



Axially Harmonic Functions and the Harmonic Functional Calculus on the S -spectrum

Fabrizio Colombo¹ · Antonino De Martino¹ · Stefano Pinton¹ · Irene Sabadini¹ 

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Abstract

The spectral theory on the S -spectrum was introduced to give an appropriate mathematical setting to quaternionic quantum mechanics, but it was soon realized that there were different applications of this theory, for example, to fractional heat diffusion and to the spectral theory for the Dirac operator on manifolds. In this seminal paper we introduce the harmonic functional calculus based on the S -spectrum and on an integral representation of axially harmonic functions. This calculus can be seen as a bridge between harmonic analysis and the spectral theory. The resolvent operator of the harmonic functional calculus is the commutative version of the pseudo S -resolvent operator. This new calculus also appears, in a natural way, in the product rule for the F -functional calculus.

Keywords Harmonic analysis · S -spectrum · Integral representation of axially harmonic functions · Harmonic functional calculus · Resolvent equation · Riesz projectors · F -functional calculus

Mathematics Subject Classification 47A10 · 47A60

1 Introduction

The notion of S -spectrum and of S -resolvent operators for quaternionic linear operators and for linear Clifford operators have been identified only in 2006 using methods in hypercomplex analysis. The original motivation for the investigation of a new notion of spectrum was the paper [8] of Birkhoff and von Neumann, where the authors showed that quantum mechanics can also be formulated using quaternions, but they did not specify what notion of spectrum one should use for quaternionic linear operators. The

✉ Irene Sabadini
irene.sabadini@polimi.it

¹ Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy

appropriate notion is that of S -spectrum, which has also been used for the spectral theorem for quaternionic linear operators, see [2], and for Clifford algebra linear operators, see [14].

The quaternionic spectral theory on the S -spectrum is systematically organised in the books [13, 20] while for the Clifford setting see [19]. In particular, the history of the discovery of the S -spectrum and the formulation of the S -functional calculus are explained in the introduction of the book [20] with a complete list of references.

Nowadays there are several research directions in the area of the spectral theory on the S -spectrum, and without claiming completeness we mention: the characteristic operator function, see [5], slice hyperholomorphic Schur analysis, see [4], and several applications to fractional powers of vector operators that describe fractional Fourier's laws for nonhomogeneous materials, see for example [6, 21, 23]. These results on the fractional powers are based on the H^∞ -functional calculus (see the seminal papers [3, 12]).

The main purpose of this paper is to show that using the Fueter mapping theorem and the spectral theory on the S -spectrum we can define a functional calculus for harmonic functions in four variables. This new calculus can be seen as the harmonic version of the Riesz-Dunford functional calculus.

Before to explain our results we need some further explanations of the setting in which we will work.

1.1 The Fueter–Sce–Qian Extension Theorem and Spectral Theories

The Fueter–Sce–Qian mapping theorem is a crucial result that constructs hyperholomorphic (in a suitable sense) quaternionic or Clifford algebra valued functions starting from holomorphic functions of one complex variable and it consists of a two steps procedure: the first step gives slice hyperholomorphic functions and the second one gives the Fueter regular functions in the case of the quaternions, or monogenic functions in the Clifford algebra setting. Prior to the introduction of slice hyperholomorphic functions, the first step was simply seen as an intermediate step in the construction. We point out that the Fueter–Sce–Qian mapping theorem has deep consequences in the spectral theories, in fact it determines their structures in the hypercomplex setting, as we shall see in the sequel.

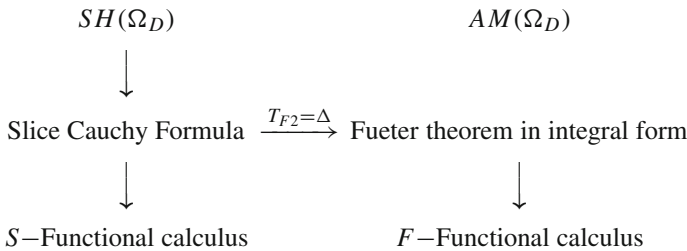
To further clarify the two steps procedure we summarize the construction in the quaternionic case (which was originally introduced by Fueter, see [28]). Denoting by $\mathcal{O}(D)$ the set of holomorphic functions on $D \subseteq \mathbb{C}^+$, by $SH(\Omega_D)$ the set of induced functions on Ω_D (which turn out to be the set of slice hyperholomorphic functions) and by $AM(\Omega_D)$ the set of axially monogenic functions on Ω_D , the Fueter construction can be visualized as:

$$\mathcal{O}(D) \xrightarrow{T_{F1}} SH(\Omega_D) \xrightarrow{T_{F2}=\Delta} AM(\Omega_D),$$

where T_{F1} denotes the first linear operator of the Fueter construction and $T_{F2} = \Delta$ is the Laplace operator in four dimensions. The Fueter mapping theorem induces two spectral theories: in the first step we have the spectral theory on the S -spectrum

associated with the Cauchy formula of slice hyperholomorphic functions; in the second step we obtain the spectral theory on the monogenic spectrum associated with the Cauchy formula of monogenic functions.

We also note that the Fueter mapping theorem allows to use slice hyperholomorphic functions to obtain the so-called F -functional calculus, see [11, 15, 16, 18], which is a monogenic functional calculus on the S -spectrum. It is based on the idea of applying the operator T_{F2} to the slice hyperholomorphic Cauchy kernel, as illustrated by the diagram:



where Δ is the Laplace operator in four dimensions.

Remark 1.1 Observe that in the above diagram the arrow from the space of axially monogenic functions $AM(\Omega_D)$ is missing because the F -functional calculus is deduced from the slice hyperholomorphic Cauchy formula. Moreover, we use the set of slice hyperholomorphic functions $SH(\Omega_D)$, that contains the set of intrinsic functions.

To proceed further we fix the notation for the quaternions, that are defined as follows:

$$\mathbb{H} = \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the imaginary units satisfy the relations

$$\begin{aligned}
 e_1^2 = e_2^2 = e_3^2 = -1 \quad \text{and} \quad e_1e_2 = -e_2e_1 = e_3, \\
 e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.
 \end{aligned}$$

Given $q \in \mathbb{H}$ we call $\text{Re}(q) := q_0$ the real part of q and $\underline{q} = q_1e_1 + q_2e_2 + q_3e_3$ the imaginary part. The modulus of $q \in \mathbb{H}$ is given by $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, the conjugate of q is defined by $\bar{q} = q_0 - \underline{q}$ and we have $|q| = \sqrt{q\bar{q}}$. The symbol \mathbb{S} denotes the unit sphere of purely imaginary quaternions

$$\mathbb{S} = \{\underline{q} = q_1e_1 + q_2e_2 + q_3e_3 \mid q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Notice that if $J \in \mathbb{S}$, then $J^2 = -1$. Therefore, J is an imaginary unit, and we denote by

$$\mathbb{C}_J = \{u + Jv \mid u, v \in \mathbb{R}\},$$

an isomorphic copy of the complex numbers. Given a non-real quaternion $q = q_0 + \underline{q} = q_0 + J_q|\underline{q}|$, we set $J_q = \underline{q}/|\underline{q}| \in \mathbb{S}$ and we associate to q the 2-sphere defined by

$$[q] := \{q_0 + J|\underline{q}| \mid J \in \mathbb{S}\}.$$

We recall that the Fueter operator \mathcal{D} and its conjugate $\overline{\mathcal{D}}$ are defined as follows

$$\mathcal{D} := \partial_{q_0} + \sum_{i=1}^3 e_i \partial_{q_i} \quad \text{and} \quad \overline{\mathcal{D}} := \partial_{q_0} - \sum_{i=1}^3 e_i \partial_{q_i}.$$

The operators \mathcal{D} and $\overline{\mathcal{D}}$ factorize the Laplace operator $\mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D} = \Delta$.

1.2 The Fine Structure of Hyperholomorphic Spectral Theory and Related Problems

In this paper we further refine the above diagram, observing that, in the case of the quaternions, the map T_{F2} can be factorized as $T_{F2} = \Delta = \overline{\mathcal{D}}\mathcal{D}$, so there is an intermediate step between slice hyperholomorphic functions and Fueter regular functions, and the intermediate class of functions that appears is the one of axially harmonic $AH(\Omega_D)$ functions, see Definition 3.3. Thus the diagram becomes as follows:

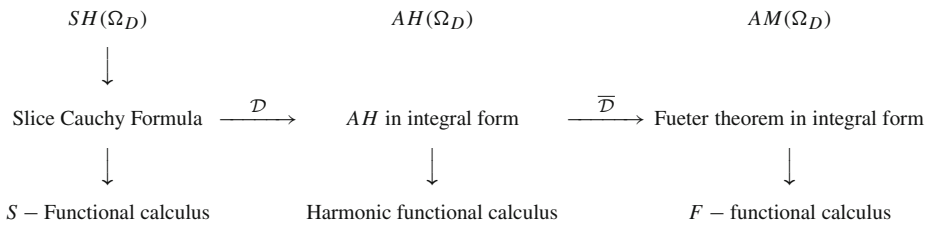
$$\mathcal{O}(D) \xrightarrow{T_{F1}} SH(\Omega_D) \xrightarrow{\mathcal{D}} AH(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} AM(\Omega_D).$$

It is important to define precisely what we mean by intermediate functional calculus between the S -functional calculus and the F -functional calculus, both from the points of view of the function theory and of the operator theory. The notions of *fine structures of the spectral theory on the S -spectrum* arise naturally from the Fueter extension theorem.

Definition 1.2 (*Fine structure of the spectral theory on the S -spectrum*) We will call *fine structure of the spectral theory on the S -spectrum* the set of functions spaces and the associated functional calculi induced by a factorization of the operator T_{F2} , in the Fueter extension theorem.

Remark 1.3 In the Clifford algebra setting the map T_{F2} becomes the Fueter–Sce operator given by $T_{FS2} = \Delta_{\frac{n-1}{n+1}}$ and its splitting is more involved. We are investigating it in general, when n is odd, and in the case $n = 5$ we have a complete description of all the possible fine structures, see [24]. When n is even the Laplace operator has a fractional power and so one has to work in the space of distributions using the Fourier multipliers, see [33].

The fine structure of the quaternionic spectral theory on the S -spectrum is illustrated in the following diagram



where the description of the central part of the diagram, i.e., the fine structure, is the main topic of this paper.

Remark 1.4 As for the space of axially monogenic functions, the arrow from the space of axially harmonic functions is missing. In fact, like the F -functional calculus, also the harmonic functional calculus is deduced from the slice hyperholomorphic Cauchy formula.

To sum up, the main problems addressed in this paper are:

Problem 1.5 In the Fueter extension theorem consider the factorization

$$SH(\Omega_D) \xrightarrow{\mathcal{D}} X(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} AM(\Omega_D),$$

and give an integral representation of the functions in the space $X(\Omega_D) := \mathcal{D}(SH(\Omega_D))$ and, using this integral transform, define its functional calculus.

Problem 1.6 Determine a product rule formula for the F - functional calculus.

As we will see in the sequel the above problems are related. In fact, the product rule of the F -functional calculus is based on the functional calculus in Problem 1.5.

