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The Poisson equation on Riemannian manifolds with weighted Poincaré inequality at infinity

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Abstract

We prove an existence result for the Poisson equation on non-compact Riemannian manifolds satisfying weighted Poincaré inequalities outside compact sets. Our result applies to a large class of manifolds including, for instance, all non-parabolic manifolds with minimal positive Green's function vanishing at infinity. On the source function, we assume a sharp pointwise decay depending on the weight appearing in the Poincaré inequality and on the behavior of the Ricci curvature at infinity. We do not require any curvature or spectral assumptions on the manifold. In comparison with previous works, we can deal with a more general setting on the curvature bounds and without any spectral assumption.

Keywords Poisson equation · Riemannian manifolds · Green's functions · Weighted Poincaré inequality

Mathematics Subject Classification 53C21 · 35R01

1 Introduction

The existence of solutions to the Poisson equation

$$-\Delta u = f$$

on a complete Riemannian manifold (M, g), for a given function f on M, is a classical problem which has been the object of deep interest in the literature. Malgrange [11] obtained solvability of the Poisson equation for any smooth function f with compact support, as a consequence of the existence of a Green's function for $-\Delta$ on every complete Riemannian manifold. Under integrability assumptions on f, existence of solutions has been established

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by Strichartz [17] and Ni–Shi–Tam [16, Theorem 3.2] (see also [15, Lemma 2.3]). Moreover, in the same paper, the authors proved an existence result for the Poisson problem on manifolds with nonnegative Ricci curvature under a sharp integral assumption involving suitable averages of f. This condition in particular is satisfied if

$$|f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{\alpha}}$$

for some $C > 0$ and $\alpha > 2$, where $r(x) := \mathrm{dist}(x, p)$ is the distance function of any $x \in M$ from a fixed reference point $p \in M$. In fact, they proved a more general result where the decay rate of f is just assumed to be of order $1 + \varepsilon$. Note that this result is sharp on the flat space $\mathbb{R}^n$.

From now on let us consider solutions u of the Poisson equation $-\Delta u = f$ which can be represented as

$$u(x) = \int_M G(x, y) f(y)\, \mathrm{d}y,$$

where $G(x,\ y)$ is a Green's function of $-\Delta$ on M (see Sect. 2 for further details). Munteanu–Sesum [12] addressed the case of manifolds with positive spectrum, i.e., $\lambda_1(M) > 0$, and Ricci curvature bounded from below, obtaining existence of solutions under the pointwise decay assumption

$$|f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{\alpha}}$$

for some $C > 0$ and $\alpha > 1$. Note that this result is sharp on $\mathbb{H}^n$. Their proof relies on very precise integral estimates on the minimal positive Green's function, which are inspired by the work of Li–Wang [10]. Note that in [12, 13] the authors also study the behavior of the solution at infinity.

In [4] the authors generalized the existence result in [12], obtaining existence of solutions on manifolds with positive essential spectrum, i.e., $\lambda_1^{\mathrm{ess}}(M) > 0$, for source functions f satisfying

$$\sum_{m=1}^{\infty} \frac{\theta_R(m+1) - \theta_R(m)}{\lambda_1\big(M \setminus B_{m-1}(p)\big)} \sup_{M \setminus B_{m-1}(p)} |f| < \infty,$$

for any $R > 0$, where $\theta_R(m)$ is a function related to a lower bound on the Ricci curvature, locally on geodesic balls with center p and radius $2R + m$. In particular, the authors showed in [4, Corollary 1.3] existence of solutions on Cartan–Hadamard manifolds with strictly negative Ricci curvature, whenever

$$-C\big(1 + r(x)\big)^{\gamma_1} \leq \mathrm{Ric} \leq -\frac{1}{C}\big(1 + r(x)\big)^{\gamma_2}, \quad |f(x)| \leq \frac{C}{\big(1 + r(x)\big)^{\alpha}},$$

for some $C > 0$ and $\gamma_1, \gamma_2 \geq 0$ with $\alpha > 1 + \frac{\gamma_1}{2} - \gamma_2$.

Observe that the results in [4, 12] cannot be used whenever the Ricci curvature tends to zero at infinity fast enough (see [19]) since, in this case, one has $\lambda_1^{\mathrm{ess}}(M) = 0$ (and so $\lambda_1(M) = 0$). In particular, the case of $\mathbb{R}^n$ is not covered. On the other hand, the result in [16] does not apply on manifolds with negative curvature. The purpose of our paper is to obtain a general result which includes, as special cases, both manifolds with strictly negative curvature and manifolds with Ricci curvature vanishing at infinity. Moreover, our result is sharp on spherically symmetric manifolds, and in particular on $\mathbb{R}^n$ and $\mathbb{H}^n$.

Note that the condition $\lambda_1(M) > 0$ is equivalent to the validity of the Poincaré inequality

$$\lambda_1(M) \int_M u^2 \, \mathrm{d}V \leq \int_M |\nabla u|^2 \, \mathrm{d}V$$

for any $u \in C_c^\infty(M)$. On the other hand, one has positive essential spectrum if and only if, for some compact subset $K \subset M$, one has $\lambda_1(M \setminus K) > 0$ and

$$\lambda_1(M \setminus K) \int_M u^2 \, \mathrm{d}V \leq \int_M |\nabla u|^2 \, \mathrm{d}V$$

for any $u \in C_c^\infty(M \setminus K)$. Generalizing the previous inequalities, one says that (M, g) satisfies a *weighted Poincaré inequality* with a nonnegative weight function ρ if

$$\int_M \rho \, v^2 \, \mathrm{d}V \leq \int_M |\nabla v|^2 \, \mathrm{d}V \tag{1}$$

for every $v \in C_c^\infty(M)$. If for any $R \geq R_0 > 0$, there exists a nonnegative function ρ_R defined on M such that (1) holds for every $v \in C_c^\infty(M \setminus B_R(p))$ and for $\rho \equiv \rho_R$, we say that (M, g) satisfies a *weighted Poincaré inequality at infinity*. In addition, inspired by [10], we say that (M, g) satisfies the property $\left(\mathcal{P}_w^\infty\right)$, if a weighted Poincaré inequality at infinity holds for the family of weights ρ_R and the conformal ρ_R-metric defined by

$$g_{\rho_R} := \rho_R \, g$$

is complete for every $R \geq R_0$. The validity of a weighted Poincaré inequality on some classes of manifolds has been investigated in the literature. It is well known that on $\mathbb{R}^n$ inequality (1) holds with $\rho(x) = \frac{(n-2)^2}{4} \frac{1}{r^2(x)}$. It is also called *Hardy inequality*. More in general, it holds on every Cartan–Hadamard manifold with $\rho(x) = \frac{C}{r^2(x)}$, for some $C > 0$ (see [1, 3] for some refinement of this result).

In order to state our main results, we need to introduce a (increasing) function $\omega(s)$ related to the value of the Ricci curvature on the annulus $B_{\frac{5}{4}s}(p) \setminus B_{\frac{3}{4}s}(p)$ (see (4) for the precise definition). In this paper, we prove the following result.

Theorem 1.1 *Let (M, g) be a complete non-compact Riemannian manifold satisfying the property $\left(\mathcal{P}_w^\infty\right)$ w.r.t. the family of weights ρ_R, $R \geq R_0$, and let f be a locally Hölder continuous function on M. If*

$$\sum_m^\infty \Big(\omega(m+1) - \omega(m) + 1\Big) \sup_{M \setminus B_m(p)} \frac{|f|}{\rho_m} < \infty,$$

then the Poisson equation

$$-\Delta u = f \quad \text{in } M$$

admits a classical solution u.

Assume that $\lambda_1^{\mathrm{ess}}(M) > 0$ and

$$\mathrm{Ric} \geq -C\big(1 + r(x)\big)^{\gamma}$$

for some $\gamma \geq 0$. Then, it is direct to see that

$$\omega(m+1) - \omega(m) \sim C\Big(\theta_R(m+1) - \theta_R(m)\Big) \sim Cm^{\frac{\gamma}{2}}$$

for every $R > 0$ and the property $\big(\mathcal{P}_w^{\infty}\big)$ w.r.t. the family of weights ρ_R, $R \geq R_0$, holds for every R with $\rho_R(x) = \lambda_1(M \setminus B_R(p))$. Thus,

$$\Big(\omega(m+1) - \omega(m) + 1\Big) \sup_{M\setminus B_m(p)} \frac{|f|}{\rho_m} \sim C\,\frac{\theta_R(m+1) - \theta_R(m)}{\lambda_1\big(M \setminus B_m(p)\big)} \sup_{M\setminus B_m(p)} |f|;$$

therefore, our existence result is in accordance with those in [4, 12].

