A multipoint vorticity mixed finite element method for incompressible Stokes flow

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Abstract

We propose a mixed finite element method for Stokes flow with one degree of freedom per element and facet of simplicial grids. The method is derived by considering the vorticity-velocity-pressure formulation and eliminating the vorticity locally through the use of a quadrature rule. The discrete solution is pointwise divergence-free and the method is pressure robust. The theoretically derived convergence rates are confirmed by numerical experiments.

Keywords: multipoint vorticity, mixed finite element method, Stokes flow, hybridization
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1. Introduction

The velocity field that weakly solves a Stokes flow problem is typically sought in the Hilbert space H^1. Stable, mixed finite element methods (MFEM) that satisfy this regularity constraint can be constructed through the use of discrete Stokes complexes [1]. However, we aim to construct a MFEM of low order using the Raviart-Thomas finite element pair, which lacks the regularity necessary to fit into the Stokes complex framework. Instead, we choose to conform to a three-field formulation of the Stokes equations, known as the *vorticity-velocity-pressure* formulation [2]. Recently, this formulation was discretized using the framework of finite element exterior calculus [3].

The three-field formulation thus introduces the vorticity as a third variable and the main idea proposed in this letter is to subsequently eliminate this variable, locally, using a low-order quadrature rule. The technique is inspired by the *multipoint flux MFEM* (MF-MFEM) [4], and we inherit the nomenclature by referring to our method as a *multipoint vorticity MFEM* (MV-MFEM). The method is linearly convergent with the velocity in $H(\nabla\cdot, \Omega)$ and the (locally post-computed) vorticity in $H(\nabla\times, \Omega)$.

We remark that multipoint stress MFEM (MS-MFEM)[5] was recently proposed for Stokes flow, based on the stress-velocity-vorticity formulation. In that case, the stress and vorticity are eliminated, which leads to a method that preserves local momentum balance but lacks local mass conservation, in contrast to MV-MFEM. The hybridization techniques in this family of methods were recently identified and generalized as local approximations of the exterior coderivative [6].

In this letter, we combine a low-order discretization of the three-field formulation from [3] with the computation of local coderivatives from [6]. Focusing on the lowest order method, we show that the use of the localized quadrature rule does not impact the linear convergence. Moreover, the pressure variable is unaffected, as is the curl of the vorticity and the divergence of the velocity. In fact, in two dimensions, the only influence of the quadrature rule is a second-order error in the velocity. These results are shown theoretically in Section 4 and experimentally in Section 5.

1.1. Notation

Let $L^2(\Omega)$ denote the space of square-integrable functions on Ω and let $H(\nabla\cdot, \Omega)$ and $H(\nabla\times, \Omega)$ be the spaces of square-integrable vector fields on Ω with square-integrable divergence and curl, respectively. The $H^s(\Sigma)$ norm is denoted $\|\cdot\|_{s,\Sigma}$ and we use the short-hand notation $\|\cdot\|_\Sigma := \|\cdot\|_{0,\Sigma}$ and $\|\cdot\| := \|\cdot\|_{0,\Omega}$. We use $\langle\cdot,\cdot\rangle_\Omega$ to represent the

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L^2-inner product on Ω for scalar and vector-valued functions. Angled brackets $\langle\cdot,\cdot\rangle$ denote duality pairings, X^* is the dual of a Hilbert space X, and B' is the adjoint of an operator B. Parentheses $(\cdot,\cdot)$ are used to represent tuples. In 2D, the curl operator is defined as $\nabla\times r = [\partial_2,\ -\partial_1]^T r$ for scalar fields r and as $\nabla\times q := \partial_1 q_2 - \partial_2 q_1$ for vector-fields q. The cross product with a vector ν is defined analogously. The notation $a \lesssim b$, respectively $a \gtrsim b$, implies that a bounded constant $C > 0$ exists, independent of the mesh size, such that $Ca \le b$, respectively $a \ge Cb$. Moreover, we denote $a \eqsim b$ if $a \lesssim b$ and $b \lesssim a$.

