_λ are all positive. Besides, in the notation of Theorem 5.3, the discriminant Δ_λ of p_λ reads

$$\Delta_\lambda = \lambda^3(\alpha\kappa\lambda^2\eta^4 + \phi_\lambda\eta^2 - \psi_\lambda),$$

having set

$$\phi_\lambda = \beta\kappa^2\lambda^3 + \gamma\kappa\lambda^2 + \alpha\lambda(\alpha\kappa^2 - \mu) + \alpha^3\kappa,$$
$$\psi_\lambda = \mu\lambda(\kappa^3\lambda + \beta\kappa\lambda + \alpha\kappa^2 + \gamma).$$

Observe that if $\mu = 0$ then $\phi_\lambda > 0$. Moreover,

$$\psi_\lambda \geq 0 \qquad \text{and} \qquad \psi_\lambda = 0 \;\Leftrightarrow\; \mu = 0.$$

After Theorem 5.3, the four roots of p_λ have negative real part if and only if $\Delta_\lambda > 0$, and this happens if and only if

$$\alpha\kappa\lambda^2\eta^4 + \phi_\lambda\eta^2 - \psi_\lambda > 0. \tag{6.1}$$

This is equivalent to ask

$$\eta^2 > x_\lambda,$$

where

$$x_\lambda = \frac{1}{2\alpha\kappa\lambda^2}\Big(\sqrt{\phi_\lambda^2 + 4\alpha\kappa\lambda^2\psi_\lambda} - \phi_\lambda\Big)$$

is independent of η. In particular, $x_\lambda = 0$ if $\mu = 0$, while $x_\lambda > 0$ if $\mu > 0$. In the latter case, letting $\lambda \to \infty$ one has

$$x_\lambda \sim \frac{\mu(\kappa^2 + \beta)}{\kappa\beta\lambda} \to 0.$$

Since x_λ is a continuous function of λ and $\sigma(A)$ is a closed set, we infer that

$$x_\star = \sup_{\lambda\in\sigma(A)} x_\lambda$$

is (nonnegative and) finite. Actually, the supremum is attained for some $\lambda_\star \in \sigma(A)$. Note also that $x_\star = 0$ if and only if $\mu = 0$.

In order to finish the proof, we observe that if $\eta^2 > x_\star$ then (6.1) holds true for every $\lambda \in \sigma(A)$, and thus all the roots of p_λ have negative real part. Conversely, if $\eta^2 \leq x_\star$ then $\Delta_{\lambda_\star} \leq 0$, and Theorem 5.3 ensures that $p_{\lambda_\star}$ admits at least one root with nonnegative real part. □

7 Final Remarks

7.1 Dependence of the Stability Threshold on the Thermal Conductivity

In the supercritical case, a natural question to ask is whether or not a large thermal conductivity κ pushes towards the dissipation of the system, which translates into having a smaller stability threshold $x_\star$. Contrary to what one might expect, this is not the case. Indeed, in the recent work [6] it was shown that for the MGT-Fourier model (1.4) in the supercritical regime one cannot hope to obtain stability by fixing η and arbitrarily increasing κ. The situation is the same for our system (2.1).

Proposition 7.1 *Let $\alpha, \beta, \gamma > 0$ be fixed, and let $\mu > 0$. Then*

$$\lim_{\kappa\to\infty} x_\star = \infty.$$

Proof For any fixed $\lambda \in \sigma(A)$, we write x_λ as in the proof of Theorem 3.3. Taking the limit $\kappa \to \infty$, we see at once that

$$x_\lambda \sim c_\lambda\kappa,$$

where the constant $c_\lambda > 0$ is given by

$$c_\lambda = \frac{\alpha^2 + \beta\lambda^2}{2\alpha\lambda}\left(\sqrt{1 + \frac{4\alpha\mu\lambda^2}{(\alpha^2 + \beta\lambda^2)^2}} - 1\right).$$

Since $x_\star \geq x_\lambda$ for every $\lambda \in \sigma(A)$, the conclusion follows. □

7.2 The Optimal Decay Rate

Once we are in the case $\omega_\star < 0$, meaning that $S(t)$ decays exponentially, it is interesting to find the optimal decay rate of the semigroup for $\alpha, \beta, \gamma, \kappa$ fixed, by properly modulating the coupling parameter η. Here the issue is to study $\omega_\star$ as a function of η, finding (if it exists) the value of η minimizing $\omega_\star$. The proof of Theorem 3.3 implicitly says that

$$\omega_\star \to 0 \qquad \text{when} \qquad |\eta| \to \sqrt{x_\star}^{\,+}.$$

The next result shows that the same is true in the limit $|\eta| \to \infty$.