We recall that by [10, Corollary 1.4, Lemma 1.5] the validity of a weighted Poincaré inequality (1) on M implies the non-parabolicity of the manifold; on the contrary, if (M, g) is non-parabolic, then a weighted Poincaré inequality holds on M, with weight

$$\rho(x) := \frac{|\nabla G(p,x)|^2}{4G^2(p,x)},$$

where G is the minimal positive Green's function on (M, g). Exploiting this result, using similar techniques as in Theorem 1.1, we obtain the following refined result on complete non-compact non-parabolic manifolds.

Theorem 1.2 *Let (M, g) be a complete non-compact non-parabolic Riemannian manifold with minimal positive Green's function G. Let $\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)}$ and let f be a locally Hölder continuous function on M. If*

$$\sum_{m}^{\infty} \Big(\omega(m+1) - \omega(m)\Big) \sup_{M\setminus B_m(p)} \frac{|f|}{\rho} < \infty,$$

then the Poisson equation

$$-\Delta u = f \quad \text{in } M$$

admits a classical solution u.

Remark 1.3 We explicitly observe that in Theorem 1.2 the completeness of the conformal metric $g_\rho = \rho g$ is not required. As it was observed in [10], the completeness of g_ρ would hold if $G(p, x) \to 0$ as $r(x) \to \infty$, a condition that we do not need to assume here.

It is well known that $\mathbb{R}^n$ is a non-parabolic manifold if $n \geq 3$, with minimal positive Green's function $G(x, y) = \frac{c_n}{|x-y|^{n-2}}$ for some positive constant c_n. Moreover, the weighted Poincaré – Hardy's inequality holds on $\mathbb{R}^n$ with

$$\rho(x) = \frac{|\nabla G(0,x)|^2}{4G^2(0,x)} = \frac{(n-2)^2}{4}\frac{1}{|x|^2}.$$

In this case, using the definition (4) of the function $\omega(s)$, it is easy to see that

$$\omega(m+1)-\omega(m)\sim C\log\left(1+\frac{1}{m}\right)\sim\frac{C}{m}.$$

Hence, we can apply Theorem 1.2, with

$$\Big(\omega(m+1)-\omega(m)\Big)\sup_{M\setminus B_m(p)}\frac{|f|}{\rho_m}\sim C\,m\sup_{M\setminus B_m(p)}|f|$$

and the convergence of the series follows, whenever $|f(x)|\leq C/(1+r(x))^{\alpha}$ for some $\alpha>2$. This condition is optimal, as it can be easily verified by explicit computations.

In general, concerning Cartan–Hadamard manifolds, by using Theorem 1.1 we improve [4, Corollary 1.3] allowing the Ricci curvature to approach zero at infinity.

Corollary 1.4 *Let (M,g) be a Cartan–Hadamard manifold, and let f be a locally Hölder continuous, bounded function on M. If*

$$-C\big(1+r(x)\big)^{\gamma_1}\leq \mathrm{Ric}\leq -\frac{1}{C}\big(1+r(x)\big)^{\gamma_2},\quad |f(x)|\leq\frac{C}{\big(1+r(x)\big)^{\alpha}},$$

for some $C\geq 1,\gamma_1,\gamma_2\in\mathbb{R},\gamma_1\geq\gamma_2,\gamma_1\geq 0$ and α satisfying

$$\alpha>\begin{cases}1+\frac{\gamma_1}{2}-\gamma_2 & \text{if } \gamma_2\geq -2\\ 3+\frac{\gamma_1}{2} & \text{if } \gamma_2<-2\end{cases},$$

then the Poisson equation

$$-\Delta u=f\quad\text{in } M$$

admits a classical solution u.

Remark 1.5 In the special case $\gamma_1=\gamma_2=\gamma\geq 0$ the condition on α in the previous corollary becomes

$$\alpha>\begin{cases}1-\frac{\gamma}{2} & \text{if } \gamma\geq -2\\ 2 & \text{if } \gamma<-2.\end{cases}$$

In particular, in (M,g) is the standard hyperbolic space $\mathbb{H}^n$, and then $\gamma=0$. Thus, we need that $\alpha>1$ and this condition is sharp as observed above. We will consider also the case $\gamma<0$ in Sect. 6.2 on model manifolds.

The paper is organized as follows: In Sect. 2 we collect some preliminary results and we define precisely the function ω; in Sect. 3 we prove a refined local gradient estimates for positive harmonic functions; in Sect. 4 we prove key estimates on the positive minimal Green's function $G(x,y)$ of a non-parabolic manifold, by means of the property $\big(\mathcal{P}_w^{\infty}\big)$ w.r.t. the family of weights ρ_R, $R\geq R_0$; in Sect. 5 we prove Theorem 1.1; finally, in Sect. 6 we prove Corollary 1.4 and show the optimality of the assumption in Theorem 1.2 for rotationally symmetric manifolds.

Finally, we note that some results concerning the Poisson equation on some manifolds satisfying a weighted Poincaré inequality have been very recently obtained in [14]. However, their assumptions and results apparently are completely different to ours.

2 Preliminaries

Let (M, g) be a complete non-compact n-dimensional Riemannian manifold. For any $x \in M$ and $R > 0$, we denote by $B_R(x)$ the geodesic ball of radius R with center x and let $\mathrm{Vol}(B_R(x))$ be its volume. We denote by Ric the Ricci curvature of g. For any $x \in M$, let $\mu(x)$ be the smallest eigenvalue of Ric at x. Thus, for any $V \in T_xM$ with $|V| = 1$, $\mathrm{Ric}(V, V)(x) \geq \mu(x)$ and we have $\mu(x) \geq -\omega(r(x))$ for some $\omega \in C([0, \infty))$, $\omega \geq 0$. Hence, for any $x \in M$, we have

$$\mathrm{Ric}(V, V)(x) \geq -(n-1)\frac{\varphi''(r(x))}{\varphi(r(x))}, \tag{2}$$

for some $\varphi \in C^\infty((0, \infty)) \cap C^1([0, \infty))$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Note that $\varphi, \varphi', \varphi''$ are positive in $(0, \infty)$. We set

$$K_R(x) := \sup_{y \in B_{r(x)+R}(p) \setminus B_{r(x)-R}(p)} \frac{\varphi''(r(y))}{\varphi(r(y))}$$

for $r(x) > R > 1$;

$$I_R(x) := \begin{cases} \sqrt{K_R(x)} \coth\left(\sqrt{K_R(x)} R/2\right) & \text{if } K_R(x) > 0 \\ \frac{2}{R} & \text{if } K_R(x) = 0; \end{cases}$$

$$Q_R(x) := \max\left\{K_R(x), \frac{I_R(x)}{R}, \frac{1}{R^2}\right\}. \tag{3}$$

Note that $Q_R(x) \equiv Q_R(r(x))$. For any $z \in M$, let γ be the minimal geodesic connecting p to z. We define the function

$$\omega(z) = \omega(r(z)) := \int_a^{r(z)} \sqrt{Q_{\frac{r(\gamma(s))}{4}}(r(\gamma(s))}\, \mathrm{d}s, \tag{4}$$

for a given $a > 0$. Note that $t \mapsto \omega(t)$ is increasing and so invertible.

Under (2), we know that

$$\mathrm{Vol}(B_R(p)) \leq C \int_0^R \varphi^{n-1}(\xi)\, \mathrm{d}\xi. \tag{5}$$

Moreover, let $\mathrm{Cut}(p)$ be the *cut locus* of $p \in M$.

It is known that every complete Riemannian manifold admits a Green's function (see [11]), i.e., a smooth function defined in $(M \times M) \setminus \{(x, y) \in M \times M \,:\, x = y\}$ such that $G(x, y) = G(y, x)$ and $\Delta_y G(x, y) = -\delta_x(y)$. We say that (M, g) is non-parabolic if there exists a minimal positive Green's function $G(x, y)$ on (M, g), and parabolic otherwise.