2. The continuous problem

Let $\Omega\subset\mathbb{R}^n$ be a simply connected, Lipschitz domain with $n\in\{2,3\}$. Let $\mu > 0$ be the constant viscosity and $g\in(L^2(\Omega))^n$ a given body force. The governing equations for incompressible Stokes flow are given by momentum and mass balance for the velocity q and pressure p, and we supplement these with the definition of the vorticity r:

$$-\nabla\cdot(2\mu\varepsilon q - pI) = g, \qquad \nabla\cdot q = 0, \qquad r = \mu\nabla\times q. \tag{2.1a}$$

Here, ε is the symmetric gradient and $I\in\mathbb{R}^{n\times n}$ the identity tensor. We consider the following boundary conditions:

$$\nu\times q = \nu\times q_0, \qquad p = p_0, \qquad \text{on } \partial_p\Omega, \qquad \nu\times r = 0, \qquad \nu\cdot q = 0, \qquad \text{on } \partial_q\Omega, \tag{2.1b}$$

in which $\partial_p\Omega\cup\partial_q\Omega$ is a disjoint decomposition of the boundary $\partial\Omega$ and p_0, q_0 are given. In order to ensure uniqueness of p and q, we assume that $\partial_p\Omega\neq\emptyset$ and $\|\nu\cdot\phi\|_{\partial_q\Omega} + \|\nu\times\phi\|_{\partial_p\Omega} > 0$ for all rigid body motions $\phi\neq 0$.

We continue by deriving the variational formulation of (2.1) in terms of (r, q, p). First, we follow [3] and note the following calculus identity for solenoidal fields q:

$$-\nabla\cdot(\varepsilon q) = \frac{1}{2}\nabla\times(\nabla\times q) - \nabla(\nabla\cdot q) = \frac{1}{2}\nabla\times(\nabla\times q).$$

Using the definition of the vorticity $r := \mu\nabla\times q$, this identity allows us to rewrite the momentum balance equation as

$$\nabla\times r + \nabla\cdot(pI) = g.$$

Second, let us define the composite Hilbert space $X := R\times Q\times P$ with

$$R := \{r\in H(\nabla\times,\Omega) \mid \nu\times r|_{\partial_q\Omega} = 0\}, \qquad Q := \{q\in H(\nabla\cdot,\Omega) \mid \nu\cdot q|_{\partial_q\Omega} = 0\}, \qquad P := L^2(\Omega).$$

We are now ready to introduce the test functions $(\tilde r,\tilde q,\tilde p)\in X$ and apply integration by parts, which leads us to the variational, three-field formulation of problem (2.1): Find $(r, q, p)\in X$ such that

$$\langle\mu^{-1}r,\tilde r\rangle_\Omega - \langle q,\nabla\times\tilde r\rangle_\Omega = \langle q_0,\nu\times\tilde r\rangle_{\partial_p\Omega}, \qquad \forall\tilde r\in R, \tag{2.2a}$$

$$\langle\nabla\times r,\tilde q\rangle_\Omega - \langle p,\nabla\cdot\tilde q\rangle_\Omega = \langle g,\tilde q\rangle_\Omega - \langle p_0,\nu\cdot\tilde q\rangle_{\partial_p\Omega}, \qquad \forall\tilde q\in Q, \tag{2.2b}$$

$$\langle\nabla\cdot q,\tilde p\rangle_\Omega = 0, \qquad \forall\tilde p\in P. \tag{2.2c}$$

