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Mathematics meets the fashion industry on path to product innovation and sustainability

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Review

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Mathematics remains an endless source of inspiration for both the conception and the fabrication of clothes and fashion products. Over the last century, the prominent role of mathematics in the fashion industry has evolved together with the progress in computational resources and fabrication technologies. Nowadays, mathematics not only provides useful theoretical and computational tools to assist design and fabrication, but also acts as a catalyst of creativeness. This survey discusses the latest advances on mathematical models and methods in the fashion industry, restricting our attention to clothing design and multi-scale modelling, and their applications for digital manufacturing of textiles and garments. Particular focus is given to the usage of mathematics to drive product innovation and to reduce the environmental impact of the fashion industry in order to meet the Sustainable Development Goals of the United Nations.

1. Introduction

Mathematics has always been an endless source of inspiration for both the conception and the fabrication of clothes and fashion products. For example, the systematic ability of geometry to quantify the arrangements and the interrelationship of shapes and patterns in space has especially proved crucial to complement the creative process in clothing. In the eighteenth century, outstanding mathematicians, such as Euler [1] and Monge [2], solved the challenge to draw useful mappings of curved surfaces leading to fundamental results and techniques of differential geometry. These mathematical

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principles immediately found applications in fashion related problems, with the pioneering contributions of Chebyshev [3] and Bianchi [4], analysing how to design woven fabrics in order to wrap onto a generic surface.

These seminal works have established the prominent role of mathematics in the fashion industry, that has evolved together with the progress in computational resources and fabrication technologies. Nowadays, mathematics not only provides useful theoretical and computational tools to assist design and fabrication, but also acts as a catalyst of new conceptual ideas.

In the following, we provide a state of the art and a survey on the latest advances in mathematical models and methods in the fashion industry, restricting our attention to clothing design and its applications for digital manufacturing of textiles and garments. Particular focus will be given to the usage of mathematics to drive product innovation and to reduce the environmental impact of the fashion industry [5], in order to meet the Sustainable Development Goals of the United Nations.[1]

This article is organized into three sections. Section 2 deals with the mathematics of clothing accounting for purely geometric models of cloth motion, as well as for mechanics-based approaches in the context of virtual simulation of garments and multi-scale modelling of textiles. Section 3 deals with mathematics-inspired design of fabrics and clothes. This part includes a discussion of clothes style and patterns followed by some examples of iconic fashion collections inspired by mathematical concepts. Finally, §4 examines the role played by physics-based mathematical modelling for digital manufacturing of garments and smart textiles.

2. Mathematics of clothing

Garments and fashion accessories can be seen as surfaces wrapped onto the three-dimensional domain occupied by the human body. The wrapping morphology depends on a number of physical interactions including gravitational, elastic, contact and self-collision forces. Thus, *clothing* is intimately related to the mathematical branches of differential geometry and continuum mechanics. In this section, we will review various mathematical theories and simulation tools that have been developed to predict the mechanical behaviour of clothes and fashion items.

(a) Clothes as developable surfaces

Most textiles and fabrics can easily deform out-of-plane but are constrained not to deform in-plane. In mathematical terms, they can be effectively modelled as developable surfaces, i.e. they can be flattened onto a plane without any stretching. Under this hypothesis, it is possible to decompose any garment as the union of patches that can be unfolded onto a plane without distortion. This methodology is generally employed in fashion design software to define the geometry of sewing patterns from three-dimensional garment models [6]. Since the mathematical characterization of a developable surface is not unique, different modelling approaches have been proposed. A first approach is based on *ruled* surfaces. The latter are generated by sweeping a straight line, called ruling, along a curve, called directrix. Mathematically, a ruled surface is defined through the parametrization $\boldsymbol{x}(v,t) = \boldsymbol{\alpha}(t) + v\,\boldsymbol{w}(t)$, where $\boldsymbol{\alpha}(t)$ is the directrix while $\boldsymbol{w}(t)$ is the direction of the rulings. Such a surface is developable if the condition $(\boldsymbol{w}' \times \boldsymbol{\alpha}') \cdot \boldsymbol{w} = \boldsymbol{0}$ holds [7]. Examples of ruled developable surfaces, referred to as *warped*, are cylinders, cones, tangent surfaces and union of pieces of these three. Using this definition, continuous developable surfaces can be generated by approximating a directrix using Bézier or B-spline polynomials [8,9]. In such a scenario, a developability condition can be imposed analytically resulting in a nonlinear problem for the control points of the boundary curves. Interacting modelling tools for generating smooth developables were also proposed based on this approach [10]. However, despite their mathematical elegance, the resulting nonlinearities have generally prevented continuous models from being used in industrial design.

[1] https://unfashionalliance.org/.

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More efficient approaches rely on discrete approximations of developables, generally using polygonal meshes. A widespread methodology is based on constructing discrete ruled surfaces by *boundary triangulation* [11]. Operationally, the boundary curve is approximated with line segments and the surface with triangles whose vertices are the boundary points. Differently from continuous methods, the developability constraint results in a geometric condition that can be readily imposed. Mitani & Suzuki [12] demonstrated that discrete developables based on boundary triangulation can be used for approximating complex surfaces by the union of developable strips. Rose *et al.* [13] extended the use of boundary triangulation to automatic generation of developables on arbitrary sketched boundaries. This algorithm was employed for the development of a interactive design tool for garments. Discrete developables can also be constructed using different methods. A considerable number of works focused on transforming a given (non-developable) polygonal surface into a developable one with minimum distortion from the original shape. Such a transformation is guided by a particular selection of the mathematical condition for developability. For instance, developable approximation of discrete surfaces can be obtained by modifying the position of the vertices such that a quasi-vanishing Gaussian curvature is obtained [14,15]. Alternative approaches apply the condition that a developable surface has a one-dimensional Gauss image, i.e it is a network of curves in the Gauss sphere [16]. Based on this definition, the recent work of Binninger *et al.* [17] proposed a novel algorithm based on iterative deformation of a given surface by 'thinning' its Gauss image.

Other methods for generating discrete developables were proposed in the realm of the so-called *discrete differential geometry* [18]. Here, a discrete counterpart of a smooth parametrization is obtained by representing the surface through quadrilateral meshes, also known as *nets*. In this way, local isometry reduces to geometrical constraints on the discrete mesh. Similarly to smooth manifolds, a given surface can be represented by different discrete nets. Therefore, various nets have been proposed due to special geometrical properties. For instance, Liu *et al.* [19] modelled developable surfaces via conjugate nets consisting of rulings and their conjugate directions. Furthermore, developable manifolds have been modelled using discrete orthogonal geodesic nets [20], discrete parallel geodesic coordinates [21] and discrete isometric mappings [22]. These methods are generally more flexible than the ones based on conjugated developable nets since rulings are not directly encoded into the mesh.