We say that (M, g) satisfies a *weighted Poincaré inequality* with a nonnegative weight function ρ if

$$\int_M \rho\, v^2\, \mathrm{d}V \leq \int_M |\nabla v|^2\, \mathrm{d}V \tag{6}$$

for every $v \in C_c^\infty(M)$. If for any $R \geq R_0 > 0$, there exists a nonnegative function ρ_R such that (1) holds for every $v \in C_c^\infty(M \setminus B_R(p))$ and for $\rho \equiv \rho_R$, we say that (M, g) satisfies a *weighted Poincaré inequality at infinity*. In addition, inspired by [10], we say that (M, g) satisfies the property $\left(\mathcal{P}^\infty_{\rho_R}\right)$ if a weighted Poincaré inequality at infinity holds for the family of weights ρ_R and the conformal ρ_R-metric defined by

$$g_\rho := \rho_R\, g$$

is complete. With this metric we consider the ρ-distance function

$$r_\rho(x, y) = \inf_\gamma l_\rho(\gamma)$$

where the infimum of the lengths is taken over all curves joining x and y, with respect to the metric g_ρ. For the fixed reference point $p \in M$, we denote by

$$r_\rho(x) = r_\rho(p, x).$$

Note that $|\nabla r_\rho(x)|^2 = \rho(x)$. Finally, we denote by

$$B^\rho_R(p) = \{x \in M \,:\, r_\rho(x) \leq R\}.$$

Let $\lambda_1(M)$ be the bottom of the L^2-spectrum of $-\Delta$. It is known that $\lambda_1(M) \in [0, +\infty)$ and it is given by the variational formula

$$\lambda_1(M) = \inf_{v \in C_c^\infty(M)} \frac{\int_M |\nabla v|^2\, \mathrm{d}V}{\int_M v^2\, \mathrm{d}V}.$$

If $\lambda_1(M) > 0$, then (M, g) is non-parabolic (see [6, Proposition 10.1]). Whenever (M, g) is non-parabolic, let $G_R(x, y)$ be the Green's function of $-\Delta$ in $B_R(z)$ satisfying zero Dirichlet boundary conditions on $\partial B_R(z)$, for some $z \in M$. We have that $R \mapsto G_R(x, y)$ is increasing and, for any $x, y \in M$,

$$G(x, y) = \lim_{R \to \infty} G_R(x, y), \tag{7}$$

locally uniformly in $(M \times M) \setminus \{(x, y) \in M \times M \,:\, x = y\}$. We define $\lambda_1(\Omega)$, with Ω an open subset of M, to be the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary conditions. It is well known that $\lambda_1(\Omega)$ is decreasing with respect to the inclusion of subsets. In particular, $R \mapsto \lambda_1(B_R(x))$ is decreasing and $\lambda_1(B_R(x)) \to \lambda_1(M)$ as $R \to \infty$.

For any $x \in M$, for any $s > 0$ and for any $0 \leq a < b \leq +\infty$, we define

$$\mathcal{L}_x(s) := \{y \in M \,:\, G(x, y) = s\},$$
$$\mathcal{L}_x(a, b) := \{y \in M \,:\, a < G(x, y) < b\}.$$

3 Local gradient estimate for harmonic functions

In this section, we improve [4, Lemma 3.1]. We set

$$k_R(z) := \sup_{B_R(z)} \frac{\varphi''(r(y))}{\varphi(r(y))}$$

for $z \in M$ and $R > 0$;

$$i_R(z) := \begin{cases} \sqrt{k_R}\coth\left(\sqrt{k_R(z)}R/2\right) & \text{if } k_R(z) > 0 \\ \frac{2}{R} & \text{if } k_R(z) = 0. \end{cases}$$

Lemma 3.1 *Let $R > 0$ and $z \in M$. Let $u \in C^2(B_R(z))$ be a positive harmonic function in $B_R(z)$. Then,*

$$|\nabla u(\xi)| \leq C\sqrt{\max\left\{k_R(z), \frac{i_R(z)}{R}, \frac{1}{R^2}\right\}}\, u(\xi) \quad \text{for any} \quad \xi \in B_{R/2}(z),$$

for some positive constant $C > 0$.

Proof Following the classical argument of Yau, let $v := \log u$. Then,

$$\Delta v = -|\nabla v|^2.$$

Let $\eta(\xi) = \eta(d(\xi))$, with $d(\xi) := \mathrm{dist}(\xi, z)$, a smooth cutoff function such that $\eta(\xi) \equiv 1$ on $B_{R/2}(z)$, with support in $B_R(z)$, $0 \leq \eta \leq 1$ and

$$-\frac{4}{R} \leq \frac{\eta'}{\eta^{1/2}} \leq 0 \quad \text{and} \quad \frac{|\eta''|}{\eta} \leq \frac{8}{R^2}.$$

Let $w = \eta^2|\nabla v|^2$. Then,

$$\frac{1}{2}\Delta w = \frac{1}{2}\eta^2\Delta|\nabla v|^2 + \frac{1}{2}|\nabla v|^2\Delta\eta^2 + \langle\nabla|\nabla v|^2, \nabla\eta^2\rangle.$$

Then, from classical Bochner–Weitzenböch formula and Newton inequality, one has

$$\begin{aligned} \frac{1}{2}\Delta|\nabla v|^2 &= |\nabla^2 v|^2 + \mathrm{Ric}(\nabla v, \nabla v) + \langle\nabla v, \nabla\Delta v\rangle \\ &\geq \frac{1}{n}(\Delta v)^2 - (n-1)\frac{\varphi''}{\varphi}|\nabla v|^2 - \langle\nabla|\nabla v|^2, \nabla v\rangle \\ &= \frac{1}{n}|\nabla v|^4 - (n-1)\frac{\varphi''}{\varphi}|\nabla v|^2 - \langle\nabla|\nabla v|^2, \nabla v\rangle. \end{aligned}$$

Moreover, by Laplacian comparison, since $\mathrm{Ric} \geq -(n-1)k_R(z)$ in $B_R(z)$, we have

$$\begin{aligned} \frac{1}{2}\Delta\eta^2 &= \eta\eta'\Delta\rho + \eta\eta'' + (\eta')^2 \\ &\geq (n-1)i_R(z)\eta\eta' + \eta\eta'' + (\eta')^2 \\ &\geq -\frac{4}{R}\left((n-1)i_R(z) + \frac{2}{R}\right)\eta \end{aligned}$$

pointwise in $B_R(z) \setminus (\{z\} \cup \mathrm{Cut}(z))$ and weakly on $B_R(z)$. Thus,

$$
\begin{aligned}
\frac{1}{2}\Delta w \geq & \frac{1}{n}\frac{w^2}{\eta^2} - (n-1)\frac{\varphi''}{\varphi}w - \frac{4}{R}\left((n-1)i_R(z) + \frac{2}{R}\right)\frac{w}{\eta} \\
& - 4\frac{|\eta'|^2}{\eta^2}w + \frac{2}{\eta}\langle\nabla w, \nabla\eta\rangle - \langle\nabla w, \nabla v\rangle + \frac{2}{\eta}\langle\nabla v, \nabla\eta\rangle w \\
\geq & \frac{1}{n}\frac{w^2}{\eta^2} - (n-1)\frac{\varphi''}{\varphi}w - \frac{4}{R}\left((n-1)i_R(z) + \frac{2}{R}\right)\frac{w}{\eta} \\
& + \frac{2}{\eta}\langle\nabla w, \nabla\eta\rangle - \langle\nabla w, \nabla v\rangle - \frac{64}{R^2}\frac{w}{\eta} - \frac{8}{R}\frac{w^{3/2}}{\eta^{3/2}} \\
\geq & \frac{1}{2n}\frac{w^2}{\eta^2} - (n-1)\frac{\varphi''}{\varphi}w - \frac{4}{R}\left((n-1)i_R(z) + \frac{18+8n}{R}\right)\frac{w}{\eta} \\
& + \frac{2}{\eta}\langle\nabla w, \nabla\eta\rangle - \langle\nabla w, \nabla v\rangle.
\end{aligned}
$$

Let q be a maximum point of w in $\overline{B}_R(z)$. Since $w \equiv 0$ on $\partial B_R(z)$, we have $q \in B_R(z)$. First assume $q \notin \mathrm{Cut}(z)$. At q, we obtain

$$
0 \geq \left[\frac{1}{2n}w - (n-1)\frac{\varphi''}{\varphi} - \frac{4}{R}\left((n-1)i_R(z) + \frac{18+8n}{R}\right)\right]w.
$$

So

$$
w(q) \leq 2n(n-1)\frac{\varphi''\big(r(q)\big)}{\varphi\big(r(q)\big)} + \frac{8n(n-1)}{R}i_R(z) + \frac{144n + 64n^2}{R^2}.
$$

Thus, for any $\xi \in B_{R/2}(z)$,

$$
\begin{aligned}
|\nabla v(\xi)|^2 & \leq 2n(n-1)\frac{\varphi''\big(r(q)\big)}{\varphi\big(r(q)\big)} + \frac{8n(n-1)}{R}i_R(z) + \frac{144n + 64n^2}{R^2} \\
& \leq 2n(n-1)k_R(z) + \frac{8n(n-1)}{R}i_R(z) + \frac{144n + 64n^2}{R^2}
\end{aligned}
$$

We get

$$
\frac{|\nabla u(\xi)|}{u(\xi)} = |\nabla v(\xi)| \leq C\sqrt{\max\left\{k_R(z), \frac{i_R(z)}{R}, \frac{1}{R^2}\right\}}.
$$

for some positive constant $C > 0$. By standard Calabi trick (see [2, 5]), the same estimate can be obtained when $q \in \mathrm{Cut}(z)$. This concludes the proof of the lemma.