1.3 Structure of the Paper and Main Results

The paper consists of 9 sections, the first one being this introduction. In Sects. 2 and 3 we give the preliminary material on spectral theories in the hyperholomorphic setting and the underlying function theories. In Sects. 3 and 4 we consider axially harmonic functions, see Definition 3.3. Using the Cauchy formulas of left (resp. right) slice hyperholomorphic functions we write an integral representation for axially harmonic functions, see Theorem 4.16. More precisely, let $W \subset \mathbb{H}$ be an open set and let U be a slice Cauchy domain such that $\overline{U} \subset W$. Then, for $J \in \mathbb{S}$ and $ds_J = ds(-J)$ we have that if f is left slice hyperholomorphic on W , then the function $\tilde{f}(q) = \mathcal{D}f(q)$ is harmonic and it admits the following integral representation

$$\tilde{f}(q) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1} ds_J f(s), \quad q \in U, \tag{1.1}$$

where

$$\mathcal{Q}_{c,s}(q)^{-1} := (s^2 - 2\text{Re}(q)s + |q|^2)^{-1}.$$

We note that the presence of $-\frac{1}{\pi}$ in front of (1.1) is justified by the computations performed in (4.5). A similar representation is obtained also for right slice hyperholomorphic functions. The integral depends neither on U nor on the imaginary unit $J \in \mathbb{S}$.

The integral representation (1.1) is the crucial point to define the harmonic functional calculus, also called *Q-functional calculus* because its resolvent operator is the commutative pseudo S -resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$.

Let $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$ be a quaternionic bounded linear operator with commuting components $T_\ell, \ell = 0, \dots, 3$. We define the S -spectrum of T as

$$\sigma_S(T) = \{ s \in \mathbb{H} \mid s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T} \text{ is not invertible as bounded linear operator} \},$$

where $\bar{T} = T_0 - T_1e_1 - T_2e_2 - T_3e_3$. The commutative pseudo S -resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$ is defined as:

$$\mathcal{Q}_{c,s}(T)^{-1} = (s^2 - s(T + \bar{T}) + T\bar{T})^{-1}$$

for $s \notin \sigma_S(T)$. The harmonic functional calculus is defined in Definition 5.7, but roughly speaking for every function $\tilde{f} = \mathcal{D}f$, with f left slice hyperholomorphic, we define the harmonic functional calculus as

$$\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s),$$

where U is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U, \bar{U} \subset \text{dom}(f)$, $ds_J = ds(-J)$ and $J \in \mathbb{S}$ is an arbitrary imaginary unit. A similar definition holds for $\tilde{f} = f\mathcal{D}$ with f right slice hyperholomorphic.

In Sect. 6 we introduce possible resolvent equations for the harmonic functional calculus and in Sects. 7 and 8 we study some of its properties. In particular, we have the Riesz projectors, see Theorem 7.2. Specifically, let $T = T_1e_1 + T_2e_2 + T_3e_3$ and assume that the operators $T_\ell, \ell = 1, 2, 3$, have real spectrum. Let $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with $\text{dist}(\sigma_1, \sigma_2) > 0$ and assume that $G_1, G_2 \subset \mathbb{H}$ are two bounded slice Cauchy domains such that $\sigma_1 \subset G_1, \bar{G}_1 \subset G_2$ and $\text{dist}(G_2, \sigma_2) > 0$. Then the operator

$$\tilde{P}(T) := \frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1}$$

is a projection, i.e., $\tilde{P}^2 = \tilde{P}$. Moreover, the operator \tilde{P} commutes with T .

Section 9 concludes the paper and contains some properties of the F -functional calculus, such as the product rule, that can be proved using the Q -functional calculus.

2 Preliminary Results on Functions and Operators

We recall some basic results and notations that we will need in the following.

2.1 Hyperholomorphic Functions and the Fueter Mapping Theorem

Definition 2.1 Let $U \subseteq \mathbb{H}$.

- We say that U is axially symmetric if, for every $u + Iv \in U$, all the elements $u + Jv$ for $J \in \mathbb{S}$ are contained in U .
- We say that U is a slice domain if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_J$ is a domain in \mathbb{C}_J for every $J \in \mathbb{S}$.

Definition 2.2 An axially symmetric open set $U \subset \mathbb{H}$ is called slice Cauchy domain if $U \cap \mathbb{C}_J$ is a Cauchy domain in \mathbb{C}_J for every $J \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if, for every $J \in \mathbb{S}$, the boundary of $U \cap \mathbb{C}_J$ is the union of a finite number of nonintersecting piecewise continuously differentiable Jordan curves in \mathbb{C}_J .

On axially symmetric open sets we define the class of slice hyperholomorphic functions.

Definition 2.3 (*Slice hyperholomorphic functions*) Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 \mid u + \mathbb{S}v \in U\}.$$

We say that a function $f : U \rightarrow \mathbb{H}$ of the form

$$f(q) = \alpha(u, v) + J\beta(u, v)$$

is left slice hyperholomorphic if α and β are \mathbb{H} -valued differentiable functions such that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \quad \text{for all } (u, v) \in \mathcal{U}, \quad (2.1)$$

and if α and β satisfy the Cauchy–Riemann system

$$\partial_u \alpha(u, v) - \partial_v \beta(u, v) = 0, \quad \partial_v \alpha(u, v) + \partial_u \beta(u, v) = 0.$$

Right slice hyperholomorphic functions are of the form

$$f(q) = \alpha(u, v) + \beta(u, v)J,$$

where α, β satisfy the above conditions.

Notation The set of left (resp. right) slice hyperholomorphic functions on U is denoted by the symbol $SH_L(U)$ (resp. $SH_R(U)$). The subset of intrinsic slice hyperholomorphic functions consists of those slice hyperholomorphic functions such that α, β are real-valued function and is denoted by $N(U)$.

Remark 2.4 If the axially symmetric set U does not intersect the real line then we can set

$$\mathcal{U} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^+ \mid u + \mathbb{S}v \in U\}.$$

and

$$f(q) = \alpha(u, v) + J\beta(u, v), \quad (u, v) \in \mathcal{U}.$$

The function f is left slice hyperholomorphic if α and β satisfy the Cauchy–Riemann system. Similarly, under the same conditions on α and β , $f(q) = \alpha(u, v) + \beta(u, v)J$ is said right slice hyperholomorphic.

Functions in the kernel of the Fueter operator are called Fueter regular functions and are defined as follows.

Definition 2.5 (*Fueter regular functions*) Let $U \subset \mathbb{H}$ be an open set. A real differentiable function $f : U \rightarrow \mathbb{H}$ is called left Fueter regular if

$$\mathcal{D}f(q) := \partial_{q_0}f(q) + \sum_{i=1}^3 e_i \partial_{q_i}f(q) = 0.$$

It is called right Fueter regular if

$$f(q)\mathcal{D} := \partial_{q_0}f(q) + \sum_{i=1}^3 \partial_{q_i}f(q)e_i = 0.$$

There are several possible definitions of slice hyperholomorphicity, that are not fully equivalent, but Definition 2.3 of slice hyperholomorphic functions is the most appropriate one for the operator theory and it comes from the Fueter mapping theorem which, inspired by [28], can be stated as follows:

Theorem 2.6 (*Fueter mapping theorem*) Let $f_0(z) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let

$$\Omega_D = \{q = q_0 + \underline{q} \mid (q_0, |\underline{q}|) \in D\}$$

be the open set induced by D in \mathbb{H} . Then the operator T_{F1} defined by

$$f(q) = T_{F1}(f_0) := \alpha(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(q_0, |\underline{q}|)$$

maps the set of holomorphic functions in the set of intrinsic slice hyperholomorphic functions. Moreover, the function

$$\check{f}(q) := T_{F2} \left(\alpha(q_0, |q|) + \frac{q}{|q|} \beta(q_0, |q|) \right),$$

where $T_{F2} = \Delta$ and Δ is the Laplacian in four real variables q_ℓ , $\ell = 0, 1, 2, 3$, is in the kernel of the Fueter operator i.e.

$$\mathcal{D}\check{f} = 0 \text{ on } \Omega_D.$$

Remark 2.7 In the late of 1950s, Sce extended the Fueter mapping theorem to the Clifford setting in the case of odd dimensions, see [35]. In this case, the operator T_{FS2} becomes $T_{FS2} := \Delta_{n+1}^{\frac{n-1}{2}}$, where Δ_{n+1} is the Laplacian in $n + 1$ dimensions, so in this case we are dealing with a differential operator. For a translation of Sce works in hypercomplex analysis with commentaries see [22]; this includes also the version of the Fueter mapping theorem for octonions. In 1997, Qian proved that the Fueter–Sce theorem holds also in the case of even dimensions. In this case the operator $\Delta_{n+1}^{\frac{n-1}{2}}$ is a fractional operator, see [33, 34].

Using the theory of monogenic function McIntosh and his collaborators introduced the spectral theory on the monogenic spectrum to define functions of noncommuting operators on Banach spaces. They developed the monogenic functional calculus and several of its applications, see the books [29] and the papers [30, 31].

We now recall the slice hyperholomorphic Cauchy formulas that are the starting point to construct the hyperholomorphic spectral theories on the S -spectrum. We will be in need the following result [20, Thm. 2.1.22], [20, Prop. 2.1.24].

Theorem 2.8 *Let $s, q \in \mathbb{H}$ with $|q| < |s|$, then*

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} = -(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}(q - \bar{s})$$

and

$$\sum_{n=0}^{+\infty} s^{-n-1} q^n = -(q - \bar{s})(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}.$$

Moreover, for any $s, q \in \mathbb{H}$ with $q \notin [s]$, we have

$$-(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}(q - \bar{s}) = (s - \bar{q})(s^2 - 2\text{Re}(q)s + |q|^2)^{-1}$$

and

$$-(q - \bar{s})(q^2 - 2\text{Re}(s)q + |s|^2)^{-1} = (s^2 - 2\text{Re}(q)s + |q|^2)^{-1}(s - \bar{q}).$$

In view of Theorem 2.8 there are two possible representations of the Cauchy kernels for both the left and the right slice hyperholomorphic functions.

Definition 2.9 Let $s, q \in \mathbb{H}$ with $q \notin [s]$ then we define

$$\mathcal{Q}_s(q)^{-1} := (q^2 - 2\text{Re}(s)q + |s|^2)^{-1}, \quad \mathcal{Q}_{c,s}(q)^{-1} := (s^2 - 2\text{Re}(q)s + |q|^2)^{-1},$$

that are called pseudo Cauchy kernel and commutative pseudo Cauchy kernel, respectively.

Definition 2.10 Let $s, q \in \mathbb{H}$ with $q \notin [s]$ then

- We say that the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s, q)$ is written in the form I if

$$S_L^{-1}(s, q) := \mathcal{Q}_s(q)^{-1}(\bar{s} - q).$$

- We say that the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s, q)$ is written in the form I if

$$S_R^{-1}(s, q) := (\bar{s} - q)\mathcal{Q}_s(q)^{-1}.$$

- We say that the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s, q)$ is written in the form II if

$$S_L^{-1}(s, q) := (s - \bar{q})\mathcal{Q}_{c,s}(q)^{-1}.$$

- We say that the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s, q)$ is written in the form II if

$$S_R^{-1}(s, q) := \mathcal{Q}_{c,s}(q)^{-1}(s - \bar{q}).$$

In this article, unless otherwise specified, we refer to $S_L^{-1}(s, q)$ and $S_R^{-1}(s, q)$ written in the form II.

