Analytical Derivation of Scattering and Admittance Rational Functions From Coupling Matrix

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***Abstract*—The coupling matrix provides an immediate tool to describe a network, widely used in the context of microwave filters. While several techniques exist to obtain and manipulate the coupling matrix for a given response, the opposite is traditionally achieved by numerically evaluating the response of the network at each frequency of interest and fitting the response to obtain the rational expressions embedding poles and zeros. This brief instead proposes an analytic way to retrieve directly the admittance and scattering functions, as ratios of polynomials, from the coupling matrix. Applications include commercial software packages and more robust microwave filter optimizations during design.**

***Index Terms*—Microwave filters, coupling matrix, scattering matrix, admittance matrix, Cauchy fitting.**

I. Introduction

MOST current techniques for the design of microwave filters rely on a common tool, namely the coupling matrix. The typical frame is described by several of the cornerstone works in filter theory, i.e., [1], [2], in which the designer usually determines the desired scattering response of the filter as a matrix of rational functions sharing a common denominator. Once the response is known, a lumped model circuit can be analytically obtained, traditionally as a transversal network composed by several parallel branches [1]. A general network, involving resonators and non-resonating nodes as shown in Fig. 1, is completely described by means of the coupling matrix and the list of resonating nodes. The coupling matrix can be manipulated according to a number of operations which do not affect the response observed at the ports, but which correspond to different topologies, more or less suitable for practical implementation (i.e., waveguide, planar, micromachining. . .); available operations [3], [4] include rotations, scaling and others descending from circuital transformations [5].

While scientific works extensively describe the derivation of the coupling matrix to match a desired rational behavior (e.g., [6]) and transformations into a suitable topology [7], no analytic method computes the polynomials of the response from an arbitrary coupling matrix. This operation in fact requires inverting a large matrix with polynomial elements, thus involving a symbolic engine, expensive and computationally intensive. Alternatively, the response of the network implementing the given coupling matrix is evaluated numerically and, via Cauchy's method [8], [9] or Vector Fitting [10], [11], the rational functions are then estimated, though subject to convergence and accuracy issues [12].

Manuscript received 9 October 2022; revised 29 December 2022; accepted 24 January 2023. Date of publication 1 February 2023; date of current version 30 June 2023. This brief was recommended by Associate Editor M. Chen. *(Corresponding author: Matteo Oldoni.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCSII.2023.3241429.

Digital Object Identifier 10.1109/TCSII.2023.3241429

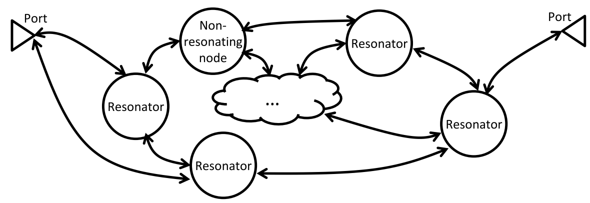


Fig. 1. Conceptual diagram of a network composed of coupled resonators and non-resonating nodes. The external response, e.g., the scattering or admittance matrix, is observed through the available ports.

The present work proposes instead an analytic procedure to compute the polynomials defining the admittance and scattering matrices observed at the ports for reciprocal lossless or lossy networks described by an arbitrary coupling matrix. Knowledge of the underlying polynomials of a network allows to tune poles and zeros in order to match the theoretical model after electromagnetic simulation [13] or to obtain insights in the behavior of a circuit or its alternative forms, e.g., [14], useful in the investigation of novel topologies or sections available for filter design.

Notation and assumptions used here are described in Section II and then developed into the expressions of admittance and scattering polynomials in Section III. A few examples are shown in Section IV, before the conclusions of Section V.