To highlight the structure of this problem, we define the operators A, B_r, B_q, and functionals f_r, f_q such that:

$$\langle Ar,\tilde r\rangle := \langle\mu^{-1}r,\tilde r\rangle_\Omega, \qquad \langle B_r r,\tilde q\rangle := \langle\nabla\times r,\tilde q\rangle_\Omega, \qquad \langle B_q q,\tilde p\rangle := \langle\nabla\cdot q,\tilde p\rangle_\Omega,$$
$$\langle f_r,\tilde r\rangle := -\langle q_0,\tilde r\rangle_{\partial_q\Omega}, \qquad \langle f_q,\tilde q\rangle := \langle g,\tilde q\rangle_\Omega - \langle p_0,\nu\cdot\tilde q\rangle_{\partial_q\Omega},$$

for all $(r, q, p), (\tilde r,\tilde q,\tilde p)\in X$. Recall that unsubscripted, angled brackets $\langle\cdot,\cdot\rangle$ denote duality pairings. We then identify problem (2.2) to be of the form $\mathcal{B}(r, q, p) = f$ with $\mathcal{B} : X\to X^*$ and $f\in X^*$ given by

$$\mathcal{B} := \begin{bmatrix} A & -B_r' & \\ B_r & & -B_q' \\ & B_q & \end{bmatrix}, \qquad f := \begin{bmatrix} f_r \\ f_q \\ 0 \end{bmatrix}.$$

For the analysis of this problem, we define the energy norm as

$$|||(r, q, p)|||^2 := \|r\|^2 + \|\nabla\times r\|^2 + \|q\|^2 + \|\nabla\cdot q\|^2 + \|p\|^2. \tag{2.3}$$

It was shown in [3, Lem. 3.4] that problem (2.2) admits a unique solution that is bounded in the energy norm (2.3).

3. Mixed finite element discretization

3.1. A three-field mixed finite element method

Let Ω_h be a shape-regular, simplicial tesselation of Ω. For $n = 3$, the finite element spaces are chosen as the linear Nédélec elements of the second kind, the Raviart-Thomas space of lowest order, and the piecewise constants:

$$R_h := \mathbb{N}_1 \cap R, \qquad Q_h := \mathbb{RT}_0 \cap Q, \qquad P_h := \mathbb{P}_0 \cap P.$$

We clarify that this choice of spaces leads to one degree of freedom per face for the velocity, one degree of freedom per element for the pressure, but two degrees of freedom per edge for the vorticity. This choice facilitates the hybridization introduced in the next section. For $n = 2$, we only adapt the vorticity space to the linear Lagrange elements, $R_h := \mathbb{L}_1 \cap R$. It is important to note that $\nabla \times R_h \subseteq Q_h$ and $\nabla \cdot Q_h = P_h$. Finally, we define the composite space as $X_h := R_h \times Q_h \times P_h$.

The *three-field mixed finite element method* (3F-MFEM) of problem (2.2) is: Find $\hat{x} \in X_h$ such that

$$\langle \mathcal{B}\hat{x}, \tilde{x} \rangle = \langle f, \tilde{x} \rangle, \qquad \forall \tilde{x} \in X_h. \tag{3.1}$$

Lemma 3.1 ([3, Thm. 3.7])**.** *The 3F-MFEM is stable and convergent with the error bounded by*

$$|||\hat{x} - x||| \lesssim \inf_{\tilde{x} \in X_h} |||\tilde{x} - x|||. \tag{3.2}$$

3.2. A quadrature rule for the vorticity

Following [6, Eq. (4.6)], we introduce a quadrature rule for the vorticity variable and define the associated norm:

$$\langle r, \tilde{r} \rangle_h := \sum_{\omega \in \Omega_h} \frac{|\omega|}{n+1} \sum_{x \in \mathcal{V}(\omega)} (r_\omega \cdot \tilde{r}_\omega)(x), \qquad \|r\|_h^2 := \langle r, r \rangle_h, \qquad \forall r, \tilde{r} \in \mathbb{P}_1^{k_n}(\Omega_h) \supseteq R_h, \tag{3.3}$$

with r_ω the restriction of r on the simplex ω and $\mathcal{V}(\omega)$ the set of vertices of ω. We use $\mathbb{P}_l(\Omega_h)$ to denote the space of discontinous, elementwise polynomials of order l and the coefficient is given by $k_3 = 3$ and $k_2 = 1$. Finally, let us define the auxiliary space $W_h := \mathbb{P}_0^{k_n}(\Omega_h)$ containing elementwise constant (vector) fields.

Two key properties of the quadrature rule are highlighted in the following lemma.