The following results will be very important in the sequel.

Lemma 2.11 *Let $s \notin [q]$. The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s, q)$ is left slice hyperholomorphic in q and right slice hyperholomorphic in s . The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s, q)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in q .*

Theorem 2.12 (The Cauchy formulas for slice hyperholomorphic functions) *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $J \in \mathbb{S}$ and set $ds_J = ds(-J)$. If f is a left slice hyperholomorphic function on a set that contains \bar{U} then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s, q) ds_J f(s), \quad \text{for any } q \in U. \tag{2.2}$$

If f is a right slice hyperholomorphic function on a set that contains \bar{U} , then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, q), \quad \text{for any } q \in U. \tag{2.3}$$

These integrals depend neither on U nor on the imaginary unit $J \in \mathbb{S}$.

Moreover, for slice hyperholomorphic functions hold a version of the Cauchy integral theorem holds.

Theorem 2.13 (Cauchy integral Theorem) *Let $U \subset \mathbb{H}$ be open, let $J \in \mathbb{S}$, and let $f \in SH_L(U)$ and $g \in SH_R(U)$. Moreover, let $D_J \subset U \cap \mathbb{C}_J$ be an open and bounded subset of the complex plane \mathbb{C}_J with $\bar{D}_J \subset U \cap \mathbb{C}_J$ such that ∂D_J is a finite union of piecewise continuously differentiable Jordan curves. Then*

$$\int_{\partial D_J} g(s) ds_J f(s) = 0,$$

where $ds_J = ds(-J)$.

Now, we recall what happens when we apply the second Fueter operator $T_{F2} := \Delta$, where $\Delta = \sum_{i=0}^3 \partial_{q_i}^2$, to the slice hyperholomorphic Cauchy kernel.

Proposition 2.14 *Let $q, s \in \mathbb{H}$ and $q \notin [s]$. Then:*

- *The function $\Delta S_L^{-1}(s, q)$ is a left Fueter regular function in the variable q and right slice hyperholomorphic in s .*
- *The function $\Delta S_R^{-1}(s, q)$ is a right Fueter regular function in the variable q and left slice hyperholomorphic in s .*

In [20, Thm. 2.2.2] there are the explicit computations of the functions

$$(s, q) \mapsto \Delta S_L^{-1}(s, q), \quad (s, q) \mapsto \Delta S_R^{-1}(s, q).$$

Theorem 2.15 *Let $q, s \in \mathbb{H}$ with $q \notin [s]$. Then we have*

$$\Delta S_L^{-1}(s, q) = -4(s - \bar{q})(s^2 - 2\text{Re}(q)s + |q|^2)^{-2}$$

and

$$\Delta S_R^{-1}(s, q) = -4(s^2 - 2\text{Re}(q)s + |q|^2)^{-2}(s - \bar{q}).$$

We recall the definition of the F -kernels.

Definition 2.16 Let $q, s \in \mathbb{H}$. We define for $s \notin [q]$, the left F -kernel as

$$F_L(s, q) := \Delta S_L^{-1}(s, q) = -4(s - \bar{q})\mathcal{Q}_{c,s}(q)^{-2},$$

and the right F -kernel as

$$F_R(s, q) := \Delta S_R^{-1}(s, q) = -4\mathcal{Q}_{c,s}(q)^{-2}(s - \bar{q}).$$

We recall the following relation between the F -kernel and the commutative pseudo Cauchy kernel $\mathcal{Q}_{c,s}(q)^{-1}$.

Theorem 2.17 *Let $s, q \in \mathbb{H}$ be such that $q \notin [s]$, then*

$$F_L(s, q)s - qF_L(s, q) = -4\mathcal{Q}_{c,s}(q)^{-1}$$

and

$$sF_R(s, q) - F_R(s, q)q = -4\mathcal{Q}_{c,s}(q)^{-1}$$

The following result plays a key role, see [20, Thm. 2.2.6].

Theorem 2.18 (The Fueter mapping theorem in integral form) *Let $U \subset \mathbb{H}$ be a slice Cauchy domain, let $J \in \mathbb{S}$ and set $ds_J = ds(-J)$.*

- *If f is a left slice hyperholomorphic function on a set W , such that $\bar{U} \subset W$, then the left Fueter regular function $\check{f}(q) = \Delta f(q)$ admits the integral representation*

$$\check{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s, q) ds_J f(s). \tag{2.4}$$

- *If f is a right slice hyperholomorphic function on a set W , such that $\bar{U} \subset W$, then the right Fueter regular function $\check{f}(q) = \Delta f(q)$ admits the integral representation*

$$\check{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J F_R(s, q). \tag{2.5}$$

The integrals depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

2.2 The S-functional Calculus

We now recall some basic facts of the S -function calculus, see [19, 20] for more details. Let X be a two sided quaternionic Banach module of the form $X = X_{\mathbb{R}} \otimes \mathbb{H}$, where $X_{\mathbb{R}}$ is a real Banach space. In this paper we consider $\mathcal{B}(X)$ the Banach space of all bounded right linear operators acting on X .

In the sequel we will consider bounded operators of the form $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$, with commuting components T_i acting on the real vector space $X_{\mathbb{R}}$, i.e., $T_i \in \mathcal{B}(X_{\mathbb{R}})$ for $i = 0, 1, 2, 3$. The subset of $\mathcal{B}(X)$ given by the operators T with commuting components T_i will be denoted by $\mathcal{BC}(X)$.

Now let $T : X \rightarrow X$ be a right (or left) linear operator. We give the following.

Definition 2.19 Let $T \in \mathcal{B}(X)$. For $s \in \mathbb{H}$ we set

$$\mathcal{Q}_s(T) := T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I}.$$

We define the S -resolvent set $\rho_S(T)$ of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X)\},$$

and we define the S -spectrum $\sigma_S(T)$ of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo S -resolvent operator of T at s .

Theorem 2.20 *Let $T \in \mathcal{B}(X)$ and $s \in \mathbb{H}$ with $\|T\| < |s|$. Then we have*

$$\sum_{n=0}^{\infty} T^n s^{-n-1} = -\mathcal{Q}_s(T)^{-1}(T - \bar{s}\mathcal{I}),$$

and

$$\sum_{n=0}^{\infty} s^{-n-1} T^n = -(T - \bar{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

According to the left or right slice hyperholomorphicity, there exist two different resolvent operators.

Definition 2.21 (*S -resolvent operators*) Let $T \in \mathcal{B}(X)$ and $s \in \rho_S(T)$. Then the left S -resolvent operator is defined as

$$S_L^{-1}(s, T) := -\mathcal{Q}_s(T)^{-1}(T - \bar{s}\mathcal{I}),$$

and the right S -resolvent operator is defined as

$$S_R^{-1}(s, T) := -(T - \bar{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

The so-called S -resolvent equation, see [1, Thm. 3.8], involves both S -resolvent operators and the Cauchy kernel of slice hyperholomorphic functions and is recalled in the next result:

Theorem 2.22 (*S -resolvent equation*) Let $T \in \mathcal{B}(X)$ then for $s, p \in \rho_S(T)$, with $q \notin [s]$, we have

$$\begin{aligned} S_R^{-1}(s, T)S_L^{-1}(p, T) &= \left[\left(S_R^{-1}(s, T) - S_L^{-1}(p, T) \right) p \right. \\ &\quad \left. - \bar{s} \left(S_R^{-1}(s, T) - S_L^{-1}(p, T) \right) \right] \mathcal{Q}_s(p)^{-1}, \end{aligned} \tag{2.6}$$

where $\mathcal{Q}_s(p) = p^2 - 2\text{Re}(s)p + |s|^2$.

To give the definition of the S -functional calculus we need the following classes of functions.

Notation Let $T \in \mathcal{B}(X)$. We denote by $SH_L(\sigma_S(T))$, $SH_R(\sigma_S(T))$ and $N(\sigma_S(T))$ the sets of all left, right and intrinsic slice hyperholomorphic functions f , respectively, with $\sigma_S(T) \subset \text{dom}(f)$.

Definition 2.23 (*S-functional calculus*) Let $T \in \mathcal{B}(X)$. Let U be a slice Cauchy domain that contains $\sigma_S(T)$ and \bar{U} is contained in the domain of f . Set $ds_J = -dsJ$ for $J \in \mathbb{S}$ so we define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s, T) ds_J f(s), \text{ for every } f \in SH_L(\sigma_S(T)) \quad (2.7)$$

and

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, T), \text{ for every } f \in SH_R(\sigma_S(T)). \quad (2.8)$$

The definition of S -functional calculus is well posed since the integrals in (2.7) and (2.8) depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$, see [17], [20, Thm. 3.2.6].

2.3 The F -functional Calculus

Let us consider $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$ such that $T \in \mathcal{BC}(X)$.

Definition 2.24 Let $T \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ we set

$$\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T},$$

where $\bar{T} = T_0 - T_1e_1 - T_2e_2 - T_3e_3$. We define the F -resolvent set as

$$\rho_F(T) = \{s \in \mathbb{H} : \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X)\}.$$

Moreover, we define the F -spectrum of T as

$$\sigma_F(T) = \mathbb{H} \setminus \rho_F(T).$$

By [18, Prop. 4.14] we have that the F -spectrum is the commutative version of the S -spectrum, i.e., we have

$$\sigma_F(T) = \sigma_S(T), \quad T \in \mathcal{BC}(X),$$

and consequently $\rho_F(T) = \rho_S(T)$.

For $s \in \rho_S(T)$ the operator $\mathcal{Q}_{c,s}(T)^{-1}$ is called the commutative pseudo S -resolvent operator of T .

It is possible to define a commutative version of the S -functional calculus.

Theorem 2.25 *Let $T \in \mathcal{BC}(X)$ and $s \in \mathbb{H}$ be such that $\|T\| < s$. Then,*

$$\sum_{m=0}^{\infty} T^m s^{-1-m} = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1},$$

and

$$\sum_{m=0}^{\infty} s^{-1-m} T^m = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

Definition 2.26 Let $T \in \mathcal{BC}(X)$ and $s \in \rho_S(T)$. We define the left commutative S -resolvent operator as

$$S_L^{-1}(s, T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1},$$

and the right commutative S -resolvent operator as

$$S_R^{-1}(s, T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

For the sake of simplicity we denote the commutative version of the S -resolvent operators with the same symbols used for the noncommutative ones. It is possible to define an S -functional calculus as done in Definition 2.23. Below, when dealing with the S -resolvent operators, we intend their commutative version.

We conclude with the definition of the F -functional calculus.

Definition 2.27 (*F-resolvent operators*) Let $T \in \mathcal{BC}(X)$. We define the left F -resolvent operator as

$$F_L(s, T) = -4(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-2}, \quad s \in \rho_S(T),$$

and the right F -resolvent operator as

$$F_R(s, T) = -4\mathcal{Q}_{c,s}(T)^{-2}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T).$$

With the above definitions and Theorem 2.18 at hand, we can recall the F -functional calculus was first introduced in [18] and then investigated in [11, 15, 16].

