

Asymptotic solutions for the modal properties of shallow cables with small bending stiffness and compliant supports

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ABSTRACT

Metallic cables are key components of large and strategic infrastructures, such as suspension or cable-stayed bridges, and are also adopted in many special structural applications, including cable-roof systems and slender footbridges. Monitoring of the cable axial force, possibly along with other relevant structural parameters, is crucial for early detection of structural damage. Vibration-based identification procedures are widely used for this purpose, and they rely on the measurement of a set of low-order natural frequencies and mode shapes of the cable, along with structural models linking these properties to the cable parameters. The accuracy of the identified parameters is inherently affected by the ability of the underlying structural model to correctly reproduce the essential features of the linear dynamics of the cable. Analytical models, if properly formulated, offer accuracy, computational efficiency, and clear insight into parameters sensitivity. This paper addresses a notable gap in the literature by developing asymptotic closed-form solutions for the modal properties of shallow cables that simultaneously account for bending stiffness, sag/extensibility effects, and rotational flexibility at the supports. The results obtained with both a classical asymptotic approach to the solution of the algebraic eigenvalue problem and the matched asymptotic expansion method are systematically compared against finite elements simulations and semi-analytical solutions, revealing remarkable accuracy for a broad range of values of Irvine's parameter. These results offer practical tools for efficient monitoring of cable structures where both bending stiffness and cable extensibility effects are relevant, reducing computational costs in inverse analysis and SHM applications.

1. Introduction

Metallic cables are key components of large and strategic infrastructures, such as suspension or cable-stayed bridges and overhead electrical lines. Due to their large strength to weight ratio and flexibility, they are also efficiently adopted in many special structural applications, such as cable-roof systems, slender footbridges and deployable structures.

The relevance of such structural applications clearly motivates the interest of researchers and engineers in developing simple yet effective techniques for the identification and monitoring of the cable axial force, possibly along with other relevant structural parameters such as the cable bending stiffness. Variations in time of the cable axial force, indeed, can allow for detection of damage phenomena at an earlier stage than the one revealed by visual inspection [1,2].

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Vibration-based axial force identification procedures are the most widespread non-destructive Structural Health Monitoring (SHM) techniques for cable structures [3–5]. They are based on: (a) the measurement of a set of natural frequencies and mode shapes of the cable through either ambient or forced vibration tests; (b) the definition of a structural model that relates the modal properties of the cable to its axial force and other relevant structural parameters, such as for example the cable bending stiffness. An optimization problem is then set up to find out the set of structural parameters that minimizes the difference between the measured modal frequencies and mode shapes and the corresponding predictions of the structural model.

Within this framework, it is clear that the accuracy of the identified parameters is inherently affected by the ability of the underlying structural model to correctly reproduce the essential features of the linear dynamics of the cable element. Besides accuracy, two additional requirements that should guide the definition of the structural model to be used within the identification procedure are the computational efficiency and the model sensitivity with respect to the variables of the optimization problem.

Analytical models, if properly formulated, can be at the same time sufficiently accurate and computationally efficient. Moreover, they typically allow to clearly recognize the role of the main structural parameters on the overall dynamic properties of the system, paving the way to a detailed analysis of the model sensitivity. Different analytical models have been developed in the literature to describe the linear vibrations of a cable element for SHM purposes. They can be tentatively classified based on the structural theory adopted to model the cable element and on the assumptions introduced to model the interaction between the cable and the surrounding structural elements, i.e. the boundary conditions of the cable model.

Due to their slenderness, cables are often modeled as “perfectly-flexible” one-dimensional continua, that can only withstand axial forces [6,7]. The linear dynamics of shallow suspended cables, i.e. cables characterized by small values of the sag-to-span ratio δ in the static equilibrium configuration (in the order of $\delta < \frac{1}{8}$), has been thoroughly investigated by Irvine and Caughey [8]. Linear vibrations of the perfectly flexible shallow cable model are characterized by the decoupling between in-plane and out-of-plane vibrations, i.e. between vibrations respectively belonging to and perpendicular to the plane of the gravity loads. Out-of-plane vibrations are governed by the same equation of the taut string model, whose natural frequencies and mode shapes can be evaluated through exact closed-form expressions. In-plane vibrations, on the other hand, turn out to be controlled by a single dimensionless parameter, i.e. the Irvine parameter λ^2 , which synthetically accounts for the relative importance between elastic and geometric stiffness effects.

Cables strung at relatively large values of axial load, such as stay cables in bridges, are associated to small values of the static sag (i.e. $\delta \ll 1$) and of the Irvine parameter (i.e. $\lambda^2 \lesssim 1$). It is worth noting that for vanishingly small values of the Irvine parameter, i.e. formally $\lambda^2 \rightarrow 0$, both the effects of the axial extensibility and of the static sag tend to become negligible and the shallow cable model asymptotically approaches the classic taut string one. Starting from this observation, the taut string model has often been employed to approximately describe both out-of-plane and in-plane linear vibrations of a cable element for SHM purposes, e.g. [9,10].

Neglecting both sag/extensibility and bending stiffness effects, however, may lead to an oversimplified structural model and, consequently, to biased estimates of the cable axial force as it has been shown, for example, in [11].

Enriched cable models accounting for the bending stiffness of the cable cross sections but neglecting sag/extensibility effects have also been proposed in the literature and applied to SHM. Within this context, shear deformability is typically neglected and the cable is modeled as a straight tensioned Euler-Bernoulli beam model, e.g. [11–18].

The introduction of the bending stiffness makes the dynamic properties of the structural model depending on the non-dimensional group $\varepsilon = \sqrt{\frac{EI}{HL^2}}$, where: EI , H and L denote the bending stiffness, the horizontal component of the axial force and the span length of the cable, respectively. Closed-form exact solutions for the eigenvalue problem of the straight cable with small bending stiffness are only available for the special case of doubly-hinged boundary conditions. Whenever dealing with the case of doubly-clamped boundary conditions or with the more general case of elastically restrained cable end sections, closed-form exact solutions are no longer available and numerical models (based, for example, on finite element discretizations of the structural problem [17]) or semi-analytical models (relying on the numerical solution of the frequency equation of the model [18]) can be adopted to evaluate the natural frequencies and mode shapes of the cable.

In typical structural applications, however, the non-dimensional bending parameter ε is a small number, i.e. $\varepsilon \ll 1$. The smallness of the bending stiffness term makes the eigenvalue problem of the cable model singularly perturbed and hints the existence of boundary layers in the mode shapes [19], i.e. portions of the cable close to the end sections where the bending moment distribution is characterized by a large gradient and the mode shapes considerably differ from the ones of the underlying taut string model (that can be recovered, as a special case, by setting $\varepsilon = 0$).

The smallness of the cable bending stiffness paves the way to the adoption of perturbation methods to find closed-form asymptotic expressions for the natural frequencies and mode shapes of the problem. Closed-form asymptotic expressions for the natural frequencies of a doubly-clamped straight cable with small bending stiffness have been presented, for example, in [14,15] and later generalized by Ceballos and Prato [13] to deal with elastically restrained rotations of the cable end sections. The problem was later reconsidered in [11], where asymptotic solutions were presented for both the natural frequencies and mode shapes of a straight cable with small bending stiffness on elastic supports. Within this context, the asymptotic solutions have clearly revealed that the flexibility of the supports affect the lower natural frequencies of the cable model through a single structural parameter. The performance of different numerical strategies for the identification of both the axial force and the bending stiffness in presence of unknown boundary conditions were then thoroughly investigated. Minimal conditions for the identification of the axial force, bending stiffness and rotational flexibility of the cable restraints, were later investigated by exploiting perturbation techniques in [16].

In some applications, however, it is important to account for both the small bending stiffness and the sag/extensibility effects to get a precise estimation of the lowest in-plane natural circular frequency of the system. This is the case, for example, of cable-roof

beams, transversally rigid footbridges or the long stabilizing cables in multi-span stayed bridges, where the in-plane dynamics is governed by low-order modes and the cables may present a non-negligible initial sag compared to their span length (i.e. $\lambda^2 = \text{ord}(1)$).

Several researchers have investigated the linear vibrations of shallow cables with small bending stiffness through numerical models, based either on finite element [20] or finite difference [21,22] discretizations of the structural problem, or semi-analytical approaches relying on the numerical solution of the frequency equations of the model [23–25]. Most of these studies assumed simple boundary conditions at the cable end sections, in the form of either perfect hinges or perfect clamps. Extensive parametric analyses were carried out (e.g. [20,22,23]) highlighting the effect of the main non-dimensional parameters that govern the problem, namely: the Irvine parameter λ^2 and the non-dimensional bending stiffness ϵ . The Irvine parameter has been shown to significantly affect the first symmetric in-plane mode of long cables, while the bending stiffness parameter has been shown to significantly affect higher-order natural frequencies of the cable.

No previous attempts have been made, to the best of the authors' knowledge, to derive approximate closed-form expressions for the spectral properties of shallow cables by leveraging the smallness of the non-dimensional bending parameter.

Whenever dealing with inverse structural analysis (e.g. SHM applications) or preliminary design problems, however, approximate closed-form expressions of the natural frequencies and mode shapes are of utmost importance. In fact, SHM methodologies typically involve iterative optimization problems, near-real-time processing, and large data sets, for which the repeated numerical evaluation of the cable's spectral properties may significantly increase the overall computational cost. To overcome this drawback, the present paper aims at developing general closed-form solutions for the spectral properties of a shallow cable with small bending stiffness and supports rotational compliance.

In doing so, this work addresses a notable gap in the literature by providing asymptotic solutions for the modal properties of cables that simultaneously account for both bending stiffness and cable extensibility effects, which have not been comprehensively treated in past studies.

To set up the basic framework, the case of perfectly flexible sagged cable is firstly considered in Section 2.1. The equation of motion of the cable is written in dimensionless form and Irvine's frequency equation is readily obtained. Then, a closed-form expression of the natural circular frequencies accounting for sagging effect is derived thanks to the application of the Rayleigh-Ritz discretization method.

In Section 2.2, the case of a straight tensioned beam/cable with small bending stiffness and rotational springs at its ends is considered. A semi-analytical solution of the eigenvalue problem is developed, and the asymptotic expansion of both natural circular frequencies and mode shapes in the small dimensionless bending stiffness parameter is briefly revised. A hybrid approach for the reconstruction of the mode shapes is also introduced, and compared against the results of a finite element simulation.

Section 3 attacks the case of a shallow sagged cable with small bending stiffness and rotational springs at its end. First, the semi-analytical solution of the eigenvalue problem is extended to the case of sagged cables, as a logical continuation of the simpler result previously obtained. The same is done for the asymptotic expansion of the natural circular frequencies, while in Section 3.4 a novel accurate solution for both mode shapes and circular frequencies is derived by employing the Matched Asymptotic Expansion (MAE) method.

Within this spirit, the eigenvalue problem is solved in the continuous form by matching an outer solution, capturing the far-field behavior of the sagged cable, with two inner solutions, expressing the corrections induced by both the bending stiffness effect and the non-homogeneous boundary conditions, in the regions close to the cable ends.

In Section 4 the accuracy of different perturbative techniques is systematically compared, both in terms of natural frequencies and mode shapes, against the semi-analytical solutions and the outcome of finite element models. Finally, Section 5 draws the conclusions of the work.

2. Preliminaries

2.1. Perfectly flexible sagged cable

Let us consider a perfectly flexible cable with axial stiffness EA , span length L , mass per unit of length μ and subject to a pretension force H . The cable is simply supported at the same level and characterized by a shallow static equilibrium configuration described by the function $y(x)$, with $x \in [0, L]$ an abscissa spanning the horizontal chord of the cable (see Fig. 1).

According to the small-sag cable theory [6], the static profile of the cable may be approximately described by the following parabolic shape:

$$y(x) = x(L - x) \frac{4d}{L^2} \quad (1)$$

where the symbol d denotes the sag at the midspan (see Fig. 1). Differentiating twice Eq. (1), the static curvature of the cable can be expressed as:

$$\frac{\partial^2 y}{\partial x^2} = -\frac{8d}{L^2} = -\frac{\mu g}{H} \quad (2)$$

g being the gravity acceleration.

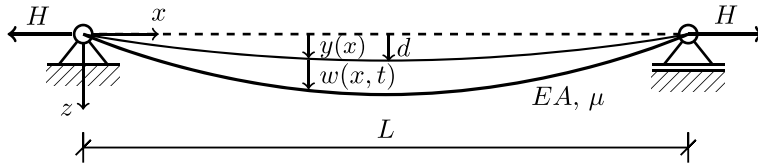


Fig. 1. Structural scheme of a sagged, perfectly flexible cable, with supports at the same level.

Small undamped planar free vibrations of the cable are governed by the following second-order partial differential equation (e.g. [6]):

$$H \frac{\partial^2 w}{\partial x^2} + h(t) \frac{d^2 y}{dx^2} = \mu \frac{\partial^2 w}{\partial t^2} \quad (3)$$

where t denotes the time, $w(x, t)$ is the dynamic displacement of the cable centerline around the static profile and $h(t)$ is the time-varying additional horizontal component of the axial force. According to [6], $h(t)$ can be expressed as:

$$h(t) = \frac{\mu g}{H} \frac{EA}{L_e} \int_0^L w(x, t) dx \quad (4)$$

where

$$L_e \simeq L \left[1 + \frac{1}{8} \left(\frac{\mu g L}{H} \right)^2 \right] \quad (5)$$

is the virtual length of the cable. Introducing the well-known elasto-geometric Irvine's parameter:

$$\lambda^2 = \left(\frac{\mu g L}{H} \right)^2 \frac{EA}{H} \frac{L}{L_e} \quad (6)$$

Eq. (4) can also be rearranged as:

$$h(t) = \frac{H^2 \lambda^2}{\mu g L^3} \int_0^L w(x, t) dx \quad (7)$$

Let us now introduce a dimensionless abscissa $\xi = x/L$ and a dimensionless time variable $\tau = \varpi_c t$, ϖ_c being the following characteristic circular frequency:

$$\varpi_c = \frac{1}{L} \sqrt{\frac{H}{\mu}} \quad (8)$$

Eq. (3) can be readily translated in non-dimensional form:

$$-\frac{\partial^2 \bar{w}}{\partial \xi^2} + \lambda^2 \int_0^1 \bar{w} d\xi + \frac{\partial^2 \bar{w}}{\partial \tau^2} = 0 \quad (9)$$

where $\bar{w}(\xi, \tau) = w(x(\xi), t(\tau))/L$ is the dimensionless dynamic transverse displacement of the cable.

The equation of motion (9) goes along with suitable initial conditions and homogenous boundary conditions:

$$\bar{w}(0, \tau) = \bar{w}(1, \tau) = 0, \quad \forall \tau \quad (10)$$

Stationary oscillatory solutions of Eq. (9) read:

$$\bar{w}(\xi, \tau) = \phi(\xi) p(\tau) \quad (11)$$

where $\phi(\xi)$ is a mode shape function and $p(\tau)$ a harmonic function of τ with (nondimensional) circular frequency ω .

Substitution of Eq. (11) into Eqs. (9) and (10) yields the second order Sturm-Liouville problem defined by the ordinary differential equation:

$$-\phi'' + \lambda^2 \int_0^1 \phi(\xi) d\xi - \omega^2 \phi = 0 \quad (12)$$

along with the homogeneous boundary conditions:

$$\phi(0) = \phi(1) = 0 \quad (13)$$

As it is well known, the problem admits countably infinite solutions: $\{\phi_n(\xi), \omega_n\}$, $n \in \mathbb{N}^+$, that will be briefly discussed in the rest of this section. Please notice that once the non-dimensional natural frequencies ω_n are known, their dimensional counterpart can be simply obtained as: $\varpi_n = \varpi_c \omega_n$, ϖ_c being the characteristic frequency defined in Eq. (8).

2.1.1. Semi-analytical solution of the eigenvalue problem

Following [6], the solutions of the eigenvalue problem defined by the Eqs. (12) and (13) are sought by distinguishing between antisymmetric and symmetric mode shape functions $\phi(\xi)$. In the first case, the integral in Eq. (12) is equal to zero and the natural frequencies and mode shapes coincide with the ones of a taut string:

$$\phi_n(\xi) = \sin(\omega_n \xi), \quad \omega_n = n\pi, \quad n = 2k, \quad k \in \mathbb{N}^+ \quad (14)$$

Whenever dealing with symmetric mode shape functions $\phi(\xi)$, instead, general solutions of the governing Eq. (12) can be expressed as:

$$\phi(\xi) = \sin(\omega \xi) + C \cos(\omega \xi) + \frac{\lambda^2}{\omega^2 - \lambda^2} \left[\frac{1}{\omega} (1 - \cos \omega) + \frac{C}{\omega} \sin \omega \right] \quad (15)$$

where C is a real integration constant that is evaluated by imposing the first boundary condition, i.e. $\phi(0) = 0$:

$$C = \frac{\lambda^2(1 - \cos \omega)}{\lambda^2(\omega - \sin \omega) - \omega^3} \quad (16)$$

Back substitution of Eq. (16) into Eq. (15) and imposition of the second boundary condition, $\phi(1) = 0$, delivers the frequency equation for the symmetric in-plane modes (cf. [6]):

$$\tan\left(\frac{\omega}{2}\right) = \frac{\omega}{2} - \frac{4}{\lambda^2} \left(\frac{\omega}{2}\right)^3 \quad (17)$$

Closed-form solutions of Eq. (17) are not available. The roots ω_n ($n = 2k + 1$, with $k \in \mathbb{N}$) of the frequency Eq. (17), hence, must be first numerically evaluated and then substituted back into Eqs. (15) and (16) to get the symmetric mode shapes of the cable.

2.1.2. Approximate closed-form expression for the natural frequencies of the cable

In order to avoid the numerical solution of the frequency Eq. (17), a simple yet effective approximate closed-form expression for the natural frequencies of the cable is presented in the following. This expression will be then used in Sec. (3.3) as a part of the solution strategy adopted to deal with the more general case of a shallow cable with small bending stiffness.