Lemma 3.2 ([6, Thm. 4.1])**.** *The quadrature rule from* (3.3) *satisfies the following properties*

$$\|r\|_h \eqsim \|r\|, \qquad \forall r \in R_h, \qquad \langle \tilde{r}, w \rangle_h = \langle \tilde{r}, w \rangle, \qquad \forall \tilde{r} \in \mathbb{P}_1^{k_n}, w \in W_h. \tag{3.4}$$

Let $A_h : R_h \to R_h^*$ be the operator associated with the quadrature rule such that $\langle A_h r, \tilde{r} \rangle := \langle \mu^{-1} r, \tilde{r} \rangle_h$. Next, we replace A by A_h in the system which leads us to define a new operator $\mathcal{B}_h : X_h \to X_h^*$:

$$\langle \mathcal{B}_h(r, q, p), (\tilde{r}, \tilde{q}, \tilde{p}) \rangle := \langle \mathcal{B}(r, q, p), (\tilde{r}, \tilde{q}, \tilde{p}) \rangle + \langle A_h r, \tilde{r} \rangle - \langle A r, \tilde{r} \rangle, \qquad \forall (r, q, p), (\tilde{r}, \tilde{q}, \tilde{p}) \in X_h. \tag{3.5}$$

With this substitution in place, we consider the augmented system: Find $x_h \in X_h$ such that

$$\langle \mathcal{B}_h x_h, \tilde{x} \rangle = \langle f, \tilde{x} \rangle, \qquad \forall \tilde{x} \in X_h. \tag{3.6}$$

Similar to (2.3), the natural energy norm for this problem is given by

$$|||(r, q, p)|||_h^2 := \|r\|_h^2 + \|\nabla \times r\|^2 + \|q\|^2 + \|\nabla \cdot q\|^2 + \|p\|^2, \tag{3.7}$$

and Lemma 3.2 directly gives us the equivalence relation $|||x||| \eqsim |||x|||_h$ for all $x \in X_h$.

Lemma 3.3 (Well-posedness)**.** *Problem* (3.6) *admits a unique and bounded solution. In particular, the operator $\mathcal{B}_h$ from* (3.5) *satisfies*

$$\sup_{\tilde{x} \in X_h} \frac{\langle \mathcal{B}_h x_h, \tilde{x} \rangle}{|||\tilde{x}|||_h} \eqsim |||x_h|||_h, \qquad \forall x_h \in X_h. \tag{3.8}$$

Proof. For the continuity bound "$\lesssim$", we note that $\mathcal{B}$ is continuous in the energy norm $|||\cdot|||$, which is equivalent to $|||\cdot|||_h$. Moreover, A_h is continuous in the norm $\|\cdot\|_h$ and so $\mathcal{B}_h$, given by (3.5), is continuous in the norm $|||\cdot|||_h$. For the lower bound "$\gtrsim$", we follow the same steps as in the proof of [3, Lem. 3.4] and use the equivalence relation from Lemma 3.2 where necessary. Finally, we follow [3, Lem. 3.5] to obtain the final requirements to invoke the Babuška-Lax-Milgram theorem for existence of the unique and bounded solution. □

3.3. A multipoint vorticity mixed finite element method

The quadrature rule (3.3) is local in the sense that $\langle r_i, \tilde{r}_j\rangle_h$ is only non-zero for basis functions $r_i, \tilde{r}_j \in R_h$ that have a degree of freedom at the same vertex. A_h can thus be inverted by solving local systems around the vertices which allows us to eliminate the vorticity r_h. After elimination, we obtain the *multipoint vorticity mixed finite element method* (MV-MFEM): Find $(q_h, p_h) \in Q_h \times P_h$ such that

$$\begin{bmatrix} B_r A_h^{-1} B_r' & -B_q' \\ B_q & \end{bmatrix} \begin{bmatrix} q_h \\ p_h \end{bmatrix} = \begin{bmatrix} f_q - B_r A_h^{-1} f_r \\ 0 \end{bmatrix}. \tag{3.9}$$