(b) Models for deformable clothes

Modelling garments as isotropic inextensible manifolds might be too restrictive for various fabrics. Some level of deformation is indeed possible depending on the specific fabric morphology. For instance, knitted fabrics readily adapt to a given surface as their fibres can undergo significant stretching. Local deformation in woven fabrics is instead granted by the alteration of the angle between warp and weft threads. The first attempt to model cloth deformation dates back to the work of Chebyshev [3]. He was interested in determining the possibility of woven fabrics to conform to a generic surface. To this end, Chebyshev developed a purely geometrical model where the fabric was idealized as a network of orthogonal inextensible fibres. Such a surface model was later referred to as *Chebyshev net*. By selecting the surface coordinates u and v as the fibres directions, a smooth parametrization $\boldsymbol{r}(u,v):\mathbb{R}^2\to\mathbb{R}^3$ is a Chebyshev net if $\boldsymbol{r}_u\cdot\boldsymbol{r}_u=1$ and $\boldsymbol{r}_v\cdot\boldsymbol{r}_v=1$ (subscripts denote partial differentiation). The metric properties of such a surface are determined by the first fundamental form

$$\mathrm{d}s^2=\mathrm{d}u^2+2\sin\gamma\,\mathrm{d}u\,\mathrm{d}v+\mathrm{d}v^2,$$

where ds is a line element on the surface while γ refers to the deformed angle between fibres. The Gaussian curvature K is linked to the angle between fibres through the differential equation

$$\phi_{uv}+K(u,v)\sin\gamma=0. \tag{2.1}$$

Note that in the case of a constant Gaussian curvature, equation (2.1) reduces to the well-known Sine-Gordon equation. In practice, the angle between fibres has to change in order to cover

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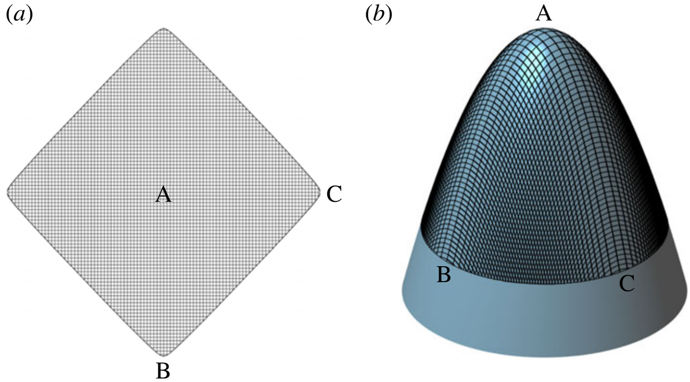

Figure 1. Plot of the solution of the clothing problem (2.2) for a paraboloidal obstacle of given height: resulting shape of the initially plane Chebyshev net (*a*) and deformation of the textile applied to the obstacle (*b*). Capital letters A, B and C mark corresponding points in the undeformed and deformed textile.

surfaces that are not developable (see figure 1). Chebyshev was able to find the shape of a net covering a hemisphere by numerical approximation of equation (2.1). The problem for the full sphere was solved only recently by Ghys [23], highlighting the existence of a singularity of the cusp type. The existence of Chebyshev nets were extensively studied in mathematical literature. On one hand, Bianchi [4] and Bieberbach [24] proved the local existence of Chebyshev net clothing any regular surface. On the other hand, sufficient conditions for the global existence are much more difficult to work out. In this context, Hazzidakis [25] showed that a rectangular Chebyshev patch can fit a given regular surface as long as the magnitude of its total Gaussian curvature is less than 2π. Despite this limit, Voss [26] showed that smooth Chebyshev patches can be generated on surfaces of revolutions that do not intersect their rotation axis. Other recent contributions regarding the global existence of Chebyshev nets include [27,28].

A rigorous approach for the problem of clothing a generic surface with an initially plane Chebyshev net was proposed by Servant [29,30]. Given a surface, specified by its parametrization $\boldsymbol{r}(x, y)$, the clothing problem reduces to express x and y as a function of the coordinates of a Chebyshev net, i.e. $x = x(u, v)$ and $y = y(u, v)$. Accordingly, the inextensibility condition for the fibres reads

$$\left.\begin{aligned} E\,x_u^2 + 2F\,x_u y_u + G\,y_u^2 &= 1 \\ \text{and} \qquad E\,x_v^2 + 2F\,x_v y_v + G\,y_v^2 &= 1, \end{aligned}\right\} \tag{2.2}$$

where $E = \boldsymbol{r}_x \cdot \boldsymbol{r}_x$, $F = \boldsymbol{r}_x \cdot \boldsymbol{r}_y$ and $G = \boldsymbol{r}_y \cdot \boldsymbol{r}_y$ are the given coefficients of the first fundamental form of the target surface. Servant's equations are then obtained by cross differentiation of equation (2.2), resulting in a second order quasi-linear hyperbolic system with the fibres as characteristics. Boundary conditions can be either specified along two intersecting fibres or along a generic curve. It should be noted that, even in the case of a regular surface, solutions of the Servant's equations might present singularities due to hyperbolic behaviour. A pathological case is found when $\gamma \to 0$, i.e. in correspondence to an envelope of fibres where the fabric turns back on itself. This case is representative of collapsed regions where the solution of Servant's equation fails to cover completely the target surface. Moreover, discontinuities across characteristics can frequently arise. The latter case is connected with the appearance of folds in the fabric.

Practical design of Chebyshev nets is largely affected by the regularity of the target surface. Analytical solutions for the clothing problem (2.2) can only be obtained for simple smooth manifolds, such as surfaces of revolutions [31,32]. Nevertheless, even in the case of regular obstacles, the solution of the clothing problem generally involves singularities [23]. The situation gets even worse in the general context of piecewise smooth surfaces. When singularities appear, the covering Chebyshev net is typically built by the union of individual patches sewn together. Aono *et al.* [33] proposed an algorithm for generating Chebyshev nets on arbitrary smooth

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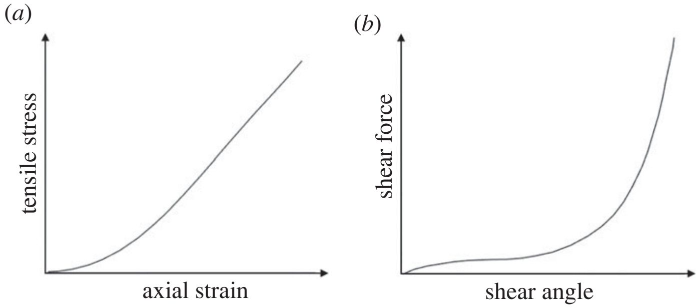

Figure 2. Plots of the typical stress/strain response of woven fabrics subjected to (*a*) tensile deformation along the yarn directions and (*b*) shear deformation.

obstacles. The method couples a numerical solver for the solution of the Servant's equations with an algorithm for preventing anomalies (e.g. gaps and wrinkles) by the insertion of darts (i.e. folds stitched into the material). Alternative numerical strategies for the problem of mapping a Chebyshev net on arbitrary surfaces were proposed using the finite-element method [34]. The clothing problem was also studied on generic manifolds using *discrete* Chebyshev nets [35]. The latter is a discrete analogue of a smooth Chebyshev net consisting of a quadrangular mesh formed by inextensible edges. In this context, Garg *et al.* [36] presented a computational approach for the clothing of a given surface. The method integrates shape optimization tools and allows for the interactive insertion of darts by the user. Finally, an algorithm for the generation of free-form surfaces composed by Chebyshev net with singularities placed automatically was proposed in [37].

Studying the clothing problem through a purely geometrical model might give incorrect results in many practical situations. First, an erroneous prediction of collapsed regions can arise from the fact that the aforementioned model does not include any shear resistance. Second, geometrical models work well as long as the fibres are subjected to tensile stresses. Indeed, the presence of compressive loads along fibres might give rise to out-of plane bucking. The appearance of wrinkles is then related to the bending stiffness of the cloth. Therefore, rigorous modelling of cloth deformation, including the analysis of instabilities, shall include the formulation of suitable constitutive laws describing the stress/strain behaviour of textiles. Generally, the response of textiles and fabrics is highly anisotropic and nonlinear. It is also characterized by large in-plane deformations and low-bending stiffness. Inelastic deformations are frequently observed even for small stresses and at room temperature [38]. The mechanical characterization of fabrics can be pursued through experimental tests. For instance, the *Kawabata Evaluation System* [39] provides a well-established texting procedure for textile mechanics. A qualitative stress/stress response of woven fabrics is reported in figure 2 for in-plane tensile and shear deformations. Because of *decrimping* phenomena, an increasing tensile stiffness is observed for increasing strain along the yarn directions. As a result, the tensile stiffness of the fabric is generally lower than the constituting yarns form small strains. The shear stiffness is relatively small in the first part of the curve. Subsequently, as the shear angle becomes larger, the shear rigidity increases significantly. Large shear angles might imply yarn contacts potentially leading to wrinkling effects known as 'shear-locking' [40].