Definition 2.28 (*The F-functional calculus for bounded operators*) Let U be a slice Cauchy domain that contains $\sigma_S(T)$ and \overline{U} is contained in the domain of f . Let $T = T_1e_1 + T_2e_2 + T_3e_3 \in \mathcal{BC}(X)$, assume that the operators T_ℓ , $\ell = 1, 2, 3$ have real spectrum and set $ds_J = ds/J$, where $J \in \mathbb{S}$. For any function $f \in SH_L(\sigma_S(T))$, we define

$$\check{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s, T) ds_J f(s). \tag{2.9}$$

For any $f \in SH_R(\sigma_S(T))$, we define

$$\check{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J F_R(s, T). \tag{2.10}$$

The definition of the F -functional calculus is well posed since the integrals in (2.9) and (2.10) depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

The left and right F -resolvent operators satisfy the equalities in the next result [20, Thm. 7.3.1]:

Theorem 2.29 (Left and right F -resolvent equations) *Let $T \in \mathcal{BC}(X)$ and let $s \in \rho_S(T)$. The F -resolvent operators satisfy the equations*

$$F_L(s, T)s - TF_L(s, T) = -4Q_{c,s}(T)^{-1}$$

and

$$sF_R(s, T) - F_R(s, T)T = -4Q_{c,s}(T)^{-1}.$$

3 Axially Harmonic Functions

In this section, we solve the first part of Problem 1.5. We begin by rewriting the Fueter mapping theorem (see Theorem 2.6) in a more refined way, considering the factorization of the Laplace operator Δ in terms of the Fueter operator \mathcal{D} and its conjugate $\overline{\mathcal{D}}$.

Theorem 3.1 (Fueter mapping theorem (refined)) *Let $f_0(z) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D = \{q = q_0 + \underline{q} \mid (q_0, |\underline{q}|) \in D\} \tag{3.1}$$

be the open set induced by D in \mathbb{H} . The operator T_{F1} defined by

$$f(q) = T_{F1}(f_0) := \alpha(q_0, |\underline{q}|) + \frac{q}{|\underline{q}|} \beta(q_0, |\underline{q}|)$$

maps the set of holomorphic functions in the set of intrinsic slice hyperholomorphic functions. Then, the function

$$\check{f}(q) := T'_{F2} \left(\alpha(q_0, |\underline{q}|) + \frac{q}{|\underline{q}|} \beta(q_0, |\underline{q}|) \right),$$

where $T'_{F2} := \mathcal{D}$ is the Fueter operator, is in the kernel of the Laplace operator, i.e.,

$$\Delta \check{f} = 0 \text{ on } \Omega_D.$$

Moreover,

$$\check{f}(q) := T''_{F_2} \tilde{f},$$

where $T''_{F_2} = \overline{\mathcal{D}}$ and $\overline{\mathcal{D}} = \partial_{q_0} - \sum_{i=1}^3 e_i \partial_{q_i}$, is in the kernel of the Fueter operator, i.e.,

$$\mathcal{D}\check{f} = 0 \text{ on } \Omega_D.$$

Remark 3.2 The consideration in Remark 2.4 holds obviously also in the case of Theorem 3.1.

In Theorem 3.1 we have applied to the slice hyperholomorphic function f firstly the Fueter operator and then the operator $\overline{\mathcal{D}}$, while in Theorem 2.6 we apply directly the Laplacian. Therefore, there is a class of functions that lies between the class of slice hyperholomorphic functions and the class of axially monogenic functions: it is the so-called class of axially harmonic functions that we introduce below.

Definition 3.3 (Axially harmonic function) Let $U \subseteq \mathbb{H}$ be an axially symmetric open set not intersecting the real line, and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^+ \mid u + \mathbb{S}v \in U\}.$$

Let $f : U \rightarrow \mathbb{H}$ be a function, of class \mathcal{C}^3 , of the form

$$f(q) = \alpha(u, v) + J\beta(u, v), \quad q = u + Jv, \quad J \in \mathbb{S},$$

where α and β are \mathbb{H} -valued functions. More in general let f be as above and let $U \subseteq \mathbb{H}$ be an axially symmetric open set and consider

$$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 \mid u + \mathbb{S}v \in U\},$$

and assume that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \quad \text{for all } (u, v) \in \mathcal{U}. \quad (3.2)$$

Let us set

$$\tilde{f}(q) := \mathcal{D}f(q), \text{ for } q \in U.$$

If

$$\Delta \tilde{f}(q) = 0, \text{ for } q \in U$$

we say that \tilde{f} is axially harmonic on U .

The axially monogenic functions satisfy a system of differential equations called Vekua system, see [25, 36, 37]. In the case of axially harmonic functions, the functions $A(q_0, r)$ and $B(q_0, r)$ satisfy a second order system of differential equations.

Theorem 3.4 *Let U be an axially symmetric open set in \mathbb{H} , not intersecting the real line, and let $\tilde{f}(q) = A(q_0, r) + \underline{\omega}B(q_0, r)$ be an axially harmonic function on U , $r > 0$ and $\underline{\omega} \in \mathbb{S}$. Then the functions $A = A(q_0, r)$ and $B = B(q_0, r)$ satisfy the following system*

$$\begin{cases} \partial_{q_0}^2 A(q_0, r) + \partial_r^2 A(q_0, r) + \frac{2}{r} \partial_r A(q_0, r) = 0 \\ \partial_{q_0}^2 B(q_0, r) + \partial_r^2 B(q_0, r) + \frac{2r \partial_r B(q_0, r) - 2B(q_0, r)}{r^2} = 0. \end{cases}$$

Proof An axially harmonic function is written as

$$\tilde{f}(q) = A(q_0, r) + \underline{\omega}B(q_0, r), \quad q = q_0 + r\underline{\omega} \in U$$

and it is in the kernel of the operator $\Delta = \mathcal{D}\overline{\mathcal{D}}$. We denote the Fueter operator as $\mathcal{D} = \partial_{q_0} + \partial_{\underline{q}}$ and $\overline{\mathcal{D}} = \partial_{q_0} - \partial_{\underline{q}}$, where $\partial_{\underline{q}} = e_1 \partial_{q_1} + e_2 \partial_{q_2} + e_3 \partial_{q_3}$. We know that (see [32])

$$\partial_{\underline{q}}(A(q_0, r) + \underline{\omega}B(q_0, r)) = \underline{\omega} \partial_r A(q_0, r) - \partial_r B(q_0, r) - \frac{2}{r} B(q_0, r). \quad (3.3)$$

This implies that

$$\begin{aligned} \overline{\mathcal{D}}f(q) &= (\partial_{q_0} - \partial_{\underline{q}})(A(q_0, r) + \underline{\omega}B(q_0, r)) \\ &= \left(\partial_{q_0} A(q_0, r) + \partial_r B(q_0, r) + \frac{2}{r} B(q_0, r) \right) + \underline{\omega} (\partial_{q_0} B(q_0, r) - \partial_r A(q_0, r)). \end{aligned}$$

By setting

$$\begin{aligned} A'(q_0, r) &:= \partial_{q_0} A(q_0, r) + \partial_r B(q_0, r) + \frac{2}{r} B(q_0, r) \quad \text{and} \\ B'(q_0, r) &:= \partial_{q_0} B(q_0, r) - \partial_r A(q_0, r), \end{aligned}$$

we get

$$\overline{\mathcal{D}}f(q_0, r) = A'(q_0, r) + \underline{\omega}B'(q_0, r).$$

Now, by applying another time formula (3.3) we obtain

$$\partial_{\underline{q}} \overline{\mathcal{D}}f(q) = \underline{\omega} \partial_r A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r} B'(q_0, r).$$

Therefore, we have

$$\begin{aligned} \Delta f(q) &= \mathcal{D}\overline{\mathcal{D}}f(q) \\ &= (\partial_{q_0} + \partial_{\underline{q}})\overline{\mathcal{D}}f(q_0, r) \\ &= \left(\partial_{q_0}A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r}B'(q_0, r) \right) \\ &\quad - \underline{\omega}(\partial_{q_0}B'(q_0, r) + \partial_r A'(q_0, r)). \end{aligned}$$

Since the function f is axially harmonic we have $\Delta f(q) = 0$, thus we get

$$\begin{cases} \partial_{q_0}A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r}B'(q_0, r) = 0 \\ \partial_{q_0}B'(q_0, r) + \partial_r A'(q_0, r) = 0. \end{cases} \tag{3.4}$$

Now, we write the system (3.4) in terms of A and B by substituting A' and B'

$$\begin{aligned} &\partial_{q_0}A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r}B'(q_0, r) \\ &= \partial_{q_0}^2 A(q_0, r) + \partial_{q_0}\partial_r B(q_0, r) + \frac{2}{r}\partial_{q_0}B(q_0, r) \\ &\quad - \partial_r\partial_{q_0}B(q_0, r) + \partial_r^2 A(q_0, r) - \frac{2}{r}\partial_{q_0}B(q_0, r) \\ &\quad + \frac{2}{r}\partial_r A(q_0, r) \\ &= \partial_{q_0}^2 A(q_0, r) + \partial_r^2 A(q_0, r) + \frac{2}{r}\partial_r A(q_0, r). \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\partial_{q_0}B'(q_0, r) + \partial_r A'(q_0, r) \\ &= \partial_{q_0}^2 B(q_0, r) - \partial_{q_0}\partial_r A(q_0, r) + \partial_r\partial_{q_0}A(q_0, r) \\ &\quad + \partial_r^2 B(q_0, r) + \frac{2}{r}\partial_r B(q_0, r) - \frac{2}{r^2}B(q_0, r) \\ &= \partial_{q_0}^2 B(q_0, r) + \partial_r^2 B(q_0, r) \\ &\quad + \frac{2r\partial_r B(q_0, r) - 2B(q_0, r)}{r^2}. \end{aligned} \tag{3.6}$$

By putting (3.5) and (3.6) in (3.4) we get the statement. □

Remark 3.5 If we suppose that a function f is harmonic over the ball $B_r(p)$ of radius r and center p , and continuous in the closure of the ball we can write

$$f(q) = \frac{r^2 - |q - p|^2}{|\partial B_r(p)|r} \int_{\partial B_r(p)} \frac{f(y)}{|y - q|^4} dy,$$

where $|\partial B_r(p)|$ is the measure of the sphere and dy is the surface element.

4 Integral Representation Axially Harmonic Functions

In this section we show how to write an axially harmonic function in integral form. The main advantage of this approach is that it is enough to compute an integral of slice hyperholomorphic functions in order to get an axially harmonic function. The crucial point to get the integral representation is to apply the Fueter operator \mathcal{D} to the slice hyperholomorphic Cauchy kernels written in second form, see Definition 2.10.