II. Notation

In a parallel model of a lowpass normalized prototype, the M network nodes are classified as:

- *from 1 to P:* external ports, with reference impedance normalized to 1 and possibly a resonator represented by a capacitor ($C_k \neq 0$, mostly encountered in lumped-element designs) or not ($C_k = 0$, more typical in distributed-element filters) and a frequency-invariant complex admittance y_k toward ground;
- *from $P + 1$ to $P + N$:* non-resonating nodes, involving a frequency-invariant complex admittance y_k toward ground but no capacitor, $C_k = 0$;
- *from $P + N + 1$ to $P + N + R = M$:* resonators, each made by a unitary capacitor $C_k = 1$ toward ground and a frequency-invariant complex admittance y_k in parallel;

Nodes are coupled by means of admittance inverters, which are described by an inversion constant, complex in general. The

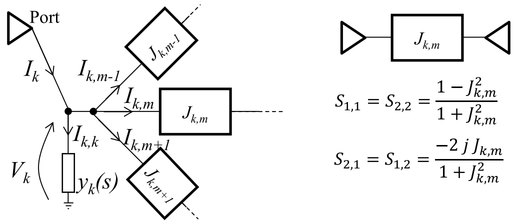


Fig. 2. Conventions of currents and voltages in an arbitrary node k (left) and scattering parameters of a generical frequency-invariant admittance inverter (right), referred to unitary impedance.

present work assumes all inverters to be frequency-invariant and with the scattering parameters indicated in Fig. 2.

By temporarily assuming, as shown in Fig. 2, that node k is coupled to every other node and that it involves a generical admittance $y_k(s)$ (so that it can represent both a resonating node by $C_k \neq 0$ for $k = P{+}N{+}1, \dots, M$ and a non-resonating one with $C_k = 0$ for $k \leq P + N$) and may also directly be an external port which injects a current I_k, the net current into the node must obey Kirchhoff law:

$$V_k y_k(s) + \sum_{m=1;\ m\neq k}^{M} jJ_{k,m}V_m = I_k \tag{1}$$

This allows to write a matrix equation:

$$\boldsymbol{y}\ \underline{V} = \underline{I} \text{ with } \underline{I} = \begin{bmatrix} I_1 & \cdots & I_P & 0 & \cdots & 0 \end{bmatrix}^T, \tag{2}$$

while $\underline{V}$ gathers all the node voltages in a column vector. The symmetrical admittance matrix $\boldsymbol{y}(s)$ can be decomposed in a frequency-variant diagonal, where the Laplace variable $s = \sigma + j\omega$ embeds the pulsation, and a frequency-invariant part, which coincides with the coupling matrix $\boldsymbol{M}$:

$$\boldsymbol{y}(s) = j\boldsymbol{M} + s \cdot \text{Diag}\{C_k\colon k \in [1;\ M]\} \tag{3}$$

The off-diagonal elements of $\boldsymbol{M}$ contain the admittance inversion constant $J_{k,m}$ of the coupling between the k-row and the m-column nodes (zero if nodes are not coupled); diagonal elements instead are the susceptances in each node:

$$\boldsymbol{M} = \underset{\left(\text{with } (k,m) \in [1;M]^2\right)}{\text{Diag}\left\{\frac{y_k}{j}\right\} + \{J_{k,m}\}} = \left[\begin{array}{c|cc} \boldsymbol{M}^{(p)} & \multicolumn{2}{c}{\boldsymbol{M}^{(p,i)}} \\ \hline \boldsymbol{M}^{(p,i)^T} & \boldsymbol{M}^{(n)} & \boldsymbol{M}^{(n,r)} \\ & \boldsymbol{M}^{(n,r)^T} & \boldsymbol{M}^{(r)} \end{array}\right] \tag{4}$$

The partitioning in eq. (4) highlights the classification of nodes, applied also to the admittance matrix to separate port nodes "p" from internal nodes "i" (non-resonant and resonant):

$$\boldsymbol{y}(s) = \begin{bmatrix} \boldsymbol{y}^{(p)}(s)_{[P\times P]} & j\boldsymbol{M}^{(p,i)}_{[P\times M-P]} \\ j\boldsymbol{M}^{(p,i)^T} & \boldsymbol{y}^{(i)}(s)_{[M-P\times M-P]} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{y}^{(p)}(s)$ involves $\boldsymbol{M}^{(p)}$ and possibly the capacitors on those nodes, whereas $\boldsymbol{y}^{(i)}(s)$ involves the bottom-right block in eq. (4) and the associated capacitors in the resonant nodes.