Let us consider, in the spirit of the classical Rayleigh-Ritz discretization method, the trigonometric series expansion of the nondimensional displacement function $\bar{w}(\xi, \tau)$:

$$\bar{w}(\xi, \tau) = \sum_{n=1}^{\infty} \sin(\bar{\omega}_n \xi) \bar{p}_n(\tau) \quad (18)$$

where $\bar{p}_n$ may assume the physical meaning of the n th dimensionless principal coordinate. It is also worth noting that, within this context, the orthogonal basis $\{\bar{\phi}_n = \sin(\bar{\omega}_n \xi)\}$ and the parameters $\bar{\omega}_n$ can be conveniently regarded as an approximation of, respectively, the exact modal basis and the nondimensional natural frequencies of the cable. More in details, the basis $\{\bar{\phi}_n\}$ provides an exact description of the antisymmetric modes of the cable ($n = 2k$, with $k \in \mathbb{N}^+$), while the symmetric ones ($n = 2k + 1$, with $k \in \mathbb{N}$) are only approximately represented (cf. Eqs. (14) and (15)).

By recalling the orthogonality properties of the sinusoidal functions, the approximate frequencies $\bar{\omega}_n$ can be easily obtained, through an application of the Rayleigh-Ritz approach, as:

$$\bar{\omega}_n = \sqrt{\frac{\partial \mathcal{U}_{[2]}}{\partial \bar{p}_n} \left(\frac{\partial \mathcal{T}}{\partial \dot{\bar{p}}_n} \right)^{-1}} \quad (19)$$

where the dot symbol denotes the total derivative with respect to the dimensionless time τ , while $\mathcal{T}$ and $\mathcal{U}_{[2]}$ are, respectively, the kinetic energy and a second-order approximate series expansion of the potential energy of the cable:

$$\mathcal{T} = \frac{\mu L^3}{4} \varpi_c^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \dot{\bar{p}}_i \dot{\bar{p}}_j \quad (20)$$

$$\mathcal{U}_{[2]} = \frac{EAL^3}{2} \left(\frac{\mu g}{H} \right)^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1 - (-1)^i}{i\pi} \right) \left(\frac{1 - (-1)^j}{j\pi} \right) \bar{p}_i \bar{p}_j + HL^3 \sum_{k=1}^{\infty} \left(\frac{k\pi}{2L} \right)^2 \bar{p}_k^2 \quad (21)$$

Substitution of Eqs. (20), (21) in Eq. (19), under the additional hypothesis that $L_e = L$ (cf. Eqs. (5) and (6)), yields:

$$\bar{\omega}_n = n\pi \sqrt{1 + \frac{2\lambda^2}{(n\pi)^4} \left[1 - (-1)^n \right]^2} \quad (22)$$

It is worth noting that, as expected, due to the particular choice herein made for the basis $\{\bar{\phi}_n\}$, the Eq. (22) delivers the exact (non-dimensional) natural frequencies of the antisymmetric modes (i.e. $n = 2k$, with $k \in \mathbb{N}^+$), cf. Eq. (14). On the other hand, the relative percentage error encountered adopting Eq. (22) for the symmetric modes (i.e. $n = 2k + 1$, with $k \in \mathbb{N}$) is depicted in Fig. 2 for a broad range of values of the Irvine parameter, up to the first crossover point (i.e. $\lambda^2 = 4\pi^2$, see [6]).

Fig. 2 clearly shows that for most structural applications (e.g. stay or suspension cables), in which the Irvine's parameter typically assumes small values (i.e., $\lambda^2 < 10$), the relative error of the approximated frequency $\bar{\omega}_1$ is below 0.5%, while being practically zero for higher symmetric modes, completely justifying the adoption of Eq. (22).

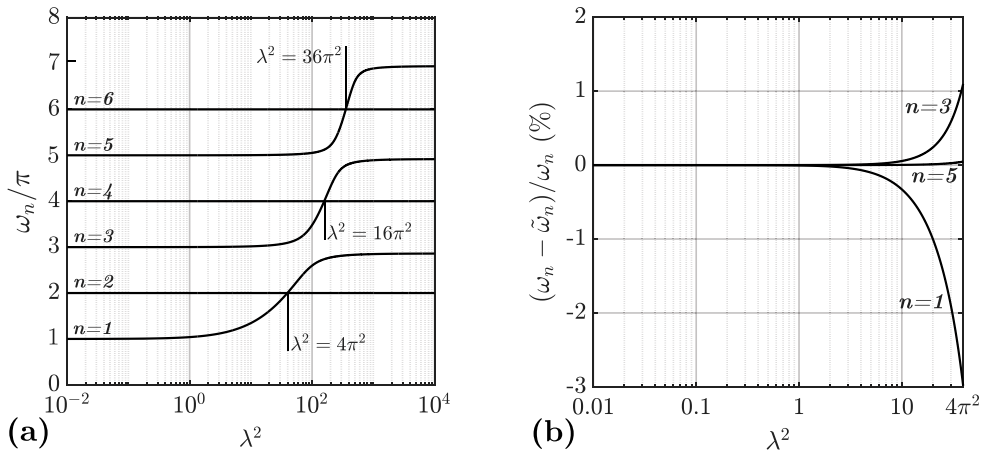


Fig. 2. (a) General dimensionless curves for the first six natural circular frequencies of the cable, obtained via numerical solution of the Irvine's equation (see Eq. (17), ω_n) for a broad range of λ^2 values. (b) Relative percentage error between the first three in-plane natural circular frequencies ($n = 1, 3, 5$) obtained via numerical solution of the Irvine's equation (see Eq. (17), ω_n) and its closed-form approximate expression (see Eq. (22), $\tilde{\omega}_n$), for a range of λ^2 values up to the first crossover point, i.e. $\lambda^2 = 4\pi^2$.

2.2. Straight cable with small bending stiffness

Let us now consider a straight cable with bending stiffness EI , span length L , mass per unit of length μ and subject to a pretension force H . The cable is simply supported at the same level and characterized by a static equilibrium configuration which coincides with the undeformed straight one. The rotations of the cable end sections are elastically restrained by two springs with stiffness K_0 and K_1 (see Fig. 3). The sag/extensibility effects are disregarded.

Small undamped free vibrations are governed by the following fourth-order partial differential equation:

$$EI \frac{\partial^4 w}{\partial x^4} - H \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial t^2} = 0 \quad (23)$$

where t is the time and $w(x, t)$ the transverse dynamic displacement of the cable centerline.

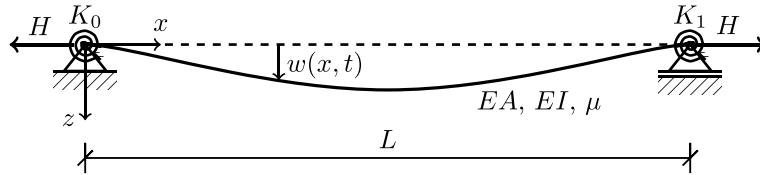


Fig. 3. Structural scheme of a straight cable with small bending stiffness and flexible rotational supports.

By recalling that $\xi = x/L$ and Eq. (8), the Eq. (23) can be restated in nondimensional form:

$$\varepsilon^2 \frac{\partial^4 \tilde{w}}{\partial \xi^4} - \frac{\partial^2 \tilde{w}}{\partial \xi^2} + \frac{\partial^2 \tilde{w}}{\partial \tau^2} = 0 \quad (24)$$

where $\tilde{w}(\xi, \tau)$ denotes the dimensionless dynamic transverse displacement of the cable and ε is the dimensionless bending parameter:

$$\varepsilon = \sqrt{\frac{EI}{HL^2}} \quad (25)$$

The equation of motion (24) goes along with suitable initial conditions and the following boundary conditions:

$$\begin{cases} \tilde{w}(0, \tau) = 0 \\ \tilde{w}(1, \tau) = 0 \\ (1 - \rho_0) \varepsilon^2 \frac{\partial^2 \tilde{w}}{\partial \xi^2}(0, \tau) - \rho_0 \frac{\partial \tilde{w}}{\partial \xi}(0, \tau) = 0 \\ (1 - \rho_1) \varepsilon^2 \frac{\partial^2 \tilde{w}}{\partial \xi^2}(1, \tau) + \rho_1 \frac{\partial \tilde{w}}{\partial \xi}(1, \tau) = 0 \end{cases} \quad (26)$$

where the flexural degree-of-fixity parameters ρ_i ($i = 0, 1$) have been introduced as in [11]:

$$\rho_i = \frac{K_i}{HL \varepsilon + K_i} \quad \text{with } i = 0, 1 \quad (27)$$

The degree-of-fixity parameters ρ_i take values in the range $[0, 1]$ with free and perfectly restrained end rotations corresponding, respectively, to the lower and upper bound values. If $\rho_0 = \rho_1 = 0$, the doubly-hinged configuration is retrieved, while if $\rho_0 = \rho_1 = 1$, the doubly-clamped configuration is obtained.

Stationary oscillatory solutions of Eq. (24) are again sought in the form of Eq. (11). Substitution of Eq. (11) in Eqs. (24) and (26) yields the fourth-order Sturm-Liouville problem defined by the ordinary differential equation:

$$\varepsilon^2 \phi'''' - \phi'' - \omega^2 \phi = 0 \quad (28)$$

along with the boundary conditions:

$$\begin{cases} \phi(0) = \phi(1) = 0 \\ (1 - \rho_0)\varepsilon^2 \phi''(0) - \rho_0 \varepsilon \phi'(0) = 0 \\ (1 - \rho_1)\varepsilon^2 \phi''(1) + \rho_1 \varepsilon \phi'(1) = 0 \end{cases} \quad (29)$$

where the apex has been used to denote the derivative with respect to the dimensionless abscissa ξ .

It is worth noting that in most structural applications (e.g. cables in stayed or suspended bridges), the dimensionless bending stiffness parameter ε is a small number (i.e. $\varepsilon \ll 1$), which makes the problem (28)–(29) singularly perturbed and hints the existence of boundary layers in the mode shapes of the cable (see, e.g., [19,26]). Typical values of ε in the case of stay cables are lower than 0.01 [4,22]. Slightly higher values can also be found in the case of cement-grouted parallel-bundle wire cables (e.g. $\varepsilon \approx 0.02 - 0.03$) or in comparatively shorter cables used in tensile structures [15,17].

2.2.1. Semi-analytical solution of the eigenvalue problem

General solutions of Eq. (28) can be expressed as:

$$\phi(\xi) = \alpha_1 \sin(z_1 \xi) + \alpha_2 \cos(z_1 \xi) + \alpha_3 \exp(-z_2 \xi) + \alpha_4 \exp(-z_2(1 - \xi)) \quad (30)$$

where $\alpha_i \in \mathbb{R}$ ($i = 1, \dots, 4$) are integration constants, while the arguments z_1 and z_2 are the two functions of the non-dimensional vibration frequency ω :

$$z_j = \frac{1}{\varepsilon \sqrt{2}} \sqrt{(-1)^j + \sqrt{1 + (2\varepsilon\omega)^2}}, \quad j = 1, 2. \quad (31)$$

Please notice that using exponential instead of hyperbolic functions in Eq. (30) is more appropriate to highlight the existence of two bending boundary layers close to the end sections of the cable, when $\varepsilon \ll 1$ and $\rho_i \neq 0$ ($i = 0, 1$). Substitution of Eqs. (30) and (31) in the boundary conditions (29) yields the algebraic eigenvalue problem:

$$\mathbf{B}(\omega; \mathcal{P}) \boldsymbol{\alpha} = \mathbf{0} \quad (32)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is a column vector listing the integration constants of Eq. (30), $\mathbf{0}$ is the four-dimensional null column vector and $\mathbf{B}$ is the 4×4 “boundary condition matrix” depending on both ω and the set of parameters: $\mathcal{P} = \{\varepsilon, \rho_0, \rho_1\}$:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & e \\ s & c & e & 1 \\ -\bar{z}_1 \rho_0 & -\bar{z}_1^2(1 - \rho_0) & \bar{z}_2^2(1 - \rho_0) + \bar{z}_2 \rho_0 & e \bar{z}_2^2(1 - \rho_0) - e \bar{z}_2 \rho_0 \\ c \bar{z}_1 \rho_1 - s \bar{z}_1^2(1 - \rho_1) & -s \bar{z}_1 \rho_1 - c \bar{z}_1^2(1 - \rho_1) & e \bar{z}_2(1 - \rho_1) - e \bar{z}_2 \rho_1 & \bar{z}_2^2(1 - \rho_1) + \bar{z}_2 \rho_1 \end{bmatrix}$$

where $c = \cos(z_1)$, $s = \sin(z_1)$, $e = \exp(-z_2)$, $\bar{z}_1 = \varepsilon z_1$ and $\bar{z}_2 = \varepsilon z_2$.

Non trivial solutions $\{\omega_n, \boldsymbol{\alpha}_n\}$, with $n \in \mathbb{N}^+$, of the algebraic eigenvalue problem (32) can be obtained by solving the characteristic equation: $\det[\mathbf{B}(\omega_n; \mathcal{P})] = 0$.

As it is well known, the latter equation admits closed-form solutions only in the special case of a doubly-hinged cable. In general, hence, the nondimensional natural frequencies ω_n must be obtained through numerical solution of the characteristic equation.

Once the frequencies ω_n are known, they are back-substituted in Eq. (32), and the vector of integration constants $\boldsymbol{\alpha}$ may be found after numerical evaluation of the eigenvectors of the so-obtained matrix $\mathbf{B}$, and their suitable normalization.

On the other hand, the vector of integration constants $\boldsymbol{\alpha}$ may also be obtained in an approximate form, by neglecting the exponentially small terms contained in the boundary condition matrix $\mathbf{B}$ (i.e. all the terms proportional to e), as it has been previously done in [16]. Eq. (32) reduces to a singular system of algebraic equations (i.e. $\text{rank}(\mathbf{B}) = 3$), and should, hence, be solved by setting the first entry of $\boldsymbol{\alpha}_n$ to 1. By doing so, the following approximate closed-form vector of integration constants is obtained:

$$\boldsymbol{\alpha}_n = \begin{bmatrix} 1 & -\mathcal{K} & +\mathcal{K} & \mathcal{K}c - s \end{bmatrix}^T \quad \forall n \in \mathbb{N}^+ \quad (33)$$

where:

$$\mathcal{K} = \frac{\rho_0 \bar{z}_1}{(1 - \rho_0)(\bar{z}_1^2 + \bar{z}_2^2) + \rho_0 \bar{z}_2} \quad (34)$$

It is worth noticing that Eq. (33) strictly satisfies only the first two boundary conditions of the original system (32), therefore leading to expected inaccuracies in the evaluation of the modal curvatures in the two boundary layers of the cable.

In both cases, the mode shapes are then obtained by direct substitution of the vector α into Eq. (30). The two aforementioned procedures for the evaluation of the mode shapes $\{\phi_n\}$ will be denoted in the following, respectively, as “semi-analytical exact” (SAE) and “semi-analytical approximate” (SAA) solutions.

Although both approaches provide accurate reconstruction of the mode shapes (see Section 2.2.3), they rely on the numerical evaluation of the cable’s natural frequencies. This requirement entails the repeated solution of the algebraic eigenvalue problem (32), which may become computationally demanding when embedded within an SHM framework.

To mitigate this drawback and improve the overall computational efficiency of mechanically-based SHM methodologies for shallow cables, an asymptotic solution of the algebraic eigenvalue problem is introduced in the following section.

2.2.2. Asymptotic solution of the algebraic eigenvalue problem

A second-order accurate asymptotic solution of the algebraic eigenvalue problem (32) has been developed in [11] and is here briefly reviewed. The following regular expansions are introduced:

$$\omega = \omega_0 + \omega_1\epsilon + \omega_2\epsilon^2 + o(\epsilon^2), \quad (35)$$

$$\alpha = \alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + o(\epsilon^2). \quad (36)$$

Substitution of Eqs. (35) and (36) in Eq. (32) yields the matrix equation:

$$(\mathbf{B}_0 + \mathbf{B}_1\epsilon + \mathbf{B}_2\epsilon^2)(\alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2) = \mathbf{0} \quad (37)$$

where $\mathbf{B}_0$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are 4×4 matrices whose components are fully reported in [11]. The individual vanishing of the coefficients of the different powers in ϵ in Eq. (37), then, leads to the system of equations

$$\begin{cases} \text{ord}(\epsilon^0) : \mathbf{B}_0\alpha_0 = \mathbf{0} \\ \text{ord}(\epsilon^1) : \mathbf{B}_0\alpha_1 + \mathbf{B}_1\alpha_0 = \mathbf{0} \\ \text{ord}(\epsilon^2) : \mathbf{B}_0\alpha_2 + \mathbf{B}_1\alpha_1 + \mathbf{B}_2\alpha_0 = \mathbf{0} \end{cases} \quad (38)$$

that can be easily solved for the variables ω_i and α_i ($i = 0, 1, 2$) through a cascaded approach, starting from the leading order problem (i.e. $\text{ord}(\epsilon^0)$) and moving towards the higher order ones. Solution of (38) is fully detailed in [11] and leads to the following second-order accurate asymptotic expression for the non-dimensional natural frequencies of the straight cable with small bending stiffness:

$$\omega_k = k\pi \left(1 + p\epsilon + \left(\frac{(k\pi)^2}{2} + p^2 \right) \epsilon^2 \right) + o(\epsilon^2), \quad k \in \mathbb{N}^+ \quad (39)$$

The leading order term in Eq. (39) coincides, as expected, with the frequencies of the taut-string model, while the effects of the bending stiffness and of the flexibility of the cable anchorages enter the first and second order correction terms through the non-dimensional variables ϵ and p , with:

$$p = \rho_0 + \rho_1 \quad (40)$$

The right eigenvector of $\mathbf{B}_0$ reads:

$$\alpha_{0,n} = [1 \quad 0 \quad 0 \quad 0]^T \quad \forall n \in \mathbb{N}^+ \quad (41)$$

While its first and second order corrections respectively read:

$$\alpha_{1,n} = [0 - \pi n \rho_0 \pi n \rho_0 (-1)^{n+1} \pi n \rho_1]^T \quad \forall n \in \mathbb{N}^+ \quad (42)$$

$$\alpha_{2,n} = [0 - \pi n p \rho_0 \pi n p \rho_0 (-1)^{n+1} \pi n p \rho_1]^T \quad \forall n \in \mathbb{N}^+ \quad (43)$$

Within this asymptotic approach (which will be hereafter identified with the acronym ASY), mode shapes are then reconstructed by back-substitution of α_0 , α_1 and α_2 into Eqs. (36) and then (30).

As an alternative, mode shapes may be also reconstructed by using the vector of integration constants calculated with Eqs. (33) and (34), in which the functions $\tilde{z}_1$ and $\tilde{z}_2$ are computed adopting the asymptotic expression for ω_n , i.e. Eq. (39). This second approach will be denoted in the following as “hybrid” solution (HYB), since it combines both the asymptotic and the semi-analytical solutions.