□

As a corollary, we have the following

Corollary 3.2 *Let* (M, g) *be non-parabolic. If* $r(z) > R > 0$, *then*

$$
|\nabla G(p, z)| \leq C\sqrt{Q_R(z)}\, G(p, z),
$$

for some positive constant $C > 0$.

4 Green's function estimates

4.1 Pointwise estimate

Lemma 4.1 *Let (M, g) be non-parabolic, and let $a > 0$ and $y \in M \setminus B_a(p)$. Then,*

$$A^{-1} \exp(-B\,\omega(y)) \leq G(p, y) \leq A \exp(B\,\omega(y)),$$

where $A := \max\left\{\max_{\partial B_a(p)} G(p, \cdot), \left(\min_{\partial B_a(p)} G(p, \cdot)\right)^{-1}\right\}$ and $B > 0$ is a positive constant depending on C from Corollary 3.2.

Proof Let $y \in M \setminus \overline{B_a(p)}$ with $a > 0$ and consider the minimal geodesic γ joining p to y and let $y_0 \in \partial B_a(p)$ be a point of intersection of γ with $\partial B_a(p)$. Since $G(p, \cdot)$ is harmonic in $B_{r(z)/4}(z)$, for every $z \in \gamma$ with $r(z) \geq a$, by Corollary 3.2 we get

$$|\nabla G(p, z)| \leq C\sqrt{Q_{r(z)/4}(z)}\, G(p, z).$$

We have

$$\begin{aligned} G(p, y) &= G(p, y_0) + \int_a^{r(y)} \langle \nabla G(p, \gamma(s)), \dot{\gamma}(s)\rangle \,\mathrm{d}s \\ &\leq G(p, y_0) + C\int_a^{r(y)} \sqrt{Q_{\frac{r(\gamma(s))}{4}}\left(r(\gamma(s))\right)}\, G(p, \gamma(s))\,\mathrm{d}s. \end{aligned}$$

By Gronwall's inequality,

$$G(p, y) \leq G(p, y_0) \exp\left(C\int_a^{r(y)} \sqrt{Q_{\frac{r(\gamma(s))}{4}}\left(r(\gamma(s))\right)}\,\mathrm{d}s\right) \leq A \exp(B\,\omega(y)),$$

with $A := \max\left\{\max_{\partial B_a(p)} G(p, \cdot), \left(\min_{\partial B_a(p)} G(p, \cdot)\right)^{-1}\right\}$ and $B = C$. Similarly,

$$G(p, y) \geq A^{-1} \exp(-B\,\omega(y)).$$

$\square$

Remark 4.2 One has

$$G(p, y) \geq A^{-1} \exp(-B\,\omega(a))$$

for any $y \in \overline{B_a(p)}$. This follows from Lemma 4.1 with $y \in \partial B_a(p)$ and the maximum principle, since $y \mapsto G(p, y)$ is (weakly) superharmonic in $B_a(p)$. In particular,

$$\mathcal{L}_p\left(0, A^{-1}\exp(-B\,\omega(a))\right) \subset M \setminus B_a(p).$$

Remark 4.3 We also note that

$$\mathcal{L}_p(A \exp(B\,\omega(a)), \infty) \subset B_a(p).$$

In fact, let $y \in M \setminus B_a(p)$ and take $j > r(y)$. Since $G_j(p, y) \leq G(p, y)$ and $G_j(p, \cdot) \equiv 0$ on $\partial B_j(p)$, by Lemma 4.1, we have

$$G_j(p,y) \le A\exp(B\omega(a)) \quad \text{on} \quad \partial\big(B_j(p)\setminus B_a(p)\big);$$

note that the right-hand side is independent of y. Since $y \mapsto G_j(x,y)$ is harmonic in $B_j(p)\setminus B_a(p)$, by maximum principle,

$$G_j(p,y) \le A\exp(B\omega(a)) \quad \text{in} \quad B_j(p)\setminus B_a(p).$$

Sending $j\to\infty$, by (7), we obtain

$$G(p,y) \le A\exp(B\omega(a)) \quad \text{in} \quad M\setminus B_a(p),$$

and the claim follows.

4.2 Auxiliary estimates

Lemma 4.4 *Let (M,g) be non-parabolic. For any $s>0$, there holds*

$$\int_{\mathcal{L}_p(s)} |\nabla G(p,y)|\,\mathrm{d}A(y) = 1$$

where $\mathrm{d}A(y)$ is the $(n-1)$-dimensional Hausdorff measure on $\mathcal{L}_x(s)$. As a consequence, by the co-area formula, for any $0<a<b$, there holds

$$\int_{\mathcal{L}_p(a,b)} \frac{|\nabla G(p,y)|^2}{G(p,y)}\,\mathrm{d}y = \log\left(\frac{b}{a}\right).$$

For the proof see [12]. Moreover, we get the following weighted integrability property for the Green's function.

Lemma 4.5 *Assume that (M,g) satisfies the property $\left(\mathcal{P}_w^\infty\right)$ w.r.t. the family of weights ρ_R, $R\ge R_0$. Fix $m\ge R_0$. Then, for any $R_1>0$ such that $B_m(p)\subset B^{\rho_m}_{R_1}(p)$, one has*

$$\int_{M\setminus B^{\rho_m}_{2R_1}(p)} \rho_m(y)\,|G(p,y)|^2\,\mathrm{d}y < \infty.$$

Remark 4.6 Note that $B_m(p)\subset B^{\rho_m}_{R_1}(p)$ for every R_1 large enough.

Proof In order to simplify the notation, let $\rho\equiv\rho_m$. Fix $R_1>0$ such that $B_m(p)\subset B^{\rho}_{R_1}(p)$ and let ϕ be defined as

$$\phi(x) := \begin{cases} 0 & \text{on } B^{\rho}_{R_1}(p) \\ \frac{r_\rho(x)-R_1}{R_1} & \text{on } B^{\rho}_{2R_1}(p)\setminus B^{\rho}_{R_1}(p) \\ 1 & \text{on } M\setminus B^{\rho}_{2R_1}(p). \end{cases}$$

Let $R>2R_1$ and $G^{\rho}_R(p,y)$ be the Green's function of $-\Delta$ in $B^{\rho}_R(p)$ satisfying zero Dirichlet boundary conditions on $\partial B^{\rho}_R(p)$. Following the proof in [10], since G^{ρ}_R is harmonic in $B^{\rho}_R(p)$, one has

$$\begin{aligned}\int_{B_R^\rho(p)} |\nabla(\phi\, G_R^\rho)|^2 \,\mathrm{d}V &= \int_{B_R^\rho(p)} |\nabla\phi|^2 \big(G_R^\rho\big)^2 \,\mathrm{d}V + \int_{B_R^\rho(p)} |\nabla G_R^\rho|^2 \phi^2 \,\mathrm{d}V \\ &\quad + 2\int_{B_R^\rho(p)} \langle \nabla\phi, \nabla G_R^\rho\rangle \phi G_R^\rho \,\mathrm{d}V \\ &= \int_{B_R^\rho(p)} |\nabla\phi|^2 \big(G_R^\rho\big)^2 \,\mathrm{d}V + \frac{1}{2}\int_{B_R^\rho(p)} \Delta\big(G_R^\rho\big)^2 \phi^2 \,\mathrm{d}V \\ &\quad + 2\int_{B_R^\rho(p)} \langle \nabla\phi, \nabla G_R^\rho\rangle \phi G_R^\rho \,\mathrm{d}V \\ &= \int_{B_R^\rho(p)} |\nabla\phi|^2 \big(G_R^\rho\big)^2 \,\mathrm{d}V\end{aligned}$$

where the last equality follows by integration by parts and the fact that $G_R^\rho(p,y)$ vanishes on $\partial B_R^\rho(p)$. Hence, the weighted Poincaré inequality yields

$$\int_{M\setminus B_{R_1}^\rho(p)} \rho\,\big(G_R^\rho\big)^2 \phi^2 \,\mathrm{d}V \le \int_{B_R^\rho(p)} |\nabla(\phi\, G_R^\rho)|^2 \,\mathrm{d}V \le \frac{1}{R_1^2}\int_{B_{2R_1}^\rho(p)\setminus B_{R_1}^\rho(p)} \rho\,\big(G_R^\rho\big)^2 \,\mathrm{d}V$$

Letting $R \to \infty$, by Fatou's lemma and uniform convergence of $G_R^\rho \to G$ on compact subsets, we get

$$\int_{M\setminus B_{2R_1}^\rho(p)} \rho\, G^2 \,\mathrm{d}V \le \frac{1}{R_1^2}\int_{B_{2R_1}^\rho(p)\setminus B_{R_1}^\rho(p)} \rho\, G^2 \,\mathrm{d}V$$

and the thesis follows. ☐

We expect a decay estimate similar to the one in [10, Theorem 2.1]. However, we leave out this refinement since it is not necessary in our arguments.