The admittance matrix observed from the external ports, $\boldsymbol{Y}(s)$, can be obtained by solving eq. (2), which gives:

$$\begin{aligned} \boldsymbol{Y}(s) &= j\boldsymbol{M}^{(p)} + s\,\text{Diag}\{C_k\colon k \in [1;\ P]\} \\ &\quad + \boldsymbol{M}^{(p,i)}\boldsymbol{z}^{(i)}(s)\boldsymbol{M}^{(p,i)^T} \text{ with } \boldsymbol{z}^{(i)}(s) = \left(\boldsymbol{y}^{(i)}(s)\right)^{-1} \end{aligned} \tag{6}$$

By inspection, $\boldsymbol{Y}(s)$ can be seen to be a $P \times P$ symmetrical matrix of rational functions, all sharing the denominator:

$$\boldsymbol{Y}(s) = \frac{\boldsymbol{Y}^{\text{Num}}(s)}{Y^{\text{Den}}(s)}. \tag{7}$$

The corresponding symmetrical $P \times P$ scattering matrix $\boldsymbol{S}(s)$ observable from the unitary-characteristic impedance ports is also a rational matrix with a common denominator:

$$\boldsymbol{S}(s) = -(\boldsymbol{Y}(s) + \boldsymbol{E})^{-1}(\boldsymbol{Y}(s) - \boldsymbol{E}) = \frac{\boldsymbol{S}^{\text{Num}}(s)}{S^{\text{Den}}(s)} \tag{8}$$

where $\boldsymbol{E}$ denotes the $P \times P$ identity matrix. Today, in order to avoid the intensive symbolic matrix inversion required by eq. (6), the equation is evaluated numerically at a finite number of frequency points and then fed to eq. (8) to obtain the evaluated response of the network. Poles and reflection/transmission zeros of the network are not however available: if needed, one must therefore perform a rational fitting of the response, with consequences related to sensitivity to number and choice of sampled points and numerical instability.

The next section instead proposes the analytical computation of the admittance polynomials ($Y^{\text{Den}}(s)$ and the elements of $\boldsymbol{Y}^{\text{Num}}(s)$) and of the scattering polynomials ($S^{\text{Den}}(s)$ and the elements of $\boldsymbol{S}^{\text{Num}}(s)$), so that their coefficients can be obtained beforehand, without relying on fitting and sampling. The procedure is analytical and thus the returned rational response's passivity or losslessness will reflect that of the original coupling matrix, without need to enforce it.

III. Derivation and Expressions of Polynomials

In order to compute the denominator and numerator polynomials of eq. (6), the first hurdle is the computation of $\boldsymbol{z}^{(i)}(s)$. Equation (3) and eq. (4), however, highlight that:

$$\boldsymbol{y}^{(i)}(s) = \text{Diag}\{sC_k\colon k = P+1, \dots, M\} + j\boldsymbol{M}_i. \tag{9}$$

The Woodbury matrix identity [15] provides the inverse of a sum of two matrices, with one of them invertible. The objective here is to easily invert the diagonal part, but the presence of non-resonating nodes violates this requirement. A further partitioning is introduced to split non-resonating nodes (null capacitors) and resonators (unitary capacitors):

$$\boldsymbol{y}^{(i)}(s) = \begin{bmatrix} j\boldsymbol{M}^{(n)} & j\boldsymbol{M}^{(n,r)} \\ j\boldsymbol{M}^{(n,r)^T} & s\boldsymbol{C} + j\boldsymbol{M}^{(r)} \end{bmatrix} \tag{10}$$

The inverse $\boldsymbol{z}^{(i)}(s) = (\boldsymbol{y}^{(i)}(s))^{-1}$ of such 2×2 block-partitioned matrix is computed according to most algebra textbooks [16] or formularies [17], if $\boldsymbol{M}^{(n)}$ is invertible:

$$\boldsymbol{z}^{(i)}(s) = \begin{bmatrix} -j\boldsymbol{M}^{(n)^{-1}} + \boldsymbol{F}\,\boldsymbol{D}(s)\,\boldsymbol{F}^T & -\boldsymbol{F}\,\boldsymbol{D}(s) \\ -\boldsymbol{D}(s)\,\boldsymbol{F}^T & \boldsymbol{D}(s) \end{bmatrix} \tag{11a}$$

$$\text{with } \boldsymbol{D}(s) = (s\boldsymbol{C} + j\boldsymbol{Q})^{-1}, \quad \boldsymbol{F} = \boldsymbol{M}^{(n)^{-1}}\boldsymbol{M}^{(n,r)} \tag{11b}$$

$$\boldsymbol{Q} = \boldsymbol{M}^{(r)} - \boldsymbol{M}^{(n,r)^T}\boldsymbol{M}^{(n)^{-1}}\boldsymbol{M}^{(n,r)} \tag{11c}$$