2.2.3. A remark on mode shapes reconstruction

Given the variety of approaches presented in the previous sections, it is both timely and instructive to assess their relative performance in the mode shapes reconstruction problem. Accordingly, in the following, a detailed comparison of the different methodologies will be carried out, with the final aim of providing a clear understanding of the advantages and limitations of each approach in the context of mode shape reconstruction.

To do so, a finite element model is set up by considering a cable characterized by $\lambda^2 = 0$ and $\epsilon = 0.1$. Please notice that, while typical structural applications are characterized by significantly smaller values of ϵ (in the order of 0.01), this relatively large value of ϵ has been deliberately selected in order to better highlight the differences among the various approaches. In fact, the accuracy of the asymptotic solution is expected to decrease as ϵ increases.

Table 1
Acronyms of the different approaches and summary of their characteristics.

Approach	Acronym	Frequencies ω_n	Mode shapes ϕ_n
Fully Numerical - Reference	FEM	Numerical	Numerical
Semi-analytical exact (Section 2.2.1)	SAE	Numerical	Numerical
Semi-analytical approximate (Section 2.2.1)	SAA	Numerical	Closed-form
Asymptotic (Section 2.2.2)	ASY	Asymptotic	Asymptotic
Hybrid (Section 2.2.2)	HYB	Asymptotic	Closed-form

Among the different combinations of physical parameters which can be selected to get the target value of non-dimensional bending stiffness, the following values have been chosen, without any loss of generality: $L = 1$ m, $\mu = 1$ kg/m, $EI = 0.01$ Nm², and $H = 1$ N, resulting in a unitary characteristic circular frequency ($\varpi_c = 1$ rad/s).

The cable self-weight has been neglected consistently with the assumption of $\lambda^2 = 0$ (see Eq. (6)).

The cable is considered to be doubly clamped (i.e. $\rho_0 = \rho_1 = 1$) at its ends and, in a fully numerical solution, it is discretized with 100 corotational beam finite elements (see [27]), non-uniformly distributed along the cable's axis, to better capture the modal shapes within the boundary layers.

Results obtained with the finite element model will be denoted in the following with the acronym FEM, whereas Table 1 collects a summary of the different approaches which will be hereafter compared.

Figs. 4–7 systematically depict the first four mode shapes of the cable, with enlargements in both boundary layer regions, and compares the solutions obtained with the different approaches collected in Table 1 against FEM results.

The exact semi-analytical approach (SAE) is clearly the most accurate compared to FEM results; however, it employs a complete numerical solution of the eigenvalue problem, which may significantly increase the overall computational cost of SHM-based methodologies.

For the present case, the approximate semi-analytical solution (SAA) practically coincides with the semi-analytical exact (SAE), revealing that the influence of correctly capturing the natural circular frequencies of the cable is by far larger than the influence of the boundary conditions, in the accuracy of mode shapes prediction.

With the asymptotic approach (ASY), the boundary conditions are only imposed in an approximate form. Therefore, the method's accuracy is rapidly decreasing as both ε and the mode number increase. It is also worth noticing that, in the spirit of classical perturbation methods, the approximation error is controlled and bounded, in the present case, by $\mathcal{O}(\varepsilon^2)$ for both ω_n and the vector of integration constants α . However, no direct conclusion can be drawn for the approximation error on the mode shapes, which is mostly due to the shift in the natural frequencies, as it is depicted in Figs. 6 and 7.

Even though the righthand boundary condition on the displacement is fulfilled with the hybrid approach (HYB), it can be noted that the accuracy of the modal curvature in the righthand boundary layer obtained is quite poor, as expected, due to the fact that Eq. (33) does not fulfill exactly the last two boundary conditions of the eigenproblem.

This motivates the derivation of a novel solution via a continuous formulation, which employs the Matched Asymptotic Expansion (MAE) perturbation method. This solution will be derived in Section 3.4, with reference to the case of a sagged cable with small bending stiffness.

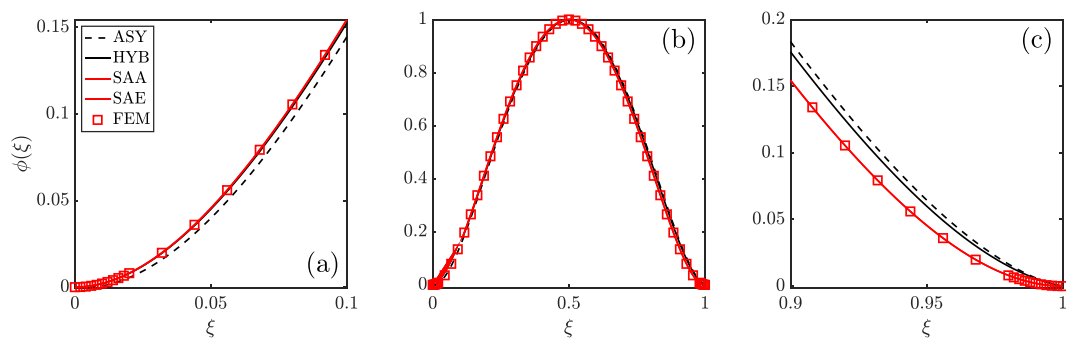


Fig. 4. First modal shape ($\phi_1(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 0$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righthand boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

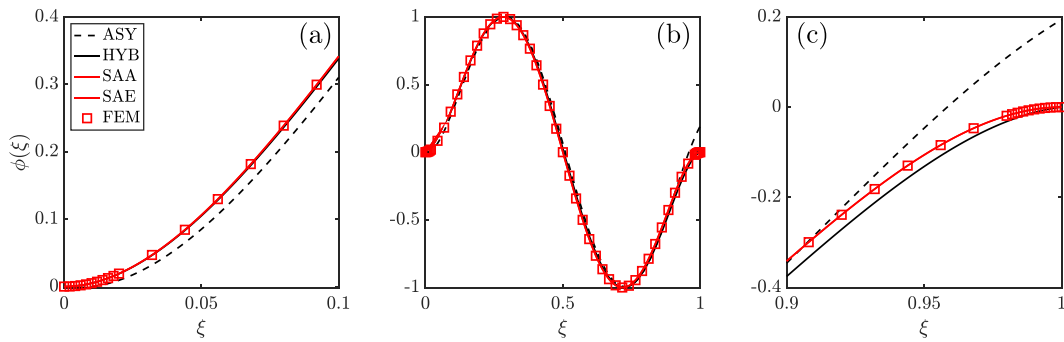


Fig. 5. Second modal shape ($\phi_2(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 0$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

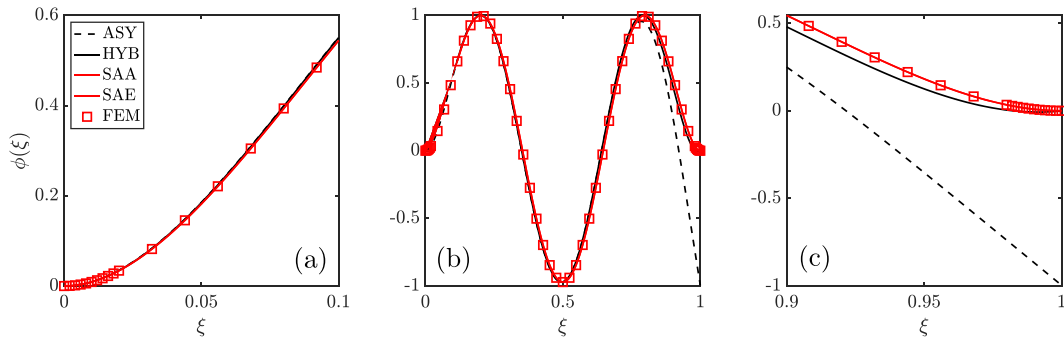


Fig. 6. Third modal shape ($\phi_3(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 0$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

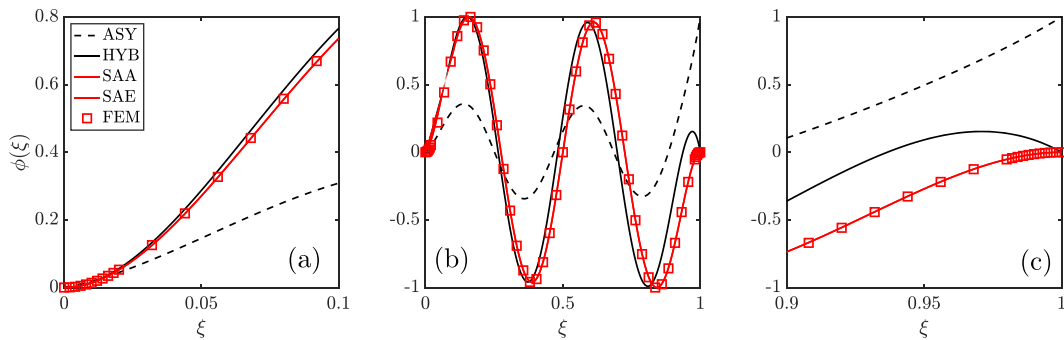


Fig. 7. Fourth modal shape ($\phi_4(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 0$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

3. Sagged cable with small bending stiffness

3.1. Problem formulation

Let us now address the complete problem, which incorporates both the sagging (see Section 2.1) and the bending stiffness (see Section 2.2) effects in the assessment of the spectral properties of the cable.

To this aim, let us make reference to Fig. 8 and consider again the case of a cable with axial stiffness EA , small bending stiffness EI , span length L , mass per unit of length μ and subject to a pretension force H . The rotations of the cable end sections are elastically restrained by two springs with stiffness K_0 and K_1 . Moreover, as it is schematically depicted in Fig. 8, let us describe the static equilibrium configuration of the cable by the parabolic-shaped profile $y(x)$ in Eq. (1) (cf. [23]).

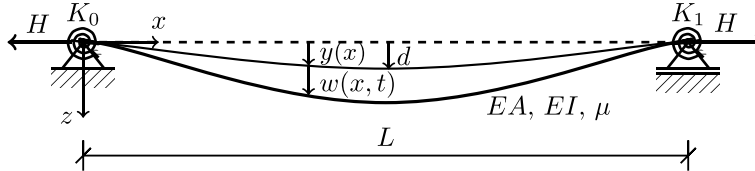


Fig. 8. Structural scheme of a sagged cable with small bending stiffness and flexible rotational supports.

Undamped free vibrations of the cable are governed by the following fourth-order partial differential equation:

$$EI \frac{\partial^4 w}{\partial x^4} - H \frac{\partial^2 w}{\partial x^2} - h(t) \frac{d^2 y}{dx^2} + \mu \frac{\partial^2 w}{\partial t^2} = 0 \quad (44)$$

$h(t)$ being the time-varying additional horizontal component of the axial force.

By recalling the definitions (7), (6), (25) and (8), as well as that $\xi = x/L$, Eq. (44) can be restated in nondimensional form:

$$\varepsilon^2 \frac{\partial^4 \bar{w}}{\partial \xi^4} - \frac{\partial^2 \bar{w}}{\partial \xi^2} + \lambda^2 \int_0^1 \bar{w} d\xi + \frac{\partial^2 \bar{w}}{\partial \tau^2} = 0 \quad (45)$$

where $\bar{w}(\xi, \tau) = w(x(\xi), t(\tau))/L$ denotes the dimensionless dynamic transverse displacement of the cable.

Inspection of Eq. (45) allows to clearly identify the terms responsible for both the sag/extensibility (i.e. proportional to the Irvine's parameter λ^2 , see Section 2.1) and the bending stiffness (i.e. proportional to ε^2 , see Section 2.2) effects, which additively contribute to the wave equation of an undamped taut string.

Substitution of Eq. (11) in Eq. (45) delivers the fourth-order Sturm-Liouville problem defined by the following non-homogeneous ordinary differential equation:

$$\varepsilon^2 \phi'''' - \phi'' + \lambda^2 \int_0^1 \phi(\xi) d\xi - \omega^2 \phi = 0 \quad (46)$$

along with the following dimensionless boundary conditions:

$$\begin{cases} \phi(0) = \phi(1) = 0 \\ (1 - \rho_0)\varepsilon^2 \phi''(0) - \rho_0 \varepsilon \phi'(0) = 0 \\ (1 - \rho_1)\varepsilon^2 \phi''(1) + \rho_1 \varepsilon \phi'(1) = 0 \end{cases} \quad (47)$$

which are the same as the ones encountered in Section 2.2, ρ_i being the flexural degree-of-fixity parameters defined in Eq. (27).

3.2. Semi-analytical solution of the eigenvalue problem

General solution of Eq. (46) can be expressed in the form:

$$\phi(\xi) = \alpha_1 \sin(z_1 \xi) + \alpha_2 \cos(z_1 \xi) + \alpha_3 \exp(-z_2 \xi) + \alpha_4 \exp(-z_2(1 - \xi)) + \frac{\lambda^2}{\omega^2} \int_0^1 \phi(\xi) d\xi \quad (48)$$

where $\alpha_i \in \mathbb{R}$ ($i = 1, \dots, 4$) are integration constants and the arguments z_1 and z_2 are the same functions of the non-dimensional vibration frequency ω introduced in Section 2.2 (see Eq. (31)).

It is worth noticing that Eq. (48) has been easily found by direct superposition of the solution of the homogeneous eigenproblem (i.e. Eq. (28), see Section 2.2) and the particular solution of the non-homogeneous eigenproblem (i.e. Eq. (46)).

The general solution of the homogeneous equation is nothing but Eq. (30), while the particular solution of the complete equation is sought in the form of a constant and it is easily found to be the last term of Eq. (48), involving the effect of cable sagging.

By integrating both sides of Eq. (48), for $\xi \in [0, 1]$, the definite integral of the mode shape $\phi(\xi)$ can be explicitly expressed as:

$$\int_0^1 \phi(\xi) d\xi = \left[\frac{\alpha_1}{z_1} (1 - \cos z_1) + \frac{\alpha_2}{z_1} \sin z_1 + \left(\frac{\alpha_3}{z_2} + \frac{\alpha_4}{z_2} \right) (1 - \exp(-z_2)) \right] \frac{\omega^2}{\omega^2 - \lambda^2} \quad (49)$$

Back-substitution of Eq. (49) into Eq. (48) leads to:

$$\begin{aligned} \phi(\xi) = & \alpha_1 \sin(z_1 \xi) + \alpha_2 \cos(z_1 \xi) + \alpha_3 \exp(-z_2 \xi) + \alpha_4 \exp(-z_2(1 - \xi)) + \\ & + \frac{\lambda^2}{\omega^2 - \lambda^2} \left[\frac{\alpha_1}{z_1} (1 - \cos z_1) + \frac{\alpha_2}{z_1} \sin z_1 + \left(\frac{\alpha_3}{z_2} + \frac{\alpha_4}{z_2} \right) (1 - \exp(-z_2)) \right] \end{aligned} \quad (50)$$

Substitution of Eqs. (50) and (31) in the boundary conditions (47) yields the algebraic eigenvalue problem:

$$\mathbf{B}(\omega; \mathcal{Q}) \boldsymbol{\alpha} = \mathbf{0} \quad (51)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is a column vector listing the integration constants of Eq. (48), $\mathbf{0}$ is the four-dimensional null column vector and $\mathbf{B}$ is the 4×4 “boundary condition matrix” depending on ω and on the set of parameters: $\mathcal{Q} = \{\varepsilon, \lambda, \rho_0, \rho_1\}$. The full expression of $\mathbf{B}$ reads:

$$\mathbf{B} = \begin{bmatrix} F_1(1 - c) & 1 + F_1 s & 1 + F_2(1 - e) & E + F_2(1 - e) \\ s + F_1(1 - c) & c + F_1 s & e + F_2(1 - e) & 1 + F_2(1 - e) \\ -\tilde{z}_1 \rho_0 & -\tilde{z}_1^2(1 - \rho_0) & \tilde{z}_2^2(1 - \rho_0) + \tilde{z}_2 \rho_0 & e \tilde{z}_2^2(1 - \rho_0) - e \tilde{z}_2 \rho_0 \\ c \tilde{z}_1 \rho_1 - s \tilde{z}_1^2(1 - \rho_1) & -s \tilde{z}_1 \rho_1 - c \tilde{z}_1^2(1 - \rho_1) & e \tilde{z}_2^2(1 - \rho_1) - e \tilde{z}_2 \rho_1 & \tilde{z}_2^2(1 - \rho_1) + \tilde{z}_2 \rho_1 \end{bmatrix}$$

with $c = \cos(z_1)$, $s = \sin(z_1)$, $e = \exp(-z_2)$, $\tilde{z}_1 = \varepsilon z_1$, $\tilde{z}_2 = \varepsilon z_2$, $F = \lambda^2/(\omega^2 - \lambda^2)$, $F_1 = F/z_1$ and $F_2 = F/z_2$.

Comparing the expression of matrix $\mathbf{B}$ obtained for the straight cable with small bending stiffness (see Section 2.2) and the one just found for the sagged cable, it can be noted that only the first two rows of the boundary condition matrix have been modified, as a consequence of the introduction of the particular solution of the non-homogeneous equation in the expression of $\phi(\xi)$ (see Eq. (50)).

The effect of cable sagging is clearly evident in the additional contributions proportional to F_1 and F_2 .

Non trivial solutions $\{\omega_n, \alpha_n\}$, with $n \in \mathbb{N}^+$, of the algebraic eigenvalue problem can be obtained by solving the characteristic equation: $\det[\mathbf{B}(\omega; \mathcal{Q})] = 0$.

In general, as already recalled in Section 3.2, the nondimensional natural frequencies ω_n must be obtained through numerical solution of the characteristic equation. Back-substitution of $\{\omega_n\}$ into Eq. (51), and numerical computations of the eigenvectors of matrix $\mathbf{B}$, then, leads to the exact vector of integration constants $\boldsymbol{\alpha}$.