Rivlin [41] formulated the equilibrium of a Chebyshev net subjected to in-plane deformation. Such a formulation was later extended to the general deformation of a curved net by application of forces [41]. In both cases, the mechanics was considered only in terms of tension across fibres. The effect of shear resistance was later included in [42,43] for both planar and general deformations. In particular, a comprehensive analysis of possible singularities arising from the solution of the resulting boundary value problem was discussed therein. Subsequently, general theoretical frameworks for a network of inextensible fibres including bending [44] and twisting [45] were also proposed. For completeness, it should be mentioned that similar models were also developed for elastic nets, i.e. by getting rid of the characteristic fibre inextensibility of Chebyshev nets (see [46,47]).

(c) Virtual simulations of textiles and garments

A cornerstone in the digitalization of fashion industry is the ability to reproduce the behaviour of clothes in operational conditions. Virtual garment simulations are at the core of the emerging virtual-try-on technology, in which customers can virtually try garments in a digital environment. The final goal is animating the shape of garments worn by a digital mannequin including aesthetic details such as drapes, folds and wrinkles. Accurate and reliable virtual modelling is a challenging task since it involves complex interactions between garment deformation, mannequin motion and external forces. Some of the main difficulties are summarized next. First, fabric is a deformable solid with a relatively small flexural rigidity compared to the in-plane stiffness. As a result, garments undergo large displacements with the formation of peculiar drapes and wrinkles triggered by out-of-plane instability. Second, the resulting garment shape depends on the application of different forces such as gravity, contact with the wearer, self-collisions and frictions. Among them, modelling repulsive forces due to cloth–cloth collisions and preventing cloth from interpenetrating solid objects require involved mathematical treatments. Finally, suitable models for virtual-try-on should be computationally fast in order for the customer to perform real-time interactive cloth manipulation.

Early approaches for cloth animation employed *physics-based continuous models* in which the fabric is idealized as an elastic surface moving in a three-dimensional space. Continuous models allow one to accurately describe cloth mechanics by the usage of engineering properties such as elastic modulus, fabric thickness and density. In their seminal paper, Terzopoulos *et al.* [48] demonstrated the potential of continuum mechanics to properly simulate cloth motion subjected to gravitational forces and collisions with impenetrable objects. Subsequently, a number of publications investigated the application of rigorous continuous descriptions based on theories of elastic plates and shells [49–52]. These works employed the finite-element method to numerically solve the resulting mechanical problem. Since the computational time of continuous models is generally high, numerical simulations were limited to the draping of relatively simple cloth patches. Continuous models were soon abandoned in favour of faster to solve *discrete models*. However, the recent rise of *isogeometric analysis* has opened new perspectives in continuum clothing animations [53]. As an alternative to the finite-element method, isogeometric analysis is based on NURBS (non-uniform rational B-spline), a standard technology used in CAD systems. This numerical method is, therefore, prone to be directly coupled with existing CAD methods for the design of garments.

Examples of discrete models are the so-called *particle methods* and *mass-spring systems*. The former category idealizes fabrics as a collection of particles that are subjected to specific forces that model internal cloth structure and the surrounding environment. Different approaches have been adopted for modelling particle interactions and then computing cloth motion. Breen *et al.* [54] treated cloth deformation as an energy minimization problem with total energy function expressed as

$$U = U_{\text{coll}} + U_{\text{gra}} + U_{\text{stretch}} + U_{\text{bend}} + U_{\text{shear}}, \tag{2.3}$$

where U_{coll} is a collision energy and U_{gra} is the gravitational potential, while $U_{\text{stretch}} + U_{\text{bend}} + U_{\text{shear}}$ accounts for cloth deformation in terms of stretching, bending and shearing. Realistic expression for the energy of deformation can be estimated from empirical data based on the Kabawata testing procedure. Alternatively, Eberhardt *et al.* [55] employed the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial x_i}, \tag{2.4}$$

to determine the trajectories of particles, where v_i and x_i refer to velocity and position of the ith particle, respectively, while $L = E_{\text{kin}} - V$ is the Lagrange function with V being a cloth-specific potential and E_{kin} the kinetic energy of the system. The resulting mathematical problem of particle methods consists of a system of ordinary differential equations in time which is solved by numerical approximation. The usage of explicit integration schemes generally requires very small time steps because of the poor conditioning of the problem. To overcome this limitation, Baraff & Witkin [56] proposed a numerical method based on the combination of implicit integration and the usage of penalty forces to handle cloth–cloth collisions.

Differently from particle methods, mass-spring systems idealize cloth by means of virtual masses, each of which is connected to its neighbours by elastic springs [57]. To describe fabric deformation, three kinds of connective springs were generally considered: structural, shear and flexural. The dynamics of cloth is governed by Newton's second law $\boldsymbol{F}_i = m_i \boldsymbol{a}_i$, with $\boldsymbol{F}_i$ being the total force, m_i the mass and $\boldsymbol{a}_i$ the acceleration of each virtual mass i. The applied force splits between internal and external contributions, $\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{int}} + \boldsymbol{F}_i^{\text{ext}}$. Internal forces account for the resulting tension of the connected springs. External contributions collect forces applied to the garment as well as self-collisions and interactions with other objects. Due to their reasonable trade-off between accuracy and simulation speed, mass-spring models have been extensively adopted and enhanced in different aspects. Traditional linear-elastic models for the mechanics of fabrics have been replaced with more realistic constitutive laws. For instance, Wang *et al.* [58] modelled the typical nonlinear behaviour of fibres using a data-driven piecewise linear approximation while Volino *et al.* [59] used the Saint Venant–Kirchoff hyperelastic law to model in-plane cloth mechanics. Significant efforts have also been made in deriving more accurate and robust treatments for collisions, contact and frictions. This specific topic is crucial for simulation speed since it can take most of the computational time. In addition, the prediction of realistic fabric drapes and wrinkles is largely affected by the accuracy of contact and collision handling. In this regard, Bridson *et al.* [60] derived an efficient model based on the combination of fail-safe geometric collision and a fast repulsion force method. A novel algorithm for incremental collision detection based on spatial hashing coupled with a GPU-based nonlinear impact zone solver has been recently proposed by Tang *et al.* [61]. An alternative fast numerical method for solving the dynamics of mass-spring systems was presented by Liu *et al.* [62]. The latter made use of a coordinate block descent method coupled with a nonlinear Newton solver.

Besides physics-based continuum and discrete methods, cloth simulations are also carried out using *geometrical models*. These approaches rely on a purely geometrical description of cloth instead of incorporating fundamental physics equations. The computational load is, therefore, much lower than physics-based models at the expense of accuracy due to lack of physical background. Garments are idealized as the union of cylindrical patches in the geometrical model for cloth folds proposed by Decaudin *et al.* [16]. The formation of folds was described by means of the 'buckling mesh' procedure, a solution strategy that combines the three characteristic buckling modes of cylinders: vertical axis align folds, diamond folds pattern and twist buckling. Finally, Chen & Tang [63] formulated an alternative geometrical approach by modelling cloth deformation through a minimization problem including collision handling.

(d) Multi-scale modelling of textiles

The composite industry has in the last decades seen a significant advancement in the mechanical modelling of textiles [64]. Textile reinforced composites are structural materials combining a matrix, usually a polymer, and a textile-based reinforcement, typically made of carbon/glass fibres. Their manufacturing is characterized by a two-step process. First, a planar textile patch is deformed into the target shape via the so-called *forming* or *preform* stage. Second, a liquid resin is injected within the reinforcement and later consolidated at high temperature. In particular, the forming process not only determines the fibres alignment of the reinforcement but may also drive the formation of flaws and defects in the composite. In order to improve the fabrication processes, a large amount of published works and software focused on the optimization of the forming, as well as the mechanical characterization of textiles at different scales [65].

