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AN OPERATIONAL APPROACH TO THE GENERALIZED RENCONTRES POLYNOMIALS

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Abstract. In this paper, we study the umbral operators J, M and N associated with the generalized rencontres polynomials $D_n^{(m)}(x)$. We obtain their representations in terms of the differential operator $\mathfrak{D}_x$ and the shift operator E. Then, by using these representations, we obtain some combinatorial and differential identities for the generalized rencontres polynomials. Finally, we extend these results to some related polynomials and, in particular, to the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$.

1. Introduction

1.1. **Generalized rencontres polynomials.** Let $m \in \mathbb{N}$. The generalized rencontres polynomials $D_n^{(m)}(x)$, the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$ are defined by (see [3, 5])

$$D_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} x^k,$$

$$P_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (m+n-k)! x^k,$$

$$A_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}^{(m)} x^k,$$

where the coefficients

$$d_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (m+k)!$$

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are the generalized derangement numbers (see [3, 5]), and the coefficients

$$a_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (m+k)!$$

are the generalized arrangement numbers. For $m = 0$, we have the ordinary derangement numbers d_n [4, page 182] and the ordinary arrangement numbers a_n [4, page 75]. Since

$$\begin{aligned} d^{(m)}(t) &= \sum_{n\geq 0} d_n^{(m)} \frac{t^n}{n!} = \frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, \\ p^{(m)}(t) &= \sum_{n\geq 0} (m+n)! \frac{t^n}{n!} = \frac{m!}{(1-t)^{m+1}}, \\ a^{(m)}(t) &= \sum_{n\geq 0} a_n^{(m)} \frac{t^n}{n!} = \frac{m!\mathrm{e}^{t}}{(1-t)^{m+1}}, \end{aligned}$$

then we have the exponential generating series

$$D^{(m)}(x;t) = \sum_{n\geq 0} D_n^{(m)}(x) \frac{t^n}{n!} = d^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}, \tag{1.1}$$

$$P^{(m)}(x;t) = \sum_{n\geq 0} P_n^{(m)}(x) \frac{t^n}{n!} = p^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{xt}}{(1-t)^{m+1}}, \tag{1.2}$$

$$A^{(m)}(x;t) = \sum_{n\geq 0} A_n^{(m)}(x) \frac{t^n}{n!} = a^{(m)}(t)\mathrm{e}^{xt} = \frac{m!\mathrm{e}^{(x+1)t}}{(1-t)^{m+1}}. \tag{1.3}$$

In particular, from these series, we also have

$$\begin{aligned} D_n^{(m)}(x) &= \sum_{k=0}^{n} \binom{n}{k} (m+k)!(x-1)^{n-k}, \\ A_n^{(m)}(x) &= \sum_{k=0}^{n} \binom{n}{k} (m+k)!(x+1)^{n-k}. \end{aligned} \tag{1.4}$$

Clearly, the polynomials $P_n^{(m)}(x)$ and $A_n^{(m)}(x)$ can be expressed in terms of the polynomials $D_n^{(m)}(x)$, namely

$$\begin{aligned} P_n^{(m)}(x) &= D_n^{(m)}(x+1), \\ A_n^{(m)}(x) &= D_n^{(m)}(x+2). \end{aligned}$$

1.2. **Sheffer sequences and umbral operators.** Given any polynomial sequence $\{p_n(x)\}_{n\geq 0}$, where each $p_n(x)$ is a polynomial with degree n, we can consider the linear operators $J, M, N : \mathbb{Q}[x] \to \mathbb{Q}[x]$ defined for every $n \in \mathbb{N}$, by

$$Jp_n(x) = np_{n-1}(x), \quad Mp_n(x) = p_{n+1}(x) \quad \text{and} \quad Np_n(x) = np_n(x),$$

where J is the umbral derivative (or lowering operator, or annihilation operator), M is the umbral shift (or raising operator or creation operator) and N is the umbral theta operator.

By Sheffer's theorem [11], every linear operator $L : \mathbb{Q}[x] \to \mathbb{Q}[x]$ can be represented by means of an exponential series in the derivative $\mathfrak{D}_x$ with respect to x. More precisely, there exists a unique polynomial sequence $\{L_n(x)\}_{n\geq 0}$, where $L_n(x) \in \mathbb{Q}[x]$ for every $n \in \mathbb{N}$, such that

$$Lp(x) = \sum_{k\geq 0} \frac{L_k(x)}{k!} \mathfrak{D}_x^k \, p(x) = \sum_{k=0}^{n} \frac{L_k(x)}{k!} p^{(k)}(x),$$

for every polynomial $p(x) \in \mathbb{Q}[x]$ of degree n. For instance, the shift operator E^λ, defined by $E^\lambda p(x) = p(x+\lambda)$, is represented by the exponential series $\mathrm{e}^{\lambda \mathfrak{D}_x}$.