Theorem 4.1 *Let $s, q \in \mathbb{H}$ be such that $q \notin [s]$ then*

$$\mathcal{D}S_L^{-1}(s, q) = -2\mathcal{Q}_{c,s}(q)^{-1}$$

and

$$S_R^{-1}(s, q)\mathcal{D} = -2\mathcal{Q}_{c,s}(q)^{-1}.$$

Proof We prove only the first equality since the second one follows with similar computations. First, we apply ∂_{q_0} and ∂_{q_i} for $i = 1, 2, 3$ to the left slice hyperholomorphic Cauchy kernel

$$S_L^{-1}(s, q) = (s - \bar{q})\mathcal{Q}_{c,s}(q)^{-1}.$$

Thus, we have

$$\begin{aligned} \partial_{q_0}S_L^{-1}(s, q) &= -\mathcal{Q}_{c,s}(q)^{-1} - (s - \bar{q})\mathcal{Q}_{c,s}(q)^{-2}(-2s + 2q_0) \\ &= -\mathcal{Q}_{c,s}(q)^{-1} - 2q_0(s - \bar{q})\mathcal{Q}_{c,s}(q)^{-2} + 2(s - \bar{q})\mathcal{Q}_{c,s}(q)^{-2}s \\ &= -\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s, q) - \frac{1}{2}F_L(s, q)s, \end{aligned}$$

where $F_L(s, q)$ is the left F -kernel, see Definition 2.16. Then, for $i = 1, 2, 3$ we get

$$\begin{aligned} \partial_{q_i}S_L^{-1}(s, q) &= e_i\mathcal{Q}_{c,s}(q)^{-1} - 2q_i(s - \bar{q})\mathcal{Q}_{c,s}(q)^{-2} \\ &= e_i\mathcal{Q}_{c,s}(q)^{-1} + \frac{1}{2}q_iF_L(s, q). \end{aligned}$$

Thus, by Theorem 2.17, we obtain

$$\begin{aligned} \mathcal{D}S_L^{-1}(s, q) &= \partial_{q_0}S_L^{-1}(s, q) + \sum_{i=1}^3 e_i\partial_{q_i}S_L^{-1}(s, q) \\ &= -\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s, q) - \frac{1}{2}F_L(s, q)s - 3\mathcal{Q}_{c,s}(q)^{-1} + \frac{q}{2}F_L(s, q) \\ &= -4\mathcal{Q}_{c,s}(q)^{-1} - \frac{1}{2}(F_L(s, q)s - qF_L(s, q)) \\ &= -2\mathcal{Q}_{c,s}(q)^{-1}. \end{aligned}$$

□

Remark 4.2 Although the slice hyperholomorphic Cauchy kernel written in form I is more suitable in various cases, like for the definition of S -functional calculus, it does not allow easy computations of $\mathcal{D}S_L^{-1}(s, q)$.

We observe that when we apply the Laplace operator to a monomial q^n we get a polynomial in terms of q and \bar{q} , see [28, page 316 formula (12)], [27, Thm. 3.2]. The same feature happens when the Fueter operator is applied to the monomial q^n , see [7, Lemma 1].

Lemma 4.3 For all $n \geq 1$ we have

$$\mathcal{D}q^n = q^n \mathcal{D} = -2 \sum_{k=1}^n q^{n-k} \bar{q}^{k-1}.$$

Remark 4.4 Since

$$\overline{\sum_{k=1}^n q^{n-k} \bar{q}^{k-1}} = \sum_{k=1}^n q^{n-k} \bar{q}^{k-1}$$

we deduce that $\mathcal{D}q^n$ is real.

Definition 4.5 Let $s, q \in \mathbb{H}$, we define the commutative Q -series as

$$-2 \sum_{m=1}^{+\infty} \sum_{k=1}^m q^{m-k} \bar{q}^{k-1} s^{-1-m} \quad \text{and} \quad -2 \sum_{m=1}^{+\infty} \sum_{k=1}^m s^{-1-m} q^{m-k} \bar{q}^{k-1}.$$

Remark 4.6 The two series in Definition 4.5 coincide, where they converge, since $\sum_{k=1}^m q^{m-k} \bar{q}^{k-1}$ is real.

Proposition 4.7 For $s, q \in \mathbb{H}$ with $|q| < |s|$, the commutative Q -series converges.

Proof To prove the convergence, it is sufficient to prove the convergence of the modulus of the series, i.e., we consider

$$\sum_{m=1}^{+\infty} 2m |q|^{m-1} |s|^{-1-m}.$$

The last series converges by the ratio test. Indeed, since $|q| < |s|$, we have

$$\lim_{m \rightarrow \infty} \frac{(m+1) |q|^m |s|^{-2-m}}{m |q|^{m-1} |s|^{-1-m}} = \lim_{m \rightarrow \infty} \frac{m+1}{m} |q| |s|^{-1} < 1.$$

□

Lemma 4.8 For $q, s \in \mathbb{H}$ such that $|q| < |s|$, we have

$$\mathcal{Q}_{c,s}(q)^{-1} = \sum_{m=1}^{+\infty} \sum_{k=1}^m q^{m-k} \bar{q}^{k-1} s^{-1-m} = \sum_{m=1}^{+\infty} \sum_{k=1}^m s^{-1-m} q^{m-k} \bar{q}^{k-1}.$$

Proof We prove the first equality since the second one can be proved in a similar way. By Theorem 2.8, we can expand the left Cauchy kernel as

$$S_L^{-1}(s, q) = \sum_{m=0}^{\infty} q^m s^{-1-m}.$$

By Theorem 4.1 and Proposition 4.7, which allows to exchange the series with the Fueter operator, we have

$$-2\mathcal{Q}_{c,s}(q)^{-1} = \mathcal{D}S_L^{-1}(s, q) = \sum_{m=0}^{\infty} (\mathcal{D}q^m) s^{-1-m}.$$

We get the statement by applying Lemma 4.3. □

Remark 4.9 Using the well-known equality

$$(a^n - b^n) = (a - b) \sum_{k=1}^n a^{n-k} b^{k-1}$$

for $a = q$ and $b = \bar{q}$, and by Lemma 4.3 we have

$$\mathcal{D}q^n = \begin{cases} -2nq^{n-1} & \text{if } \text{Im}(q) = 0, \\ -(q)^{-1}(q^n - \bar{q}^n) & \text{if } \text{Im}(q) \neq 0. \end{cases}$$

With this result, we can prove Theorem 4.1 by using the series expansion of the kernel in the following way: if $|q| < |s|$ and $\underline{q} \neq 0$ then

$$\begin{aligned} \mathcal{D}S_L^{-1}(s, q) &= \sum_{m=0}^{\infty} (\mathcal{D}q^m) s^{-1-m} \\ &= -(q)^{-1} \left(\sum_{m=1}^{\infty} q^m s^{-1-m} - \sum_{m=1}^{\infty} \bar{q}^m s^{-1-m} \right) \\ &= -(q)^{-1} (S_L^{-1}(s, q) - S_L^{-1}(s, \bar{q})) \\ &= -(q)^{-1} (2q \mathcal{Q}_{c,s}(q)^{-1}) \\ &= -2\mathcal{Q}_{c,s}(q)^{-1}, \end{aligned}$$

if $|q| < |s|$ and $\underline{q} = 0$, we have

$$\begin{aligned}
 \mathcal{Q}_{c,s}(q)\mathcal{D}S_L^{-1}(s, q) &= (s^2 - 2qs + q^2) \left(-2 \sum_{m=1}^{\infty} mq^{m-1}s^{-1-m} \right) \\
 &= -2 \sum_{m=1}^{\infty} mq^{m-1}s^{1-m} + 4 \sum_{m=1}^{\infty} mq^m s^{-m} - 2 \sum_{m=1}^{\infty} mq^{m+1}s^{-m-1} \\
 &= -2 \sum_{m=0}^{\infty} q^m s^{-m} + 2 \sum_{m=1}^{\infty} mq^m s^{-m} - 2 \sum_{m=2}^{\infty} mq^m s^{-m} \\
 &\quad + 2 \sum_{m=2}^{\infty} q^m s^{-m} \\
 &= -2.
 \end{aligned}$$

Now, we study the regularity of the function $\mathcal{D}S_L^{-1}(s, q)$ in both variables.

Proposition 4.10 *Let $s, q \in \mathbb{H}$ be such that $q \notin [s]$. The function $\mathcal{D}S_L^{-1}(s, q)$ is an intrinsic slice hyperholomorphic function in s .*

Proof This follows by Theorem 4.1 and the shape of the commutative pseudo Cauchy kernel. □

Remark 4.11 The function $\mathcal{D}(S_L^{-1}(s, q))$ is not left slice hyperholomorphic in the variable q . Indeed, let $q = u + Iv$ for an arbitrary $I \in \mathbb{S}$ then $\mathcal{Q}_{c,s}(q)^{-1} = (s^2 - 2us + u^2 + v^2)^{-1}$ and we have the following two relations

$$\frac{\partial}{\partial u} \mathcal{Q}_{c,s}(u + Iv)^{-1} = -(-2s + 2u)\mathcal{Q}_{c,s}(u + Iv)^{-2}$$

and

$$\frac{\partial}{\partial v} \mathcal{Q}_{c,s}(u + Iv)^{-1} = -2v\mathcal{Q}_{c,s}(u + Iv)^{-2},$$

which yield

$$\begin{aligned}
 \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) \mathcal{Q}_{c,s}(u + Iv)^{-1} &= -(-2s + 2u + 2Iv)\mathcal{Q}_{c,s}(u + Iv)^{-2} \\
 &= 2(s - q)\mathcal{Q}_{c,s}(u + Iv)^{-2} = -\frac{1}{2}F_L(s, \bar{q}).
 \end{aligned}$$

The function $\mathcal{D}S_L^{-1}(s, q)$ turns out to be harmonic in q , as proved in the following result.

Proposition 4.12 *Let $s, q \in \mathbb{H}$ be such that $q \notin [s]$. Then the function $\mathcal{D}S_L^{-1}(s, q)$ is harmonic in the real components of q .*

Proof Since the left slice hyperholomorphic Cauchy kernel is a C^∞ function for any $q \notin [s]$, we can apply to it a differential operator of any order. Then the result follows by the facts that the Laplacian is a real operator, thus it commutes with \mathcal{D} , and by Proposition 2.14. Indeed

$$\Delta \mathcal{D}S_L^{-1}(s, q) = \mathcal{D}\Delta S_L^{-1}(s, q) = \mathcal{D}F_L(s, q) = 0. \quad \square$$

Finally as a consequence of the definition of the F -kernel we have:

Lemma 4.13 *Let $s, q \in \mathbb{H}$ be such that $q \notin [s]$, then*

$$\mathcal{D}^2 S_L^{-1}(s, q) = F_L(s, \bar{q}).$$

Proof By Theorem 4.1 we have

$$\mathcal{D}^2 S_L^{-1}(s, q) = -2\mathcal{D}\mathcal{Q}_{c,s}(q)^{-1}. \quad (4.1)$$

Firstly, we apply the derivatives with respect to q_0 and q_i , with $i = 1, 2, 3$ to the commutative pseudo Cauchy kernel

$$\frac{\partial}{\partial q_0} \mathcal{Q}_{c,s}(q)^{-1} = -2(-s + q_0)(s^2 - 2q_0s + |q|^2)^{-2},$$

and for $i = 1, 2, 3$ we get

$$\frac{\partial}{\partial q_i} \mathcal{Q}_{c,s}(q)^{-1} = -2q_i(s^2 - 2q_0s + |q|^2)^{-2}.$$

Thus we obtain

$$\begin{aligned} \mathcal{D}\mathcal{Q}_{c,s}(q)^{-1} &= \frac{\partial}{\partial q_0} \mathcal{Q}_{c,s}(q)^{-1} + \sum_{i=1}^3 e_i \frac{\partial}{\partial q_i} \mathcal{Q}_{c,s}(q)^{-1} \\ &= -2(-s + q_0 + \underline{q})(s^2 - 2q_0s + |q|^2)^{-2} \\ &= 2(s - q)(s^2 - 2q_0s + |q|^2)^{-2}. \end{aligned}$$

Therefore by (4.1) we get

$$\mathcal{D}^2 S_L^{-1}(s, q) = -4(s - q)(s^2 - 2q_0s + |q|^2)^{-2} = F_L(s, \bar{q}). \quad \square$$

Remark 4.14 By Proposition 2.14 it is clear that the function $F_L(s, \bar{q})$ is axially anti-monogenic. This observation together with Lemma 4.13 imply the construction of the following diagram

$$\mathcal{O}(D) \xrightarrow{T_{F1}} SH(\Omega_D) \xrightarrow{\mathcal{D}} AH(\Omega_D) \xrightarrow{\mathcal{D}} \overline{AM(\Omega_D)}, \quad (4.2)$$

where Ω_D is defined as in (3.1) and $\overline{AM(\Omega_D)}$ is the set of axially anti-monogenic functions. In order to avoid this set of functions in the constructions like the one in (4.2) we impose that the composition of the operators appearing in (4.2) must be the Fueter map (in the case of this paper Δ). This is very important when we increase the dimension of the algebra, see [24].