The mechanical response of engineering textiles is inherently *multi-scale*, since it involves an interplay of (i) macro-, (ii) meso- and (iii) micro-scale effects (see figure 3).

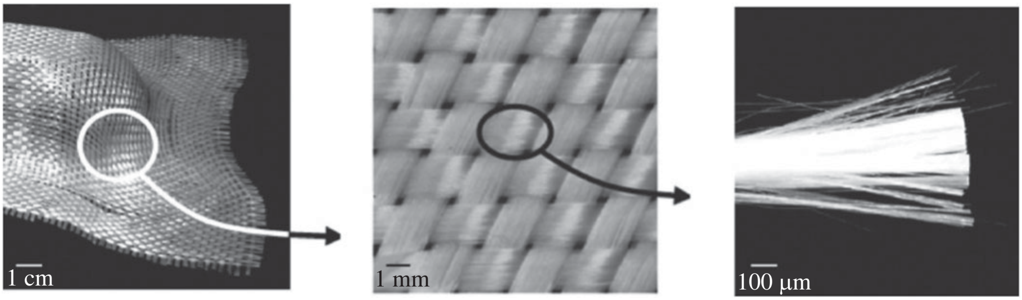

Figure 3. Sketch of the multi-scale composition of an engineering textile (adapted from [65]).

The *macro-scale* is the composite part whereby the dry reinforcement can be described as a continuous medium. This description is generally preferred to predict textile draping and wrinkling in the preform stage [66]. Early approaches used purely geometrical approaches similar to those described in §2(b) (see [67,68] for instance). Subsequently, more accurate approaches were developed using solid shell theories to include the elastic response of textiles. In this regard, a number of constitutive laws—generally of hyperelastic [69,70] and hypoelastic [71,72] type—have been proposed to capture the peculiar mechanical behaviour of these engineering fabrics.

Meso-scale approaches aim at simulating the response of textiles at the intermediate scale of interlacing yarns [73]. The main benefit of this approach is the explicit description of the topology of a given textile (e.g. woven, knitted, braided). Meso-scale models can be classified into *large* meso- and *unit-cell* meso-models [74]. In the first category, the overall preform stage is simulated with full representation of textile yarns. Therefore, reinforcement deformation can be predicted precisely, especially at the locations characterized by high distortion, where continuous macro-scale models generally fail [75]. On the other hand, unit-cell meso-models are used to determine the homogenized properties of textiles through the simulation of a representative unit-cell of the textile [76].

The accuracy of meso-scale simulations depends mainly on the digital reconstruction of the textile structure and on the constitutive laws used for the yarns. A significant number of meso-scale models considers yarns as continuous media whose interlacing structure is generated by either *geometric* methods or *microscanning* reconstruction [77]. In geometric methods, virtual textiles are generated according to the given interlacing structure (typically of woven/knitted type) and yarn properties. More advanced geometric routines also account for the bending/compressive forces applied to the yarns in order to mimic the actual textile production [78,79]. Geometric methods offer the possibility of exploring novel interlacing structures and patterns in a fully digital environment [80]. This capability is nowadays implemented in a variety of commercial and open-source platforms such as *TexGen* [81] and *WiseTex* [82]. Alternatively, the fabric structure can be reconstructed from real composites through their microscanning [83]. This methodology has two main benefits: first, it is possible to reconstruct the exact meso-structure of a textile composite; second, the material parameters of the yarns can be calibrated via comparison between virtual tests and sample microscanning.

Discrete meso-scale models have also been proposed in order to reduce the computational burden of continuous meso-models. Here, threads are modelled using either elastic springs or one-dimensional structural elements such as trusses and beams [84–86]. Compared to continuum meso-models, discrete elements enhance the treatment of yarns–yarn contact thus avoiding spurious yarn interpenetration [87].

Micro-scale models consider textiles at the scale of a single constituting fibre [88–90]. In conventional and engineering textiles, each yarn consists indeed of a bundle made by micrometre

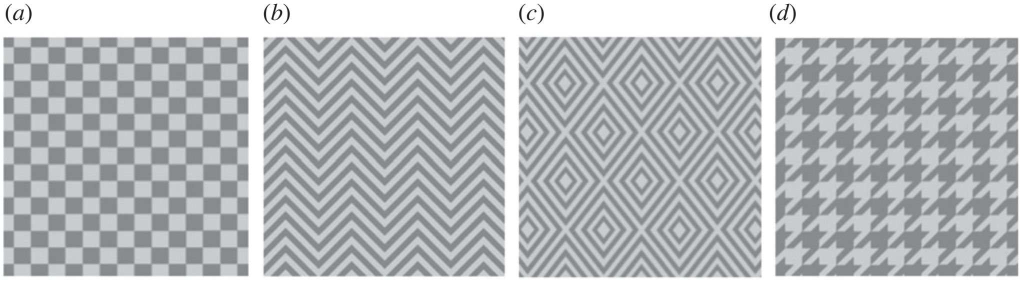

Figure 4. Some examples of geometric patterns arising from fabric weaving: (*a*) matt fabric, (*b*) herringbone fabric, (*c*) diamond twill fabric and (*d*) houndstooth fabric.

fibres. In this approach, fibres shall be distinguished in order to evaluate their relative motion within a yarn. The main difficulties in fibres modelling are related to their geometry, mechanical behaviour as well as their mutual contact/friction interactions. Numerical simulations at such small scales can be used to predict the geometry of the yarn cross-section as a function on the waving process, as well as the mechanical properties of a fibre bundle or a textile unit-cell. The potential of micro-scale simulations has also been exploited in virtual simulations of cloth. For instance, Kaldor *et al.* [91] proposed a yarn-level model for the animation of knitted cloth, simulating its mechanical response using different interlocking patterns. The treatment of collision was later enhanced by Otaduy and co-workers using a persistent contact method for both woven [92] and knitted fabrics [93].

3. Mathematics-inspired fashion style

Clothing can serve different functions in everyday life. Besides insulating humans, it has a remarkable social impact, being a sort of non-verbal communication which may identify many socioeconomic factors. These aspects are strictly connected to the clothing *style*, which encompasses a combination of aesthetic features, such as fabric type, design, fitting, colouring and patterning. This section deals with the mathematical principles proposed in the creative design of fabrics and clothes. It pertains to the so-called *mathematical art*, a practice in which artists and designers conceive visually attracting shapes and motifs inspired by mathematical concepts.

(a) Tilings and patterns for fabrics

Tilings and patterns are forms of surface decoration that are often used to make pleasant and visual attracting designs. Specifically, a tiling, or tessellation, is the covering of the plane or the space by closed shapes, called *tiles*, that have disjoint interiors. Patterns are created by the repetition of some *motifs* in a more or less systematic manner. Tilings and patterns originated very early in human history for architectural and artistic purposes. Such geometric decorations have also been extensively experimented in textile design. In this regard, tilings and patterns can be created in two different ways. First, the production of fabrics itself leads inevitably to the formation of a geometrical structure. Moreover, fabrics are often subjected to embellishment through colouring and drawing of geometrical shapes and motifs.

Some classic examples of tilings and patterns created through weaving are reported in figure 4. Textile fabrication through waving is generally performed with three distinct techniques: plain weave, satin and twill. Matt (figure 4*a*) is a variant of plain wave that creates a checkerboard effect similar to a tessellation by square tiles. Twill fabrics are characterized by diagonal lines as typically shown in herringbone and diamond twill fabrics (4*b*,*c*, respectively). Houndstooth fabric (known also as *pied-de-poule*) is a further example of classic tessellation created by weaving. In this case, the tiling is characterized by the assembly of equal tiles in the shape of broken checks.