The latter may be also approximately obtained by disregarding exponentially small terms (i.e. containing e) in the matrix $\mathbf{B}$ and solving Eq. (51) setting the first entry of $\boldsymbol{\alpha}_n$ to 1 (see Section 2.2.1). According to this procedure, the following vector of integration constants is found:

$$\boldsymbol{\alpha}_n = [1 \quad \bar{J}_2 \quad \bar{J}_3 \quad \bar{J}_4]^T \quad \forall n \in \mathbb{N}^+ \quad (52)$$

The expressions of $\bar{J}_j$ $j = 2, 3, 4$ are given by:

$$\bar{J}_2 = \frac{(F_2 s + F_1(c - 1)) \tilde{z}_2 (\tilde{z}_2(1 - \rho_0) + \rho_0) - (1 + 2F_2) \tilde{z}_1 \rho_0}{(1 + F_1 s + F_2(1 - c)) \tilde{z}_2 (\tilde{z}_2(1 - \rho_0) + \rho_0) + (1 + 2F_2) \tilde{z}_1^2(1 - \rho_0)} \quad (53)$$

$$\bar{J}_3 = \frac{(F_2 s + F_1(c - 1)) \tilde{z}_1^2(1 - \rho_0) + \tilde{z}_1 \rho_0 (1 + F_1 s + F_2(1 - c))}{(1 + F_1 s + F_2(1 - c)) \tilde{z}_2 (\tilde{z}_2(1 - \rho_0) + \rho_0) + (1 + 2F_2) \tilde{z}_1^2(1 - \rho_0)} \quad (54)$$

$$\begin{aligned} \bar{J}_4 = & \frac{(F_1 s^2 + s + F_1(c - 1)^2) \tilde{z}_2 (\tilde{z}_2(\rho_0 - 1) - \rho_0) + \tilde{z}_1 (\tilde{z}_1 s(\rho_0 - 1) + c \rho_0)}{(1 + F_1 s + F_2(1 - c)) \tilde{z}_2 (\tilde{z}_2(1 - \rho_0) + \rho_0) + (1 + 2F_2) \tilde{z}_1^2(1 - \rho_0)} + \\ & + \frac{F_2 \tilde{z}_1 (\tilde{z}_1 s(\rho_0 - 1) + \rho_0(c - 1)) + F_1 \tilde{z}_1 (\tilde{z}_1(\rho_0 - 1)(1 - c) + s \rho_0)}{(1 + F_1 s + F_2(1 - c)) \tilde{z}_2 (\tilde{z}_2(1 - \rho_0) + \rho_0) + (1 + 2F_2) \tilde{z}_1^2(1 - \rho_0)} \end{aligned} \quad (55)$$

In the case of $\lambda^2 = 0$, the constants F_1 and F_2 vanish and Eqs. (53–55) reduce respectively to $\bar{J}_2 = -\mathcal{K}$, $\bar{J}_3 = +\mathcal{K}$ and $\bar{J}_4 = \mathcal{K}c - s$, retrieving Eq. (33) as a special case.

Finally, mode shapes are then obtained by back-substitution of Eq. (52) into Eq. (50).

3.3. Asymptotic solution of the algebraic eigenvalue problem

An asymptotic solution of the algebraic eigenvalue problem (Eq. (51)) may be obtained applying the methodology already discussed in Section 2.2.2.

This time, however, there are two dimensionless variables (i.e., λ^2 and ε^2) which should, in principle, be ordered. In order to build the most general solution possible, it is necessary to enforce that $\lambda = \text{ord}(1)$, while keeping the dimensionless bending stiffness ε as the only small parameter, in the spirit of the perturbation method.

The only difference with respect to the procedure collected in Section 2.2.2 lies in the fact that, this time, the leading order problem is no more the one of a taut string, but is the problem of a perfectly flexible sagged cable, which has already been treated in Section 2.1.

Accordingly, the leading order circular frequencies should be obtained via numerical solution of Irvine’s equation (see Eq. (17)), losing the advantages of a closed-form asymptotic solution. To overcome this limitation, it is therefore convenient to introduce the

approximate solution of the natural frequencies for the perfectly flexible cable, which is given by Eq. (22) (see also Section 2.1.2). By doing so, the following second-order accurate asymptotic expression for the non-dimensional natural frequencies of the cable is obtained:

$$\omega_n = n\pi \left(\sqrt{1 + \frac{2\lambda^2}{(n\pi)^4} [1 - (-1)^n]^2} + p\epsilon + \left(\frac{(n\pi)^2}{2} + p^2 \right) \epsilon^2 \right) + o(\epsilon^2) \quad (56)$$

which only differs from Eq. (39) for the leading order term. It also worth noticing that Eq. (39) is retrieved as a special case of Eq. (56) for $\lambda^2 \rightarrow 0$, i.e. when the sagged cable degenerates in the taut-string.

The very same solution can be obtained by recognizing that the effect of the cable sagging on the dimensionless circular frequencies may be included at leading order regardless the magnitude of the Irvine's parameter. Therefore, within this pragmatic approach, there is no need for the assessment of the ordering of the two dimensionless quantities (i.e. λ^2 and ϵ^2) and Eq. (56) may be retrieved combining the asymptotic solution developed in Section 2.2 with the leading-order closed-form expression derived in Section 2.1.

As already recalled in Section 2.2.3, the asymptotic approach (ASY) may lead to significant errors in the reconstruction of the mode shapes. To overcome this issue, the closed-form solution for the integration constants may be used (see Eqs. (52–55) and Table 1). However, this hybrid approach (HYB) is inherently less accurate in capturing the modal curvatures within the boundary layers, as both the mode number n and the dimensionless bending parameter ϵ increase (cf. Section 2.2.3).

This again motivates the derivation of a novel closed-form solution via a continuous formulation, which is presented in the next section.

3.4. Asymptotic solution of the continuous eigenvalue problem

In this section, the continuous eigenvalue problem is attacked with the Matched Asymptotic Expansion (MAE) perturbation method. In the spirit of the classical MAE method, the composite solution of the problem is obtained as the superposition of an outer solution, which describes the far-field unperturbed behavior, and two inner solutions, which capture the singularly-perturbed response within the two bending boundary layers.

Let us adopt the following regular expansions for the eigenvectors $\phi(\xi)$ and the eigenvalues $\Omega = \omega^2$:

$$\phi(\xi) = \phi_0(\xi) + \epsilon \phi_1(\xi) + o(\epsilon^2) \quad (57)$$

$$\Omega = \Omega_0 + \epsilon \Omega_1 + o(\epsilon^2) \quad (58)$$

Please notice that, although higher order terms could be introduced in the regular ansatz (Eqs. (57)–(58)) only first-order terms are herein retained in order to keep the analytical tractability of the solution. The eigenvalue problem is described by Eqs. (46–47)

Substitution of Eqs. (57) and (58) into Eq. (46) yields the cascade of outer problems:

$$\begin{cases} \text{ord}(\epsilon^0) : & -\phi_0'' + \lambda^2 \int_0^1 \phi_0(\xi) d\xi = \Omega_0 \phi_0 \\ \text{ord}(\epsilon^1) : & -\phi_1'' + \lambda^2 \int_0^1 \phi_1(\xi) d\xi = \Omega_0 \phi_1 + \Omega_1 \phi_0 \end{cases} \quad (59)$$

The inner solutions are instead captured with the stretched coordinates $\zeta_1 = \xi/\epsilon$ (for the lefthand boundary layer) and $\zeta_2 = (1 - \xi)/\epsilon$ (for the righthand boundary layer), respectively:

$$\Phi(\zeta_1)'''' - \Phi(\zeta_1)'' = \epsilon^2 \Omega \Phi(\zeta_1) - \epsilon^3 \lambda^2 \int_0^{1/\epsilon} \Phi(\zeta_1) d\zeta_1 \quad (60)$$

$$\Phi(\zeta_2)'''' - \Phi(\zeta_2)'' = \epsilon^2 \Omega \Phi(\zeta_2) + \epsilon^3 \lambda^2 \int_{1/\epsilon}^0 \Phi(\zeta_2) d\zeta_2 \quad (61)$$

Moreover, the integral can be rewritten, for the sake of convenience, by making reference to the original variable of the problem (i.e. $\phi(\xi)$):

$$\Phi(\zeta_i)'''' - \Phi(\zeta_i)'' = \epsilon^2 \Omega \Phi(\zeta_i) - \epsilon^2 \lambda^2 \int_0^1 \phi(\xi) d\xi, \quad \text{with } i = 1, 2 \quad (62)$$

A regular expansion for both the left and the right inner solutions is then adopted:

$$\Phi(\zeta_1) = L_0(\zeta_1) + \epsilon L_1(\zeta_1) + o(\epsilon^2) \quad (63)$$

$$\Phi(\zeta_2) = R_0(\zeta_2) + \epsilon R_1(\zeta_2) + o(\epsilon^2) \quad (64)$$

while Eq. (57) is again used for $\phi(\xi)$. Back-substitution of Eqs. (58) and (63–64) into Eqs. (60–61) leads to the cascade of inner problems:

$$\begin{cases} \text{ord}(\epsilon^0) : & L_0'''' - L_0'' = 0 \\ \text{ord}(\epsilon^1) : & L_1'''' - L_1'' = 0 \end{cases} \quad (65)$$

for the lefthand boundary layer, and

$$\begin{cases} \text{ord}(\epsilon^0) : & R_0'''' - R_0'' = 0 \\ \text{ord}(\epsilon^1) : & R_1'''' - R_1'' = 0 \end{cases} \quad (66)$$

for the righthand boundary layer. It is worth noticing that the inner problems are homogeneous both at leading order and at first order, while the non-homogenous term enters only at second order, if additional terms are added to the regular ansatz (see Eqs. (63-64)).

3.4.1. Leading order problem

First, the leading order problem is considered:

$$\begin{cases} \text{Outer} & -\phi_0'' + \lambda^2 \int_0^1 \phi_0(\xi) d\xi = \Omega_0 \phi_0 \\ \text{Inner, Left} & L_0'''' - L_0'' = 0 \\ \text{Inner, Right} & R_0'''' - R_0'' = 0 \end{cases} \quad (67)$$

The boundary conditions will not be imposed at this stage, but only when the final composite solution is assembled.

The solution of the outer problem is exactly the one already obtained in Section 2.1, for a perfectly flexible sagged cable i.e.:

$$\phi_0(\xi) = \sin(\sqrt{\Omega_0}\xi) + B_0 \cos(\sqrt{\Omega_0}\xi) + \frac{\lambda^2}{\Omega_0 - \lambda^2} \left[\frac{1}{\sqrt{\Omega_0}} (1 - \cos \sqrt{\Omega_0}) + \frac{B_0}{\sqrt{\Omega_0}} \sin \sqrt{\Omega_0} \right] \quad (68)$$

Eq. (68) specializes to the solution of the taut-string for $\lambda^2 \rightarrow 0$.

The solutions of the leading-order inner problems are expressed as follow:

$$L_0(\zeta_1) = \varepsilon^q (A_1 + B_1 \zeta_1 + C_1 \exp(-\zeta_1)) \quad (69)$$

$$R_0(\zeta_2) = \varepsilon^q (A_2 + B_2 \zeta_2 + C_2 \exp(-\zeta_2)) \quad (70)$$

where $q \in \mathbb{R}^+$ is real number which accounts for the possibility that the outer solution and the inner solutions have orders of magnitude different from 1. Indeed, the matching between the left inner solution and the outer solution should be carried out at an intermediate scale of ε , namely, by introducing a new stretched coordinate:

$$\eta = \frac{\xi}{\sqrt{\varepsilon}} = \sqrt{\varepsilon} \zeta_1 \quad (71)$$

Performing a Taylor series expansion of both the outer and the left inner solutions, and neglecting exponentially small terms (i.e. $C_1 = 0$), allows to conclude that $q = 1$ for balancing requirements.

Moreover, matching of the leading order term leads to:

$$B_0 = \frac{\lambda^2 (\cos \sqrt{\Omega_0} - 1)}{\lambda^2 (\sin \sqrt{\Omega_0} - \sqrt{\Omega_0}) + \Omega_0^{3/2}} \quad (72)$$

Similarly, the matching between the outer and the right inner solutions is carried out considering the following stretched coordinate:

$$\eta = \frac{1 - \xi}{\sqrt{\varepsilon}} = \sqrt{\varepsilon} \zeta_2 \quad (73)$$

and neglecting exponentially small terms (i.e. $C_2 = 0$). The matching of the leading order terms provides exactly Irvine's transcendental equation, i.e. Eq. (17), whose solutions can be effectively approximated with Eq. (22), as long as $\lambda^2 < 4\pi^2$, as it has been discussed in Section 2.1. Therefore, with the aim of avoiding numerical integration of Irvine's equation, and keeping the solution fully analytical, the following expression of the leading-order eigenvalues is adopted:

$$\Omega_0 = (n\pi)^2 + \frac{2\lambda^2}{(n\pi)^2} [1 - (-1)^n]^2 \quad (74)$$

The leading order outer solution can be then expressed as

$$\begin{aligned} \phi_0(\xi) = & \frac{\lambda^2}{\sqrt{\Omega_0}(-\lambda^2 + \Omega_0)} - \frac{\lambda^2 \cos \sqrt{\Omega_0}}{\sqrt{\Omega_0}(-\lambda^2 + \Omega_0)} + \frac{\lambda^2 (\cos \sqrt{\Omega_0} - 1) \cos(\sqrt{\Omega_0}\xi)}{-\lambda^2 \sqrt{\Omega_0} + \Omega_0^{3/2} + \lambda^2 \sin \sqrt{\Omega_0}} \\ & + \frac{\lambda^4 (\cos \sqrt{\Omega_0} - 1) \sin \sqrt{\Omega_0}}{\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) (-\lambda^2 \sqrt{\Omega_0} + \Omega_0^{3/2} + \lambda^2 \sin \sqrt{\Omega_0})} + \sin(\sqrt{\Omega_0}\xi) \end{aligned} \quad (75)$$

which is a sinusoidal function with additional correction terms accounting for the nonzero sagging of the cable. Again, for $\lambda^2 \rightarrow 0$, Eqs. (74-75) simply deliver the taut-string solution (see Eq. (14)).

Moreover, it can be concluded that the leading order term in the inner solutions is of $\text{ord}(\varepsilon)$, as expected (see Eqs. (69-70) for $q = 1$), which allows to state that the dimensionless length of the boundary layers is given by the parameter ε itself, regardless the considered mode number and the boundary conditions. Finally, the eigenvalues at leading order are given by Eq. (74).

3.4.2. First order problem

Let us now consider the first-order problem:

$$\begin{cases} \text{Outer} & -\phi_1'' + \lambda^2 \int_0^1 \phi_1(\xi) d\xi = \Omega_0 \phi_1 + \Omega_1 \phi_0 \\ \text{Inner, Left} & L_1'''' - L_1'' = 0 \\ \text{Inner, Right} & R_1'''' - R_1'' = 0 \end{cases} \quad (76)$$

Again, boundary conditions will be imposed only after the composite solution has been assembled. Back-substitution of Eq. (75) in the first of Eq. (76) allows to get the solution of the outer problem, which reads:

$$\phi_1(\xi) = a_0 \sin(\sqrt{\Omega_0} \xi) + b_0 \cos(\sqrt{\Omega_0} \xi) + K_0 + K_1 \xi \sin(\sqrt{\Omega_0} \xi) + K_2 \xi \cos(\sqrt{\Omega_0} \xi) \quad (77)$$

where the constants K_i , $i = 0, 1, 2$ are given by

$$K_0 = -\frac{h_1 \left(\lambda^2 \Omega_0^{3/2} - \lambda^4 \sqrt{\Omega_0} + \lambda^4 \sin \sqrt{\Omega_0} \right) + \lambda^2 \Omega_1 \left(\cos \sqrt{\Omega_0} - 1 \right)}{\Omega_0 \left(-\lambda^2 \sqrt{\Omega_0} + \Omega_0^{3/2} + \lambda^2 \sin \sqrt{\Omega_0} \right)} \quad (78)$$

$$K_1 = \frac{\lambda^2 \Omega_1 \sqrt{\Omega_0} \left(2\xi - 2\xi \cos \sqrt{\Omega_0} + 1 \right) - \Omega_1 \left(\Omega_0^{3/2} + \lambda^2 \sin \sqrt{\Omega_0} \right)}{4\xi \Omega_0 \left(-\lambda^2 \sqrt{\Omega_0} + \Omega_0^{3/2} + \lambda^2 \sin \sqrt{\Omega_0} \right)} \quad (79)$$

$$K_2 = \frac{\Omega_1}{2\sqrt{\Omega_0}} \quad (80)$$

and

$$h_1 = \int_0^1 \phi_1(\xi) d\xi \quad (81)$$

indicates the integral of the additional dynamic component of the horizontal tension at first order, for brevity. The full expression of h_1 is collected in Appendix A. It depends on a_0 , b_0 , Ω_0 , Ω_1 and λ^2 , and will in the following be substituted into Eq. (78) to get an explicit expression of K_0 , which will be useful for the subsequent derivations.

Finally, it is worth noticing that Eq. (77) is the complete solution of the non-homogeneous outer problem, which accounts for a particular solution that has been determined according to the principle of similarity.

The inner problems at first order are homogeneous. Therefore, their solutions read:

$$L_1(\zeta_1) = \varepsilon^s (a_1 + b_1 \zeta_1 + c_1 \exp(-\zeta_1)) \quad (82)$$

$$R_1(\zeta_2) = \varepsilon^s (a_2 + b_2 \zeta_2 + c_2 \exp(-\zeta_2)) \quad (83)$$

where $s \in \mathbb{R}^+$. This time, the matching can only be achieved for $\xi = \varepsilon^r \eta$, with $1/2 < r < 1$, as it is usual in higher order matching with intermediate scales. The following stretched coordinate is adopted for the lefthand matching conditions:

$$\eta = \frac{\xi}{\varepsilon^{3/4}} = \varepsilon^{1/4} \zeta_1 \quad (84)$$

Performing the usual Taylor series expansion of both the inner and the left inner solutions, and neglecting exponentially small terms (i.e. $C_1 = 0$ and $c_1 = 0$), allows to conclude that $s = 1$ for balancing requirements. Matching of the terms in $\varepsilon^{3/4}$ and ε allow to obtain the constants b_0 and B_1 . The latter is given by:

$$B_1 = \sqrt{\Omega_0} \quad (85)$$

while the expression of b_0 is collected in Appendix B.

Similarly, the matching between the outer and righthand inner solutions is carried out considering the following stretched coordinate:

$$\eta = \frac{1 - \xi}{\varepsilon^{3/4}} = \varepsilon^{1/4} \zeta_2 \quad (86)$$

Neglecting exponentially small terms (i.e. $C_2 = 0$ and $c_2 = 0$), vanishing of the coefficients at the different orders delivers the expression of constants A_2 and B_2 . The latter is given by:

$$B_2 = \frac{(\lambda^2 - \Omega_0) \Omega_0 \cos \sqrt{\Omega_0} - \lambda^2 \sqrt{\Omega_0} \sin \sqrt{\Omega_0}}{\sqrt{\Omega_0} (-\lambda^2 + \Omega_0) + \lambda^2 \sin \sqrt{\Omega_0}} \quad (87)$$

while the expression of A_2 is collected in Appendix B.