We observe that if we set $q = u + Iv$ and we apply the 2-dimensional Laplacian

$$\Delta_2 := \overline{\partial_I} \partial_I = \left(\frac{\partial}{\partial u} - I \frac{\partial}{\partial v} \right) \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right),$$

to the commutative pseudo Cauchy kernel we get its square:

Lemma 4.15 *Let $s, q = u + Iv \in \mathbb{H}$ be such that $q \notin [s]$, then*

$$\Delta_2 \mathcal{Q}_{c,s}(q)^{-1} = 4 \mathcal{Q}_{c,s}(q)^{-2}.$$

Proof We set $q = u + Iv, I \in \mathbb{S}$. By Remark 4.11 we know that

$$\partial_I \mathcal{Q}_{c,s}(u + Iv)^{-1} = 2(s - u - Iv) \mathcal{Q}_{c,s}(q)^{-2}.$$

Now, we have

$$\begin{aligned} \frac{\partial}{\partial u} \partial_I \mathcal{Q}_{c,s}(u + Iv)^{-1} &= -2 \mathcal{Q}_{c,s}(u + Iv)^{-2} \\ &\quad -4(s - u - Iv) \mathcal{Q}_{c,s}(u + Iv)^{-3}(-2s + 2u), \end{aligned}$$

and

$$\frac{\partial}{\partial v} \partial_I \mathcal{Q}_{c,s}(u + Iv)^{-1} = -2I \mathcal{Q}_{c,s}(u + Iv)^{-2} - 8(s - u - Iv) \mathcal{Q}_{c,s}(u + Iv)^{-3}v.$$

By definition of the 2-dimensional laplacian and since the variable s commute with $\mathcal{Q}_{c,s}(u + Iv)$, we get

$$\begin{aligned} \Delta_2 \mathcal{Q}_{c,s}(q)^{-1} &= \left(\frac{\partial}{\partial u} - I \frac{\partial}{\partial v} \right) \partial_I \mathcal{Q}_{c,s}(u + Iv)^{-1} \\ &= -4(s - u - Iv) \mathcal{Q}_{c,s}(u + Iv)^{-3}(-2s + 2u) \\ &\quad + 8I(s - u - Iv) \mathcal{Q}_{c,s}(u + Iv)^{-3}v \\ &\quad - 4 \mathcal{Q}_{c,s}(u + Iv)^{-2} \\ &= 8(s - u - Iv)(s - u) \mathcal{Q}_{c,s}(u + Iv)^{-3} \\ &\quad + 8I(s - u - Iv)v \mathcal{Q}_{c,s}(u + Iv)^{-2} \\ &\quad - 4 \mathcal{Q}_{c,s}(u + Iv)^{-2} \\ &= 8(s^2 - su - us + u^2 - Isv + Iuv + Isv - Iuv + v^2) \\ &\quad \mathcal{Q}_{c,s}(u + Iv)^{-3} - 4 \mathcal{Q}_{c,s}(u + Iv)^{-2} \end{aligned}$$

$$\begin{aligned}
 &= 8\mathcal{Q}_{c,s}(u + Iv)\mathcal{Q}_{c,s}(u + Iv)^{-3} - 4\mathcal{Q}_{c,s}(u + Iv)^{-2} \\
 &= 8\mathcal{Q}_{c,s}(u + Iv)^{-2} - 4\mathcal{Q}_{c,s}(u + Iv)^{-2} = 4\mathcal{Q}_{c,s}(u + Iv)^{-2}.
 \end{aligned}$$

□

We conclude this section with an integral representation of axially harmonic functions that will allow us to define the harmonic functional calculus based on the S -spectrum.

Theorem 4.16 (Integral representation of axially harmonic functions) *Let $W \subset \mathbb{H}$ be an open set. Let U be a slice Cauchy domain such that $\bar{U} \subset W$. Then for $J \in \mathbb{S}$ and $ds_J = ds(-J)$ we have:*

- (1) *If $f \in SH_L(W)$, then the function $\tilde{f}(q) = \mathcal{D}f(q)$ is harmonic and it admits the following integral representation*

$$\tilde{f}(q) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1} ds_J f(s), \quad q \in U. \tag{4.3}$$

- (2) *If $f \in SH_R(W)$, then the function $\tilde{f}(q) = f(q)\mathcal{D}$ is harmonic and it admits the following integral representation*

$$\tilde{f}(q) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(q)^{-1}, \quad q \in U. \tag{4.4}$$

The integrals depend neither on U nor on the imaginary unit $J \in \mathbb{S}$.

Proof We prove only the first statement because the other proof is similar. We can write the function f by using the Cauchy formula for slice hyperholomorphic functions, see Theorem 2.12. Now, by applying the left Fueter operator to $f(q)$ and by Theorem 4.1 we get

$$\begin{aligned}
 \tilde{f}(q) = \mathcal{D}f(q) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{D}S_L^{-1}(s, q) ds_J f(s) \\
 &= -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1} ds_J f(s).
 \end{aligned} \tag{4.5}$$

Since $\tilde{f}(q) = \mathcal{D}f(q)$ and by Proposition 4.12, it is immediately verified that $\tilde{f}(q)$ is a harmonic function. The independence of integral in (4.3) from the set U and the imaginary unit $J \in \mathbb{S}$ follows by the Cauchy formula. □

In this section we have described the central part of the following diagram

$$\mathcal{O}(D) \xrightarrow{T_{F1}} SH(\Omega_D) \xrightarrow{\mathcal{D}} AH(\Omega_D) \xrightarrow{\bar{\mathcal{D}}} AM(\Omega_D) \tag{4.6}$$

where Ω_D is defined as in (3.1).

Remark 4.17 In the quaternionic setting it is possible to obtain another diagram besides the one in (4.6). This comes from the factorization $\Delta = \mathcal{D}\overline{\mathcal{D}}$ and is called second fine structure in the quaternionic setting, see Definition 1.2. The set of functions that lies between the set of slice hyperholomorphic functions and the set of axially monogenic functions is the set of axially polyanalytic functions of order 2, for more details see [26].

5 The Harmonic Functional Calculus on the S -spectrum

In this section we introduce the harmonic functional calculus on the S -spectrum, which is based on the integral representation of axially harmonic functions. Recall that X denotes a two-sided quaternionic Banach space.

We give meaning to the substitution of the variable q with the operator T in the power series introduced in Definition 4.5.

Definition 5.1 Let $T = T_0 + \sum_{i=1}^3 e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$, we formally define the commutative pseudo S -resolvent series as

$$-2 \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \overline{T}^{k-1} s^{-1-m} \quad \text{and} \quad -2 \sum_{m=1}^{\infty} \sum_{k=1}^m s^{-1-m} T^{m-k} \overline{T}^{k-1}.$$

Remark 5.2 The two series in Definition 5.1 coincide, where they converge.

Proposition 5.3 Let $T = T_0 + \sum_{i=1}^3 e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$ and $\|T\| < |s|$, the series in the Definition 5.1 converges. Moreover, we have

$$\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \overline{T}^{k-1} s^{-1-m} = \sum_{m=1}^{\infty} \sum_{k=1}^m s^{-1-m} T^{m-k} \overline{T}^{k-1} = \mathcal{Q}_{c,s}(T)^{-1}. \quad (5.1)$$

Proof For the convergence of the series it is sufficient to prove the convergence of the series of the operator norm:

$$\sum_{m=1}^{\infty} m \|T\|^{m-1} |s|^{-1-m}. \quad (5.2)$$

Since

$$\lim_{m \rightarrow \infty} \frac{(m+1) \|T\|^m |s|^{-2-m}}{m \|T\|^{m-1} |s|^{-1-m}} = \lim_{m \rightarrow \infty} \frac{m+1}{m} \|T\| |s|^{-1} < 1,$$

by the ratio test the series (5.2) is convergent. To prove the equality (5.1), we show that

$$\begin{aligned} & \mathcal{Q}_{c,s}(T) \left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) \\ &= \left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) \mathcal{Q}_{c,s}(T) = \mathcal{I}. \end{aligned} \tag{5.3}$$

The first equality in (5.3) is a consequence of the following facts: for any positive integer m the operator $\sum_{k=1}^m T^{m-k} \bar{T}^{k-1}$ does not contain any imaginary units, so it is real and then it commutes with any power of s . Secondly, the components of T are commuting among them and the operator $\mathcal{Q}_{c,s}(T)$, see Definition 2.24, can be written as: $s^2\mathcal{I} - 2sT_0 + \sum_{i=0}^3 T_i^2$.

Now we prove the second equality in (5.3). First we observe that

$$\begin{aligned} & \left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) \mathcal{Q}_{c,s}(T) \\ &= \left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) (s^2 - s(T + \bar{T}) + T\bar{T}) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{1-m} - \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m+1-k} \bar{T}^{k-1} s^{-m} - \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^k s^{-m} \\ & \quad + \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k+1} \bar{T}^k s^{-1-m}. \end{aligned}$$

Making the change of index $m' = 1 + m$ in the second and fourth series, we have

$$\begin{aligned} & \left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) \mathcal{Q}_{c,s}(T) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{1-m} - \sum_{m'=2}^{\infty} \sum_{k=1}^{m'-1} T^{m'-k} \bar{T}^{k-1} s^{1-m'} - \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^k s^{-m} \\ & \quad + \sum_{m'=2}^{\infty} \sum_{k=1}^{m'-1} T^{m'-k} \bar{T}^k s^{-m'} \\ &= \mathcal{I} + \sum_{m=2}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{1-m} - \sum_{m'=2}^{\infty} \sum_{k=1}^{m'-1} T^{m'-k} \bar{T}^{k-1} s^{1-m'} + \\ & \quad - \bar{T} s^{-1} - \sum_{m=2}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^k s^{-m} + \sum_{m'=2}^{\infty} \sum_{k=1}^{m'-1} T^{m'-k} \bar{T}^k s^{-m'}. \end{aligned}$$

Simplifying the opposite terms in the first and second series and in the third and fourth series, we finally get

$$\left(\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} \right) \mathcal{Q}_{c,s}(T) = \mathcal{I} + \sum_{m=2}^{\infty} \bar{T}^{m-1} s^{1-m} - \sum_{m=2}^{\infty} \bar{T}^{m-1} s^{1-m} = \mathcal{I}.$$

□

Lemma 5.4 *Let $T \in \mathcal{BC}(X)$. The commutative pseudo S -resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$ is a $\mathcal{B}(X)$ -valued right and left slice hyperholomorphic function of the variable s in $\rho_S(T)$.*

Proof It follows by Proposition 4.10. □

Remark 5.5 We point out an important difference between the commutative and the noncommutative pseudo S -resolvent operator. For $T \in \mathcal{B}(X)$ with noncommuting components the operator $\mathcal{Q}_{c,s}(T)$ is not well defined because $T\bar{T} \neq \bar{T}T$. But in the case $T \in \mathcal{BC}(X)$ then it turns out to be well defined and the inverse is $\mathcal{B}(X)$ -valued slice hyperholomorphic function for $s \in \rho_S(T)$.