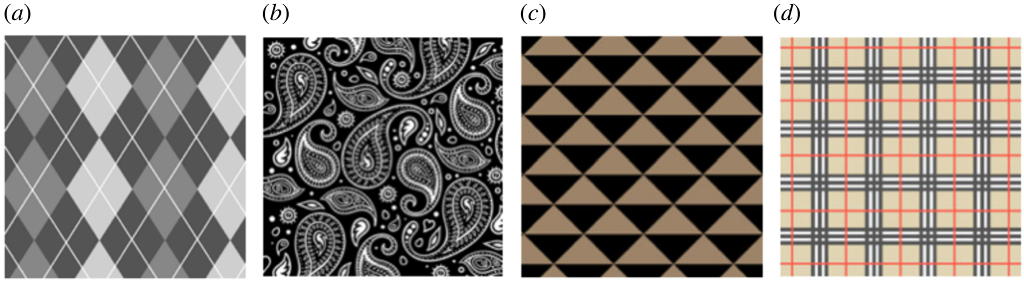

Figure 5. Some examples of textile tilings and patterns: (*a*) Argyle style, (*b*) Paisley style, (*c*) Prada pattern and (*d*) Burberry tartan pattern.

Figure 5 collects some iconic patterns and tailing styles in the fashion industry. The Argyle and Paisley styles (figure 5*a*,*b*) are traditional examples of the usage of tessellation and pattern concepts, respectively. The former reproduces a tessellation by diamonds. Paisley is Persian ornamental pattern created by the repetition of the *boteh*, a stylized teardrop shaped motif. Textile tilings and patterns are also used by fashion brands to mark their clothes. In this regard, Prada's triangle pattern and Burberry's tartan motif are among the most iconic and recognizable shapes in the fashion world (see figure 5*c*,*d*).

Despite the fact that tilings and patterns are well-established forms of art and design, the study of their mathematical properties is relatively recent. Extensive investigations on the subject started only at the beginning of the twentieth century. A comprehensive mathematical treatment of the topic is contained in the classic book by Grünbaum & Shephard [94]. During the last century, the beauty and mathematical structure of tilings and patterns stimulated the production of fascinating styles. Escher, Fathauer and Penrose are among the most notable tessellation artists and mathematicians. Some of their iconic creations inspired the production of innovative fabric styles as well. For instance, artist Sam Kerr collaborated with fashion designer Paul Smith and accessories brand Marwood to design tilings for T-shirts, ties, bow ties and pocket squares [95]. Creative tessellations can be designed by adopting a well-established methodology that consists in replacing the edges of polygonal tiles by suitable curves. An example of this geometric method is Escher's famous lizard, whose construction procedure is sketched in figure 6*a*. In this case, a lizard is obtained from a regular hexagon by cutting shapes of some sides of the hexagon that are subsequently rotated and reapplied to other sides. To ensure that the created shapes tessellate (as shown in figure 6*b*), precise geometrical rules must be followed. Using a similar strategy, Park [97] explored the use of this technique to create various patterns based on traditional Korean motifs.

It is also possible to tessellate non-Euclidean geometries, such as the hyperbolic plane. In the Euclidean plane the only regular edge-to-edge monohedral tessellations by regular polygons are the ones that have as prototiles an equilateral triangle, a square and a regular hexagon. If one gives up on the requirements of Euclidean geometry, many more options and styles are available. As a form of mathematical art, Escher created five works based on tessellations of the hyperbolic plane. A practical way to craft hyperbolic planes is by sewing curved edges polygons made of fabric. Heleman Ferguson, an American artist, used this technique to make a quilt that approximates a uniform pentapentagonal tiling of the hyperbolic plane (figure 6*c*). Crocheting is another building technique that can be used to create hyperbolic geometrical shapes as illustrated by mathematician Daina Taimina [98] (see figure 6*d*).

Mathematical theories can inspire the generation of alternative graphical shapes and motifs. For instance, Neves *et al.* [99] pioneered the usage of computer generated fractals for printed textile patterns. The term *fractal* refers to geometric motifs that can replicate their shapes at smaller, or larger, scales due to inherent self-similarity [100]. By the usage of fractal geometry it is possible to generate visual attracting motifs that resemble the shape of nature, that found many applications in the fashion industry [101]. At first, individual fractal motifs are generated using a

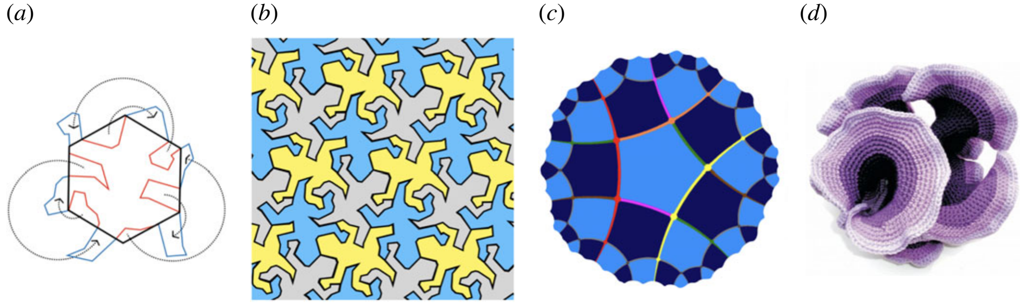

Figure 6. (*a*) Graphical construction of famous Escher's lizard tile out of a regular hexagon. (*b*) Reproduction of monohedral Escher tessellation by lizard shaped tiles. (*c*) Representation on the Poicaré disc of the pentapentagonal tessellation of the hyperbolic plane used by Heleman Ferguson for crafting an hyperbolic quilt (courtesy of Jeffrey R. Weeks[2]). (*d*) Hyperbolic geometry made via crocheting [96] (courtesy of Daina Taimina).

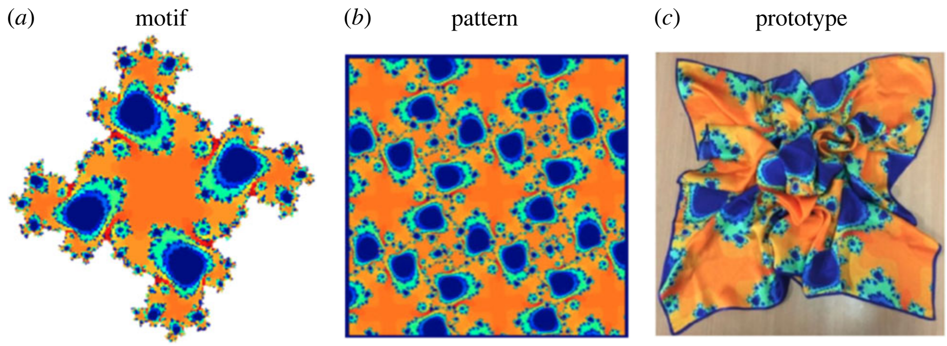

Figure 7. Scarf fractal pattern designed in [101]. (*a*) Computer generated fractal motif. (*b*) Fabric pattern obtained by the spatial repetition of individual motifs. (*c*) Picture of the three-dimensional configuration of a silk scarf prototype digitally printed with the obtained fractal pattern.

combination of fractal theory and computer graphic technology. Once a motif has been generated, a fabric pattern is created by repeating individual motifs in a structured manner. Finally the designed pattern is reported on fabrics by means of advanced printing technology. Figure 7 reports a fractal pattern generated in [101] to be printed on a silk scarf. Further contributions in the research of fractal art for textile design include [102,103].

Not surprisingly,[2] the beauty of fractal art for textile design was also researched by notable fashion designers. Starting from the early 1990s, American stylist Jhane Barnes made extensive use of fractal patterns generated by an in-house computer software developed by mathematician William Jones and Dana Cartwright [104]. Her creations demonstrated the capability of mathematical art to readily inspire an endless spectrum of colourful textile designs.

(b) Geometrical ideas for creative clothes

Besides textile embellishment through unconventional patterns, fashion designers have also devised creative garment shapes inspired by mathematics. In particular, some complex ideas developed in geometry and topology became sources of inspiration for designing garment forms and shapes. One of the most iconic examples is the collaboration between Japanese stylist Issey Miyake and American mathematician William Thurston in 2010 [105]. The latter gave a major

[2] www.ams.org/publicoutreach/ams-news-releases/thurston-miyake.