The dependence on s thus comes from the only inverse matrix needed, $\boldsymbol{D}(s)$. The eigendecomposition of the constant complex symmetrical matrix $\boldsymbol{Q} = \boldsymbol{V}\,\boldsymbol{\Lambda}\,\boldsymbol{V}^{-1}$, then gives:

$$\boldsymbol{D}(s) = \left(s\boldsymbol{C} + j\boldsymbol{V}\,\boldsymbol{\Lambda}\,\boldsymbol{V}^{-1}\right)^{-1}. \tag{12}$$

By virtue of the Woodbury matrix identity, and the equivalent forms reported by [18], $D(s)$ is given by:

$$\boldsymbol{D}(s)=\frac{\boldsymbol{C}^{-1}-\boldsymbol{C}^{-1}j\boldsymbol{V}\left(s\boldsymbol{\Lambda}^{-1}+\boldsymbol{V}^{-1}\boldsymbol{C}^{-1}j\boldsymbol{V}\right)^{-1}\boldsymbol{V}^{-1}\boldsymbol{C}^{-1}}{s}$$

$\boldsymbol{C}$ coincides with the identity matrix $\boldsymbol{E}$ (unitary capacitors in internal resonating nodes) and $\boldsymbol{\Lambda}$ is the diagonal matrix containing the eigenvalues λ_r, yielding:

$$\boldsymbol{D}(s)=\frac{\boldsymbol{E}-j\boldsymbol{V}\left(s\,\mathrm{Diag}\left\{\frac{1}{\lambda_r}\right\}+j\boldsymbol{E}\right)^{-1}\boldsymbol{V}^{-1}}{s}$$

The term in parentheses is a diagonal matrix $\mathrm{Diag}\left\{\frac{s}{\lambda_r}+j\right\}$ with $r\in[1;R]$, easily invertible element-by-element:

$$\boldsymbol{D}(s)=\frac{\boldsymbol{E}-j\boldsymbol{V}\,\mathrm{Diag}\left\{\frac{\lambda_r}{s+j\lambda_r}\right\}\boldsymbol{V}^{-1}}{s}$$

Since the objective is reaching rational expressions, the common denominator is isolated, apparently of degree $R+1$:

$$\boldsymbol{D}(s)=\frac{\prod_r(s+j\lambda_r)\boldsymbol{E}-j\boldsymbol{V}\,\mathrm{Diag}\left\{\lambda_k\prod_{r\neq k}(s+j\lambda_r)\right\}\boldsymbol{V}^{-1}}{s\prod_r(s+j\lambda_r)}$$

The numerators of $\boldsymbol{D}(s)$ however all contain at least one root in $s=0$, verifiable evaluating the numerator in the origin:

$$\prod_r(j\lambda_r)\boldsymbol{E}-\boldsymbol{V}\,\mathrm{Diag}\left\{\prod_{r=1}^{N}(j\lambda_r)\right\}\boldsymbol{V}^{-1}=\prod_r(j\lambda_r)\left(\boldsymbol{E}-\boldsymbol{V}\boldsymbol{V}^{-1}\right)$$

which is identically $\boldsymbol{0}$. Therefore, a common s term can be gathered in the numerators and simplified with the corresponding term in the denominator, to obtain that $\boldsymbol{D}(s)$ is indeed a matrix of rational functions of degree at most R, whose denominator is indicated as $K(s)$:

$$\boldsymbol{D}(s)=\frac{\left(\boldsymbol{E}\prod_r(s+j\lambda_r)-j\boldsymbol{V}\,\mathrm{Diag}\left\{\lambda_k\prod_{r\neq k}(s+j\lambda_r)\right\}\boldsymbol{V}^{-1}\right)/s}{\boxed{\prod_{r=1}^{R}(s+j\lambda_r)}\rightarrow=K(s)} \tag{13}$$