A Sheffer sequence [2, 7–11] with spectrum $(g(t), f(t))$ is a polynomial sequence $\{s_n(x)\}_{n\geq 0}$ having exponential generating series

$$s(x;t) = \sum_{n\geq 0} s_n(x) \frac{t^n}{n!} = g(t)\mathrm{e}^{xf(t)},$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series, with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The umbral operators J, M and N associated with a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ with spectrum $(g(t), f(t))$ are given by [9, page 49, 50]

$$\begin{aligned} J &= \widehat{f}(\mathfrak{D}_x), \\ M &= \frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + xf'(\widehat{f}(\mathfrak{D}_x)), \\ N = MJ &= \left(\frac{g'(\widehat{f}(\mathfrak{D}_x))}{g(\widehat{f}(\mathfrak{D}_x))} + xf'(\widehat{f}(\mathfrak{D}_x)) \right) \widehat{f}(\mathfrak{D}_x), \end{aligned}$$

where $\widehat{f}(t)$ is the compositional inverse of $f(t)$. In particular, for an Appell sequence [1, 7, 9, 10], i.e., a Sheffer sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ with spectrum $(g(t), t)$ (where $f(t) = \widehat{f}(t) = t$), we have

$$J = \mathfrak{D}_x, \tag{1.5}$$

$$M = \frac{g'(\mathfrak{D}_x)}{g(\mathfrak{D}_x)} + x, \tag{1.6}$$

$$N = MJ = \frac{g'(\mathfrak{D}_x)}{g(\mathfrak{D}_x)} \mathfrak{D}_x + x \mathfrak{D}_x . \tag{1.7}$$

By identity (1.5), we have $a_n'(x) = na_{n-1}(x)$ for every $n \in \mathbb{N}$.

By series (1.1), (1.2) and (1.3), the generalized rencontres polynomials $D_n^{(m)}(x)$, the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$ form an Appell sequence, respectively with spectrum

$$\left(\frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, t\right), \quad \left(\frac{m!}{(1-t)^{m+1}}, t\right) \quad \text{and} \quad \left(\frac{m!\mathrm{e}^{t}}{(1-t)^{m+1}}, t\right). \tag{1.8}$$

More generally, the shifted polynomials $D_n^{(m)}(x+\alpha)$ form an Appell sequence with spectrum

$$\left(\frac{m!\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}, t\right). \tag{1.9}$$

In this paper, we will determine the representation of the main umbral operators associated with the generalized rencontres polynomials and then, by using these representations, we obtain some combinatorial and differential identities for the generalized rencontres polynomials. Finally, we extend these results to the shifted polynomials $D_n^{(m)}(x+\alpha)$ and, in particular, to the generalized permutation polynomials $P_n^{(m)}(x)$ and the generalized arrangement polynomials $A_n^{(m)}(x)$.

2. Operators for the Generalized Rencontres Polynomials

Since the generalized rencontres polynomials form an Appell sequence, by identity (1.5), we have

$$\mathfrak{D}_x\, D_n^{(m)}(x) = nD_{n-1}^{(m)}(x), \quad \text{for all } n \in \mathbb{N},$$

and, more generally,

$$\mathfrak{D}_x^k\, D_n^{(m)}(x) = \binom{n}{k} k! D_{n-1}^{(m)}(x), \quad \text{for all } n, k \in \mathbb{N}.$$

For the second operator M, we have the following result.

Theorem 2.1. *The operator M is given by*

$$M = \frac{m+\mathfrak{D}_x}{1-\mathfrak{D}_x} + x. \tag{2.1}$$

Proof. The operator M is given by formula (1.6). By the first spectrum in (1.8), we have

$$g(t) = \frac{m!\mathrm{e}^{-t}}{(1-t)^{m+1}}, \quad g'(t) = \frac{m+t}{1-t} g(t) \quad \text{and} \quad \frac{g'(t)}{g(t)} = \frac{m+t}{1-t}.$$

This implies at once formula (2.1). □

From this theorem, we can obtain the following recurrence (already obtained in [3, (10)] by using the exponential series techniques).