The noncommutative pseudo S -resolvent operator $\mathcal{Q}_s(T)^{-1}$ turns out to be well defined for operators $T \in \mathcal{B}(X)$ with noncommuting components, but it is not a $\mathcal{B}(X)$ -valued slice hyperholomorphic function.

Remark 5.6 The functional calculus based on axially harmonic functions in integral form will be called *harmonic functional calculus (on the S -spectrum)* or, since it is based on the commutative pseudo S -resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$, Q -functional calculus.

Definition 5.7 (*Harmonic functional calculus on the S -spectrum*) Let $T \in \mathcal{BC}(X)$ and set $ds_J = ds(-J)$ for $J \in \mathbb{S}$. For every function $\tilde{f} = \mathcal{D}f$ with $f \in SH_L(\sigma_S(T))$, we set

$$\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s), \tag{5.4}$$

where U is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\bar{U} \subset \text{dom}(f)$ and $J \in \mathbb{S}$ is an arbitrary imaginary unit.

For every function $\tilde{f} = f\mathcal{D}$ with $f \in SH_R(\sigma_S(T))$, we set

$$\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(T)^{-1}, \tag{5.5}$$

where U and J are as above.

Theorem 5.8 *The harmonic functional calculus on the S -spectrum is well-defined, i.e., the integrals in (5.4) and (5.5) depend neither on the imaginary unit $J \in \mathbb{S}$ nor on the slice Cauchy domain U .*

Proof Here we show only the case $\tilde{f} = \mathcal{D}f$ with $f \in SH_L(\sigma_S(T))$, since the other one follows by analogous arguments.

Since $\mathcal{Q}_{c,s}(T)^{-1}$ is a right slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from the set U follows by the Cauchy integral formula, see Theorems 2.12 and 2.13.

Now, we want to show the independence from the imaginary unit. Let us consider two imaginary units $J, I \in \mathbb{S}$ with $J \neq I$ and two bounded slice Cauchy domains U_q, U_s with $\sigma_s(T) \subset U_q, \bar{U}_q \subset U_s$ and $\bar{U}_s \subset \text{dom}(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_J)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Recall that $\mathcal{Q}_{c,q}(T)^{-1}$ is right slice hyperholomorphic on $\rho_S(T)$, also at infinity, since $\lim_{q \rightarrow +\infty} \mathcal{Q}_{c,q}(T)^{-1} = 0$. Thus, the Cauchy formula implies

$$\begin{aligned} \mathcal{Q}_{c,s}(T)^{-1} &= \frac{1}{2\pi} \int_{\partial((\mathbb{H} \setminus U_q) \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I S_R^{-1}(q, s) \\ &= \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I S_L^{-1}(s, q). \end{aligned} \tag{5.6}$$

The last equality is due to the fact that $\partial((\mathbb{H} \setminus U_q) \cap \mathbb{C}_I) = -\partial(U_q \cap \mathbb{C}_I)$ and $S_R^{-1}(q, s) = -S_L^{-1}(s, q)$. Combining (5.4) and (5.6) we get

$$\begin{aligned} \tilde{f}(T) &= -\frac{1}{\pi} \int_{\partial(U_s \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s) \\ &= -\frac{1}{\pi} \int_{\partial(U_s \cap \mathbb{C}_J)} \left(\frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I S_L^{-1}(s, q) \right) ds_J f(s). \end{aligned}$$

Due to Fubini’s theorem we can exchange the order of integration and by the Cauchy formula we obtain

$$\begin{aligned} \tilde{f}(T) &= -\frac{1}{\pi} \int_{\partial(U_q \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I \left(\frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_J)} S_L^{-1}(s, q) ds_J f(s) \right) \\ &= -\frac{1}{\pi} \int_{\partial(U_q \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I f(q). \end{aligned}$$

This proves the statement. □

Problem 5.9 Let U be a slice Cauchy domain. It might happen that $f, g \in SH_L(U)$ (resp. $f, g \in SH_R(U)$) and $\mathcal{D}f = \mathcal{D}g$ (resp. $f\mathcal{D} = g\mathcal{D}$). Is it possible to show that for any $T \in \mathcal{BC}(X)$, with $\sigma_S(T) \subset U$, we have $\tilde{f}(T) = \tilde{g}(T)$?

We start to address this problem by observing that $\mathcal{D}(f - g) = 0$ (resp. $(f - g)\mathcal{D} = 0$). Therefore it is necessary to study the set

$$\begin{aligned} (\ker \mathcal{D})_{SH_L(U)} &:= \{f \in SH_L(U) : \mathcal{D}(f) = 0\} \quad \text{resp.} \quad (\ker \mathcal{D})_{SH_R(U)} \\ &:= \{f \in SH_R(U) : (f)\mathcal{D} = 0\}. \end{aligned}$$

Theorem 5.10 *Let U be a connected slice Cauchy domain of \mathbb{H} , then*

$$\begin{aligned} (\ker \mathcal{D})_{SH_L(U)} &= \{f \in SH_L(U) : f \equiv \alpha \text{ for some } \alpha \in \mathbb{H}\} \\ &= \{f \in SH_R(U) : f \equiv \alpha \text{ for some } \alpha \in \mathbb{H}\} = (\ker D)_{SH_R(U)}. \end{aligned}$$

Proof We prove the result in the case $f \in SH_L(U)$ since the case $f \in SH_R(U)$ follows with similar arguments. We proceed by double inclusion. The fact that

$$(\ker D)_{SH_L(U)} \supseteq \{f \in SH_L(U) : f \equiv \alpha \text{ for some } \alpha \in \mathbb{H}\}$$

is obvious. The other inclusion can be proved observing that if $f \in (\ker \mathcal{D})_{SH_L(U)}$, after a change of variable if needed, there exists $r > 0$ such that the function f can be expanded in a convergent series at the origin

$$f(q) = \sum_{k=0}^{\infty} q^k \alpha_k \text{ for } (\alpha_k)_{k \in \mathbb{N}_0} \subset \mathbb{H} \text{ and for any } q \in B_r(0)$$

where $B_r(0)$ is the ball centered at 0 and of radius r . By Lemma 4.3, we have

$$0 = \mathcal{D}f(q) \equiv \sum_{k=1}^{\infty} \mathcal{D}(q^k) \alpha_k = -2 \sum_{k=1}^{\infty} \sum_{s=1}^k q^{k-s} \bar{q}^{s-1} \alpha_k, \quad \forall q \in B_r(0),$$

If we restrict the previous series in a neighborhood Ω of 0 of the real line we get

$$0 = \sum_{k=1}^{\infty} q_0^{k-1} \alpha_k \quad \forall q_0 \in \Omega$$

and this implies

$$\alpha_k = 0, \quad \forall k \geq 1,$$

which yields $f(q) \equiv \alpha_0$ in Ω and since U is connected $f(q) \equiv \alpha_0$ for any $q \in U$. \square

We solve the problem 5.9 in the case in which U is connected.

Proposition 5.11 *Let $T \in \mathcal{BC}(X)$ and let U be a connected slice Cauchy domain with $\sigma_S(T) \subset U$. If $f, g \in SH_L(U)$ (resp. $f, g \in SH_R(U)$) satisfy the property $\mathcal{D}f = \mathcal{D}g$ (resp. $f\mathcal{D} = g\mathcal{D}$) then $\tilde{f}(T) = \tilde{g}(T)$.*

Proof We prove the theorem in the case $f, g \in SH_L(U)$ since the case $f, g \in SH_R(U)$ follows by similar arguments. By definition of the harmonic functional calculus on the S -spectrum, see Definition 5.7, we have

$$\tilde{f}(T) - \tilde{g}(T) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J (f(s) - g(s)).$$

Since $\mathcal{Q}_{c,s}(T)^{-1}$ is slice hyperholomorphic in the variable s by Theorem 2.12, we can change the domain of integration to $B_r(0) \cap \mathbb{C}_J$ for some $r > 0$ with $\|T\| < r$. Moreover, by hypothesis we have that $f(s) - g(s) \in (\ker D)_{SH_L(U)}$, thus by Theorem 5.10 and Proposition 5.3 we get

$$\begin{aligned} \tilde{f}(T) - \tilde{g}(T) &= -\frac{1}{\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J (f(s) - g(s)) \\ &= \int_{\partial(B_r(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J \alpha \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-1-m} ds_J \alpha = 0. \end{aligned}$$

□

In order to solve Problem 5.9, in the case U is not connected, we need the following lemma, which is based on the monogenic functional calculus developed by McIntosh and collaborators, see [29–31].

Lemma 5.12 *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ , $\ell = 1, 2, 3$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G) \cap \sigma_S(T) = \emptyset$. For every $J \in \mathbb{S}$ we have*

$$\int_{\partial(G \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J = 0. \tag{5.7}$$

Proof Since $\Delta(1) = 0$ and $\Delta(q) = 0$, by Theorem 2.18 we also have

$$\int_{\partial(G \cap \mathbb{C}_J)} F_L(s, q) ds_J = \Delta(1) = 0, \tag{5.8}$$

and

$$\int_{\partial(G \cap \mathbb{C}_J)} F_L(s, q) ds_J s = \Delta(q) = 0, \tag{5.9}$$

for all $q \notin \partial G$ and $J \in \mathbb{S}$. By the monogenic functional calculus [29–31] we have

$$F_L(s, T) = \int_{\partial\Omega} G(\omega, T) \mathbf{D}\omega F_L(s, \omega),$$

where $\mathbf{D}\omega$ is a suitable differential form, the open set Ω contains the left spectrum of T and $G(\omega, T)$ is the Fueter resolvent operator. By Theorem 2.29 we can write

$$\mathcal{Q}_{c,s}(T)^{-1} = -\frac{1}{4}(F_L(s, T)s - TF_L(s, T)),$$

thus we have

$$\begin{aligned}
 \int_{\partial(G \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J &= -\frac{1}{4} \int_{\partial(G \cap \mathbb{C}_J)} F_L(s, T)s - T F_L(s, T) ds_J \\
 &= -\frac{1}{4} \left(\int_{\partial(G \cap \mathbb{C}_J)} \int_{\partial\Omega} G(\omega, T) \mathbf{D}\omega F_L(s, \omega) s ds_J \right. \\
 &\quad \left. - T \int_{\partial(G \cap \mathbb{C}_J)} \int_{\partial\Omega} G(\omega, T) \mathbf{D}\omega F_L(s, \omega) ds_J \right) \\
 &= -\frac{1}{4} \left(\int_{\partial\Omega} G(\omega, T) \mathbf{D}\omega \left(\int_{\partial(G \cap \mathbb{C}_J)} F_L(s, \omega) ds_J s \right) \right. \\
 &\quad \left. - T \int_{\partial\Omega} G(\omega, T) \mathbf{D}\omega \left(\int_{\partial(G \cap \mathbb{C}_J)} F_L(s, \omega) ds_J \right) \right) \\
 &= 0
 \end{aligned}$$

where the second equality is a consequence of the Fubini’s Theorem and the last equality is a consequence of formulas (5.8) and (5.9). □

Finally in the following result we give an answer to Problem 5.9.