The element in row $v\in[1;R]$ and column $w\in[1;R]$ of the numerator of the symmetrical $\boldsymbol{D}(s)$ is a polynomial:

$$D(s)^{\mathrm{Num}}_{v,w\neq v}=-\frac{j}{s}\sum_q V_{v,q}\lambda_q\prod_{r\neq q}(s+j\lambda_r)\left(\boldsymbol{V}^{-1}\right)_{q,w} \tag{14}$$

$$D(s)^{\mathrm{Num}}_{v,v}=\frac{K(s)-j\sum_q V_{v,q}\lambda_q\prod_{r\neq q}(s+j\lambda_r)\left(\boldsymbol{V}^{-1}\right)_{q,v}}{s} \tag{15}$$

With this knowledge, eq. (11a) can be thus solved to obtain the rational matrix representing the impedance matrix at the internal nodes, $\boldsymbol{z}^{(i)}(s)$. The denominator is $K(s)$, visible in eq. (13). The numerator polynomial in row $v\in[1;N+R]$ and column $w\in[1;N+R]$ instead is:

$$z^{(i)}(s)^{\mathrm{Num}}_{v>N,w>N}=D(s)^{\mathrm{Num}}_{v-N,w-N} \tag{16}$$

$$z^{(i)}(s)^{\mathrm{Num}}_{v\leq N,w>N}=-\sum_{q=1}^{R}F_{v,q}D(s)^{\mathrm{Num}}_{q,w-N} \tag{17}$$

$$z^{(i)}(s)^{\mathrm{Num}}_{v\leq N,w\leq N}=-j\left(\boldsymbol{M}^{(n)^{-1}}\right)_{v,w}\cdot K(s)+\sum_{q,t=1}^{R}F_{v,q}\cdot D(s)^{\mathrm{Num}}_{q,t}\cdot F_{w,t}$$
$$z^{(i)}(s)^{\mathrm{Num}}_{v>N,w\leq N}=z(s)^{\mathrm{Num}}_{w,v}\quad\text{(for symmetry)} \tag{18}$$

Once the elements of $\boldsymbol{z}^{(i)}(s)$ are ready, the rational expressions of all the elements of eq. (6) can be computed. The R-degree denominator of the desired $\boldsymbol{Y}(s)$ admittance parameters observed at the ports is still $Y^{\mathrm{Den}}(s)=K(s)$, given in eq. (13). The rational admittance element in row $v\in[1;P]$ and column $w\in[1;P]$ is $Y(s)_{v,w}=Y^{\mathrm{Num}}(s)_{v,w}/K(s)$ with:

$$Y^{\mathrm{Num}}(s)_{v,w}=\begin{Bmatrix}jM^{(p)}_{v,w}\cdot K(s) & \text{if } v\neq w\\ \left(jM^{(p)}_{v,w}+sC_v\right)\cdot K(s) & \text{if } v=w\end{Bmatrix}$$
$$+\sum_{q,t=1}^{N+R}M^{(p,i)}_{v,q}\cdot z^{(i)}(s)^{\mathrm{Num}}_{q,t}\cdot M^{(p,i)}_{w,t} \tag{19}$$

On the diagonal, the numerator polynomials may increase to $R+1$, if capacitors are present on the ports.

As final consideration, the admittance parameters can be converted to scattering parameters via eq. (8). For $P=1$, the usual scalar formula can be used:

$$S_{1,1}(s)=\frac{K(s)-Y^{\mathrm{Num}}(s)}{K(s)+Y^{\mathrm{Num}}(s)} \tag{20}$$

For $P=2$, suppressing the s-dependency:

$$S_{1,1}=\frac{-H+\left(Y^{\mathrm{Num}}_{2,2}-Y^{\mathrm{Num}}_{1,1}\right)K+(K)^2}{L} \tag{21a}$$

$$S_{2,2}=\frac{-H+\left(Y^{\mathrm{Num}}_{1,1}-Y^{\mathrm{Num}}_{2,2}\right)K+(K)^2}{L} \tag{21b}$$

$$S_{1,2}=\frac{-2Y^{\mathrm{Num}}_{1,2}\cdot K}{L}=S_{2,1} \tag{21c}$$

where $H=Y^{\mathrm{Num}}_{1,1}\cdot Y^{\mathrm{Num}}_{2,2}-Y^{\mathrm{Num}}_{1,2}\cdot Y^{\mathrm{Num}}_{2,1}$, and L is the denominator of the scattering parameters $S^{\mathrm{Den}}(s)$:

$$S^{\mathrm{Den}}(s)=L{=}H+\left(Y^{\mathrm{Num}}_{1,1}+Y^{\mathrm{Num}}_{2,2}\right)K+(K)^2 \tag{22}$$