Therefore, the solutions at first-order are now determined in a fully-explicit manner.

3.4.3. Composite solution

The composite solution can be finally obtained as:

$$\phi(\xi) = \phi_0(\xi) + \varepsilon \phi_1(\xi) + L_0\left(\frac{\xi}{\varepsilon}\right) + \varepsilon L_1\left(\frac{\xi}{\varepsilon}\right) + R_0\left(\frac{1-\xi}{\varepsilon}\right) + \varepsilon R_1\left(\frac{1-\xi}{\varepsilon}\right) - O_L(\xi) - O_R(\xi) \quad (88)$$

where $O_L(\xi)$ and $O_R(\xi)$ indicate the overlapping solutions on the lefthand and righthand sides, respectively, which have been considered twice during the matching procedures, and should therefore be subtracted from the complete expression of the composite solution.

The overlapping solutions can be expressed as:

$$O_L(\xi) = A_1\varepsilon + B_1\xi + b_1\varepsilon\xi \quad (89)$$

$$O_R(\xi) = A_2\varepsilon + B_2(1-\xi) + b_2\varepsilon(1-\xi) \quad (90)$$

Back-substitution of Eqs. (75, 77, 69–70, 82–83), with $q = s = 1$ and the determined constants into Eq. (88) allows to state that the modes shapes do not depend on b_1 and b_2 . Moreover, terms of $\text{ord}(\varepsilon^2)$ are not legitimate at this stage and should be, therefore, discarded by setting to zero the coefficients a_1 , a_2 , c_1 and c_2 .

The remaining free constants are C_1 , C_2 , A_1 , a_0 and Ω_1 . These should be determined by imposing the four boundary conditions on the composite solution.

Vanishing of the $\text{ord}(\varepsilon^1)$ terms of the displacement boundary conditions $\phi(0) = 0$ and $\phi(1) = 0$ delivers the constant C_1 and C_2 .

Please note that leading order terms of these boundary conditions are automatically zero, because of the matching, which provided the condition for the solution of Irvine's equation. By disregarding exponentially small terms, one has from the first boundary condition:

$$C_1 = -A_1 \quad (91)$$

while from the second boundary condition the expression of the coefficient C_2 is found, which is reported in Appendix C, for brevity.

The third boundary condition of system (47) provides two equations. Vanishing of $\text{ord}(\varepsilon^1)$ and $\text{ord}(\varepsilon^2)$ allows to determine, respectively, A_1 and a_0 :

$$A_1 = -\rho_1 \sqrt{\Omega_0} \quad (92)$$

$$a_0 = a_0^* \left(4\rho_0\Omega_0 \left(-\lambda^2 \sqrt{\Omega_0} + \lambda^2 \sin \sqrt{\Omega_0} + \Omega_0^{3/2} \right) \right)^{-1} \quad (93)$$

with

$$a_0^* = 4\lambda^2 \rho_0 \Omega_0^{3/2} \left(\cos \sqrt{\Omega_0} - 1 \right) + \lambda^2 \rho_0 \Omega_1 \left(\sqrt{\Omega_0} - \sin \sqrt{\Omega_0} \right) + 4\lambda^2 \Omega_0^{3/2} \left(1 - \cos \sqrt{\Omega_0} \right) - \rho_0 \Omega_0^{3/2} \Omega_1 \quad (94)$$

Finally, the last boundary condition (i.e. the bending moment at the flexible support in $\xi = 1$) allows to determine the remaining free constant, which is nothing but the first-order correction of the eigenvalues, i.e. Ω_1 . The latter is obtained by imposing the vanishing of the $\text{ord}(\varepsilon^1)$ terms of this boundary condition, and its expression is reported in Appendix D. Vanishing of $\text{ord}(\varepsilon^2)$ terms is strictly ensured, for aforementioned expression of Ω_1 , only if $\lambda^2 \rightarrow 0$. In the general case, $\text{ord}(\varepsilon^2)$ errors on this boundary condition are expected, depending on the values of λ^2 , ρ_0 and ρ_1 .

The final asymptotic expression of the mode shapes of the cable is then reported in Appendix E. It is worth noticing that for $\lambda^2 \rightarrow 0$, the following results are obtained:

$$\Omega_0 = (n\pi)^2 \quad \Omega_1 = 2p(n\pi)^2 \quad h_1 = 2p \frac{(-1)^n - 1}{n\pi} \quad A_1 = -n\pi\rho_0 \quad a_0 = -\frac{p}{2} \quad C_1 = n\pi\rho_0 \quad C_2 = (-1)^{n+1} n\pi\rho_1$$

and the expression of the mode shapes drastically simplifies as:

$$\phi_n(\xi) = \varepsilon n\pi\rho_0 \exp\left(-\frac{\xi}{\varepsilon}\right) - \varepsilon n\pi\rho_1 (-1)^n \exp\left(\frac{\xi-1}{\varepsilon}\right) + \varepsilon n\pi(p\xi - \rho_0) \cos(n\pi\xi) + (1 - p\varepsilon) \sin(n\pi\xi) \quad (95)$$

which is the MAE solution of the continuous eigenvalue problem for a straight cable with small bending stiffness and rotational springs at its end (see Section 2.2 and (Eq. 28)), which is obtained at no extra cost, as a particular case of the solution of the complete problem.

The derivation of this continuous solution was primarily aimed at improving the accuracy in the closed-form calculation of mode shapes of the cable, particularly within the bending boundary layers. As a matter of fact, the continuous formulation herein reported is intrinsically not capable of capturing the beam-like asymptotic term in the frequencies progression, while the standard asymptotic approach presented in Section 3.3 automatically does it.

A pragmatical approach to account for the beam-like term in the progression of the cable natural frequencies is to perform an a-posteriori matching of the eigenvalues. This simply reads:

$$\omega_n = \sqrt{\Omega_n^c} = \sqrt{\Omega_n^c + \Omega_n^b} \quad (96)$$

where Ω_c and Ω_b are, respectively, the eigenvalues of the cable (see Eq. (58)), i.e. $\Omega_n^c = \Omega_0 + \varepsilon\Omega_1$ and the ones of a simply-supported beam, i.e. $\Omega_n^b = \varepsilon^2 (n\pi)^4$.

It is worth noticing the progression of natural frequencies at high order modes is not influenced by the boundary conditions of the cable, as expected. For the case of a straight horizontal cable (i.e. $\lambda^2 \rightarrow 0$) Eq. (96) delivers $\omega_n = n\pi\sqrt{1 + 2p\epsilon + (n\pi\epsilon)^2}$.

Finally, contrary to the standard asymptotic approach presented in Section 3.3 (which predicts $\Omega_n \propto (n\pi)^6$ for high-order modes, see Eq. (56)), the aforementioned matching procedure makes it possible to recover the correct asymptotic convergence of the natural frequencies at high mode numbers, namely $\Omega_n \propto (n\pi)^4$ (see also Ref. [16]).

4. Numerical applications

This section presents several numerical applications to assess the capabilities of the different modeling strategies in reproducing both the natural circular frequencies and mode shapes of a sagged cable with small bending stiffness and rotational flexibility of the supports.

As a first example, the outcomes of the different approaches presented in the work are compared with the results of a finite element model for a cable characterized by moderately large value of the Irvine parameter (i.e. $\lambda^2 = 1$) and a very large value of ϵ (cf. [22]).

The finite element model is set up by considering a cable with $\mu = 1$ kg/m, and $EA = 25000$ N, and $EI = 1.3578$ Nm².

In order to correctly impose the searched slackness level (i.e. the prescribed value of Irvine's parameter $\bar{\lambda}^2$), the following steps are used to set up the finite element model:

1. The cable is initially straight and its developed length, which is set to $L_0 = 1$ m, coincides with the span length L . The FE mesh is constructed on this undeformed configuration, and the cable is discretized with 100 corotational beam finite elements.
2. The self-weight is linearly applied within a static analysis.
3. One support is then displaced of a quantity Δ , to obtain the correct span length $L = L_0 + \Delta$ and the corresponding level of sag, specified by the Irvine's parameter $\bar{\lambda}^2$.
4. The modal analysis is finally performed on the static equilibrium configuration reached at point 3.

Therefore, the problem is now shifted to the evaluation of the modified span length L of the cable such that the searched amount of slackness (described by $\bar{\lambda}^2$) is reached. To this aim, the elastic catenary equation is adopted:

$$L = \frac{HL_0}{EA} + \frac{HL_0}{W} \left[\sinh^{-1} \left(\frac{V}{H} \right) - \sinh^{-1} \left(\frac{V-W}{H} \right) \right] \quad (97)$$

where $W = \mu g L_0$ is the total weight of the cable and V denotes the vertical reaction force at the two supports, which is simply expressed as $V = W/2$, given the fact that the two supports are at the same level. A nonlinear optimization problem is then solved, which formally reads:

$$\begin{cases} F(\Delta, H) = \frac{(\mu g(L_0 + \Delta))^2}{H^3} - \frac{EA}{1 + \frac{1}{8} \frac{(\mu g(L_0 + \Delta))^2}{H^2}} - \bar{\lambda}^2 = 0 \\ G(\Delta, H) = L_0 + \Delta - \frac{HL_0}{EA} - \frac{HL_0}{W} \left[\sinh^{-1} \left(\frac{W}{2H} \right) - \sinh^{-1} \left(-\frac{W}{2H} \right) \right] = 0 \end{cases} \quad (98)$$

$F(\Delta, H)$ and $G(\Delta, H)$ being two nonlinear functions of both Δ and H . The first of Eqs. (98) is definition (6), where $L = L_0 + \Delta$, while the second of Eq. (98) is Eq. (97) specialized for a horizontal cable. The displacement Δ is assumed to be positive if it induces traction forces in the cable.

Once both the support displacement Δ and the axial force H have been determined from Eq. (98), an a-posteriori evaluation of both the characteristic frequency ϖ_c and the bending stiffness ϵ^2 is required, given the fact that H is not unitary, in general.

For the present application example, a value of $\bar{\lambda}^2 = 1$ is prescribed, leading to $H = 134.39$ N, $\Delta = 5.2$ mm, i.e. $L = 1.0052$ m. Furthermore, the bending stiffness was selected so that $\epsilon = 0.1$.

Figs. 9–12 depict the first four modal shapes of a doubly clamped cable characterized by $\epsilon = 0.1$ and $\lambda^2 = 1$. These values of the parameters have been deliberately chosen to be relatively large in order to better highlight the differences among the various approaches presented in Section 3. The results obtained with both the hybrid approach (HYB) and the matched asymptotic expansion method (MAE) are systematically compared against the semi-analytical solution (SAE) and the outcome of the finite element model (FEM). As it can be clearly assessed, the MAE solution is in excellent agreement with the FEM results, especially for the first two modal shapes. As the mode number increases, the accuracy slightly deteriorates, with some appreciable discrepancies in the global behavior of the fourth modal shape. However, both boundary conditions on displacements and curvatures in $\xi = 0$ and $\xi = 1$ are correctly captured. On the contrary, significant inaccuracies on the behavior of the mode shapes in the right-hand boundary layer can be observed already from the second mode, for what concerns the hybrid solution (HYB).

Figs. 13–16 depict the first four modal shapes of a cable characterized by $\epsilon = 0.05$ and $\lambda^2 = 1$, for varying degree-of-fixity parameters, which are selected to be representative of a doubly hinged configuration (i.e. $\rho_0 = \rho_1 = 0$), a doubly-clamped configuration (i.e. $\rho_0 = \rho_1 = 1$) and an intermediate configuration (i.e. $\rho_0 = \rho_1 = 0.5$). It is worth noticing that the expression of the first correction of the eigenvalues Ω_1 (see Eq. (D.1)) is singular for $\rho_0 = 0$. Therefore, a very small value of ρ_0 is adopted in the following plots. Moreover, for clarity purposes, FEM results are deliberately omitted from the subsequent figures, being precisely aligned with the semi-analytical solution of the eigenvalue problem (SAE results).

Results obtained with the MAE approach agree very well with the numerical solution of the eigenvalue problem. A reduction of the accuracy can be observed on both boundary layers as the mode number increases, and for intermediate values of the degree of fixity parameters (i.e. $\rho_0 = \rho_1 = 0.5$). The hybrid approach (HYB), instead, is inherently less accurate in the righthand boundary layer compared to the lefthand one and, especially for the clamped-clamped boundary conditions case.

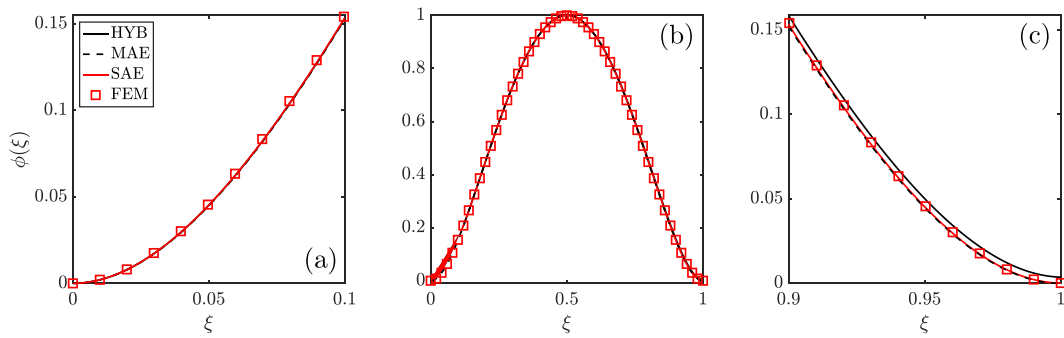


Fig. 9. First modal shape ($\phi_1(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

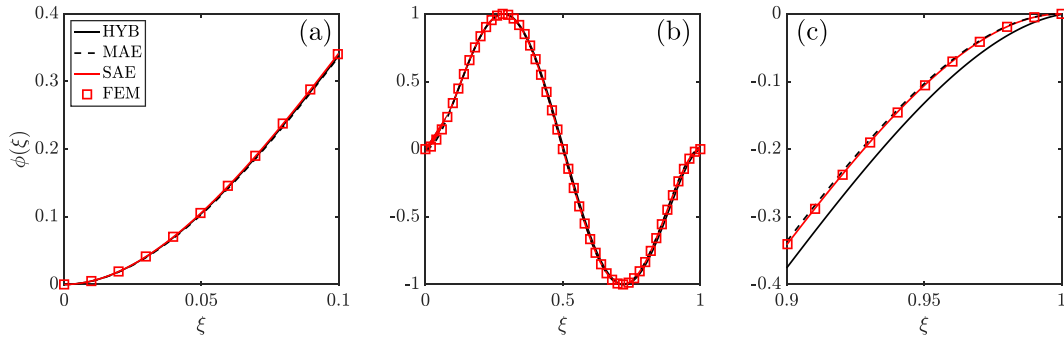


Fig. 10. Second modal shape ($\phi_2(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

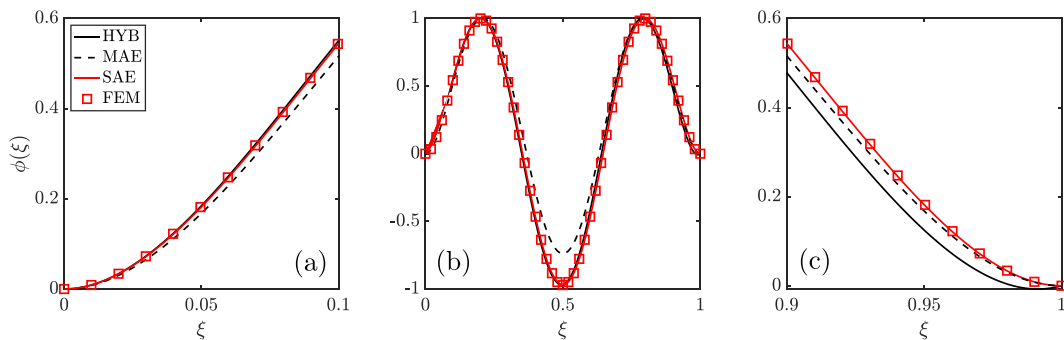


Fig. 11. Third modal shape ($\phi_3(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

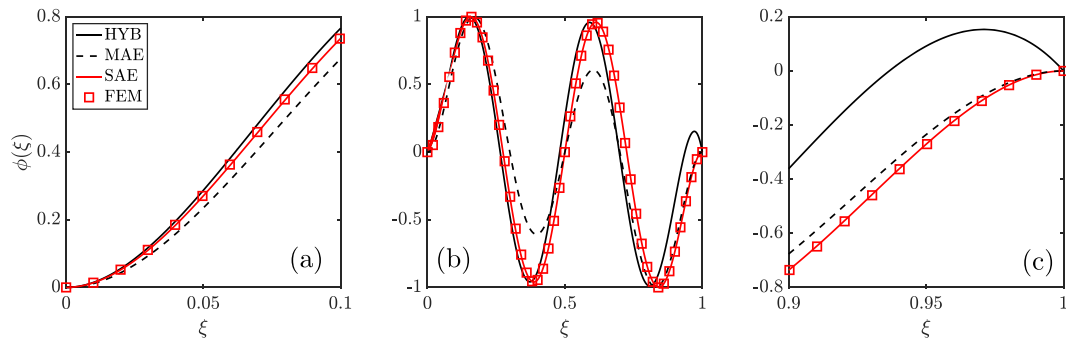


Fig. 12. Fourth modal shape ($\phi_4(\xi)$) for a doubly clamped cable (i.e. $\rho_0 = \rho_1 = 1$) with $\varepsilon = 0.1$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). The different approaches (see Table 1) are systematically compared with the results of the finite element model (FEM).

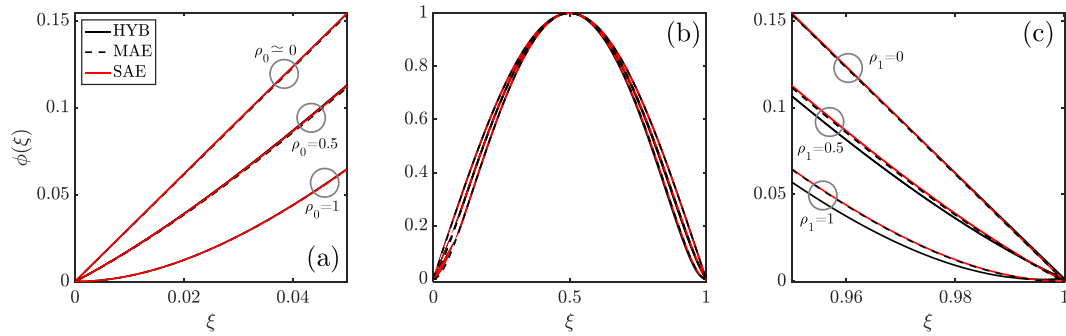


Fig. 13. First modal shape ($\phi_1(\xi)$) for a cable with $\varepsilon = 0.05$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). Results obtained with both the hybrid approach (HYB) and the Matched Asymptotic Expansion (MAE) are plotted for different values of the degree of fixity parameters $\rho_i = [0, 0.5, 1]$, $i = 0, 1$ and compared with the Semi-analytical solution (SAE).