4.3 Integral estimates on level sets

We begin by noting that using Remark 4.3 and the fact that $G(p,\cdot) \in L^1_{\mathrm{loc}}(M)$ one has the following integral estimate on large level sets.

Proposition 4.7 *Let (M, g) be non-parabolic. Choose A, B as in Lemma 4.1. Then,*

$$\int_{\mathcal{L}_p(A\exp(B\,\omega(a)),\infty)} G(p,y)\,\mathrm{d}y < \infty.$$

For intermediate levels sets, we get the following key inequality.

Proposition 4.8 *Assume that (M, g) satisfies the property $\big(\mathcal{P}_w^\infty\big)$ w.r.t. the family of weights ρ_R, $R \ge R_0$. Then, there exists a positive constant C such that for any function f and any $0 < \delta < 1$, $\varepsilon > 0$ satisfying $\mathcal{L}_p\Big(\frac{\delta\varepsilon}{2}, 2\varepsilon\Big) \subset M \setminus B_m(p)$ for some $m > R_0$, one has*

$$\left|\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} G(p,y) f(y)\,\mathrm{d}y\right| \le C(-\log\delta + 1) \sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|f|}{\rho_m}.$$

Proof We follow the general argument in [10, 12]; however, some relevant differences are in order, due to the use of the property $\left(\mathcal{P}_w^\infty\right)$. Let $\phi := \chi\psi$ with

$$\chi(y) := \begin{cases} \frac{1}{\log 2}\log\left(\frac{2G(p,y)}{\delta\epsilon}\right) & \text{on } \mathcal{L}_p\left(\frac{\delta\varepsilon}{2}, \delta\varepsilon\right) \\ 1 & \text{on } \mathcal{L}_p(\delta\varepsilon, \varepsilon) \\ \frac{1}{\log 2}\log\left(\frac{2\varepsilon}{G(p,y)}\right) & \text{on } \mathcal{L}_p(\varepsilon, 2\varepsilon) \\ 0 & \text{elsewhere} \end{cases}$$

and for any fixed $R > 0$

$$\psi(y) := \begin{cases} 1 & \text{on } B_R^{\rho_m}(p) \\ R + 1 - r_{\rho_m}(y) & \text{on } B_{R+1}^{\rho_m}(p) \setminus B_R^{\rho_m}(p) \\ 0 & \text{on } M \setminus B_{R+1}^{\rho_m}(p). \end{cases}$$

By the weighted Poincaré inequality at infinity, we get

$$\begin{aligned} \left|\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)\cap B_R^{\rho_m}(p)} G(p,y) f(y)\,\mathrm{d}y\right| &\leq \int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)\cap B_R^{\rho_m}(p)} G(p,y)\,|f(y)|\,\mathrm{d}y \\ &\leq \sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)\cap B_R^{\rho_m}(p)} \frac{|f|}{\rho_m} \int_M \rho_m(y)\, G(p,y)\phi^2(y)\,\mathrm{d}y \\ &\leq \sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)\cap B_R^{\rho_m}(p)} \frac{|f|}{\rho_m} \int_M \left|\nabla\left(\sqrt{G(p,y)}\phi(y)\right)\right|^2 \mathrm{d}y. \end{aligned}$$

We estimate

$$\begin{aligned} \int_M \left|\nabla\left(\sqrt{G(p,y)}\phi(y)\right)\right|^2 \mathrm{d}y &\leq \frac{1}{2}\int_{\mathcal{L}_p(\frac{\delta\varepsilon}{2},2\varepsilon)} \frac{|\nabla G(p,y)|^2}{G(p,y)}\,\mathrm{d}y + 2\int_M G(p,y)|\nabla\phi|^2\,\mathrm{d}y \\ &= C(-\log\delta + 1) + 2\int_M G(p,y)|\nabla\phi|^2\,\mathrm{d}y \end{aligned}$$

where we used Lemma 4.4 in the last equality. On the other hand,

$$\begin{aligned} \int_M G(p,y)|\nabla\phi|^2\,\mathrm{d}y &\leq 2\int_M G(p,y)|\nabla\chi|^2\psi^2\,\mathrm{d}y + 2\int_M G(p,y)|\nabla\psi|^2\chi^2\,\mathrm{d}y \\ &\leq 2(\log 2)^2\int_{\mathcal{L}_p(\frac{\delta\varepsilon}{2},2\varepsilon)} \frac{|\nabla G(p,y)|^2}{G(p,y)}\,\mathrm{d}y \\ &\quad + 2\int_{B_{R+1}^{\rho}(p)\setminus B_R^{\rho}(p)} \rho_m(y)\, G(p,y)\chi^2\,\mathrm{d}y \\ &\leq C(-\log\delta + 1) + \frac{4}{\delta\varepsilon}\int_{B_{R+1}^{\rho_m}(p)\setminus B_R^{\rho_m}(p)} \rho_m(y)\, G^2(p,y)\,\mathrm{d}y. \end{aligned}$$

Now we let $R \to \infty$ and use Lemma 4.5. The thesis now follows. □

In the special case when M is non-parabolic with positive minimal Green's function G and with weight $\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)}$, we have the following refinement of Proposition 4.8.

Proposition 4.9 *Assume that* (M, g) *is non-parabolic with positive minimal Green's function* G *and with weight* $\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)}$. *Then, there exists a positive constant* C *such that for any function* f *and any* $0 < \delta < 1$, $\varepsilon > 0$ *one has*

$$\left|\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} G(p,y)f(y)\,\mathrm{d}y\right| \leq C(-\log\delta) \sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|f|}{\rho}.$$

Proof We have

$$\begin{aligned}\left|\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} G(p,y)f(y)\,\mathrm{d}y\right| &\leq \sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|f|}{\rho}\left(\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} G(p,y)\,\rho(y)\,\mathrm{d}y\right)\\ &= \frac{1}{4}\sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|f|}{\rho}\left(\int_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|\nabla G(p,y)|^2}{G(p,y)}\,\mathrm{d}y\right)\\ &= \frac{1}{4}(-\log\delta)\sup_{\mathcal{L}_p(\delta\varepsilon,\varepsilon)} \frac{|f|}{\rho},\end{aligned}$$

where we have used Lemma 4.4 in the last equality. ◻

5 Proof of Theorem 1.1

In order to prove Theorem 1.1, we will show that

$$|u(x)| = \left|\int_M G(x,y)f(y)\,\mathrm{d}y\right| \leq v(x),$$

with $v \in C^0(M)$. We divide the proof in two parts, we first consider the case when (M, g) is non-parabolic, and then, the case when it is parabolic.

Proof of Theorem 1.1 **Case 1:** (M, g) *non-parabolic.*

By assumption, (M, g) satisfies $\left(\mathcal{P}^{\infty}_{w}\right)$ w.r.t. the family of weights ρ_R, $R \geq R_0$. Let $x \in M$ and choose $R = R(x) > R_0$ large enough so that $x \in B_R(p)$. One has

$$\begin{aligned}\left|\int_M G(x,y)f(y)\,\mathrm{d}y\right| &\leq \left|\int_{B_R(p)} G(x,y)f(y)\,\mathrm{d}y\right| + \left|\int_{M\setminus B_R(p)} G(x,y)f(y)\,\mathrm{d}y\right|\\ &\leq C_1(x) + \int_{M\setminus B_R(p)} G(x,y)\,|f(y)|\,\mathrm{d}y\end{aligned}$$

since $G(x,\cdot) \in L^1_{\mathrm{loc}}(M)$. Hence, by Harnack's inequality, we have

$$\begin{aligned}\left|\int_M G(x,y)f(y)\,\mathrm{d}y\right| &\leq C_1(x) + C_2(x)\int_{M\setminus B_R(p)} G(p,y)\,|f(y)|\,\mathrm{d}y\\ &\leq C_1(x) + C_2(x)\int_M G(p,y)\,|f(y)|\,\mathrm{d}y,\end{aligned} \tag{8}$$

where $C_2(x)$ can be chosen as the constant in the Harnack's inequality for the ball $B_{r(x)+1}(p)$. For any $a > 0$, we estimate

$$\int_M G(p,y)\,|f(y)|\,\mathrm{d}y = \int_{\mathcal{L}_p(0,A\exp(B\,\omega(a)))} G(p,y)\,|f(y)|\,\mathrm{d}y + \int_{\mathcal{L}_p(A\exp(B\,\omega(a)),\infty)} G(p,y)\,|f(y)|\,\mathrm{d}y.$$