While the polynomials obtained for the scattering matrix may in general have twice the degree of the admittance, if the transfer function has minimum McMillan degree [19], [20], simplifications occur thanks to H being proportional to K via another polynomial T. This occurs in most practical cases including all lossless transfer functions, yielding:

$$S_{1,1}=\frac{-T+Y^{\mathrm{Num}}_{2,2}-Y^{\mathrm{Num}}_{1,1}+K}{\hat{L}} \tag{23a}$$

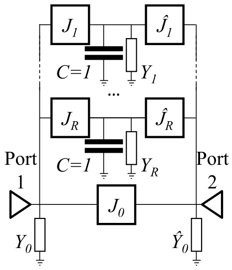
$$S_{2,2}=\frac{-T+Y^{\mathrm{Num}}_{1,1}-Y^{\mathrm{Num}}_{2,2}+K}{\hat{L}} \tag{23b}$$

$$S_{1,2}=\frac{-2Y^{\mathrm{Num}}_{1,2}}{\hat{L}}=S_{2,1} \tag{23c}$$

with degree equal to the admittance's: $\hat{L}=T+Y^{\mathrm{Num}}_{1,1}+Y^{\mathrm{Num}}_{2,2}+K$.

IV. Examples

As first application examples, a 2-port transversal network without non-resonating nodes is analyzed, assuming all unitary



$$M = \left[\begin{array}{cc|ccc} Y_0/j & J_0 & J_1 & J_i & J_R \\ J_0 & \hat{Y}_0/j & \hat{J}_1 & \hat{J}_i & \hat{J}_R \\ \hline J_1 & \hat{J}_1 & Y_1/j & 0 & 0 \\ J_i & \hat{J}_i & 0 & Y_i/j & 0 \\ J_R & \hat{J}_R & 0 & 0 & Y_R/j \end{array}\right] = \begin{bmatrix} M^{(p)} & M^{(p,i)} \\ M^{(p,i)T} & M^{(r)} \end{bmatrix}$$

Fig. 3. Transversal network (left), as typically obtained in the first step of a design of a microwave filter based on coupling matrix approaches, and its corresponding coupling matrix (right) with the used partition.

capacitors and where the external ports are nodes 1 and R+2, both with unitary reference impedance, as shown in Fig. 3.

The coupling matrix can be written and sorted to have the two ports as nodes 1 and 2, as shown in Fig. 3 (right).

The lines highlight the partitioning applied according to eq. (10). Since there are no non-resonating nodes, $\boldsymbol{M}^{(n)}$ and $\boldsymbol{M}^{(n,r)}$ are empty, respectively 0×0 and $0 \times R$. Matrix $\boldsymbol{Q}$ hence coincides with $\boldsymbol{M}^{(r)}$, from eq. (11c), and the overall $\boldsymbol{z}^{(i)}(s)$ coincides with $\boldsymbol{D}(s)$. The eigenvalue decomposition of $\boldsymbol{Q}$ is trivial, as $\boldsymbol{M}^{(r)}$ is already diagonal: the eigenvector matrix coincides with the identity $\boldsymbol{V} = \boldsymbol{E}$ and the eigenvalues are $\lambda_r = Y_r/j$. The denominator $K(s)$ of $\boldsymbol{D}(s)$, which is also the denominator of $\boldsymbol{z}^{(i)}(s)$ and of the admittance matrix $\boldsymbol{Y}(s)$, can hence be already obtained from eq. (13): $K(s) = (s+Y_1)\cdot \ldots \cdot (s+Y_R)$. The polynomial numerators of $\boldsymbol{D}(s)$ are computed via eq. (13) and they also coincide with $\boldsymbol{z}^{(i)}(s)^{\mathrm{Num}}$ by eq. (16):

$$z^{(i)}(s)^{\mathrm{Num}}_{[v,w\neq v]} = D(s)^{\mathrm{Num}}_{[v,w\neq v]} = 0$$

$$z^{(i)}(s)^{\mathrm{Num}}_{[v,v]} = D(s)^{\mathrm{Num}}_{[v,v]} = \frac{K(s) - Y_v \prod_{r\neq v}(s+Y_r)}{s} = \prod_{r\neq v}(s+Y_r)$$