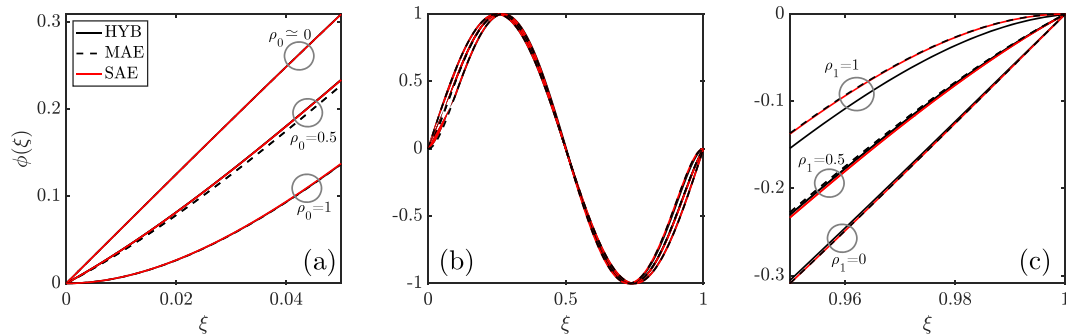


Fig. 14. Second modal shape ($\phi_2(\xi)$) for a cable with $\varepsilon = 0.05$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righten boundary layer (c). Results obtained with both the hybrid approach (HYB) and the Matched Asymptotic Expansion (MAE) are plotted for different values of the degree of fixity parameters $\rho_i = [0, 0.5, 1]$, $i = 0, 1$ and compared with the Semi-analytical solution (SAE).

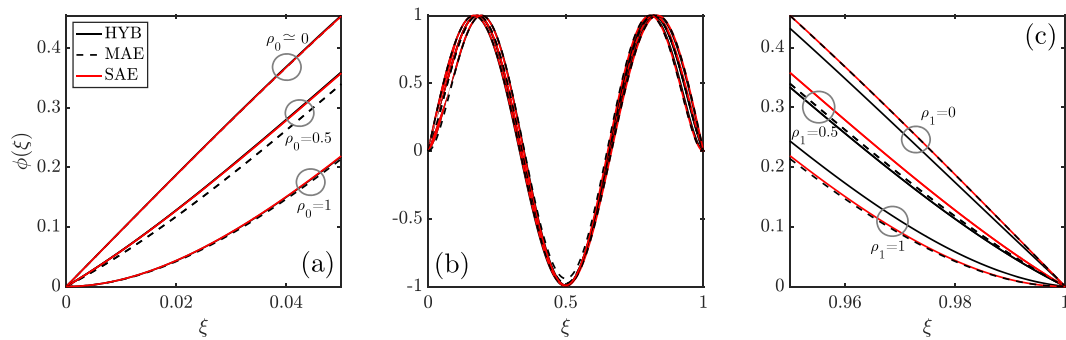


Fig. 15. Third modal shape ($\phi_3(\xi)$) for a cable with $\varepsilon = 0.05$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righthand boundary layer (c). Results obtained with both the hybrid approach (HYB) and the Matched Asymptotic Expansion (MAE) are plotted for different values of the degree of fixity parameters $\rho_i = [0, 0.5, 1]$, $i = 0, 1$ and compared with the Semi-analytical solution (SAE).

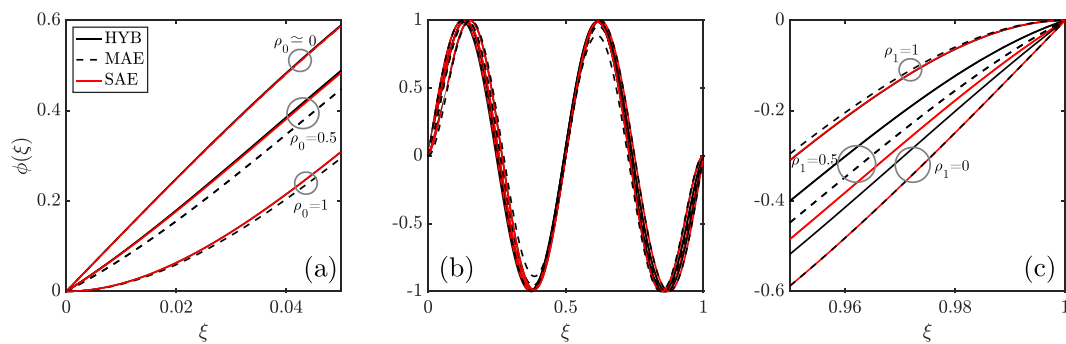


Fig. 16. Fourth modal shape ($\phi_4(\xi)$) for a cable with $\varepsilon = 0.05$ and $\lambda^2 = 1$: close-up of the lefthand boundary layer (a), global mode shape (b) and close-up of the righthand boundary layer (c). Results obtained with both the hybrid approach (HYB) and the Matched Asymptotic Expansion (MAE) are plotted for different values of the degree of fixity parameters $\rho_i = [0, 0.5, 1]$, $i = 0, 1$ and compared with the Semi-analytical solution (SAE).

To better investigate the convergence properties of the modal shapes, the following global metric is introduced: $\|\phi_n - \phi_n^{SAE}\|_2$, which represents the Euclidean norm of the difference between the predictions obtained with each approach and the semi-analytical reference solution (SAE), along the whole cable length.

Figs. 17–20 depict the value of this norm for both the HYB and the MAE solutions, as a function of ε and λ^2 , for $\rho_0 = \rho_1 = 0.50$, which is representative of an intermediate degree-of-fixity of the cable supports. For the first mode ($n = 1$, see Fig. 17) larger discrepancies with respect to the SAE solution are observed for the HYB approach in the range $\lambda^2 \approx 2 - 10$, especially for large values of ε (see Fig. 17(a)). On the contrary, the norm computed for the MAE solution indicates excellent accuracy over the entire parameter domain considered (see Fig. 17(b)).

As the mode number increases, the maximum discrepancies become confined to the region of the parameter plane near the largest value of the dimensionless bending parameter, namely $\varepsilon = 0.1$, and are distributed almost uniformly with respect to the Irvine parameter λ^2 . In addition, the largest discrepancies are consistently found for the HYB approach for all mode numbers except $n = 2$ (see Fig. 18). The analysis of such global metric up to $n = 4$ confirms the accuracy of the MAE solution and its better performance over the hybrid formulation, particularly at larger values ε .

Figs. 21–23 depict the progression of the scaled non-dimensional circular frequencies, computed with the different approaches, as a function of the degree of fixity parameters and of the Irvine parameter.

A very good accuracy is reached by the matched asymptotic expansion solution (MAE, see Eq. (96)), for the whole range of mode numbers herein considered. As the boundary conditions get stiffer in rotation, however, a slight reduction in the accuracy can be noted, which can be attributed to the absence of the second-order correction term in the derived expression.

On the other hand, the asymptotic solution (ASY) of the algebraic eigenvalue problem (see Eq. (56)) is extremely accurate up to $n = 4$ for all the different cases herein considered, while departing from the semi-analytical solution as the mode number further increases.

Most importantly, both closed-form solutions are able to precisely capture the increase of the first natural circular frequency due to the cable sagging (i.e. for $\lambda^2 > 0$), which results in a clear modification of the progression of the scaled frequencies (cf. subfigs. (a) with subfigs. (b)). It is worth noticing that this behavior is quite challenging to capture by means of finite elements simulations, due to the sensitivity of the first natural frequency on the procedures adopted to reach the sagged cable configuration under self-weight, revealing once again the importance of closed-form approximate solutions for its quick and reliable estimation.

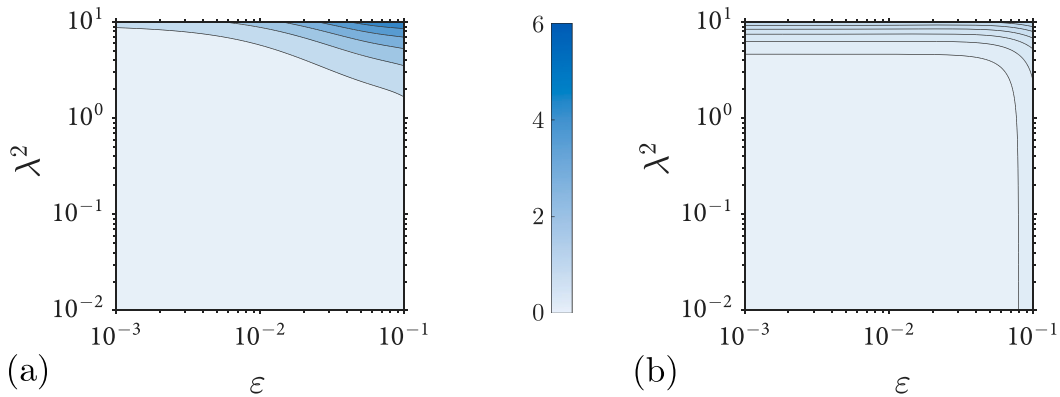


Fig. 17. Values of the norm $\|\phi_1^{HYB} - \phi_1^{SAE}\|_2$ (a) and $\|\phi_1^{MAE} - \phi_1^{SAE}\|_2$ (b) as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.50$. The reader is referred to the online version of the manuscript for colors.

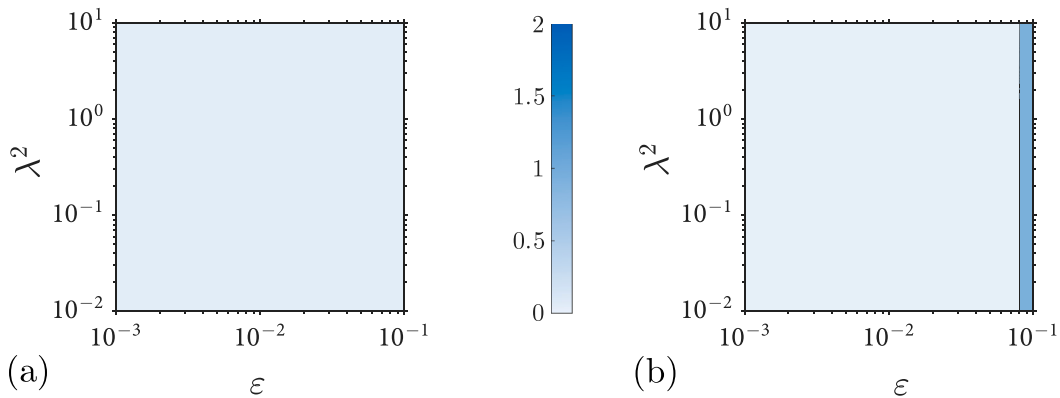


Fig. 18. Values of the norm $\|\phi_2^{HYB} - \phi_2^{SAE}\|_2$ (a) and $\|\phi_2^{MAE} - \phi_2^{SAE}\|_2$ (b) as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.50$. The reader is referred to the online version of the manuscript for colors.

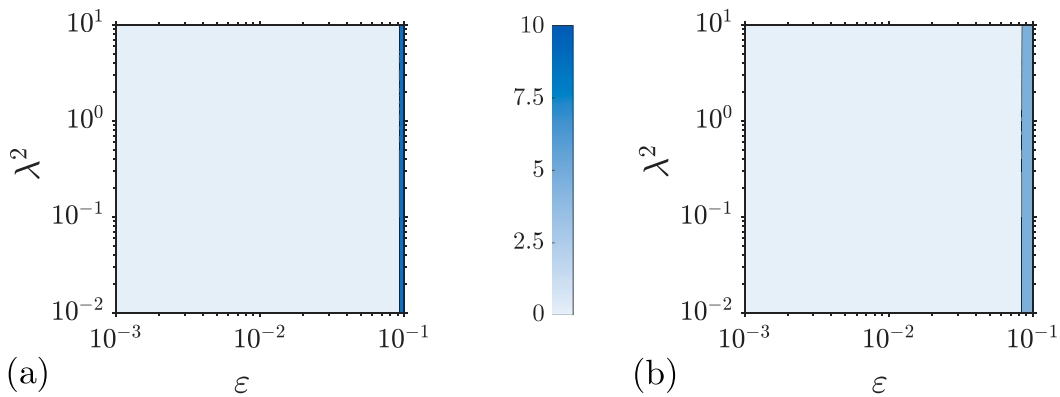


Fig. 19. Values of the norm $\|\phi_3^{HYB} - \phi_3^{SAE}\|_2$ (a) and $\|\phi_3^{MAE} - \phi_3^{SAE}\|_2$ (b) as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.50$. The reader is referred to the online version of the manuscript for colors.

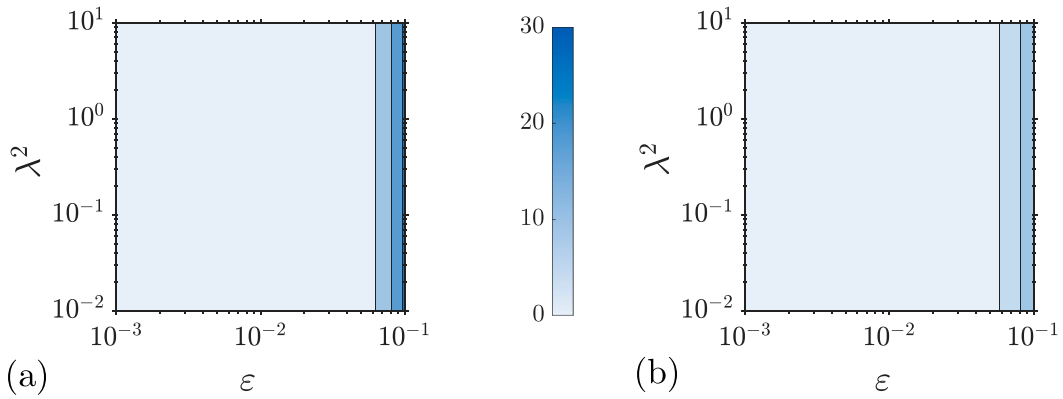


Fig. 20. Values of the norm $\|\phi_4^{HYB} - \phi_4^{SAE}\|_2$ (a) and $\|\phi_4^{MAE} - \phi_4^{SAE}\|_2$ (b) as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.50$. The reader is referred to the online version of the manuscript for colors.

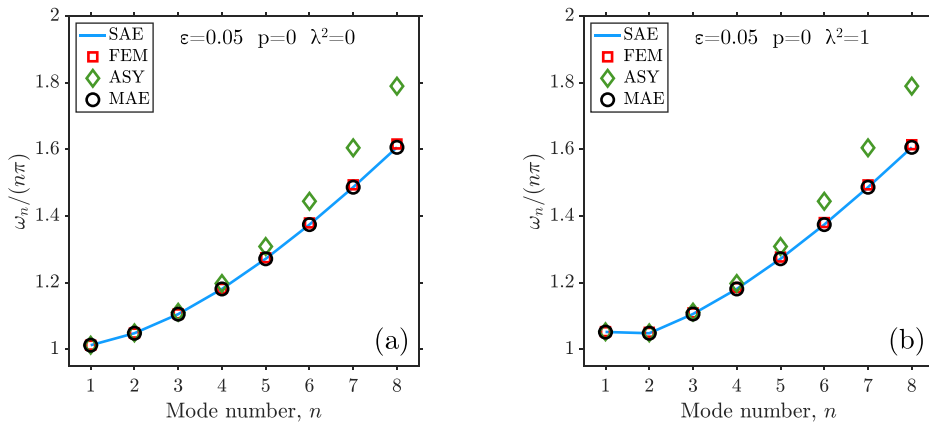


Fig. 21. Scaled non-dimensional natural circular frequencies $\omega_n/(n\pi)$ of a doubly hinge cable ($\rho_0 = \rho_1 = 0$, $p = 0$) with small bending stiffness ($\varepsilon = 0.05$) as a function of the mode number n . Results obtained with both the asymptotic approach (ASY - see Eq. (56)) and the Matched Asymptotic Expansion (MAE - see Eq. (96)) are systematically compared with the Semi-analytical solution (SAE) and the outcome of a Finite Element Model (FEM). The Irvine's parameter is set to $\lambda^2 = 0$ (a) and $\lambda^2 = 1$ (b).

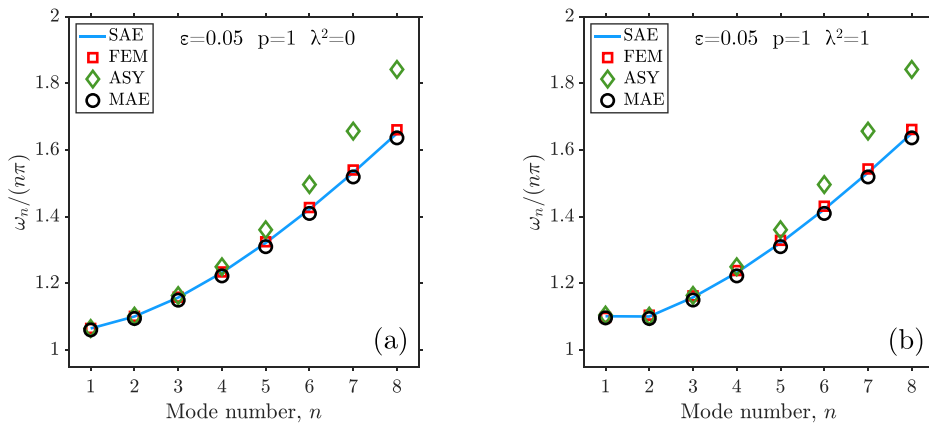


Fig. 22. Scaled non-dimensional natural circular frequencies $\omega_n/(n\pi)$ of a cable with rotational springs ($\rho_0 = \rho_1 = 0.5$, $p = 1$) and small bending stiffness ($\varepsilon = 0.05$) as a function of the mode number n . Results obtained with both the asymptotic approach (ASY - see Eq. (56)) and the Matched Asymptotic Expansion (MAE - see Eq. (96)) are systematically compared with the Semi-analytical solution (SAE) and the outcome of a Finite Element Model (FEM). The Irvine's parameter is set to $\lambda^2 = 0$ (a) and $\lambda^2 = 1$ (b).