Proposition 7.2 *Let $\alpha, \beta, \gamma, \kappa$ be fixed. Then*

$$\lim_{|\eta|\to\infty} \omega_\star = 0.$$

Proof All we need to show is that $\sigma(\mathbb{A})$ contains an element z_η whose real part tends to zero as $|\eta| \to \infty$. In the notation of the proof of Theorem 3.3, this amounts to finding $z_\eta \in \mathbb{C}$ such that

$$p_\lambda(z_\eta) = 0,$$

for some $\lambda \in \sigma(A)$. Indeed, for any fixed $\lambda \in \sigma(A)$, we have

$$p_\lambda(0) = \gamma\kappa\lambda^3 > 0.$$

Besides, choosing

$$\varepsilon_\eta = \frac{\gamma\kappa\lambda^3 + 1}{\alpha\lambda^2}\,\frac{1}{\eta^2} \to 0,$$

it is readily seen that

$$\lim_{|\eta|\to\infty} p_\lambda(-\varepsilon_\eta) = -1.$$

Thus, for $|\eta|$ large enough, $p_\lambda(-\varepsilon_\eta)$ becomes negative, implying that p_λ has a negative real root $z_\eta \in (-\varepsilon_\eta, 0)$. □

Roughly speaking, we cannot arbitrarily increase the decay rate of the semigroup by acting on the coupling parameter solely. Indeed, Proposition 7.2 actually establishes the existence of the best coupling constant

$$\eta_{\mathrm{b}} = \eta_{\mathrm{b}}(\alpha, \beta, \gamma, \kappa)$$

minimizing the value $\omega_\star$. In fact, since $\omega_\star$ depends on the square of η, we have two minimizing values of opposite sign and equal modulus.

A further question is if this can be done instead by increasing the thermal conductivity κ. More precisely, for α, β, γ fixed, we consider for every κ the minimum of $\omega_\star$, obtained by suitably choosing η. Calling this number $\omega_{\mathfrak{b}}(\kappa)$, we wonder if

$$\lim_{\kappa \to \infty} \omega_{\mathfrak{b}}(\kappa) = -\infty.$$

Again, this is false: arguing exactly as in [6, Proposition 8.3], one can prove that

$$\liminf_{\kappa \to \infty} \omega_{\mathfrak{b}}(\kappa) > -\infty.$$

The details are left to the interested reader.

7.3 Longterm Behavior Below the Threshold

We conclude our analysis with some comments about the longterm behavior in the supercritical regime when the coupling parameter η is not sufficiently large to comply with the condition $\eta^2 > x_\star$. With reference to the proof of Theorem 3.3, let us distinguish two cases:
(i) If $\eta^2 < x_\star$, then the discriminant $\Delta_{\lambda_\star}$ of the polynomial $p_{\lambda_\star}$ is strictly negative. By Theorem 5.3, there is a root with strictly positive real part, implying that $\omega_\star > 0$, that is, the operator norm of $S(t)$ blows up exponentially.
(ii) In the limit situation $\eta^2 = x_\star$ we have the equality $\Delta_{\lambda_\star} = 0$, and Theorem 5.3 tells that the spectrum of $\mathbb{A}$ contains at least two elements $\pm ir$ with $r > 0$, implying in turn that $\omega_\star = 0$. Clearly, if one between $\pm ir$ is an eigenvalue of $\mathbb{A}$, then $S(t)$ cannot be stable. On the contrary, if both $\pm ir$ are not eigenvalues, one might be tempted to apply a famous result from [1], saying that if the infinitesimal generator $\mathbb{A}$ of a bounded semigroup $S(t) = e^{t\mathbb{A}}$ has no eigenvalues on the imaginary axis and $\sigma(\mathbb{A}) \cap i\mathbb{R}$ is countable, then $S(t)$ is stable. However, we do not know in advance that $S(t)$ is bounded, and removing the boundedness assumption the result just cited is utterly false, even for analytic semigroups. An instructive example in this direction is given below.