The 2×2 admittance matrix at the ports is found by eq. (19):

$$Y(s)^{\mathrm{Num}}_{1,1} = Y_0 K(s) + \sum_{q,t=1}^{R} J_q z_i(s)^{\mathrm{Num}}_{[q,t]} J_t = Y_0 K(s) + \sum_{q=1}^{R} J_q^2 \prod_{r\neq q}(s+Y_r)$$

$$Y(s)^{\mathrm{Num}}_{2,2} = \hat{Y}_0 K(s) + \sum_{q=1}^{R} \hat{J}_q^2 \prod_{r\neq q}(s+Y_r)$$

$$Y(s)^{\mathrm{Num}}_{1,2} = jJ_0 K(s) + \sum_{q=1}^{R} J_q \hat{J}_q \prod_{r\neq q}(s+Y_r)$$

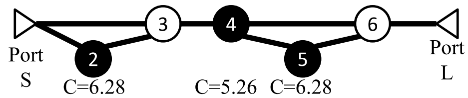
To check the validity of the results provided by the proposed technique, they are compared against the analysis of the network, easily obtained as parallel connection of R branches:

$$\boldsymbol{Y}(s) = \begin{bmatrix} Y_0 & jJ_0 \\ jJ_0 & \hat{Y}_0 \end{bmatrix} + \sum_{r=1}^{R} \frac{\begin{bmatrix} J_r^2 & J_r\hat{J}_r \\ J_r\hat{J}_r & \hat{J}_r^2 \end{bmatrix}}{s+Y_r} \tag{24}$$

which indeed coincides with the derived expressions.

A second example reports a 3rd order elliptical prototype filter, with scattering parameters defined by:

$$\hat{\mathbf{S}} = \frac{\begin{bmatrix} s^3 + 0.75869s & 0.17446s^2 + 2.5755 \\ 0.17446s^2 + 2.5755 & s^3 + 0.75869s \end{bmatrix}}{s^3 + 2.304s^2 + 3.3978s + 2.5755} \tag{25}$$



0.677	8.019	1.000	0	0	0	0
\	40.17	8.019	0	0	0	0
\	\	1.476	1.000	0	0	0
\	\	\	3.100	3.818	1.000	0
\	\	\	\	38.72	3.818	0
\	\	\	\	\	0.413	-0.280
\	\	\	\	\	\	0.189

Fig. 4. Topology and symmetric coupling matrix of the 3rd order filter.

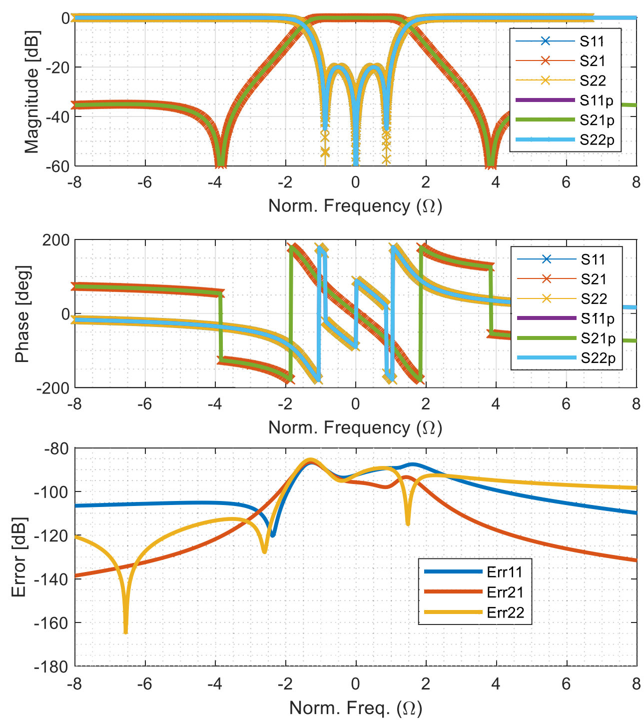


Fig. 5. Response of the scattering parameters obtained from the initial filter polynomials and from the reconstruction via the coupling matrix using the proposed technique: they are visually indistinguishable (top: magnitude, center: phase). The error in the response (bottom), defined at each frequency as $20\,\mathrm{Log}_{10}\left|S_{v,w} - \hat{S}_{v,w}\right|$, is below $-85\,$dB in the whole sampling range.