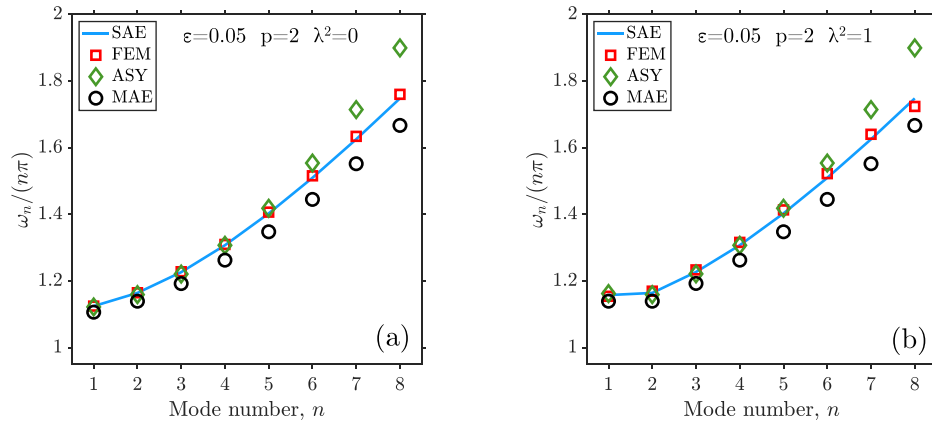


Fig. 23. Scaled non-dimensional natural circular frequencies $\omega_n/(n\pi)$ of a doubly-clamped cable ($\rho_0 = \rho_1 = 1$, $p = 2$) with small bending stiffness ($\varepsilon = 0.05$) as a function of the mode number n . Results obtained with both the asymptotic approach (ASY - see Eq. (56)) and the Matched Asymptotic Expansion (MAE - see Eq. (96)) are systematically compared with the Semi-analytical solution (SAE) and the outcome of a Finite Element Model (FEM). The Irvine's parameter is set to $\lambda^2 = 0$ (a) and $\lambda^2 = 1$ (b).

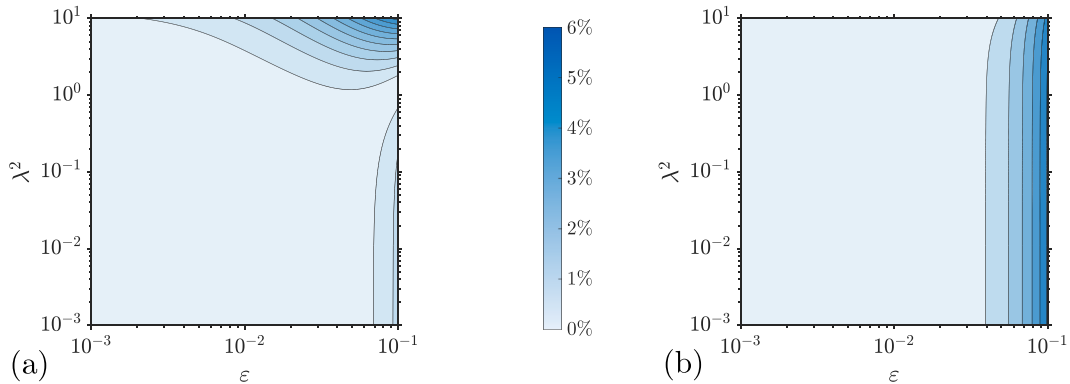


Fig. 24. Modulus of the relative percentage error of the first natural circular frequency (ω_1) computed with the asymptotic approach (see Eq. (56)) (a) and the matched asymptotic expansion (see Eq. (96)) (b), compared to numerical solution of the characteristic equation (see Section 3.2). Results are plotted as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 1$. The reader is referred to the online version of the manuscript for colors.

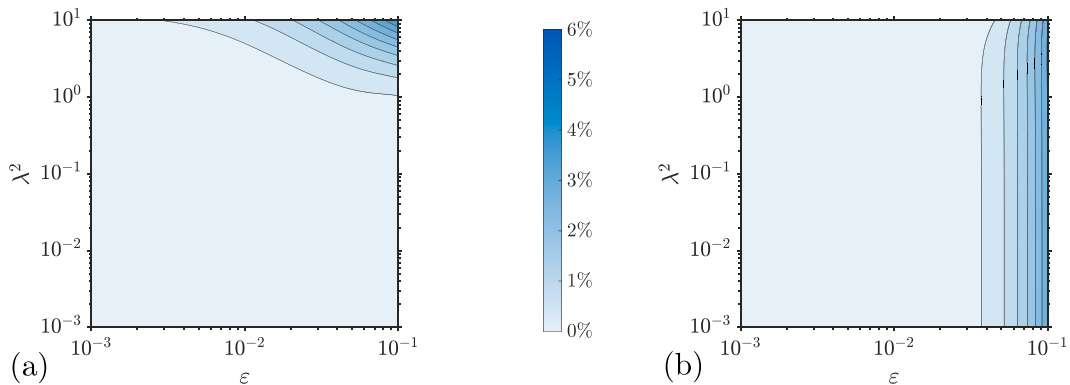


Fig. 25. Modulus of the relative percentage error of the first natural circular frequency (ω_1) computed with the asymptotic approach (see Eq. (56)) (a) and the matched asymptotic expansion (see Eq. (96)) (b), compared to numerical solution of the characteristic equation (see Section 3.2). Results are plotted as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.75$. The reader is referred to the online version of the manuscript for colors.

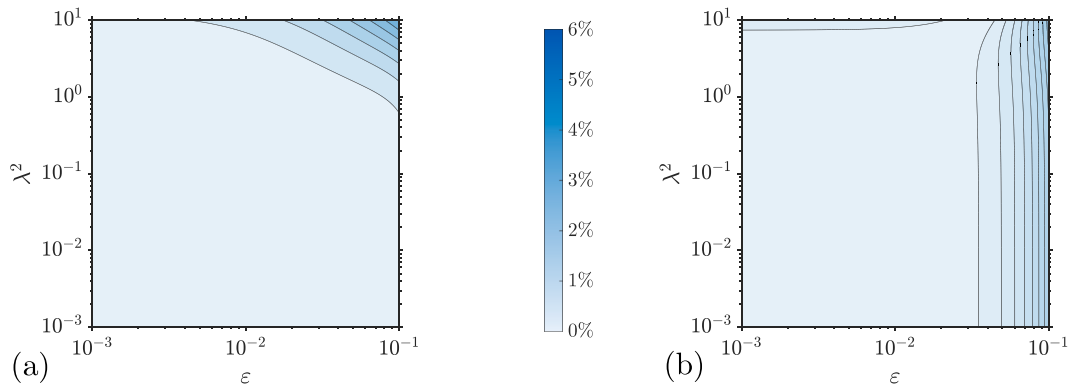


Fig. 26. Modulus of the relative percentage error of the first natural circular frequency (ω_1) computed with the asymptotic approach (see Eq. (56)) (a) and the matched asymptotic expansion (see Eq. (96)) (b), compared to numerical solution of the characteristic equation (see Section 3.2). Results are plotted as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.50$. The reader is referred to the online version of the manuscript for colors.

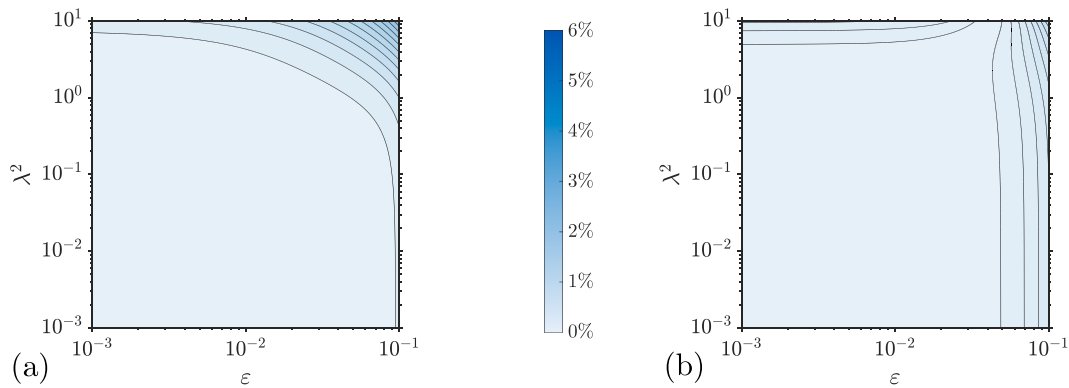


Fig. 27. Modulus of the relative percentage error of the first natural circular frequency (ω_1) computed with the asymptotic approach (see Eq. (56)) (a) and the matched asymptotic expansion (see Eq. (96)) (b), compared to numerical solution of the characteristic equation (see Section 3.2). Results are plotted as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.25$. The reader is referred to the online version of the manuscript for colors.

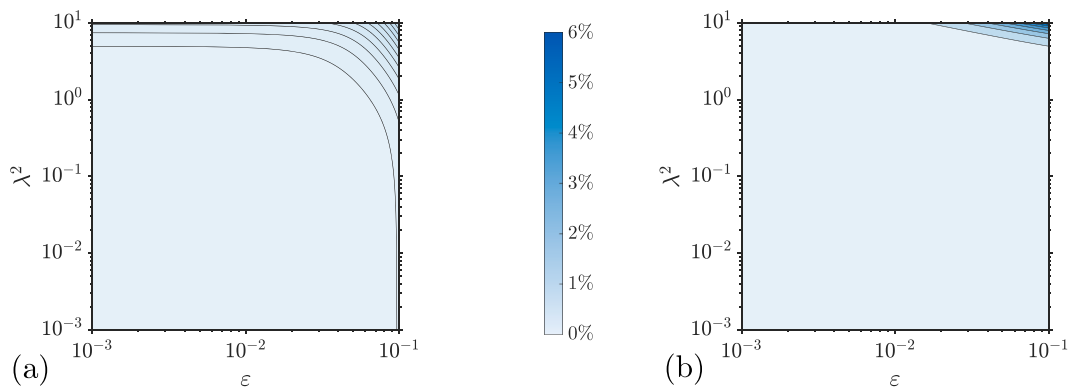


Fig. 28. Modulus of the relative percentage error of the first natural circular frequency (ω_1) computed with the asymptotic approach (see Eq. (56)) (a) and the matched asymptotic expansion (see Eq. (96)) (b), compared to numerical solution of the characteristic equation (see Section 3.2). Results are plotted as a function of both ε and λ^2 , and for $\rho_0 = \rho_1 = 0.01$. The reader is referred to the online version of the manuscript for colors.

In order to better highlight the accuracy of the proposed closed-form solutions, the modulus of the relative percentage error of the first natural circular frequency computed with both MAE and ASY approaches with respect to the semi-analytical solution is depicted in Figs. (24–28), for different values of the degree-of-fixity parameters ρ_i and for a broad range of both ε and λ^2 parameters.

Maximum percentage errors (in the order of 6%) are encountered for the doubly-clamped case, and are associated with largest values of ε parameter, consistently with the spirit of perturbation methods, and with large values of the Irvine's parameter as well.

While the asymptotic approach generally delivers largest errors in the northern-east region of the contour plot (i.e. highest values of both ε and λ^2), the accuracy of MAE solution appears to be more affected by the dimensionless bending stiffness compared to the Irvine's parameter, typically localizing errors in a narrow uniform strip around $\varepsilon = 0.1$.

As the boundary conditions gradually approach the doubly-hinged cable case, however, both solutions deliver very precise estimates across the entire space of investigated parameters, with relative percentage errors below 2% (see Figs. 27 and 28).

5. Conclusions

In this work, the spectral properties of shallow cables with small bending stiffness have been investigated by means of analytical methods.

To set up the basic framework, the case of perfectly flexible sagged cable has been firstly considered, and a closed-form expression for the dimensionless natural frequencies accounting for sag/extensibility has been derived, based on the Rayleigh-Ritz discretization procedure.

Then, the case of a straight tensioned cable with small bending stiffness and rotational springs at its ends has been addressed, and the asymptotic solutions of both natural frequencies and mode shapes have been briefly revised.

The complete problem of a sagged cable with small bending stiffness and rotational springs at its end has been subsequently considered and originally solved with a semi-analytical approach.

Leveraging on the smallness of the dimensionless bending stiffness parameter, novel asymptotic solutions for both natural circular frequencies and mode shapes of shallow cables have been also derived. Accordingly, the first asymptotic solution (ASY) considers a classical regular expansion of the algebraic eigenvalue problem; the second, instead, has been obtained by employing the Matched Asymptotic Expansion (MAE) method.

Within this spirit, the eigenvalue problem was solved in the continuous form by matching an outer solution, capturing the far-field behavior of the sagged cable, with two inner solutions, expressing the corrections induced by both the bending stiffness effect and the non-homogeneous boundary conditions, in the regions close to the cable ends.

The accuracy of the different perturbative solutions has been systematically compared against the semi-analytical results and the outcome of finite element models. For what concerns the natural frequencies, both the ASY and the MAE solutions proved to be quite accurate for a broad range of the dimensionless parameters governing the problem, namely the dimensionless bending stiffness ε , the Irvine parameter λ^2 , and the sum-of-degree-of-fixity parameters p .

Most importantly, both closed-form solutions were able to precisely capture the increase of the first natural circular frequency due to the cable sagging (i.e. for $\lambda^2 = \text{ord}(1)$), which can be quite challenging to reproduce with standard finite element models.

In terms of modal shape reconstructions, though, the MAE solution proved superior in accuracy to both the asymptotic (ASY) and hybrid (HYB) approaches, exhibiting remarkable agreement with the finite element simulations even for relatively large values of both ε and λ^2 , up to the fourth mode shape of the cable. Additionally, the MAE solution yields closed-form expressions for both the circular frequencies and the mode shapes as functions of the dimensionless parameters governing the cable response, thereby providing physical insight into the underlying dynamic behavior.

In conclusion, the obtained formulation provides practical and robust tools for the efficient monitoring of cable structures in which both bending stiffness and cable extensibility play a significant role. The proposed analytical solutions enable accurate characterization of the spectral properties of shallow cables while dramatically reducing computational burden, making them particularly well suited for inverse analysis problems and SHM applications.

CRedit authorship contribution statement

Stefano Corazza: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization; **Francesco Foti:** Writing – review & editing, Writing – original draft, Supervision, Project administration, Methodology, Investigation, Formal analysis, Conceptualization; **Vincent Denoël:** Writing – review & editing, Supervision, Project administration, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix A. Expression of h_1

The expression of the constant h_1 reads:

$$h_1 = h_1^* \left(8\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) \left(\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) + \lambda^2 \sin \sqrt{\Omega_0} \right) \right)^{-1} \quad (\text{A.1})$$

where:

$$\begin{aligned} h_1^* = & 4b_0\lambda^2\Omega_0 + 2\sqrt{\Omega_0}(4a_0\Omega_0(-\lambda^2 + \Omega_0) + (\lambda^2 - 3\Omega_0)\Omega_1) + \lambda^2(-4a_0\Omega_0 + \Omega_1)\sin(2\sqrt{\Omega_0}) \\ & - 2\sqrt{\Omega_0}(4a_0\Omega_0(-\lambda^2 + \Omega_0) + (\lambda^2 - 3\Omega_0)\Omega_1)\cos\sqrt{\Omega_0} - 4b_0\lambda^2\Omega_0\cos(2\sqrt{\Omega_0}) \\ & + 2(4a_0\lambda^2\Omega_0 + 4b_0\Omega_0^{3/2}(-\lambda^2 + \Omega_0) + 2\Omega_0^2\Omega_1 - \lambda^2(\Omega_1 + 2\Omega_0\Omega_1))\sin\sqrt{\Omega_0} \end{aligned}$$

Appendix B. Expression of A_2 and b_0

The expression of the constant A_2 reads:

$$A_2 = A_2^* \left(8\Omega_0^{3/2}(-\lambda^2 + \Omega_0) \left(\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) + \lambda^2 \sin\sqrt{\Omega_0} \right) \right)^{-1} \quad (\text{B.1})$$

with:

$$\begin{aligned} A_2^* = & \lambda^2\sqrt{\Omega_0}(4b_0\lambda^2\sqrt{\Omega_0} + 12a_0\Omega_0(-\lambda^2 + \Omega_0) + (11\lambda^2 - 15\Omega_0)\Omega_1) + 2\sqrt{\Omega_0}(4(a_0\lambda^2 + b_0(\lambda^2 - \Omega_0)\sqrt{\Omega_0})(\lambda^2 - \Omega_0)\Omega_0) \cos\sqrt{\Omega_0} \\ & + 2\sqrt{\Omega_0}(\lambda^2(7 - 4\Omega_0)\Omega_0 + 2\Omega_0^3 + \lambda^4(-5 + 2\Omega_0))\Omega_1 \cos\sqrt{\Omega_0} + \lambda^2\sqrt{\Omega_0}(-4b_0\lambda^2\sqrt{\Omega_0} + (\lambda^2 - \Omega_0)(4a_0\Omega_0 - \Omega_1)) \cos(2\sqrt{\Omega_0}) \\ & - 2(4b_0\lambda^2(\lambda^2 - \Omega_0)\Omega_0^{3/2} - 4a_0\Omega_0(-2\lambda^2\Omega_0^2 + \Omega_0^3 + \lambda^4(1 + \Omega_0))) \sin\sqrt{\Omega_0} + 2(\Omega_0^3 - 6\lambda^2\Omega_0^2 + \lambda^4(1 + 5\Omega_0))\Omega_1 \sin(\sqrt{\Omega_0}) \\ & + \lambda^2(-4a_0\lambda^2\Omega_0 + 4b_0\Omega_0^{3/2}(-\lambda^2 + \Omega_0) + \lambda^2\Omega_1) \sin(2\sqrt{\Omega_0}) \end{aligned}$$