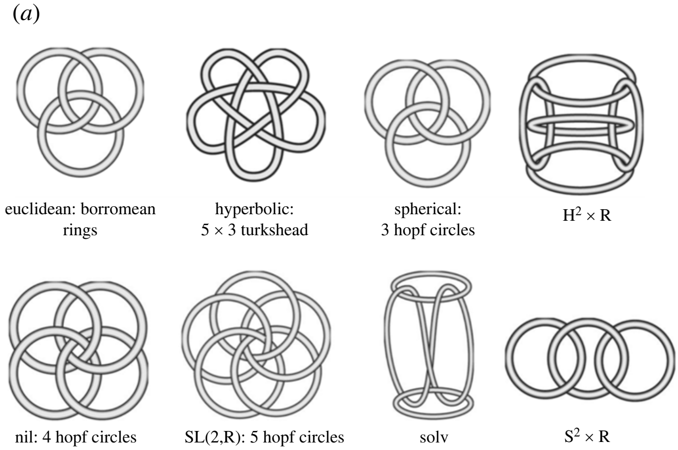

Figure 8. (*a*) Thurston's illustration of orbifolds representative of the eight geometries (Courtesy of Kelly Delp). (*b*) Issey Miyake's 2010 Autumn/Winter collection on the runway at Paris Fashion Week (Photograph by Frédérique Dumoulin).

breakthrough in the study of topology in two and three dimensions for which he received the Fields Medal in 1982. In his famous *geometrization conjecture*, Thurston claimed that all 3-manifolds result from the combination of eight types of geometrical structures, prevalently of hyperbolic type [106]. The conjecture was formally proved in 2002 by Perelman [107]. The connection between Miyake and Thurston was established by creative director Dai Fujiwara. He was intrigued by Thurston's treatment of geometry and convinced that it could provide new ideas for Miyake's *Prêt-à-porter* fashion line. This collaboration gave rise to the 2010 Autumn/Winter collection entitled '8 Geometry Link Models as Metaphor of the Universe'.[3] The concept clothes therein took inspiration from Thurston's drawings of the eight 3-manifolds geometry (see figure 8*a*). These geometrical artefacts were rearranged by Fujiwara to create garments formed by coloured linked scarves as shown in figure 8*b*. It is worth mentioning that there is much more than mere beauty in this collection. It is indeed the attempt to capture the underlying mathematical concepts as expressions of space.

A further example within the Miyake fashion house is the 2010 brand entitled '132 5'. This collection was inspired from the work on *origami* by Japanese computer scientist Jun Mitani. Origami is a form of art in which three-dimensional forms are created by folding a paper sheet. In a research paper [108], Mitani presented a novel method for designing origami starting from a target three-dimensional shape. The method automatically generates the crease pattern of the target shape by inserting appropriate flaps between the constituent polygonal faces. An important property of this construction is that the resulting surface is developable. Within a collaboration between Mitani and Miyake, this technique was adapted to develop a new production method for foldable clothes. The idea, similarly to paper origami, is to create garments by folding a single piece of cloth as shown in figure 9*a*. The mathematical foundation of this collection is well described by the brand name '132 5'. A single piece of one-dimensional cloth takes a three-dimensional form and then is folded into a flat surface (2). (5D) refers to the way that wearing finally transforms it.

Miyake's continuous research on paper-like developability had been experienced even before the launch of the '132 5' brand. Concepts of cloth folding and creasing are indeed the pillars of his famous brand 'Pleats Please'. In this collection, the crafting of garments was revolutionized by the usage of a single piece of permanently pleated fabric that is folded with a specific process. The result is a dress with a sophisticated structure similar to origami clothes. An iconic, yet representative, example of Miyake's pleating style is the concept dress entitled 'Minaret'

[3] www.3dsystems.com/learning-center/case-studies/timberland-company/.

Figure 9. (*a*) Origami dress from the Issey Miyake '123 5' collection. The unfolding mechanism is illustrated next to the mannequin (photographer Hiroshi Iwasaki) and (*b*) Issey Miyake's 'Minaret Dress' presented at the 1995 Spring/Summer collection in Paris [109] (Powerhouse collection, photographer Sue Stafford).

(see figure 9*b*). Presented at the 1995 Spring/Summer collection, this lantern-shaped dress demonstrated the ability of the Japanese stylist to master the geometry of ruled surfaces. The dress consists of the union of developable ruled patches generated from circular rulings. As a result of this geometrical construction, the garment is completely foldable. As a formidable example of the interplay between art, technology and science, the dress is exposed at the Museum of Applied Art and Science of Australia.

4. Mathematics of digital manufacturing and smart textiles

Digital manufacturing is one of the pillars of the current industrial revolution, rapidly changing the way we design, produce and distribute nearly all products. It relies on the integration of computer-aided design and simulation tools to simultaneously create products and manufacturing procedures. This section discusses the prominent role played by physics-based mathematical modelling for digital manufacturing of garments and for the design of smart textiles.

(a) Additive manufacturing of textiles and garments

Additive manufacturing is a digital methodology of making three-dimensional objects by adding material in a layer-by-layer fashion [110]. After creating a three-dimensional model with CAD programmes, this manufacturing technique is realized through the usage of proper three-dimensional printers and applies to a wide range of materials, including polymers, ceramics and metals. Available three-dimensional printing techniques depend on the phase of the material used in the process such as powder, solid and liquid-based methods. Widespread examples of three-dimensional printing methods include fused deposition modelling (FDM), selective laser sintering (SLS), stereolithography (SLA) and inject binding [111], offering inherent advantages over conventional manufacturing [112].

Not surprisingly, additive manufacturing is revolutionizing the fashion industry. For instance, fashion brands Cheng[4] and Timberland[5] partnered with three-dimensional printing companies to produce shoes with printed plastic midsoles. This technological evolution will soon enable

[4]https://www.forbes.com/sites/andriacheng/2018/05/22/with-adidas-3d-printingmay-finally-see-its-mass-retail-potential/?sh=4c12dd834a60.

[5]www.materialise.com/en/inspiration/cases/iris-van-herpen/.

the production of personalized on-demand fashion items as already established in the jewellry industry [113]. However, additive manufacturing of textiles and garments is still in its infancy as it has mainly explored for the production of concept clothes. For example, Dutch fashion designer Iris van Herpen made use of SLA to craft a semitransparent dress with a highly complex geometrical structure.[6] Using the same printing technology, a flexible floor-length nylon gown was created by Michael Schmidt and Francis Bitonti for burlesque dancer Dita Von Teese[7]. In such a case, the printed cloth embodies a sophisticated grid-like structure that provides flexibility and readily adapts to the wearer's body shape.

The research on three-dimensionally-printed textiles and garments has lagged behind other applications because of some intrinsic difficulties. The main one is recreating the essential properties of conventional textiles such as flexibility, softness, strength and porosity [114]. Flexibility of printed fabrics depends on the combination between the material employed and textile structure. Looking at the body of published literature, three-dimensionally-printed textiles are generally produced by replicating conventional interlocking and interlacing structures. For example, Bingham *et al.* [115] designed and printed a conformal garment using a particular chain mail structured textile that can be folded and bent like conventional fabrics. Using a similar methodology, Bloomfield & Borstrock [116] created a printed textile from individual links joined together. The printing of textiles with weft-knit structure was investigated by Melnikova *et al.* [117] using both SLS ans FDM. Single-faced and double-faced knit structured sheets were also fabricated in [118]. In the realm of woven structures, Partsch *et al.* [119] focused on three-dimensional printing of fabrics with various woven configurations, yarn cross-sections and yarn densities. More recently, novel strategies for printed woven and quasi-woven flexible textiles have been proposed in [120,121]. Flexible three-dimensionally-printed garments can also be designed using non-conventional textile structures. For instance, the kinematic dress by Nervous System consists of rigid triangular panels linked by hinges [122]. Thanks to the kinematic concept underneath, the garment drapes like conventional cloth and can be folded to a smaller shape for efficient fabrication.