This filter has been synthesized by section extraction in a purposefully unusual configuration involving 3 resonators, 2 non-resonant nodes, 2 ports with non-resonating susceptances, and some cross-couplings, obtaining the topology and coupling matrix in Fig. 4. The first observation is that the capacitors in the internal nodes are not unitary, and hence are scaling accordingly. After that, the nodes are sorted to have first the 2 external port nodes, then the 2 non-resonating internal nodes and finally the 3 resonating nodes with unitary capacitors. Then the procedure is carried out as explained in Section III, obtaining the admittance polynomials and then converted to scattering via eqs. (23a) to (23c). The obtained rational scattering parameters can be evaluated in the normalized frequency $s = j\Omega$ and the responses are compared against those of the initial polynomials in Fig. 5, showing a negligible error.

A quantitative estimation of the error in the roots location of the estimated polynomials shows that, reflection zeros and poles are all within $6 \cdot 10^{-8}$ while transmission zeros are within $2 \cdot 10^{-14}$. Such negligible errors prove that the

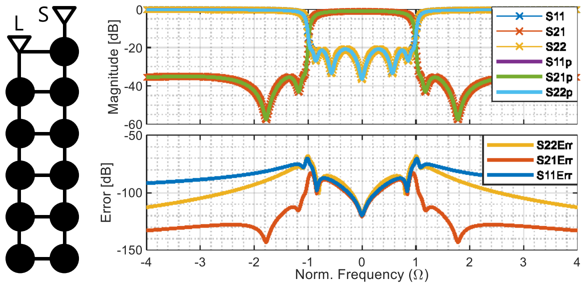


Fig. 6. 11-th order filter example. Left: topology with lossy resonators. Right top: response of the network and reconstructed rational response; not all reflection and transmission zeros are distinguishable, due to proximity and finite quality factor. Right bottom: reconstruction error below −65 dB.

technique robustly computes the required polynomials even including roundoff and numerical errors introduced by the section extraction algorithm which created the topology.

A third example instead concerns an elliptic characteristic of 11th order, with 10 finite transmission zeros and implemented in folded form. A quality factor of 30 is artificially applied to obtain a lossy coupling matrix, to prove that the proposed technique does not rely on losslessness.

The results are shown in Fig. 6. The reconstructed polynomials are found to implement roots very close to the original ones (including losses): the errors on the poles and reflection zeros are all below $1 \cdot 10^{-4}$ for both $S_{1,1}$ and $S_{2,2}$; the errors on the transmission zeros are below $1 \cdot 10^{-9}$. The evaluated response of the reconstructed polynomials, shown in Fig. 6, is visually indistinguishable from the response of the original polynomials (including losses, estimated via [21] to correspond to a quality factor of 30, confirming the initial imposition). For comparison, standard Cauchy fitting over the original noiseless response uniformly sampled in 100000 points between −10 and + 10 gives an unacceptable rational model, including one pole with positive real part and a very high fitting error in the reconstructed response (up to 0 dB) and in the location of roots (up to 0.11 even excluding the unstable pole): especially with lossy data, the traditional fitting cannot detect zeros in close proximity and the accuracy quickly degrades.

V. Conclusion

The technique proposed in this brief allows to analytically compute the polynomials defining the rational response of a reciprocal electrical network involving resonators and non-resonating nodes, arbitrarily coupled. Leveraging the form of the involved matrices, it avoids complicated operations (e.g., the inverse of a dense polynomial matrix), which otherwise hamper direct computations on the coupling matrix; only inversions of numerical blocks of the coupling matrix and simple polynomial additions and multiplications are required.

The assumptions involve the invertibility of the coupling matrix between non-resonant nodes and the absence of frequency-variant couplings. Both lossy and lossless networks can be treated, returning the corresponding rational response from their coupling matrix.

This technique offers accurate analysis of coupled-resonator structures, based on the underlying polynomials rather than today's fitting of the sampled response, in the end providing a direct assessment of the location of poles and zeros of the network, not dependent upon sampling and convergence. This constitutes a palatable feature for electromagnetic simulators as well as computer-assisted design tools.

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Open Access funding provided by 'Politecnico di Milano' within the CRUI CARE Agreement