By Proposition 4.7, Remark 4.3 we get

$$\int_M G(p,y)\,|f(y)|\,\mathrm{d}y \le \int_{\mathcal{L}_p(0,A\exp(B\,\omega(a)))} G(p,y)\,|f(y)|\,\mathrm{d}y + C_3(a) \tag{9}$$

for some positive constant $C_3(a)$. To estimate the first integral, we observe that for any $m_0 = m_0(x) \ge a$ one has

$$\begin{aligned}\int_{\mathcal{L}_p(0,A\exp(B\,\omega(a)))} G(x,y)\,|f(y)|\,\mathrm{d}y &= \int_{\mathcal{L}_p\left(0,(2A)^{-1}\exp(-B\omega(m_0))\right)} G(x,y)\,|f(y)|\,\mathrm{d}y \\ &\quad + \int_{\mathcal{L}_p\left((2A)^{-1}\exp(-B\omega(m_0)),A\exp(B\,\omega(a))\right)} G(x,y)\,|f(y)|\,\mathrm{d}y.\end{aligned} \tag{10}$$

We need the following lemma.

Lemma 5.1 *Choose A, B as in Lemma 4.1. For any $m \ge m_0 \ge a$ one has*

$$\mathcal{L}_p\left(0,A^{-1}\exp(-B\omega(m))\right) \subset M \setminus B_m(p). \tag{11}$$

Proof Since $m_0 \ge a$, Remark 4.2 implies

$$\mathcal{L}_p\left(0,A^{-1}\exp(-B\omega(m_0))\right) \subset \mathcal{L}_p\left(0,A^{-1}\exp(-B\,\omega(a))\right) \subset M \setminus B_a(p). \tag{12}$$

If

$$z \in \mathcal{L}_p\left(0,A^{-1}\exp(-B\omega(m))\right) \subset M \setminus B_a(p),$$

then by Lemma 4.1

$$A^{-1}\exp(-B\omega(m)) \ge G(p,z) \ge A^{-1}\exp(-B\omega(z)).$$

Thus,

$$\omega(z) \ge \omega(m)$$

and, by monotonicity of ω, we obtain $r(z) \ge m$. □

In particular, we get

$$\mathcal{L}_p\left(0,(2A)^{-1}\exp(-B\omega(m_0))\right) \subset \mathcal{L}_p\left(0,A^{-1}\exp(-B\omega(m_0))\right) \subset M \setminus B_{m_0}(p).$$

Thus,

$$\mathcal{L}_p\big((2A)^{-1}\exp(-B\omega(m_0)),\, A\exp(B\,\omega(a))\big) \subset B_{m_0}(p)$$

Then, since $G(x,\cdot) \in L^1_{\mathrm{loc}}(M)$, we get

$$\int_{\mathcal{L}_p\left((2A)^{-1}\exp(-B\omega(m_0)),\, A\exp(B\,\omega(a))\right)} G(x,y)\,|f(y)|\,\mathrm{d}y \le C_4(a, m_0). \tag{13}$$

Now, for any $m \ge m_0$, let

$$\varepsilon := (2A)^{-1}\exp(-B\omega(m)), \qquad \delta := \exp(B\omega(m) - B\omega(m+1)). \tag{14}$$

By Lemma 5.1,

$$\mathcal{L}_p(0, 2\varepsilon) \subset M \setminus B_m(p).$$

Hence, we can apply Proposition 4.8 obtaining

$$\begin{aligned}
&\int_{\mathcal{L}_p\left(0,(2A)^{-1}\exp(-B\omega(m_0))\right)} G(x,y)\,|f(y)|\,\mathrm{d}y \\
&\quad = \sum_{m\ge m_0} \int_{\mathcal{L}_p\left((2A)^{-1}\exp(-B\omega(m+1)),(2A)^{-1}\exp(-B\omega(m))\right)} G(x,y)\,|f(y)|\,\mathrm{d}y \\
&\quad \le C \sum_{m\ge m_0}^{\infty} (\omega(m+1) - \omega(m) + 1) \sup_{\mathcal{L}_p\left((2A)^{-1}\exp(-B\omega(m+1)),(2A)^{-1}\exp(-B\omega(m))\right)} \frac{|f|}{\rho_m} \\
&\quad \le C \sum_{m\ge m_0}^{\infty} (\omega(m+1) - \omega(m) + 1) \sup_{\mathcal{L}_p\left(0,A^{-1}\exp(-B\omega(m))\right)} \frac{|f|}{\rho_m} \\
&\quad \le C \sum_{m\ge m_0}^{\infty} (\omega(m+1) - \omega(m) + 1) \sup_{M\setminus B_m(p)} \frac{|f|}{\rho_m} < \infty,
\end{aligned} \tag{15}$$

where in the last inequality we used Lemma 5.1. The proof of Theorem 1.1 is complete in this case.

Case 2: (M, g) *parabolic.*

Let $G(x, y)$ be a Green's function on M (which is positive inside a certain ball, and negative outside). Fix any $R > 0$ and let $\rho \equiv \rho_{R_0}$. Note that arguing as in the proof of (8), it is sufficient to estimate

$$\begin{aligned}
\int_M |G(p,y)||f(y)|\,\mathrm{d}y &= \int_{M\setminus B^\rho_R(p)} |G(p,y)||f(y)|\,\mathrm{d}y + \int_{B^\rho_R(p)} |G(p,y)||f(y)|\,\mathrm{d}y \\
&\le \int_{M\setminus B^\rho_R(p)} |G(p,y)||f(y)|\,\mathrm{d}y + C,
\end{aligned}$$

since $G(p,\cdot) \in L^1_{\mathrm{loc}}(M)$ and f is locally bounded. We have that

$$M \setminus B^\rho_R(p) = \bigcup_{i=1}^{N} E_i,$$

where each E_i is an end with respect to $B^\rho_R(p)$. Note that every end E_i is parabolic. In fact, if at least one end E_i is non-parabolic, then (M, g) is non-parabolic (see [8] for a nice overview), but we are in the case that (M, g) is parabolic. Since every E_i is parabolic, every E_i has finite weighted volume (see [9]), i.e.,

$$\int_{E_i} \rho \, dy < \infty.$$

Now choose R large enough so that we can apply Lemma 4.5 obtaining

$$\begin{aligned}
&\int_{M\setminus B_R^\rho(p)} |G(p,y)|\,|f(y)|\, dy \\
&\quad \le \left(\int_{M\setminus B_R^\rho(p)} \rho(y) |G(p,y)|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{M\setminus B_R^\rho(p)} \rho(y) \left(\frac{|f(y)|}{\rho(y)} \right)^2 dy \right)^{\frac{1}{2}} \\
&\quad \le C \sup_{M\setminus B_{R_0}(p)} \frac{|f|}{\rho} \int_{M\setminus B_R^\rho(p)} \rho \, dy < \infty.
\end{aligned}$$

This concludes the proof of Theorem 1.1.