Theorem 2.2. *The generalized rencontres polynomials satisfy the recurrence*

$$D_{n+2}^{(m)}(x) = (x+m+n+1)D_{n+1}^{(m)}(x) - (n+1)(x-1)D_n^{(m)}(x). \tag{2.2}$$

Proof. Since $MD_n^{(m)}(x) = D_{n+1}^{(m)}(x)$, by (2.1), we have

$$D_{n+1}^{(m)}(x) = \frac{m + \mathfrak{D}_x}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x),$$

that is

$$(1 - \mathfrak{D}_x)D_{n+1}^{(m)}(x) = (m + \mathfrak{D}_x)D_n^{(m)}(x) + (1 - \mathfrak{D}_x)xD_n^{(m)}(x).$$

Hence, we have

$$D_{n+1}^{(m)}(x) - \mathfrak{D}_x D_{n+1}^{(m)}(x) = mD_n^{(m)}(x) + \mathfrak{D}_x D_n^{(m)}(x) + xD_n^{(m)}(x) - D_n^{(m)}(x) - x\,\mathfrak{D}_x D_n^{(m)}(x).$$

Now, since the generalized rencontres polynomials form an Appell sequence, we have

$$D_{n+1}^{(m)}(x) - (n+1)D_n^{(m)}(x) = mD_n^{(m)}(x) + nD_{n-1}^{(m)}(x) + xD_n^{(m)}(x) - D_n^{(m)}(x) - nxD_{n-1}^{(m)}(x),$$

that is

$$D_{n+1}^{(m)}(x) = (x + m + n)D_n^{(m)}(x) - n(x - 1)D_{n-1}^{(m)}(x).$$

Replacing n by $n + 1$, we obtain recurrence (2.2). ☐

In a similar way, Theorem 2.1 implies the following result.

Theorem 2.3. *The generalized rencontres polynomials satisfy the recurrence*

$$D_{n+1}^{(m)}(x) = (x - 1)D_n^{(m)}(x) + (m + 1)\sum_{k=0}^{n} \binom{n}{k} k!D_{n-k}^{(m)}(x). \tag{2.3}$$

Proof. Since $MD_n^{(m)}(x) = D_{n+1}^{(m)}(x)$, by formula (2.1) we have

$$\begin{aligned} D_{n+1}^{(m)}(x) &= \frac{m + \mathfrak{D}_x}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x) \\ &= m\frac{1}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + \frac{\mathfrak{D}_x}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + xD_n^{(m)}(x) \\ &= m\sum_{k\geq 0} \mathfrak{D}_x^k D_n^{(m)}(x) + \sum_{k\geq 1} \mathfrak{D}_x^k D_n^{(m)}(x) + xD_n^{(m)}(x). \end{aligned}$$

Since $D_n^{(m)}(x)$ is a polynomial of degree n, we have

$$\begin{aligned} D_{n+1}^{(m)}(x) &= m\sum_{k=0}^{n} \mathfrak{D}_x^k D_n^{(m)}(x) + \sum_{k=1}^{n} \mathfrak{D}_x^k D_n^{(m)}(x) + xD_n^{(m)}(x) \\ &= (x + m)D_n^{(m)}(x) + (m + 1)\sum_{k=1}^{n} \mathfrak{D}_x^k D_n^{(m)}(x) \\ &= (x + m)D_n^{(m)}(x) + (m + 1)\sum_{k=0}^{n} \mathfrak{D}_x^k D_n^{(m)}(x) - (m + 1)D_n^{(m)}(x) \\ &= (x - 1)D_n^{(m)}(x) + (m + 1)\sum_{k=0}^{n} \mathfrak{D}_x^k D_n^{(m)}(x). \end{aligned}$$

Since the generalized rencontres polynomials form an Appell sequence, we have recurrence (2.3). ☐

Finally, as a direct consequence of (1.7) and Theorem 2.1, we have the following result for the operator N.

Theorem 2.4. *The operator N is given by*

$$N = \frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1 - \mathfrak{D}_x} + x\,\mathfrak{D}_x\,. \tag{2.4}$$

Theorem 2.4 immediately implies the following differential equation.

Theorem 2.5. *The generalized rencontres polynomials satisfy the differential equation*

$$(x-1)D_n''(x) - (x+m+n-1)D_n'(x) + nD_n(x) = 0, \tag{2.5}$$

where, for simplicity, we write $D_n(x) = D_n^{(m)}(x)$.