4. A priori analysis of MV-MFEM

For the analysis of the MV-MFEM (3.9), we consider its equivalent formulation (3.6). We start by introducing suitable interpolants. Let Π_R and Π_Q be the canonical interpolants onto the finite element spaces R_h, Q_h, respectively, defined for sufficiently regular $r \in R$ and $q \in Q$. Similarly, let Π_P and Π_W be the L^2 projections onto P_h and W_h, respectively. These interpolants satisfy $\langle \nabla \times (I - \Pi_R) r, q\rangle_\Omega = 0$ and $\langle \nabla \cdot (I - \Pi_Q) q, p\rangle_\Omega = 0$ for all $(r, q, p) \in X_h$, and have the approximation properties

$$\|(I - \Pi_R) r\| + \|\nabla \times ((I - \Pi_R) r)\| \lesssim h\,(\|r\|_{1,\Omega} + \|\nabla \times r\|_{1,\Omega}), \qquad \|(I - \Pi_P) p\| \lesssim h\,\|p\|_{1,\Omega}, \tag{4.1a}$$

$$\|(I - \Pi_Q) q\| + \|\nabla \cdot ((I - \Pi_Q) q)\| \lesssim h\,(\|q\|_{1,\Omega} + \|\nabla \cdot q\|_{1,\Omega}), \qquad \|(I - \Pi_W) r\| \lesssim h\,\|r\|_{1,\Omega}. \tag{4.1b}$$

In turn, the composite interpolant Π_X is given by $\Pi_X(r, q, p) := (\Pi_R r, \Pi_Q q, \Pi_P p)$ for sufficiently regular $(r, q, p) \in X$.

Theorem 4.1 (Convergence)**.** *If the continuous solution $x \in X$ to* (2.2) *is sufficiently regular, then the discrete solution $x_h \in X_h$ to* (3.6) *converges linearly in the energy norm* (2.3)*:*

$$|||x_h - x||| \lesssim h.$$

Proof. We closely follow [6] and start, as in [6, Eq. (3.11)], by evaluating the difference between $x_h := (r_h, q_h, p_h) \in X_h$, the solution to (3.6), and $\hat{x} := (\hat{r}, \hat{q}, \hat{p}) \in X_h$, the solution to (3.1). Using the equivalences from Lemma 3.2 and Lemma 3.3, we derive

$$|||\hat{x} - x_h||| \eqsim |||\hat{x} - x_h|||_h \eqsim \sup_{\tilde{x} \in X_h} \frac{\langle \mathcal{B}_h(\hat{x} - x_h), \tilde{x}\rangle}{|||\tilde{x}|||_h} = \sup_{\tilde{x} \in X_h} \frac{\langle \mathcal{B}_h \hat{x}, \tilde{x}\rangle - \langle f, \tilde{x}\rangle}{|||\tilde{x}|||_h} = \sup_{\tilde{r} \in R_h} \frac{\langle A_h \hat{r}, \tilde{r}\rangle - \langle A\hat{r}, \tilde{r}\rangle}{|||(\tilde{r}, 0, 0)|||_h}. \tag{4.2a}$$

Next, we follow the steps from [6, Thm. 3.2] to further bound the final term. Scaling the numerator by μ for ease of presentation, we use Lemma 3.2 and a Cauchy-Schwarz inequality to obtain

$$\mu\,(\langle A_h \hat{r}, \tilde{r}\rangle - \langle A\hat{r}, \tilde{r}\rangle) = \langle \hat{r}, \tilde{r}\rangle_h - \langle \hat{r}, \tilde{r}\rangle_\Omega = \langle \hat{r} - \Pi_W r, \tilde{r}\rangle_h - \langle \hat{r} - \Pi_W r, \tilde{r}\rangle_\Omega \lesssim \|\hat{r} - \Pi_W r\|\,\|\tilde{r}\|_h. \tag{4.2b}$$