$\square$

Proof of Theorem 1.2 We start as in the proof of Theorem 1.1 using (8), (9), (10) and (13). Then, similar to (15), using Proposition 4.9, we obtain

$$\begin{aligned}
&\int_{\mathcal{L}_p\left(0,(2A)^{-1}\exp(-B\omega(m_0))\right)} G(x,y)\, |f(y)| \, dy \\
&\quad = \sum_{m\ge m_0} \int_{\mathcal{L}_p\left((2A)^{-1}\exp(-B\omega(m+1)),(2A)^{-1}\exp(-B\omega(m))\right)} G(x,y)\, |f(y)|\, dy \\
&\quad \le C \sum_{m\ge m_0}^{\infty} (\omega(m+1) - \omega(m)) \sup_{M\setminus B_m(p)} \frac{|f|}{\rho} < \infty,
\end{aligned}$$

Then

$$\left| \int_M G(x,y) f(y)\, dy \right| < \infty$$

and the proof of Theorem 1.2 is complete. $\square$

6 Cartan–Hadamard and model manifolds

We consider Cartan–Hadamard manifolds, i.e., complete, non-compact, simply connected Riemannian manifolds with non-positive sectional curvatures everywhere. Observe that on Cartan–Hadamard manifolds the cut locus of any point p is empty. Hence, for any $x \in M \setminus \{p\}$ one can define its polar coordinates with pole at p, namely $r(x) = \mathrm{dist}(x, p)$ and $\theta \in \mathbb{S}^{n-1}$. We have

$$\mathrm{meas}\big(\partial B_r(p)\big) \;=\; \int_{\mathbb{S}^{n-1}} A(r,\theta)\,\mathrm{d}\theta^1 d\theta^2 \ldots \mathrm{d}\theta^{n-1},$$

for a specific positive function A which is related to the metric tensor [6, Sect. 3]. Moreover, it is direct to see that the Laplace–Beltrami operator in polar coordinates has the form

$$\Delta \;=\; \frac{\partial^2}{\partial r^2} + m(r,\theta)\,\frac{\partial}{\partial r} + \Delta_\theta\,,$$

where $m(r,\theta) := \frac{\partial}{\partial r}(\log A)$ and Δ_θ is the Laplace–Beltrami operator on $\partial B_r(p)$. We have

$$m(r,\theta) = \Delta r(x).$$

Let

$$\mathcal{A} := \big\{f \in C^\infty((0,\infty)) \cap C^1([0,\infty)) \,:\, f'(0)=1, f(0)=0, f>0 \text{ in } (0,\infty)\big\}.$$

We say that $(M,\,g)$ is a rotationally symmetric manifold or a model manifold if the Riemannian metric is given by

$$g \;=\; \mathrm{d}r^2 + \varphi(r)^2\,\mathrm{d}\theta^2,$$

where $\mathrm{d}\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$ and $\varphi \in \mathcal{A}$. In this case,

$$\Delta \;=\; \frac{\partial^2}{\partial r^2} + (n-1)\,\frac{\varphi'}{\varphi}\,\frac{\partial}{\partial r} + \frac{1}{\varphi^2}\,\Delta_{\mathbb{S}^{n-1}}\,.$$

Note that $\varphi(r)=r$ corresponds to $M=\mathbb{R}^n$, while $\varphi(r)=\sinh r$ corresponds to $M=\mathbb{H}^n$, namely the n-dimensional hyperbolic space. The Ricci curvature in the radial direction is given by

$$\mathrm{Ric}(\nabla r,\nabla r)(x) = -(n-1)\frac{\varphi''(r(x))}{\varphi(r(x))}.$$

6.1 Cartan–Hadamard manifolds

Concerning the validity of the property $\left(\mathcal{P}_w^\infty\right)$ w.r.t. the family of weights ρ_R, $R \geq R_0$ on a Cartan–Hadamard manifold we have the following result.

Lemma 6.1 *Let $(M,\,g)$ be a Cartan–Hadamard manifold with*

$$\mathrm{Ric}(\nabla r,\nabla r)(x) \leq -C\big(1+r(x)\big)^\gamma$$

for some $\gamma \in \mathbb{R}$, $C>0$ and any $x \in M\setminus\{p\}$. Then $(M,\,g)$ satisfies the property $\left(\mathcal{P}_w^\infty\right)$ with

$$\rho_R(x) = \begin{cases} C'\,r(x)^\gamma & \text{if } \gamma \geq -2 \\ C'\,r(x)^{-2} & \text{if } \gamma < -2 \end{cases}$$

for all $R>0$ large enough and some $C'>0$.

Remark 6.2 As it will be clear from the proof, we have a weighted Poincaré inequality on M if $\gamma \leq 0$ and a the weighted Poincaré inequality for functions with compact support in $M \setminus B_1(p)$ if $\gamma > 0$.

Proof We can find $\varphi \in \mathcal{A}$ given by

$$\varphi(r) = \begin{cases} \exp\left(B\, r^{1+\frac{\gamma}{2}}\right) & \text{if } \gamma > -2 \\ r^{\delta} & \text{if } \gamma = -2 \\ r & \text{if } \gamma < -2 \end{cases} \tag{16}$$

for r large enough, $B > 0$ small, $\delta = \delta(C) > 1$ such that $\mathrm{Ric}(\nabla r, \nabla r)(x) \leq -\frac{\varphi''(r(x))}{\varphi(r(x))}$. By the Laplacian comparison in a strong form, which is valid only on Cartan–Hadamard manifolds (see [18, Theorem 2.15]), one has

$$\Delta r(x) \geq \begin{cases} C\, r(x)^{\gamma/2} & \text{if } \gamma \geq -2 \\ C r(x)^{-1} & \text{if } \gamma < -2. \end{cases}$$

Suppose $\gamma \leq 0$ and let $\alpha := \max\{\gamma, -2\} \leq 0$. For any $u \in C_c^\infty(M)$, since $|\nabla r|^2 = 1$, we have

$$\begin{aligned} C \int_M r(y)^\alpha\, u(y)^2\, \mathrm{d}y & \\ \leq \int_M u(y)^2 r(y)^{\alpha/2} \Delta r(y)\, \mathrm{d}y & \\ = -2 \int_M \langle \nabla u, \nabla r \rangle u(y) r(y)^{\alpha/2}\, \mathrm{d}y + \frac{\alpha}{2} \int_M u(y)^2 r(y)^{\alpha/2-1} |\nabla r(y)|^2\, \mathrm{d}y & \\ \leq 2 \int_M |u(y)||\nabla u(y)| r(y)^{\alpha/2}\, \mathrm{d}y & \\ \leq \frac{C}{2} \int_M r(y)^\alpha\, u(y)^2\, \mathrm{d}y + \frac{2}{C} \int_M |\nabla u(y)|^2\, \mathrm{d}y. & \end{aligned}$$

Thus,

$$\int_M r(y)^\alpha\, u(y)^2\, \mathrm{d}y \leq \frac{4}{C^2} \int_M |\nabla u(y)|^2\, \mathrm{d}y$$

and the weighted Poincaré inequality on M follows in this case.

Suppose now $\gamma > 0$. By a Barta-type argument (see, e.g., [7, Theorem 11.17]),

$$\lambda_1(M \setminus B_R(p)) \geq [C R^{\frac{\gamma}{2}}]^2 \quad \text{in } \; M \setminus B_R(p).$$

Thus, the Poincaré inequality reads

$$C R^\gamma \int_M u(y)^2\, \mathrm{d}y \leq \int_M |\nabla u(y)|^2\, \mathrm{d}y \tag{17}$$

for any u with compact support in $M \setminus B_R(p)$. Now let $R > 1$ and, for every $k \in \mathbb{N}$, define the cutoff functions

$$\varphi_k(x) := \begin{cases} r(x) - k + 1, & r(x) \in [k-1, k) \\ k + 1 - r(x), & r(x) \in [k, k+1) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $|\nabla \varphi_k| \leq 1$ and for all $x \in M \setminus B_1(p)$, $\sum_k \varphi_k = 1$ and $x \in \operatorname{supp}\varphi_k$ at most for two integers k. If $\operatorname{supp} u \subset M \setminus B_1(p)$, we have

$$\begin{aligned} \int_M r(y)^\gamma \, u(y)^2 \, \mathrm{d}y &= \int_M r(y)^\gamma \left(\sum_k \varphi_k(y) u(y) \right)^2 \mathrm{d}y \\ &\leq 2 \sum_k \int_M r(y)^\gamma \, \varphi_k(y)^2 u(y)^2 \, \mathrm{d}y \\ &\leq C \sum_k (k-1)^\gamma \int_M \varphi_k(y)^2 u(y)^2 \, \mathrm{d}y \\ &\leq C \sum_k \int_M |\nabla \big(\varphi_k(y) u(y)\big)|^2 \, \mathrm{d}y, \end{aligned}$$

where in the last passage we used (17) with $R = k - 1$. Thus,

$$\begin{aligned} \int_M r(y)^\gamma \, u(y)^2 \, \mathrm{d}y &\leq C \sum_k \left(\int_M u(y)^2 |\nabla \varphi_k(y)|^2 \, \mathrm{d}y + \int_M \varphi_k(y)^2 |\nabla u(y)|^2 \, \mathrm{d}y \right) \\ &\leq C \int_M u(y)^2 \, \mathrm{d}y + C \int_M |\nabla u(y)|^2 \, \mathrm{d}y \\ &\leq C \int_M |\nabla u(y)|^2 \, \mathrm{d}y, \end{aligned}$$

where in the last passage we used (17) with $R = 1$. Hence, the weighted Poincaré inequality holds for functions with support in $M \setminus B_1(p)$.