Theoretical and numerical modelling can play a valuable role in the rational design of textiles with tailored functionality. By contrast to trial-and-error methodologies, the use of predictive tools allows for virtual prototyping and rapid design exploration. Their potential is emphasized in the case of additive manufacturing, whereby the printed product reproduces the feature of a digital model with high fidelity. Three-dimensionally-printed textiles are a special class of flexible mechanical *metamaterials* [123]. Their structure thus consists of periodically arranged building blocks whose collective interaction determines the overall textile behaviour. Suitable modelling approaches of such structured materials rely on homogenization theories, since direct numerical simulations tend to be computationally hard to solve. This particular strategy provides a way of replacing a small scale structure with an average, homogeneous medium. Among the various developed methodologies, computational homogenization represents one of the most powerful techniques [124]. The averaged mechanical properties therein are computed through the numerical solution of a boundary value problem of an adequate digital representation of the micro-structure.

A general modelling strategy applicable to three-dimensionally-printed textiles is the two-scale model proposed by Geers *et al.* [125]. As schematized in figure 10, the method relies on the nested solution of two problems at different scales. At the so-called *macro-scale* the textile is idealized as a homogeneous solid shell with no assumptions on its constitutive behaviour. The macro-scale constitutive response, for each material point, is computed from the averaged solution of a *micro-scale* problem. The latter consists of a boundary value problem on a representative volume element (RVE) of the structured sheet. At this scale, the geometry of the structure is fully described and suitable closed-form constitutive laws model the behaviour of the constituting material phases. Loading and boundary conditions for the micro-scale

[6] www.gzinnovation.eu/material/7/shape-memory-materials/.

[7] https://www.dezeen.com/2013/03/07/3d-printed-dress-dita-von-teese-michaelschmidt-francis-bitonti/.

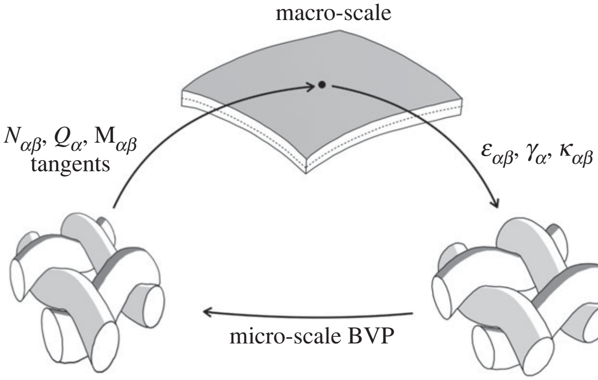

Figure 10. Schematic of a shell-based computational homogenization scheme redrawn from [125]. The macro-scale kinematics provide boundary and loading conditions for the micro-scale boundary value problem (BVP). Symbols $\varepsilon_{\alpha\beta}$, γ_α, $\kappa_{\alpha\beta}$ refer to shell in-plane strains, transverse shear strains and curvature components, respectively. After solving the micro-scale BVP, generalized shell forces ($N_{\alpha\beta}$, Q_α, $M_{\alpha\beta}$) and tangents are extracted and sent back to the macro-scale.

problem depend on the kinematics of the corresponding macroscopic points. In the simple case of first-order computational homogenization, periodic boundary conditions are generally employed.

The selection of a proper RVE for the micro-scale problem is not a trivial task. In the case of perfectly periodic structures, the RVE usually coincides with the unit-cell, i.e. the smallest tileable unit of the micro-structure. However, in the case of slender structures, compressive loads may give rise to out-of-plane buckling that can alter the initial periodicity of the micro-structure [126]. To predict such a morphological transformation, the computational homogenization scheme can be complemented with a Bloch-type stability analysis [127].

In recent years, various articles made use of computational homogenization for the mechanical characterization of conventional and three-dimensionally-printed textiles. To reduce the computational cost, the constitutive behaviour at the macro-scale is generally computed offline by means of a large number of numerical simulations at the micro-scale for a prescribed deformation space. For instance, Sperl *et al.* [128] performed yarn-level numerical simulations using the theory of elastic rods to build a model for the potential energy of the cloth at the macro-scale. The resulting homogenized model therein was used to efficiently simulate draping of various knitted and woven fabrics. Using a different approach, the homogenized response of woven fabrics was calculated through micro-scale analyses based on standard continuum mechanics in [129]. Focusing on three-dimensionally-printed textiles, Schumacher *et al.* [130] characterized the macro-scale response of printed structured sheets represented as a network of thin Kirchoff rods. Finally, Li *et al.* [131] developed a dedicated computational framework for computing the mechanical response of printed textile-like materials with sliding connections that resemble the yarn structure of conventional fabrics.

(b) Smart textiles and four-dimensional printing

Over the past few decades, there have been a tremendous development of smart textiles for the production of garments with enhanced functionalities [132]. In a broad sense, smart textiles are unconventional fabrics that are either fully or partly made with *stimuli-responsive* materials. The great advantage of this class of textiles is their ability to change aesthetic and functional properties as a function of environmental conditions [133]. Temperature and moisture regulation, self-motion, self-cleaning and colour change are some of the functional feedbacks that have been developed so far [134]. The behaviour of the employed stimuli-responsive materials, as well as the smart fabric production process, are key factors for the emergence of such advanced functionalities.

Shape-change and colour-change responses are the stimuli that have found a wider application in the textile and clothing industry. Shape-memory-alloys (SMAs) and shape-memory-polymers (SMPs) are two important examples of smart materials. Their peculiar behaviour comes from the *shape-memory* effect, i.e. the ability to recover predetermined, programmed, shapes in response to the right stimulus [135]. The shape-memory effect in SMAs typically stems from a thermally activated phase transformation between the high temperature austenite and low temperature martensite phase. Depending on the programming procedure, SMAs can be distinguished between one-way SMAs, in which only the high temperature shape is recovered, and two-way SMAs, where both high and low temperature shapes can be remembered. On the other hand, shape-memory effects in SMPs can be programmed thanks to the glass transition mechanism. Polymer based smart materials are generally more versatile than metal based [136]. Some of their advantages include higher recoverable strains, easy shape programming and multi-shape-memory effect. Besides thermal activation, a shape-change response in polymers can be triggered by water/moisture stimuli as well, as in the case of hydrogels. Other examples of smart materials for potential applications in textiles include photo-chromic [137] and electro-responsive materials.

One of the first documented uses of smart textiles in the fashion world is the 'Oricalco' shirt designed by Italian brand Corpo Nove.[8] This smart shirt actively reacts to temperature changes thanks to the use of Nilitol yarns, a shape-memory alloy made of nickel and titanium. The sleeves can be programmed to shorten as the temperature increases and no ironing is needed as the original shape can be recovered by just a flux of hot air. A Nilitol-based active textile was also developed by Bezowska & Coelho [138] to fabricate two heat-responsive concept garments. Simple aesthetic functionalities were programmed therein in the form of animated felt flowers and moving hemlines. Chan Vili [139] investigated the incorporation of shape-memory-materials in various woven structures to enhance the functionality of interior textiles. Smart curtains were programmed to open and close their woven structure depending on sunlight conditions. Water-responsive polymers can find application in the sportwear sector to improve body thermo-regulation. For instance, sport brand Nike launched a smart T-shirt named 'Sphere React Shirt' that embodies small rear vents that can open up when the wearer sweats [140]. Colour-changing fashion items have been commercialized as well. For example, a colour-changing fabric technology named 'ChroMorphus' was developed by researchers at the University of Central Florida.[9] Thanks to the usage of smart wires waved into the clothing, the fabric can be programmed to change its coloured pattern. Finally, the emergence of new conductive fabrics and soft sensors has enabled the rapid development of electronic-based textiles, or *e-textiles* [141]. Widespread applications of e-textiles include smart garments that can monitor physiological parameters such as heart-rate, breathing-rate and muscle activity [142].