Combining (4.2), the triangle inequality leads us to

$$|||\hat{x} - x_h||| \lesssim \mu^{-1} \sup_{\tilde{r} \in R_h} \frac{\|\hat{r} - \Pi_W r\|\,\|\tilde{r}\|_h}{|||(\tilde{r}, 0, 0)|||_h} \lesssim \|\hat{r} - \Pi_W r\| \le \|\hat{r} - r\| + \|r - \Pi_W r\|.$$

Next, we use another triangle inequality and Lemma 3.1 to derive

$$|||x_h - x||| \le |||x_h - \hat{x}||| + |||\hat{x} - x||| \lesssim \|r - \Pi_W r\| + \inf_{\tilde{x} \in X_h} |||x - \tilde{x}||| \le \|r - \Pi_W r\| + |||x - \Pi_X x|||.$$

Finally, the properties of the interpolants (4.1) and the assumed regularity of x provides the result. □

Several advantageous properties of 3F-MFEM are preserved despite the use of the quadrature rule and we summarize these in the following two lemmas.

Lemma 4.2 (Solenoidal)**.** *The discrete solution $q_h \in R_h$ is point-wise solenoidal.*

Proof. The solution satisfies $\langle \nabla \cdot q_h, \tilde{p} \rangle_\Omega = 0$, for all $\tilde{p} \in P_h$. Since $\nabla \cdot R_h \subseteq P_h$, the result follows. □

Lemma 4.3 (Pressure-robust). *If the force term g is perturbed by $\delta g = \nabla\phi$ for some ϕ, then the solution x_h is only perturbed in the component p_h.*

Proof. The proof of [3, Thm 5.3] is unaffected by the substitution of A by A_h, so it directly applies. □

In addition to the divergence of the velocity, certain components of the solution remain unchanged after including the quadrature rule. We formally present these invariants in the following theorem.

Theorem 4.4 (Invariants). *The introduction of the quadrature rule A_h does not influence the pressure variable, nor the curl of the vorticity, i.e. $p_h = \hat{p}$ and $\nabla \times r_h = \nabla \times \hat{r}$.*

Proof. First, since $\nabla \times R_h \subseteq Q_h$, we can construct a function $\tilde{q}_r := \nabla \times (\hat{r} - r_h)$. Setting $\tilde{x} := (0, \tilde{q}_r, 0)$, we consider the difference between systems (3.1) and (3.6) and use the fact that $\nabla \cdot \tilde{q}_r = 0$ to derive:

$$0 = \langle \mathcal{B}\hat{x} - \mathcal{B}_h x_h, \tilde{x} \rangle = \langle \nabla \times (\hat{r} - r_h), \tilde{q}_r \rangle_\Omega - \langle \hat{p} - p_h, \nabla \cdot \tilde{q}_r \rangle_\Omega = \|\nabla \times (\hat{r} - r_h)\|^2.$$

Secondly, since $\nabla \cdot Q_h = P_h$, we can construct a function $\tilde{q}_p$ such that $\nabla \cdot \tilde{q}_p = \hat{p} - p_h$. Setting now $\tilde{x} := (0, \tilde{q}_r, 0)$ and using $\nabla \times (\hat{r} - r_h) = 0$, we obtain

$$0 = \langle \mathcal{B}\hat{x} - \mathcal{B}_h x_h, \tilde{x} \rangle = -\langle \hat{p} - p_h, \nabla \cdot \tilde{q}_p \rangle_\Omega = -\|\hat{p} - p_h\|^2.$$ □

4.1. The two-dimensional case

Corollary 4.4.1. *If $n = 2$, then the vorticity is unaffected by the quadrature rule A_h, i.e. $r_h = \hat{r}$.*