Example 7.3 We provide an example of an analytic semigroup $S(t) = e^{t\mathbb{A}}$ with the following properties:

- $S(t)$ is not bounded;
- the spectrum of $\mathbb{A}$ is contained in the left complex half plane, with a single element on the imaginary axis which is not an eigenvalue of $\mathbb{A}$.

To this end, let us consider the Hilbert space ℓ^2, and define the linear (diagonal-block) operator

$$\mathbb{A} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{3} & 1 \\ 0 & -\frac{1}{3} \end{pmatrix} \oplus \cdots .$$

The spectrum of $\mathbb{A}$ is given by $\{-\frac{1}{n}\} \cup \{0\}$, where all the numbers $-\frac{1}{n}$ are eigenvalues of $\mathbb{A}$, whereas 0 belongs to the continuous spectrum. Being a bounded operator, $\mathbb{A}$ generates a uniformly continuous (hence analytic) semigroup $S(t)$ on ℓ^2, given by the formula

$$S(t) = e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \oplus e^{-\frac{t}{2}} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \oplus e^{-\frac{t}{3}} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \oplus \cdots .$$

Taking the vector

$$\boldsymbol{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus 2^{-\frac{3}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus 3^{-\frac{3}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \cdots \in \ell^2,$$

we have

$$S(t)\boldsymbol{u} = e^{-t}\begin{pmatrix} t \\ 1 \end{pmatrix} \oplus 2^{-\frac{3}{4}} e^{-\frac{t}{2}} \begin{pmatrix} t \\ 1 \end{pmatrix} \oplus 3^{-\frac{3}{4}} e^{-\frac{t}{3}} \begin{pmatrix} t \\ 1 \end{pmatrix} \oplus \cdots .$$

Thus, for every $t \geq 0$ and every n,

$$\|S(t)\boldsymbol{u}\|_{\ell^2} \geq n^{-\frac{3}{4}} t e^{-\frac{t}{n}}.$$

In particular, for $t = n$ we get

$$\|S(n)\boldsymbol{u}\|_{\ell^2} \geq n^{\frac{1}{4}} e^{-1} \to \infty,$$

as $n \to \infty$. This proves that $S(t)$ is not bounded.

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Appendix

The purpose of this final Appendix is to discuss some abstract operator theoretical results involving a strictly positive selfadjoint operator A acting on a complex Hilbert space H. We will use the same notation of Sect. 2.1. Although some of the theorems might be known, we were not able to locate a precise reference. This is also the occasion to provide all the details about some facts that have been used in previous papers without entering into the proofs (see, e.g., [11, 12]) For the reader's convenience, we first recall some general facts, referring to the book [27] for a complete presentation.

I. Theoretical Background

A spectral measure on a closed set $\Sigma \subset \mathbb{R}$ is a map

$$E : \mathfrak{B}(\Sigma) \to P(H)$$

defined on the Borel σ-algebra $\mathfrak{B}(\Sigma)$ with values in the space $P(H)$ of selfadjoint projections in H, and satisfying the following properties:

- $E(\emptyset) = 0$ and $E(\Sigma) = I$ (the identity operator on H).
- $E(\sigma) \neq 0$ for every nonempty open set $\sigma \subset \Sigma$.
- $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$, for all $\sigma_1, \sigma_2 \in \mathfrak{B}(\Sigma)$.
- If $\sigma_1 \cap \sigma_2 = \emptyset$ then $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)$.
- For every $u, v \in H$ the set function $\mu_{u,v} : \mathfrak{B}(\Sigma) \to \mathbb{C}$ defined as

$$\mu_{u,v}(\sigma) = \langle E(\sigma)u, v\rangle$$

is a complex measure.