Proof. Since $ND_n^{(m)}(x) = nD_n^{(m)}(x)$, by formula (2.4), we have

$$\frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + x\,\mathfrak{D}_x\, D_n^{(m)}(x) = nD_n^{(m)}(x),$$

that is

$$(m\,\mathfrak{D}_x + \mathfrak{D}_x^2)D_n^{(m)}(x) + (1-\mathfrak{D}_x)x\,\mathfrak{D}_x\, D_n^{(m)}(x) = n(1-\mathfrak{D}_x)D_n^{(m)}(x).$$

Hence, setting $D_n(x) = D_n^{(m)}(x)$, we have

$$(m\,\mathfrak{D}_x + \mathfrak{D}_x^2)D_n(x) + (1-\mathfrak{D}_x)xD_n'(x) = n(1-\mathfrak{D}_x)D_n(x)$$

or

$$mD_n'(x) + D_n''(x) + xD_n'(x) - D_n'(x) - xD_n''(x) = nD_n(x) - nD_n'(x).$$

This relation simplifies in the differential equation (2.5). □

Notice that, due to the fact that the generalized rencontres polynomials form an Appell sequence, the differential equation (2.5) is equivalent to recurrence (2.2).

Finally, we have the following theorem.

Theorem 2.6. *The generalized rencontres polynomials satisfy the identity*

$$(m+n+1)D_n^{(m)}(x) = (m+1)\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + n(x-1)D_{n-1}^{(m)}(x). \tag{2.6}$$

Proof. Since $ND_n^{(m)}(x) = nD_n^{(m)}(x)$, by (2.4), we have

$$\begin{aligned}
nD_n^{(m)}(x) =& \frac{m\,\mathfrak{D}_x + \mathfrak{D}_x^2}{1 - \mathfrak{D}_x} D_n^{(m)}(x) + x\,\mathfrak{D}_x\, D_n^{(m)}(x) \\
=& m\sum_{k\geq 1}\mathfrak{D}_x^k\, D_n^{(m)}(x) + \sum_{k\geq 2}\mathfrak{D}_x^k\, D_n^{(m)}(x) + x\,\mathfrak{D}_x\, D_n^{(m)}(x) \\
=& m\sum_{k=1}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + \sum_{k=2}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) + nxD_{n-1}^{(m)}(x) \\
=& m\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x) - mD_n^{(m)}(x)
\end{aligned}$$

$$+\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x)-D_n^{(m)}(x)-nD_{n-1}^{(m)}(x)+nxD_{n-1}^{(m)}(x)$$
$$=(m+1)\sum_{k=0}^{n}\binom{n}{k}k!D_{n-k}^{(m)}(x)-(m+1)D_n^{(m)}(x)+n(x-1)D_{n-1}^{(m)}(x),$$

and this simplifies in identity (2.6). □

Notice that identities (2.3) and (2.6) imply recurrence (2.2).

3. Rodrigues-Like Formulas

In this section, we find a Rodrigues-like formula for the generalized rencontres polynomials. We start by proving the following simple result, generalizing identity (1.4).

Lemma 3.1. *We have the identity*

$$D_n^{(m)}(x)=\sum_{k=0}^{n}\binom{n}{k}D_k^{(m)}(\alpha)(x-\alpha)^{n-k}. \tag{3.1}$$

Proof. By series (1.1), we have

$$D^{(m)}(x;t)=\frac{m!\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}=\frac{m!\mathrm{e}^{(x-\alpha)t}\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}=\frac{m!\mathrm{e}^{(\alpha-1)t}}{(1-t)^{m+1}}\mathrm{e}^{(x-\alpha)t}$$

or

$$D^{(m)}(x;t)=D^{(m)}(\alpha;t)\mathrm{e}^{(x-\alpha)t}.$$

This identity is equivalent to identity (3.1). □

Remark 3.1. Since $D_n^{(m)}(0)=d_n^{(m)}$, $D_n^{(m)}(1)=(m+n)!$ and $D_n^{(m)}(2)=a_n^{(m)}$ for $\alpha=1$ identity (3.1) reduces to identity (1.4), while for $\alpha=0$ and $\alpha=2$ identity (3.1) becomes

$$D_n^{(m)}(x)=\sum_{k=0}^{n}\binom{n}{k}d_k^{(m)}x^{n-k},$$
$$D_n^{(m)}(x)=\sum_{k=0}^{n}\binom{n}{k}a_k^{(m)}(x-2)^{n-k}.$$

Now we can prove the following result.