Proof. Since $\nabla \times (\hat{r} - r_h) = 0$ by Theorem 4.4 and the curl is a rotated gradient in 2D, it follows that $\hat{r} - r_h$ is constant. Computing the difference of the two systems and setting now the test function as $\tilde{x} := (\hat{r} - r_h, 0, 0)$, we derive

$$0 = \langle \mathcal{B}\hat{x} - \mathcal{B}_h x_h, \tilde{x} \rangle = \langle A\hat{r} - A_h r_h, \hat{r} - r_h \rangle - \langle \hat{q} - q_h, \nabla \times (\hat{r} - r_h) \rangle_\Omega = \langle A(\hat{r} - r_h), \hat{r} - r_h \rangle = \mu^{-1}\|\hat{r} - r_h\|^2,$$

where we used that $\hat{r} - r_h \in \mathbb{R} \subset W_h$ and Lemma 3.2. □

In 2D, the definition of the curl and the use of $R_h = \mathbb{L}_1$ provide the following properties of the interpolants for sufficiently regular $r \in R$:

$$\|(I - \Pi_W)r\| \lesssim h\,\|\nabla \times r\|, \qquad \|(I - \Pi_R)r\| \lesssim h^2\,\|r\|_{2,\Omega}. \tag{4.3}$$

Lemma 4.5. *If $n = 2$, then x_h converges quadratically to $\hat{x}$. In particular, $|||\hat{x} - x_h||| = \|\hat{q} - q_h\| \lesssim h^2$.*

Proof. The equality follows from Lemma 4.2, Theorem 4.4 and Corollary 4.4.1. For the final estimate, we note that $\hat{q} - q_h$ is solenoidal, which allows us to take $\tilde{r}_q$ such that $\nabla \times \tilde{r}_q = \hat{q} - q_h$. Setting $\tilde{x} := (\tilde{r}_q, 0, 0)$, we derive

$$0 = \langle \mathcal{B}\hat{x} - \mathcal{B}_h x_h, \tilde{x} \rangle = \langle A\hat{r} - A_h r_h, \tilde{r}_q \rangle - \langle \hat{q} - q_h, \nabla \times \tilde{r}_q \rangle_\Omega = \langle (A - A_h) r_h, \tilde{r}_q \rangle - \|\hat{q} - q_h\|^2. \tag{4.4a}$$

Hence, we continue by using (4.3) to obtain the following bound

$$\langle (A - A_h) r_h, \tilde{r}_q \rangle = \langle (A - A_h)(I - \Pi_W) r_h, (I - \Pi_W)\tilde{r}_q \rangle \lesssim \|(I - \Pi_W) r_h\|\,\|(I - \Pi_W)\tilde{r}_q\| \lesssim h^2 \|\nabla \times r_h\|\,\|\hat{q} - q_h\|. \tag{4.4b}$$

Finally, we combine (4.4) and use the stability of the 3F-MFEM [3, Lem. 3.4] to bound $\|\nabla \times r_h\|$.

Theorem 4.6 (Improved estimate). *If $n = 2$ and r is sufficiently regular, then r_h converges as $\|r_h - r\| \lesssim h^2$.*

Proof. We first consider the error equations for the 3F-MFEM with test function $\tilde{x} = \hat{x} - \Pi_X x$. Due to the properties of the canonical interpolants, the off-diagonal components cancel and we obtain:

$$0 = \langle \mathcal{B}(\hat{x} - x), \tilde{x} \rangle = \mu^{-1} \langle \hat{r} - r, \hat{r} - \Pi_R r \rangle_\Omega = \mu^{-1} \left(\|\hat{r} - r\|^2 + \langle \hat{r} - r, r - \Pi_R r \rangle_\Omega \right).$$

Next, Corollary 4.4.1 gives us that $r_h = \hat{r}$ and so we conclude, using (4.3), that

$$\|r_h - r\| = \|\hat{r} - r\| = \frac{-\langle \hat{r} - r, (I - \Pi_R) r \rangle_\Omega}{\|\hat{r} - r\|} \leq \|(I - \Pi_R) r\| \lesssim h^2 \|r\|_2.$$

5. Numerical results

In this section, we perform a convergence study for $n = 2$ and 3 to verify the theoretical results of Section 4. The results are obtained with the library PorePy [**?**] and PyGeoN [**?**]. Let $\Omega := (0, 1)^n$, with the known solutions

$$
\begin{aligned}
q(x, y) &= \nabla \times (x^2 y^2 (x-1)^2 (y-1)^2), & p(x, y) &= xy(1-x)(1-y), & n &= 2,\\
q(x, y, z) &= \nabla \times \begin{bmatrix}(1-x)x(1-y)^2 y^2 (1-z)^2 z^2 & 0 & 0\end{bmatrix}^T, & p(x, y, z) &= xyz(1-x)(1-y)(1-z), & n &= 3.
\end{aligned}
$$

We set boundary conditions $q_0 = 0$ and $p_0 = 0$ on $\partial\Omega = \partial_p\Omega$, let $\mu := 1$ and compute the force term as $g = \nabla\cdot(2\varepsilon q - pI)$.