The second property above tells that the spectral measure is supported on Σ; namely, if σ is any closed set strictly contained in Σ, then $E(\sigma) \neq I$.

Notation When $u = v$, we write for short $\mu_{u,u} = \mu_u$. In this case, μ_u becomes a positive Borel measure of total mass

$$\mu_u(\Sigma) = \|u\|^2.$$

If u is a unit vector then μ_u is a probability measure.

Being A strictly positive, its spectrum $\sigma(A)$ is a closed and nonempty subset of $\mathbb{R}^+$. Then, the Spectral Theorem for selfadjoint operators ensures the existence of a unique spectral measure E_A on $\sigma(A)$, called the spectral measure of A, such that

$$\langle Au, v\rangle = \int_{\sigma(A)} \lambda d\mu_{u,v}(\lambda)$$

for every $u \in \mathfrak{D}(A)$ and $v \in H$. Sometimes, the formula above is written for short as

$$A = \int_{\sigma(A)} \lambda dE_A(\lambda).$$

Remark A.1 An element $\lambda \in \sigma(A)$ is an eigenvalue of A if and only if the spectral measure E_A has an atom in λ, that is, if $E_A(\{\lambda\})$ is not the null projection. In particular, every isolated point of the spectrum of A is an eigenvalue.

For every measurable function $p : \sigma(A) \to \mathbb{C}$, it is possible to define via the functional calculus the operator $p(A)$ as

$$p(A) = \int_{\sigma(A)} p(\lambda) dE_A(\lambda)$$

with domain

$$\mathfrak{D}(p(A)) = \Big\{u \in H : \int_{\sigma(A)} |p(\lambda)|^2 d\mu_u(\lambda) < \infty\Big\}.$$

Such an operator turns out to be densely defined and closed. Besides, $p(A)$ is a normal operator, and is selfadjoint whenever p is real-valued. Further properties of $p(A)$ read as follows:

- For every $u \in \mathfrak{D}(p(A))$, one has the equality

$$\|p(A)u\|^2 = \int_{\sigma(A)} |p(\lambda)|^2 d\mu_u(\lambda).$$

- $p(A)u \in H^r$, with $r \in \mathbb{R}$, if and only if

$$\|p(A)u\|_r^2 = \int_{\sigma(A)} \lambda^r |p(\lambda)|^2 d\mu_u(\lambda) < \infty.$$

- If $q : \sigma(A) \to \mathbb{C}$ is another measurable function, then the equality

$$p(A)q(A) = q(A)p(A) = (p \cdot q)(A) = \int_{\sigma(A)} p(\lambda)q(\lambda) dE_A(\lambda)$$

  holds in the common domain of the three operators.

- If p is a continuous function, then $p(A)$ is a bounded operator if and only if p is bounded. In which case, the operator norm of $p(A)$ is given by

$$\|p(A)\|_{L(H)} = \sup_{\lambda\in\sigma(A)} |p(\lambda)|.$$

- If λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$.

Remark A.2 The functional calculus presented above continues to hold with no substantial changes for a normal operator B in place of A. The only notable difference is that $\sigma(B)$ is a subset of the complex plane.

II. Solvability of a Certain Functional Equation

Let $p : \sigma(A) \to \mathbb{C}$ be a continuous function. For $r, s \in \mathbb{R}$, we study the functional equation

$$p(A)u = \psi \tag{A.1}$$

in the unknown $u \in H^r$, where $\psi \in H^s$ is a given vector. To this end, let us consider the zero set of p

$$\mathsf{Z} = \big\{\lambda \in \sigma(A) :\ p(\lambda) = 0\big\}.$$

The following results hold.