Theorem 3.1. *For the generalized rencontres polynomials we have the Rodrigues-like formula*

$$D_n^{(m)}(x)=D^{(m)}(\alpha,\mathfrak{D}_x)(x-\alpha)^n. \tag{3.2}$$

In particular, we have

$$D_n^{(m)}(x)=d^{(m)}(\mathfrak{D}_x)x^n, \tag{3.3}$$
$$D_n^{(m)}(x)=p^{(m)}(\mathfrak{D}_x)(x-1)^n, \tag{3.4}$$
$$D_n^{(m)}(x)=a^{(m)}(\mathfrak{D}_x)(x-2)^n. \tag{3.5}$$

Proof. From identity (3.1), we have

$$D_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} D_k^{(m)}(\alpha)(x-\alpha)^{n-k} = \sum_{k\geq 0} \frac{D_k^{(m)}(\alpha)}{k!}\,\mathfrak{D}_x^k (x-\alpha)^n$$
$$= \left(\sum_{k\geq 0} D_k^{(m)}(\alpha) \frac{\mathfrak{D}_x^k}{k!} \right) (x-\alpha)^n = D^{(m)}(\alpha; \mathfrak{D}_x)(x-\alpha)^n.$$

This is (3.2). Then, (3.3), (3.4) and (3.5) can be obtained for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, respectively. □

4. Final Remarks

As already noted in the Introduction, the shifted polynomials $D_n^{(m)}(x+\alpha)$ form an Appell sequence with spectrum (1.9). From this simple observation, it is easy to see that the associated umbral operators J_α, M_α and N_α are given by

$$J_\alpha = J, \quad M_\alpha = M + \alpha \quad \text{and} \quad N_\alpha = N + \alpha\, \mathfrak{D}_x\,.$$

All the properties obtained for the generalized rencontres polynomials can be reformulated for the shifted polynomials $D_n^{(m)}(x+\alpha)$, and, in particular, for the polynomials $P_n^{(m)}(x)$ and $A_n^{(m)}(x)$. For instance, from recurrence (2.2), we obtain the recurrences

$$P_{n+2}^{(m)}(x) = (x+m+n+2)P_{n+1}^{(m)}(x) - (n+1)xP_n^{(m)}(x),$$
$$A_{n+2}^{(m)}(x) = (x+m+n+3)A_{n+1}^{(m)}(x) - (n+1)(x+1)A_n^{(m)}(x)$$

and from differential equation (2.5), we obtain the differential equations

$$xP_n''(x) - (x+m+n)P_n'(x) + nP_n(x) = 0,$$
$$(x+1)A_n''(x) - (x+m+n+1)A_n'(x) + nA_n(x) = 0,$$

where, always for simplicity, we write $P_n(x) = P_n^{(m)}(x)$ and $A_n(x) = A_n^{(m)}(x)$. Similarly, from recurrence (2.3), we obtain the recurrences

$$P_{n+1}^{(m)}(x) = xP_n^{(m)}(x) + (m+1)\sum_{k=0}^{n} \binom{n}{k} k!P_{n-k}^{(m)}(x),$$
$$A_{n+1}^{(m)}(x) = (x+1)A_n^{(m)}(x) + (m+1)\sum_{k=0}^{n} \binom{n}{k} k!A_{n-k}^{(m)}(x),$$

and, from identity (2.6), we obtain the identities

$$(m+n+1)P_n^{(m)}(x) = (m+1)\sum_{k=0}^{n} \binom{n}{k} k!P_{n-k}^{(m)}(x) + nxP_{n-1}^{(m)}(x),$$
$$(m+n+1)A_n^{(m)}(x) = (m+1)\sum_{k=0}^{n} \binom{n}{k} k!A_{n-k}^{(m)}(x) + n(x+1)A_{n-1}^{(m)}(x).$$

Finally, from identity (3.1), we obtain the identities

$$P_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} P_k^{(m)}(\alpha)(x-\alpha)^{n-k},$$
$$A_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(m)}(\alpha)(x-\alpha)^{n-k}.$$

and, consequently, we have the Rodrigues-like formulas

$$P_n^{(m)}(x) = P^{(m)}(\alpha, \mathfrak{D}_x)(x-\alpha)^n,$$
$$A_n^{(m)}(x) = A^{(m)}(\alpha, \mathfrak{D}_x)(x-\alpha)^n.$$

As a concluding remark, notice that in [6, 7] we have considered a slight variant of the polynomials considered in this paper, namely the polynomials $D_n^{(\nu)}(x)$ and $A_n^{(\nu)}(x)$ defined by the exponential generating series

$$D^{(\nu)}(x;t) = \sum_{n\geq 0} D_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{v+1}},$$
$$A^{(\nu)}(x;t) = \sum_{n\geq 0} A_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{(x+1)t}}{(1-t)^{\nu+1}},$$

where ν is an arbitrary symbol. Also these polynomials form Appell sequences and the umbral operators J, M and N are the same, except for the fact that m is replaced by ν.

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