The combination between smart materials and additive manufacturing gives rise to the so-called *four-dimensional printing*[10]. In essence, four-dimensional printing can be defined as the targeted evolution of four-dimensional printed structures, in terms of shape, property and functionality [143]. The fourth dimension thus refers to the time scale needed by smart, three-dimensionally-printed, structures to develop a targeted shape dynamics in the presence of a tailored stimulus. Multiple shape-morphing behaviours can be obtained though four-dimensional printing. Typical examples include bending, folding, twisting and the generation of surface topological features such as wrinkles, creases and buckles. Programmed self-assembly of such structures enhances significantly the potential of additive manufacturing. Indeed, complicated geometries can be first printed on the plane and then actuated in the location of use. This aspect can impact positively on storage and transportation of goods. On the other hand, current limitations of four-dimensional printing are related to the printability of smart materials and the development of multimaterial printers [144].

Except for a few examples, four-dimensional printing has not yet been explored in the fashion

[8]www.cnet.com/tech/mobile/color-changing-smart-clothes-smart-fabrics-purse-ucf-chromorphous/.

[9]www.jpl.nasa.gov/news/space-fabric-links-fashion-and-engineering/.

[10]https://www.ted.com/talks/skylar_tibbits_the_emergence_of_4d_printing.

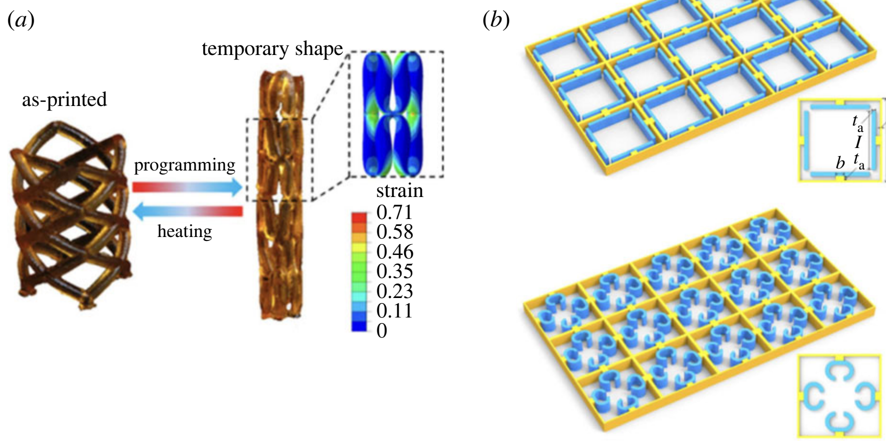

Figure 11. Modelling of smart structures: (*a*) Comparison between three-dimensionally-printed SMPs stents and finite-element numerical simulation of the thermo-activated shape-morphing [149]. (*b*) Prediction of the final configuration (right) of a thermo-responsive lattice material with initial straight beams (left) [150].

industry. A comprehensive discussion on the potential of four-dimensional printing for textiles and fashion goods is contained in [145]. One notable prototype of four-dimensionally-printed textiles is the 'Space Fabric' designed by NASA Jet Propulsion Laboratory.[11] The printed textile structure looks like a chain mail with small silver squares joined together. Thanks to its particular structure, the space fabric can fold and adapt to shapes while maintaining tensile strength. In addition, the use of stimuli-responsive materials allows for thermal regulation through combined light reflectivity and passive heat management.

Tailored shape-morphing of four-dimensionally-printed structures is generally realized using a combination of smart and conventional materials. Mathematical modelling plays a key role in optimal design of smart structures in order to achieve the desired change in shape, property and functionality [143]. A simple idea of shape-shifting material is based on kirigami tessellations, regular planar patterns formed by cut portion of thin sheets, whose number, size and orientation can be inversely defined to conform approximately to any prescribed target shape [146]. Combined with the possibility to three-dimensionally-printed materials with controlled physical and geometrical properties at the micro-scale, it is now possible to solve the geometric inverse problem for a complex structured material to grow into a any target shape [147,148]. Another crucial feature regards the implementation of suitable physical models for the mechanical response of the smart materials. Such an aspect includes the development of realistic constitutive laws, the estimation of the relevant material parameters and the description of the stimulus properties (such as type, intensity and duration).

Modelling of smart structures is inherently *multi-physical* since actuation results from the interaction of different physical phenomena (e.g. mechanical, thermal, chemical and electrical). Rigorous multi-physics models are derived from the application of a few fundamental laws, e.g. the balance equations, the constitutive theory and the laws of thermodynamics, in terms of energy balance and entropy imbalance. Moreover, in multi-scale homogenization theories, the so-called Hill–Mandel condition imposes the conservation of energy across scales. The model governing equations resulting from this sequence of tasks are typically nonlinear partial differential equations. Therefore, the solution of such models is sought by numerical approximation.

In recent years, various scientific articles proposed combined simulation and experiments procedures for the fabrication of smart structures. For example, Ge *et al.* [149], modelled

[11] www.geometrygames.org/HyperbolicBlanket/.

the actuation of four-dimensionally-printed structures made of thermally activated SMPs. As reported in figure 11*a*, shape morphing was accurately predicted by means of three-dimensional thermo-mechanical numerical simulations. The constitutive behaviour of the SMP was adapted from the multi-branch model of Yu *et al.* [151]. The latter uses a multi-branch model to simulate the multi-shape memory effect of thermo-activated polymers. Zhang *et al.* [150] focused on lightweight smart sheets obtained by three-dimensional-printing of SMPs. The thermo-mechanical behaviour of the printed structures was modelled using two different methods. An extension of the thermo-mechanical beam model was adopted for grid-like structures. A thermo-mechanical continuum model was instead used for predicting shape-morphing of two-dimensional lattice structures (see figure 11*b*). Concerning water-responsive polymers, Gladman *et al.* [152] modelled and printed composite hydrogels encoded with localized anisotropic swelling to develop complex shape-changes. The theoretical model was developed based on a thermo-mechanical theory for anisotropic surfaces.

5. Concluding remarks

In conclusion, we have outlined the evolution of the role played by mathematics in the fashion industry over the last century. On one hand, even more complex topological and geometrical principles have permeated the creative processes of designers, inspiring the innovation of garments and textiles. On the other hand, the radical switch to digital manufacturing has found in mathematics a key enabling tool for product innovation. Hence, mathematical models and methods are nowadays widely used in the fashion industry not only to assist the item conception and the material design, but also to improve the methods of production and fabrication.

Sustainability is undoubtedly the most important fallback of such novel applications of mathematics in fashion. From a production perspective, we have shown how mathematical models allow us to optimize fabrication processes for reducing material waste and environmental impact; to propose new ideas for recycling, reusing and repurposing that support the conversion from traditional to more sustainably sourced materials; to provide software tools, such as virtual-try-on and digital items, that have a lower carbon footprint and minimize in-shop returns; to personalize the production on-demand for shortening the supply chain. Finally, although out of the scope of this article, it is worth mentioning that math-based models are at the basis of any blockchain-based platform that are currently used to promote a circular business model, aiming to promote trust and transparency for customers, innovation and sustainability. These tools allow us to trace the source of raw materials and their transformation along the whole supply chain. Thus, customers gain not only a proof of authenticity against counterfeiting, promoting re-selling in second-hand markets, but also awareness of the environmental impact of their choices.

Data accessibility. This article has no additional data.

Authors' contributions. M.M.: conceptualization, formal analysis, investigation, methodology, software, writing—original draft, writing—review and editing; P.C.: conceptualization, formal analysis, funding acquisition, investigation, methodology, project administration, resources, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

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