Theorem A.1 *If Z is nonempty, then there exists $\psi \in H^s$ for which (A.1) does not admit a unique solution $u \in H^r$.*

Proof Let Z be nonempty. If $E_A(\mathsf{Z}) \neq 0$, then any nonzero vector $u \in E_A(\mathsf{Z})H$ is an eigenvector of $p(A)$ corresponding to the eigenvalue 0, and the claim follows immediately by choosing $\psi = 0$. Conversely, let $E_A(\mathsf{Z}) = 0$. Selecting $\lambda_0 \in \mathsf{Z}$, consider the set

$$\mathsf{S} = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap \sigma(A),$$

for some $\varepsilon > 0$ small enough that $|p(\lambda)| < 1$ for all $\lambda \in \mathsf{S}$. This is possible since p is continuous and $p(\lambda_0) = 0$. Then define

$$V_0 = E_A\big(\mathsf{S}\big)H.$$

Notice that V_0 is a (nontrivial) subspace of H^s for every s. We will reach the desired conclusion by constructing an element $\psi \in V_0$ for which equation (A.1) does not admit a solution u in any space H^r. To this end, for n positive integer, we introduce the sets

$$\mathsf{S}_n = \Big\{\lambda \in \mathsf{S} :\ |p(\lambda)| \in \Big[\tfrac{1}{n+1}, \tfrac{1}{n}\Big)\Big\}.$$

Since $\bigcup_n \mathsf{S}_n = \mathsf{S} \setminus \mathsf{Z}$, it follows that

$$E_A\Big(\bigcup_n \mathsf{S}_n\Big) = E_A(\mathsf{S}) \neq 0.$$

Moreover, $E_A(\mathsf{S}_n) \neq 0$ for infinitely many n: if not, by the continuity of p together with the fact that $E_A(\mathsf{Z}) = 0$, we could exhibit a neighborhood of λ_0 with null spectral measure. Up to passing to a subsequence, we can certainly assume $E_A(\mathsf{S}_n) \neq 0$ for every n. Hence, we can select a sequence of unit vectors

$$\psi_n \in E_A(\mathsf{S}_n)H \subset V_0,$$

which turn out to be mutually orthogonal, for the sets S_n are disjoint. Finally, call

$$\psi = \sum_n \frac{\psi_n}{n} \in V_0.$$

For every Borel set $\sigma \subset \mathsf{S}_n$ we have

$$\mu_\psi(\sigma) = \langle E_A(\sigma)\psi, \psi\rangle = \frac{1}{n^2}\langle E_A(\sigma)\psi_n, \psi_n\rangle = \frac{1}{n^2}\mu_{\psi_n}(\sigma),$$

where μ_{ψ_n} is a probability measure on S_n. If $u \in H^r$ solves (A.1) for this particular ψ, then

$$u = q(A)\psi \qquad \text{where} \qquad q(\lambda) = \frac{1}{p(\lambda)}.$$

On the other hand, noting that by construction

$$\inf_{\lambda \in \mathsf{S}_n} |q(\lambda)| \geq n,$$

we get

$$\begin{aligned} \|q(A)\psi\|_r^2 &= \int_{\sigma(A)} \lambda^r |q(\lambda)|^2 d\mu_\psi(\lambda) \\ &= \sum_n \frac{1}{n^2} \int_{\mathsf{S}_n} \lambda^r |q(\lambda)|^2 d\mu_{\psi_n}(\lambda) \\ &\geq \inf_{\lambda \in \mathsf{S}} \lambda^r \sum_n \frac{1}{n^2} \int_{\mathsf{S}_n} |q(\lambda)|^2 d\mu_{\psi_n}(\lambda), \end{aligned}$$

and

$$\frac{1}{n^2} \int_{\mathsf{S}_n} |q(\lambda)|^2 d\mu_{\psi_n}(\lambda) \geq \int_{\mathsf{S}_n} d\mu_{\psi_n}(\lambda) = 1, \quad \forall n.$$

This tells that $q(A)\psi$ cannot belong to H^r. □

Theorem A.2 *Let Z be empty, and let there exist* $\varepsilon > 0$ *such that*

$$\mathfrak{c} = \sup_{\lambda \in \sigma(A)} \lambda^{s+\varepsilon-r} |p(\lambda)|^2 < \infty.$$

If ψ *does not belong to* $H^{s+\varepsilon}$ *then* (A.1) *does not admit any solution* $u \in H^r$.