The expression of the constant b_0 reads:

$$b_0 = b_0^* \left(8\Omega_0 \left(\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) + \lambda^2 \sin\sqrt{\Omega_0} \right)^2 \right)^{-1} \quad (\text{B.2})$$

where:

$$\begin{aligned} b_0^* = & 2\sqrt{\Omega_0}(4a_0\lambda^2(\lambda^2 - \Omega_0)\Omega_0) + 4A_1(\lambda^2 - \Omega_0)^2\Omega_0^{3/2} - 5\lambda^4\Omega_1 + 7\lambda^2\Omega_0\Omega_1 \\ & + (2\lambda^2\sqrt{\Omega_0}(4a_0\Omega_0(-\lambda^2 + \Omega_0) + (5\lambda^2 - 7\Omega_0)\Omega_1) \cos\sqrt{\Omega_0} + \lambda^4(4a_0\Omega_0 - \Omega_1)\sin(2\sqrt{\Omega_0})) \\ & + 2\lambda^2(-4a_0\lambda^2\Omega_0 + 4A_1\Omega_0^{3/2}(-\lambda^2 + \Omega_0) - 2\Omega_0^2\Omega_1 + \lambda^2(1 + 2\Omega_0)\Omega_1) \sin\sqrt{\Omega_0} \end{aligned}$$

Appendix C. Expression of C_2

The expression of the constant C_2 reads:

$$C_2 = C_2^* \left(8\Omega_0 \left(\sqrt{\Omega_0}(-\lambda^2 + \Omega_0) + \lambda^2 \sin\sqrt{\Omega_0} \right)^2 \right)^{-1} \quad (\text{C.1})$$

with:

$$\begin{aligned} C_2^* = & -4A_1\lambda^4\Omega_0 + 20a_0\lambda^4\Omega_0^{3/2} - 20a_0\lambda^2\Omega_0^{5/2} - 19\lambda^4\sqrt{\Omega_0}\Omega_1 + 23\lambda^2\Omega_0^{3/2}\Omega_1 \\ & - 4\sqrt{\Omega_0}(4a_0\lambda^2(\lambda^2 - \Omega_0)\Omega_0 + 2A_1(\lambda^2 - \Omega_0)^2\Omega_0^{3/2}) \cos\sqrt{\Omega_0} \\ & - 4\sqrt{\Omega_0}(\lambda^4(-5 + \Omega_0) + \lambda^2(7 - 2\Omega_0)\Omega_0 + \Omega_0^3)\Omega_1 \cos\sqrt{\Omega_0} \\ & + \lambda^2\sqrt{\Omega_0}(4A_1\lambda^2\sqrt{\Omega_0} + 4a_0\Omega_0(-\lambda^2 + \Omega_0) - (\lambda^2 - 5\Omega_0)\Omega_1) \cos(2\sqrt{\Omega_0}) \\ & - 16a_0\lambda^4\Omega_0 \sin\sqrt{\Omega_0} + 8A_1\lambda^4\Omega_0^{3/2} \sin\sqrt{\Omega_0} - 8a_0\lambda^4\Omega_0^2 \sin\sqrt{\Omega_0} - 8A_1\lambda^2\Omega_0^{5/2} \sin\sqrt{\Omega_0} \\ & + 16a_0\lambda^2\Omega_0^3 \sin\sqrt{\Omega_0} - 8a_0\Omega_0^4 \sin\sqrt{\Omega_0} + 4\lambda^4\Omega_1 \sin\sqrt{\Omega_0} + 10\lambda^4\Omega_0\Omega_1 \sin\sqrt{\Omega_0} \\ & - 12\lambda^2\Omega_0^2\Omega_1 \sin\sqrt{\Omega_0} + 2\Omega_0^3\Omega_1 \sin\sqrt{\Omega_0} + 8a_0\lambda^4\Omega_0 \sin(2\sqrt{\Omega_0}) \\ & + 4A_1\lambda^4\Omega_0^{3/2} \sin(2\sqrt{\Omega_0}) - 4A_1\lambda^2\Omega_0^{5/2} \sin(2\sqrt{\Omega_0}) - 2\lambda^4\Omega_1 \sin(2\sqrt{\Omega_0}) \end{aligned}$$

Appendix D. Expression of Ω_1

The expression of the first-order correction of the eigenvalue reads:

$$\Omega_1 = \Omega_1^* D^{-1} \quad (\text{D.1})$$

where:

$$\begin{aligned} \Omega_1^* = & \Omega_0^{3/2} \left(2 \left(-2\lambda^2 \rho_0 (\rho_0 + \rho_1) \Omega_0^2 + \rho_0 (\rho_0 + \rho_1) \Omega_0^3 + \lambda^4 (4 + \rho_0^2 \Omega_0 + \rho_0 (-4 + \rho_1 \Omega_0)) \right) \cos \sqrt{\Omega_0} \right. \\ & + \lambda^2 \left(-6\lambda^2 + 6\lambda^2 \rho_0 + \lambda^2 \rho_0^2 + \lambda^2 \rho_0 \rho_1 - \lambda^2 (2 + \rho_0^2 + \rho_0 (-2 + \rho_1)) \cos \left(2\sqrt{\Omega_0} \right) \right. \\ & + 2(-1 + \rho_0 + \rho_0^2 + \rho_0 \rho_1) \sqrt{\Omega_0} (-\lambda^2 + \Omega_0) \sin \sqrt{\Omega_0} - \lambda^2 \sqrt{\Omega_0} \sin \left(2\sqrt{\Omega_0} \right) + \lambda^2 \rho_0 \sqrt{\Omega_0} \sin \left(2\sqrt{\Omega_0} \right) \\ & - \lambda^2 \rho_0^2 \sqrt{\Omega_0} \sin \left(2\sqrt{\Omega_0} \right) - \lambda^2 \rho_0 \rho_1 \sqrt{\Omega_0} \sin \left(2\sqrt{\Omega_0} \right) + \Omega_0^{3/2} \sin \left(2\sqrt{\Omega_0} \right) \\ & \left. \left. - \rho_0 \Omega_0^{3/2} \sin \left(2\sqrt{\Omega_0} \right) + \rho_0^2 \Omega_0^{3/2} \sin \left(2\sqrt{\Omega_0} \right) + \rho_0 \rho_1 \Omega_0^{3/2} \sin \left(2\sqrt{\Omega_0} \right) \right) \right) \\ D = & \rho_0 \left(6\lambda^4 \sqrt{\Omega_0} - 7\lambda^2 \Omega_0^{3/2} + \sqrt{\Omega_0} (\lambda^4 (-6 + \Omega_0) - 2\lambda^2 (-4 + \Omega_0) \Omega_0 + \Omega_0^3) \cos \left(\sqrt{\Omega_0} \right) \right. \\ & \left. - \lambda^2 \Omega_0^{3/2} \cos \left(2\sqrt{\Omega_0} \right) - 2\lambda^4 \sin \sqrt{\Omega_0} - 3\lambda^4 \Omega_0 \sin \sqrt{\Omega_0} + 4\lambda^2 \Omega_0^2 \sin \sqrt{\Omega_0} - \Omega_0^3 \sin \sqrt{\Omega_0} + \lambda^4 \sin \left(2\sqrt{\Omega_0} \right) \right) \end{aligned}$$

Appendix E. Expression of the mode shape

The complete expression of the mode shape can be casted as:

$$\phi(\xi) = \phi^{(0)}(\xi) + \epsilon \phi^{(1)}(\xi) \quad (\text{E.1})$$

The $\text{ord}(\epsilon^0)$ part of the solution is given by:

$$\phi^{(0)}(\xi) = \phi_*^{(0)}(\xi) \left(2 \left(\sqrt{\Omega_0} (-\lambda^2 + \Omega_0) + \lambda^2 \sin \sqrt{\Omega_0} \right)^2 \right)^{-1}$$

where:

$$\begin{aligned} \phi_*^{(0)}(\xi) = & -2\lambda^4 \sqrt{\Omega_0} + 2\lambda^2 \Omega_0^{3/2} + 2\lambda^2 (\lambda^2 - \Omega_0) \sqrt{\Omega_0} \cos \sqrt{\Omega_0} - 3\lambda^2 (\lambda^2 - \Omega_0) \sqrt{\Omega_0} \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) \\ & + 2\lambda^4 \sqrt{\Omega_0} \cos \left(\sqrt{\Omega_0} \xi \right) - 2\lambda^2 \Omega_0^{3/2} \cos \left(\sqrt{\Omega_0} \xi \right) + \lambda^4 \sqrt{\Omega_0} \cos \left(\sqrt{\Omega_0} (1 + \xi) \right) - \lambda^2 \Omega_0^{3/2} \cos \left(\sqrt{\Omega_0} (1 + \xi) \right) \\ & + 2\lambda^4 \sin \sqrt{\Omega_0} - \lambda^4 \sin \left(2\sqrt{\Omega_0} \right) - \lambda^4 \sin \left(\sqrt{\Omega_0} (\xi - 2) \right) + \lambda^4 \sin \left(\sqrt{\Omega_0} (\xi - 1) \right) + \lambda^4 \sin \left(\sqrt{\Omega_0} \xi \right) \\ & + 2\lambda^4 \Omega_0 \sin \left(\sqrt{\Omega_0} \xi \right) - 4\lambda^2 \Omega_0^2 \sin \left(\sqrt{\Omega_0} \xi \right) + 2\Omega_0^3 \sin \left(\sqrt{\Omega_0} \xi \right) - \lambda^4 \sin \left((1 + \xi) \sqrt{\Omega_0} \right) \end{aligned}$$

It can be clearly noted, that the global behavior of the mode shape does not depend on the boundary conditions (i.e. on ρ_0 and ρ_1), as expected. Moreover, it is easy to verify that, as $\lambda^2 \rightarrow 0$, the taut-string solution is retrieved, i.e. $\phi^{(0)}(\xi) = \sin \left(\sqrt{\Omega_0} \xi \right)$.

The $\text{ord}(\epsilon^1)$ part of the solution is the correction introduced by the bending stiffness effects, and it is given by:

$$\phi^{(1)}(\xi) = \phi_*^{(1)}(\xi) \left(8\Omega_0 \left(\sqrt{\Omega_0} (-\lambda^2 + \Omega_0) + \lambda^2 \sin \sqrt{\Omega_0} \right)^2 \right)^{-1}$$

where

$$\begin{aligned} \phi_*^{(1)}(\xi) = & 4A_1 \lambda^4 \Omega_0 + 4C_2 e^{\frac{-1+\xi}{\epsilon}} \lambda^4 \Omega_0 + 4C_1 e^{\frac{-\xi}{\epsilon}} \lambda^4 \Omega_0 - 8a_0 \lambda^4 \Omega_0^{3/2} + 8C_2 e^{\frac{-1+\xi}{\epsilon}} \lambda^4 \Omega_0^2 + 8C_1 e^{\frac{-\xi}{\epsilon}} \lambda^4 \Omega_0^2 \\ & + 8a_0 \lambda^2 \Omega_0^{5/2} - 16C_2 e^{\frac{-1+\xi}{\epsilon}} \lambda^2 \Omega_0^3 - 16C_1 e^{\frac{-\xi}{\epsilon}} \lambda^2 \Omega_0^3 + 8C_2 e^{\frac{-1+\xi}{\epsilon}} \Omega_0^4 + 8C_1 e^{\frac{-\xi}{\epsilon}} \Omega_0^4 + 10\lambda^4 \sqrt{\Omega_0} \Omega_1 \\ & - 14\lambda^2 \Omega_0^{3/2} \Omega_1 + 2\lambda^2 \sqrt{\Omega_0} (4a_0 (\lambda^2 - \Omega_0) \Omega_0 - 5\lambda^2 \Omega_1 + 7\Omega_0 \Omega_1) \cos \sqrt{\Omega_0} - 2\Omega_0^3 \Omega_1 \sin \left(\sqrt{\Omega_0} \xi \right) \\ & - 4e^{\frac{-1+\xi}{\epsilon}} \left(C_1 e^{\frac{1}{\epsilon}} + C_2 e^{\frac{2\xi}{\epsilon}} + A_1 e^{\frac{1+\xi}{\epsilon}} \right) \lambda^4 \Omega_0 \cos^2 \sqrt{\Omega_0} - 2\lambda^4 \xi \sqrt{\Omega_0} \Omega_1 \cos \left(\sqrt{\Omega_0} (\xi - 2) \right) \\ & - 12a_0 \lambda^4 \Omega_0^{3/2} \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) + 12a_0 \lambda^2 \Omega_0^{5/2} \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) + 7\lambda^4 \sqrt{\Omega_0} \Omega_1 \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) \\ & + 2\lambda^4 \xi \sqrt{\Omega_0} \Omega_1 \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) - 9\lambda^2 \Omega_0^{3/2} \Omega_1 \cos \left(\sqrt{\Omega_0} (\xi - 1) \right) + 8a_0 \lambda^4 \Omega_0^{3/2} \cos \left(\sqrt{\Omega_0} \xi \right) \end{aligned}$$

$$\begin{aligned}
& + 8A_1\lambda^4\Omega_0^2\cos\left(\sqrt{\Omega_0}\xi\right) - 8a_0\lambda^2\Omega_0^{5/2}\cos\left(\xi\sqrt{\Omega_0}\right) - 16A_1\lambda^2\Omega_0^3\cos\left(\sqrt{\Omega_0}\xi\right) + 8A_1\Omega_0^4\cos\left(\sqrt{\Omega_0}\xi\right) \\
& - 10\lambda^4\sqrt{\Omega_0}\Omega_1\cos\left(\sqrt{\Omega_0}\xi\right) + 2\lambda^4\xi\sqrt{\Omega_0}\Omega_1\cos\left(\sqrt{\Omega_0}\xi\right) + 14\lambda^2\Omega_0^{3/2}\Omega_1\cos\left(\sqrt{\Omega_0}\xi\right) \\
& + 4\lambda^4\xi\Omega_0^{3/2}\Omega_1\cos\left(\xi\sqrt{\Omega_0}\right) - 8\lambda^2\xi\Omega_0^{5/2}\Omega_1\cos\left(\xi\sqrt{\Omega_0}\right) + 4\xi\Omega_0^{7/2}\Omega_1\cos\left(\sqrt{\Omega_0}\xi\right) \\
& + 4a_0\lambda^4\Omega_0^{3/2}\cos\left((1+\xi)\sqrt{\Omega_0}\right) - 4a_0\lambda^2\Omega_0^{5/2}\cos\left(\sqrt{\Omega_0}(1+\xi)\right) + 3\lambda^4\sqrt{\Omega_0}\Omega_1\cos\left(\sqrt{\Omega_0}(1+\xi)\right) \\
& - 2\lambda^4\xi\sqrt{\Omega_0}\Omega_1\cos\left(\sqrt{\Omega_0}(1+\xi)\right) - 5\lambda^2\Omega_0^{3/2}\Omega_1\cos\left(\sqrt{\Omega_0}(1+\xi)\right) + 8a_0\lambda^4\Omega_0\sin\sqrt{\Omega_0} \\
& - 8A_1\lambda^4\Omega_0^{3/2}\sin\sqrt{\Omega_0} - 16C_2e^{-\frac{1+\xi}{\varepsilon}}\lambda^4\Omega_0^{3/2}\sin\sqrt{\Omega_0} - 16C_1e^{-\frac{\xi}{\varepsilon}}\lambda^4\Omega_0^{3/2}\sin\sqrt{\Omega_0} + 8A_1\lambda^2\Omega_0^{5/2}\sin\sqrt{\Omega_0} \\
& + 16C_2e^{-\frac{1+\xi}{\varepsilon}}\lambda^2\Omega_0^{5/2}\sin\sqrt{\Omega_0} + 16C_1e^{-\frac{\xi}{\varepsilon}}\lambda^2\Omega_0^{5/2}\sin\sqrt{\Omega_0} - 2\lambda^4\Omega_1\sin\sqrt{\Omega_0} - 4\lambda^4\Omega_0\Omega_1\sin\sqrt{\Omega_0} \\
& + 4\lambda^2\Omega_0^2\Omega_1\sin\sqrt{\Omega_0} + 4A_1\lambda^4\Omega_0\sin^2\sqrt{\Omega_0} + 4C_2e^{-\frac{1+\xi}{\varepsilon}}\lambda^4\Omega_0\sin^2\sqrt{\Omega_0} + 4C_1e^{-\frac{\xi}{\varepsilon}}\lambda^4\Omega_0\sin^2\sqrt{\Omega_0} \\
& - 4a_0\lambda^4\Omega_0\sin\left(2\sqrt{\Omega_0}\right) + \lambda^4\Omega_1\sin\left(2\sqrt{\Omega_0}\right) - 4a_0\lambda^4\Omega_0\sin\left(\sqrt{\Omega_0}(\xi-2)\right) + \lambda^4\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-2)\right) \\
& + 4a_0\lambda^4\Omega_0\sin\left(\sqrt{\Omega_0}(\xi-1)\right) + 4A_1\lambda^4\Omega_0^{3/2}\sin\left(\sqrt{\Omega_0}(\xi-1)\right) - 4A_1\lambda^2\Omega_0^{5/2}\sin\left((-1+\xi)\sqrt{\Omega_0}\right) \\
& - \lambda^4\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-1)\right) - 2\lambda^4\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-1)\right) + 6\lambda^4\xi\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-1)\right) \\
& + 2\lambda^2\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-1)\right) - 6\lambda^2\xi\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}(\xi-1)\right) + 4a_0\lambda^4\Omega_0\sin\left(\sqrt{\Omega_0}\xi\right) \\
& + 8a_0\lambda^4\Omega_0^2\sin\left(\sqrt{\Omega_0}\xi\right) - 16a_0\lambda^2\Omega_0^3\sin\left(\sqrt{\Omega_0}\xi\right) + 8a_0\Omega_0^4\sin\left(\sqrt{\Omega_0}\xi\right) - \lambda^4\Omega_1\sin\left(\sqrt{\Omega_0}\xi\right) \\
& - 2\lambda^4\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}\xi\right) - 4\lambda^4\xi\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}\xi\right) + 4\lambda^2\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}\xi\right) \\
& + 4\lambda^2\xi\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}\xi\right) - 4a_0\lambda^4\Omega_0\sin\left(\sqrt{\Omega_0}(1+\xi)\right) - 4A_1\lambda^4\Omega_0^{3/2}\sin\left(\sqrt{\Omega_0}(1+\xi)\right) \\
& + 4A_1\lambda^2\Omega_0^{5/2}\sin\left(\sqrt{\Omega_0}(1+\xi)\right) + \lambda^4\Omega_1\sin\left(\sqrt{\Omega_0}(1+\xi)\right) + 2\lambda^4\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}(1+\xi)\right) \\
& - 2\lambda^4\xi\Omega_0\Omega_1\sin\left(\sqrt{\Omega_0}(1+\xi)\right) - 2\lambda^2\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}(1+\xi)\right) + 2\lambda^2\xi\Omega_0^2\Omega_1\sin\left(\sqrt{\Omega_0}(1+\xi)\right)
\end{aligned}$$

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