Finally, the completeness of the metric $g_{\rho_R} := \rho_R \, g$ follows. In fact, for any curve $\eta(s)$ parametrized by arclength with $0 \leq s \leq T$, the length of η with respect tp g_{ρ_R} is given by

$$\int_\eta \sqrt{\rho_R} \, \mathrm{d}s \to \infty \quad \text{as } T \to \infty.$$

□

Let us write some estimates which will be useful both in the proof of Corollary 1.4 and in Sect. 6.2. Choose $\varphi \in \mathcal{A}$ as in (16) with $\gamma = \gamma_1$ obtaining

$$\frac{\varphi'(r(x))}{\varphi(r(x))} = \begin{cases} C \, r(x)^{\gamma_1/2} & \text{if } \gamma_1 \geq -2 \\ C \, r(x)^{-1} & \text{if } \gamma_1 < -2 \end{cases}$$

and

$$\frac{\varphi''(r(x))}{\varphi(r(x))} = \begin{cases} C \, r(x)^{\gamma_1} + C' r(x)^{\gamma_1/2-1} & \text{if } \gamma_1 \geq -2 \\ 0 & \text{if } \gamma_1 < -2 \end{cases}$$

for $r(x) > R > 1$. A simple computation shows that for $R = r(x)/4$, one has

$$K_R(x) = \begin{cases} C\, r(x)^{\gamma_1/2} & \text{if } \gamma_1 \geq -2 \\ 0 & \text{if } \gamma_1 < -2, \end{cases}$$
$$\frac{I_R(x)}{R} = \begin{cases} C\, r(x)^{\gamma_1/2-1} \coth\left(C' r(x)^{\gamma_1/2+1}\right) & \text{if } \gamma_1 \geq -2 \\ \frac{2}{r(x)^2} & \text{if } \gamma_1 < -2 \end{cases}$$

and

$$Q_R(x) = \begin{cases} C\, r(x)^{\gamma_1} & \text{if } \gamma_1 \geq -2 \\ \frac{2}{r(x)^2} & \text{if } \gamma_1 < -2. \end{cases}$$

Thus,

$$\omega(r) = \begin{cases} C\, r^{\gamma_1/2+1} & \text{if } \gamma_1 \geq -2 \\ C \log r & \text{if } \gamma_1 < -2, \end{cases}$$

and, as $m \to \infty$,

$$\omega(m+1) - \omega(m) \sim \begin{cases} C\, m^{\gamma_1/2} & \text{if } \gamma_1 \geq -2 \\ C m^{-1} & \text{if } \gamma_1 < -2. \end{cases} \tag{18}$$

On the other hand, using Lemma 6.1 with $\gamma = \gamma_2$, we get the estimate

$$\sup_{M \setminus B_m(p)} \frac{1}{\rho_m} \leq \begin{cases} C\, m^{-\gamma_2} & \text{if } \gamma_2 \geq -2 \\ C\, m^2 & \text{if } \gamma_2 < -2 \end{cases}.$$

Proof of Corollary 1.4 For $\gamma_1 \geq \gamma_2$ and $\gamma_1 \geq 0$, we get

$$\sum_m^\infty \left(\omega(m+1) - \omega(m) + 1\right) \sup_{M \setminus B_m(p)} \frac{|f|}{\rho_m} \leq \begin{cases} C \sum_m^\infty m^{\gamma_1/2-\gamma_2-\alpha} & \text{if } \gamma_2 \geq -2 \\ C \sum_m^\infty m^{2+\gamma_1/2-\alpha} & \text{if } \gamma_2 < -2. \end{cases}$$

and the thesis immediately follows. □

6.2 Optimality on rotationally symmetric manifolds

We show that the assumptions in Theorem 1.2 are sharp on model manifolds. Let (M, g) be a rotationally symmetric manifold with $\varphi \in \mathcal{A}$ defined as in (16) for any $r > 1$. One has

$$\int_M G(x, y) f(y)\, \mathrm{d}y < \infty \qquad \text{for any } x \in M \quad \Longleftrightarrow \quad \int_M G(p, y) f(y)\, \mathrm{d}y < \infty.$$

Hence, a solution of $-\Delta u = f$ in M exists if and only if

$$u(p) = \int_0^\infty \left(\int_r^\infty \frac{1}{\varphi(t)^{n-1}} \mathrm{d}t \right) f(r)\, \varphi(r)^{n-1}\, \mathrm{d}r < \infty.$$

Case 1: $\gamma > -2$. With our choice of φ, by the change of variable $s = t^{1+\frac{\gamma}{2}}$, it is easily seen that for any $r > 0$ sufficiently large

$$\int_r^{\infty} \frac{1}{\varphi(t)^{n-1}} \mathrm{d}t \sim C r^{-\frac{\gamma}{2}} \exp\left(-(n-1) r^{1+\frac{\gamma}{2}}\right). \tag{19}$$

Hence,

$$\frac{1}{C} \int_1^{\infty} r^{-\frac{\gamma}{2}} \exp\left(-(n-1) r^{1+\frac{\gamma}{2}}\right) \frac{1}{\left(1+r\right)^{\alpha}} \exp\left((n-1) r^{1+\frac{\gamma}{2}}\right) \mathrm{d}r \leq |u(p)|$$
$$\leq C \int_1^{\infty} r^{-\frac{\gamma}{2}} \exp\left(-(n-1) r^{1+\frac{\gamma}{2}}\right) \frac{1}{\left(1+r\right)^{\alpha}} \exp\left((n-1) r^{1+\frac{\gamma}{2}}\right) \mathrm{d}r$$

Therefore,

$$\frac{1}{C} \int_1^{\infty} \frac{1}{r^{\alpha+\frac{\gamma}{2}}} \mathrm{d}r \leq |u(p)| \leq C \int_1^{\infty} \frac{1}{r^{\alpha+\frac{\gamma}{2}}} \mathrm{d}r.$$

This yields that

$$|u(p)| < \infty \quad \text{ if and only if } \quad \alpha > 1 - \frac{\gamma}{2}.$$

On the other hand, a direct computation, using (19), shows that

$$\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)} \sim C r(x)^{\gamma}.$$

Furthermore, from (18), the assumption of Theorem 1.2 is satisfied if and only if

$$\alpha > 1 - \frac{\gamma}{2},$$

and the optimality follows in this case.

Case 2: $\gamma = -2$. We have,

$$\int_r^{\infty} \frac{1}{\varphi(t)^{n-1}} \mathrm{d}t = C\, r^{-\delta(n-1)+1}. \tag{20}$$

Thus,

$$\frac{1}{C} \int_1^{\infty} r^{-\delta(n-1)+1} \frac{1}{\left(1+r\right)^{\alpha}} r^{\delta(n-1)} \mathrm{d}r \leq |u(p)| \leq C \int_1^{\infty} r^{-\delta(n-1)+1} \frac{1}{\left(1+r\right)^{\alpha}} r^{\delta(n-1)} \mathrm{d}r$$

Therefore,

$$\frac{1}{C} \int_1^{\infty} \frac{1}{r^{\alpha-1}} \mathrm{d}r \leq |u(p)| \leq C \int_1^{\infty} \frac{1}{r^{\alpha-1}} \mathrm{d}r,$$

and

$$|u(p)| < \infty \quad \text{ if and only if } \quad \alpha > 2.$$

On the other hand, a direct computation, using (20), shows that

$$\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)} \sim Cr(x)^{-2}.$$

Furthermore, from (18), the assumption of Theorem 1.2 is satisfied if and only if

$$\alpha > 2,$$

and the optimality follows in this case.

Case 3: $\gamma < -2$. We have,

$$\int_r^\infty \frac{1}{\varphi(t)^{n-1}} \mathrm{d}t = C\, r^{2-n}. \tag{21}$$

Thus,

$$\frac{1}{C}\int_1^\infty r^{2-n} \frac{1}{\left(1+r\right)^\alpha} r^{n-1}\,\mathrm{d}r \le |u(p)| \le C\int_1^\infty r^{2-n} \frac{1}{\left(1+r\right)^\alpha} r^{n-1}\,\mathrm{d}r$$

Therefore,

$$\frac{1}{C}\int_1^\infty \frac{1}{r^{\alpha-1}}\,\mathrm{d}r \le |u(p)| \le C\int_1^\infty \frac{1}{r^{\alpha-1}}\,\mathrm{d}r,$$

and

$$|u(p)| < \infty \quad \text{if and only if} \quad \alpha > 2.$$

On the other hand, a direct computation, using (21), shows that

$$\rho(x) = \frac{|\nabla G(p,x)|^2}{4G^2(p,x)} \sim Cr(x)^{-2}.$$

Furthermore, from (18), the assumption of Theorem 1.2 is satisfied if and only if

$$\alpha > 2,$$

and the optimality follows in this last case.

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