Proof If $u \in H^r$ is a solution to (A.1), then $u = q(A)\psi$ where, as before, $q(\lambda) = 1/p(\lambda)$. Since $\mathsf{Z} = \emptyset$, it follows that q is continuous on $\sigma(A)$. Then,

$$\|\psi\|_{s+\varepsilon}^2 = \int_{\sigma(A)} \lambda^{s+\varepsilon} d\mu_\psi(\lambda) \leq \mathfrak{c} \int_{\sigma(A)} \lambda^r |q(\lambda)|^2 d\mu_\psi(\lambda) = \mathfrak{c}\|q(A)\psi\|_r^2 < \infty,$$

meaning that $\psi \in H^{s+\varepsilon}$. □

Theorem A.3 *Let Z be empty. If*

$$\mathfrak{m} = \inf_{\lambda \in \sigma(A)} \lambda^{s-r} |p(\lambda)|^2 > 0,$$

then (A.1) *admits a unique solution* $u \in H^r$*, for any given* $\psi \in H^s$.

Proof Arguing as in the previous proof, all we need to show is that $u = q(A)\psi \in H^r$. But

$$\|q(A)\psi\|_r^2 = \int_{\sigma(A)} \lambda^r |q(\lambda)|^2 d\mu_\psi(\lambda) \leq \frac{1}{\mathfrak{m}} \int_{\sigma(A)} \lambda^s d\mu_\psi(\lambda) = \frac{1}{\mathfrak{m}} \|\psi\|_s^2 < \infty,$$

as desired. □

III. An Eigenvalue Problem

We already mentioned that if λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$. Now we want to explore the converse.

Theorem A.4 *Let $p : \sigma(A) \to \mathbb{C}$ be a continuous function, and let $\zeta \in \mathbb{C}$ be an eigenvalue of $p(A)$. If the equation*

$$p(\lambda) = \zeta$$

has a finite number of solutions $\lambda_1, \ldots, \lambda_n$, then at least one of the numbers λ_j is an eigenvalue of A.

The proof of the theorem is based on the following lemma.

Lemma A.5 *Let $f : \sigma(A) \to \mathbb{R}$ be a continuous function, and let 1 be an eigenvalue of $f(A)$. Assume in addition that*

$$f(\lambda) \leq 1, \quad \forall \lambda \in \sigma(A),$$

and the equation

$$f(\lambda) = 1$$

has a finite number of solutions $\lambda_1, \ldots, \lambda_n$. Then at least one of the numbers λ_j is an eigenvalue of A.

Proof For some unit vector $u \in H$, we have the equality

$$f(A)u = \int_{\sigma(A)} f(\lambda) dE_A(\lambda)\, u = u.$$

Accordingly,

$$\langle f(A)u, u\rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda) = 1.$$

Since $f(\lambda) \leq 1$, this yields the equality

$$f(\lambda) = 1,$$

almost everywhere with respect to μ_u, which is possible if and only if

$$\mu_u = \alpha_1 \delta_{\lambda_1} + \ldots + \alpha_n \delta_{\lambda_n},$$

where δ_{λ_j} is the Dirac delta centered at λ_j and $\alpha_1 + \ldots + \alpha_n = 1$. Therefore, at least one α_j must be nonzero, implying that λ_j is an atom of E_A, that is, an eigenvalue of A. □

Proof of Theorem A.4 Define the function $h : \mathbb{C} \to \mathbb{R}$ as

$$h(z) = 1 - |z - \zeta|.$$

Then h is continuous and satisfies:

- $h(\zeta) = 1$; and
- $h(z) < 1$ for all $z \neq \zeta$.

Denoting then the normal operator $p(A)$ by B, we deduce that $1 = h(\zeta)$ is an eigenvalue of $h(B)$. Again, $h(B)$ is defined via the functional calculus, this time for normal operators. Calling

$$f(\lambda) = h(p(\lambda)),$$

we conclude that 1 is an eigenvalue of $f(A)$. Moreover, by construction,

$$f(\lambda) = h(p(\lambda)) \leq 1, \quad \forall \lambda \in \sigma(A),$$

and

$$f(\lambda) = 1 \quad \Leftrightarrow \quad \lambda \in \{\lambda_1, \ldots, \lambda_n\}.$$

A direct application of Lemma A.5 entails the desired result. □

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