Table 1 compares MV-MFEM with 3F-MFEM in terms of numbers of degrees of freedom and relative L^2-errors. For fairness of comparison in the case $n = 3$, we choose R_h in 3F-MFEM as the lowest-order Nédélec elements of the first kind, that has one degree of freedom per mesh edge. The numbers of degrees of freedom in MV-MFEM are nevertheless smaller due to the complete elimination of the vorticity. The error rates are in agreement with Theorem 4.1, exhibiting at least linear convergence of all variables with respect to h. As proven in Theorem 4.4, we observe numerically that the pressure, and therewith Error_p, is unaffected by the quadrature rule.

For $n = 2$, Corollary 4.4.1 is reflected in the fact that the columns Error_r are identical. The observed quadratic convergence in r was shown in Theorem 4.6. Finally, superconvergence of the pressure in the cell centers was observed (not reported).

		3F-MFEM							MV-MFEM						
	h	N_{dof}	Error_r	Rate_r	Error_q	Rate_q	Error_p	Rate_p	N_{dof}	Error_r	Rate_r	Error_q	Rate_q	Error_p	Rate_p
$n = 2$	6.42e-2	1.91e3	7.73e-3	-	6.62e-2	-	1.12e-1	-	1.57e3	7.73e-3	-	6.95e-2	-	1.12e-1	-
	3.17e-2	7.33e3	1.94e-3	1.96	3.28e-2	0.99	5.67e-2	0.97	6.06e3	1.94e-3	1.96	3.33e-2	1.04	5.67e-2	0.97
	1.57e-2	2.88e4	4.86e-4	1.96	1.65e-2	0.98	2.84e-2	0.98	2.39e4	4.86e-4	1.96	1.65e-2	0.99	2.84e-2	0.98
	7.83e-3	1.14e5	1.22e-4	1.99	8.26e-3	0.99	1.42e-2	0.99	9.52e4	1.22e-4	1.99	8.27e-3	1.00	1.42e-2	0.99
	3.91e-3	4.56e5	3.05e-5	2.00	4.13e-3	1.00	7.12e-3	1.00	3.80e5	3.05e-5	2.00	4.13e-3	1.00	7.12e-3	1.00
$n = 3$	1.81e-1	1.59e4	2.30e-1	-	1.43e-1	-	3.27e-1	-	1.10e4	1.42e-1	-	1.68e-1	-	3.27e-1	-
	1.65e-1	2.11e4	2.11e-1	0.90	1.32e-1	0.83	2.92e-1	1.19	1.46e4	1.25e-1	1.34	1.48e-1	1.31	2.92e-1	1.19
	1.48e-1	2.86e4	1.89e-1	1.03	1.17e-1	1.13	2.58e-1	1.16	1.99e4	1.08e-1	1.32	1.28e-1	1.35	2.58e-1	1.16
	1.36e-1	3.62e4	1.73e-1	1.03	1.07e-1	1.05	2.38e-1	0.90	2.52e4	1.00e-1	0.90	1.16e-1	1.15	2.38e-1	0.90
	1.27e-1	4.43e4	1.61e-1	1.00	1.01e-1	0.83	2.22e-1	1.06	3.09e4	9.28e-2	1.14	1.09e-1	0.97	2.22e-1	1.06

Table 1: Relative $L^2(\Omega)$-errors and convergence rates for the vorticity r, velocity q, and pressure p. The presented results are in agreement with the developed theory and show the potential of the proposed approach.

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