Proposition 5.13 *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ , $\ell = 1, 2, 3$, have real spectrum. Let U be a slice Cauchy domain with $\sigma_S(T) \subset U$. If $f, g \in SH_L(U)$ (resp. $f, g \in SH_R(U)$) satisfy the property $\mathcal{D}f = \mathcal{D}g$ (resp. $f\mathcal{D} = g\mathcal{D}$) then $\tilde{f}(T) = \tilde{g}(T)$.*

Proof If U is connected we can use Proposition 5.11. If U is not connected then $U = \cup_{l=1}^n U_l$ where the U_l are the connected components of U . Hence, we have $f(s) - g(s) = \sum_{l=1}^n \chi_{U_l}(s)\alpha_l$ and we can write

$$\tilde{f}(T) - \tilde{g}(T) = -\sum_{l=1}^n \frac{1}{\pi} \int_{\partial(U_l \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J \alpha_l.$$

The last summation is zero by Lemma 5.12. □

We conclude this section with some algebraic properties of the harmonic functional calculus.

Proposition 5.14 *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ , $\ell = 1, 2, 3$, have real spectrum.*

- If $\tilde{f} = \mathcal{D}f$ and $\tilde{g} = \mathcal{D}g$ with $f, g \in SH_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then

$$(\tilde{f}a + \tilde{g})(T) = \tilde{f}(T)a + \tilde{g}(T).$$

- If $\tilde{f} = f\mathcal{D}$ and $\tilde{g} = g\mathcal{D}$ with $f, g \in SH_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then

$$(a\tilde{f} + \tilde{g})(T) = a\tilde{f}(T) + \tilde{g}(T).$$

Proof The above identities follow immediately from the linearity of the integrals in (5.4), resp. (5.5). \square

Proposition 5.15 *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ , $\ell = 1, 2, 3$, have real spectrum.*

- *If $\tilde{f} = \mathcal{D}f$ with $f \in SH_L(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^\infty q^m a_m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\tilde{f}(T) = -2 \sum_{m=1}^\infty \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} a_m.$$

- *If $\tilde{f} = f\mathcal{D}$ with $f \in SH_R(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^\infty a_m q^m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\tilde{f}(T) = -2 \sum_{m=1}^\infty \sum_{k=1}^m a_m T^{m-k} \bar{T}^{k-1}.$$

Proof We prove the first assertion since the second one can be proven similarly. We choose an imaginary unit $J \in \mathbb{S}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Then the series expansion of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_J)$, and so

$$\begin{aligned} \tilde{f}(T) &= -\frac{1}{\pi} \int_{\partial(B_R(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J \sum_{l=0}^\infty s^l a_l \\ &= -\frac{1}{\pi} \sum_{l=0}^\infty \int_{\partial(B_R(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^l a_l. \end{aligned}$$

By replacing $\mathcal{Q}_{c,s}(T)^{-1}$ with its series expansion, see Proposition 5.3, we further obtain

$$\begin{aligned} \tilde{f}(T) &= -\frac{1}{\pi} \int_{\partial(B_R(0) \cap \mathbb{C}_J)} \sum_{m=1}^\infty \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} ds_J \sum_{l=0}^\infty s^l a_l \\ &= -\frac{1}{\pi} \sum_{m=1}^\infty \sum_{k=1}^m \sum_{l=0}^\infty T^{m-k} \bar{T}^{k-1} \int_{\partial(B_R(0) \cap \mathbb{C}_J)} s^{-1-m} ds_J s^l a_l \\ &= -2 \sum_{m=1}^\infty \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} a_m. \end{aligned}$$

The last equality is due to the fact that $\int_{\partial(B_R(0) \cap \mathbb{C}_J)} s^{-1-m} ds_J s^l$ is equal to 2π if $l = m$, and 0 otherwise. \square

6 The Resolvent Equations for Harmonic Functional Calculus

In this section we prove various resolvent equations for the pseudo S -resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$. The first version of this equation is written in terms of $\mathcal{Q}_{c,s}(T)^{-1}$ and of the S -resolvent operators.

Theorem 6.1 (The \mathcal{Q} -resolvent equation with S -resolvent operators) *Let $T \in \mathcal{BC}(X)$. Then, for $p, s \in \rho_S(T)$ with $s \notin [p]$, the following equalities hold*

$$\begin{aligned} \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} = & \left\{ \left[\mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) - S_R^{-1}(s, T) \mathcal{Q}_{c,p}(T)^{-1} \right] p \right. \\ & \left. - \bar{s} \left[\mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) - S_R^{-1}(s, T) \mathcal{Q}_{c,p}(T)^{-1} \right] \right\} \\ & (p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} = & (p^2 - 2s_0p + |s|^2)^{-1} \left\{ s \left[\mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) \right. \right. \\ & \left. \left. - S_R^{-1}(s, T) \mathcal{Q}_{c,p}(T)^{-1} \right] \right. \\ & \left. - \left[\mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) - S_R^{-1}(s, T) \mathcal{Q}_{c,p}(T)^{-1} \right] \bar{p} \right\}. \end{aligned} \tag{6.2}$$

Proof By the definition of left S -resolvent operator we have

$$\mathcal{Q}_{c,p}(T)^{-1} p = \bar{T} \mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T). \tag{6.3}$$

By iterating (6.3) we get

$$\begin{aligned} & \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} (p^2 - 2s_0p + |s|^2) \\ &= \mathcal{Q}_{c,s}(T)^{-1} [\mathcal{Q}_{c,p}(T)^{-1} p] p - 2s_0 \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} p \\ & \quad + |s|^2 \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1} [\bar{T} \mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T)] p \\ & \quad - 2s_0 \mathcal{Q}_{c,s}(T)^{-1} [\bar{T} \mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T)] \\ & \quad + |s|^2 \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1} \bar{T} [\mathcal{Q}_{c,p}(T)^{-1} p] + \mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) p \\ & \quad - 2s_0 \mathcal{Q}_{c,s}(T)^{-1} [\bar{T} \mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T)] \\ & \quad + |s|^2 \mathcal{Q}_{c,s}(T)^{-1} \mathcal{Q}_{c,p}(T)^{-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1} \bar{T} [\bar{T} \mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T)] + \mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) p \\ & \quad - 2s_0 \mathcal{Q}_{c,s}(T)^{-1} [\bar{T} \mathcal{Q}_{c,p}(T)^{-1} \end{aligned}$$

$$+S_L^{-1}(p, T)] + |s|^2 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1}.$$

Now, by the definition of the right S -resolvent operator we have

$$Q_{c,s}(T)^{-1} \bar{T} = s Q_{c,s}(T)^{-1} - S_R^{-1}(s, T). \tag{6.4}$$

This equality implies

$$\begin{aligned} & Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} (p^2 - 2s_0p + |s|^2) \\ &= [Q_{c,s}(T)^{-1} \bar{T}] \bar{T} Q_{c,p}(T)^{-1} \\ &\quad + [Q_{c,s}(T)^{-1} \bar{T}] S_L^{-1}(p, T) + Q_{c,s}(T)^{-1} S_L^{-1}(p, T) p \\ &\quad - 2s_0 [Q_{c,s}(T)^{-1} \bar{T}] Q_{c,p}(T)^{-1} - 2s_0 Q_{c,s}(T)^{-1} S_L^{-1}(p, T) \\ &\quad + |s|^2 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} \\ &= [s Q_{c,s}(T)^{-1} - S_R^{-1}(s, T)] \bar{T} Q_{c,p}(T)^{-1} \\ &\quad + [s Q_{c,s}(T)^{-1} - S_R^{-1}(s, T)] S_L^{-1}(p, T) \\ &\quad + Q_{c,s}(T)^{-1} S_L^{-1}(p, T) p - 2s_0 [s Q_{c,s}(T)^{-1} - S_R^{-1}(s, T)] Q_{c,p}(T)^{-1} \\ &\quad - 2s_0 Q_{c,s}(T)^{-1} S_L^{-1}(p, T) \\ &\quad + |s|^2 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} \\ &= s [Q_{c,s}(T)^{-1} \bar{T}] Q_{c,p}(T)^{-1} - S_R^{-1}(s, T) \bar{T} Q_{c,p}(T)^{-1} \\ &\quad + s Q_{c,s}(T)^{-1} S_L^{-1}(p, T) - S_R^{-1}(s, T) S_L^{-1}(p, T) \\ &\quad + Q_{c,s}(T)^{-1} S_L^{-1}(p, T) p - 2s_0 s Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} \\ &\quad + 2s_0 S_R^{-1}(s, T) Q_{c,p}(T)^{-1} \\ &\quad - 2s_0 Q_{c,s}(T)^{-1} S_L^{-1}(p, T) + |s|^2 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} \\ &= s [s Q_{c,s}(T)^{-1} - S_R^{-1}(s, T)] Q_{c,p}(T)^{-1} - S_R^{-1}(s, T) \bar{T} Q_{c,p}(T)^{-1} \\ &\quad + s Q_{c,s}(T)^{-1} S_L^{-1}(p, T) \\ &\quad - S_R^{-1}(s, T) S_L^{-1}(p, T) + Q_{c,s}(T)^{-1} S_L^{-1}(p, T) p - 2s_0 s Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} \\ &\quad + 2s_0 S_R^{-1}(s, T) Q_{c,p}(T)^{-1} \\ &\quad - 2s_0 Q_{c,s}(T)^{-1} S_L^{-1}(p, T) + |s|^2 Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1}. \end{aligned}$$

Now, since $s^2 - 2s_0s + |s|^2 = 0$ we get

$$\begin{aligned} & Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} (p^2 - 2s_0p + |s|^2) \\ &= (s^2 - 2s_0s + |s|^2) Q_{c,s}(T)^{-1} Q_{c,p}(T)^{-1} - s S_R^{-1}(s, T) Q_{c,p}(T)^{-1} \\ &\quad - S_R^{-1}(s, T) \bar{T} Q_{c,p}(T)^{-1} \\ &\quad + s Q_{c,s}(T)^{-1} S_L^{-1}(p, T) - S_R^{-1}(s, T) S_L^{-1}(p, T) + Q_{c,s}(T)^{-1} S_L^{-1}(p, T) p \\ &\quad + 2s_0 S_R^{-1}(s, T) Q_{c,p}(T)^{-1} \\ &\quad - 2s_0 Q_{c,s}(T)^{-1} S_L^{-1}(p, T) \\ &= -s S_R^{-1}(s, T) Q_{c,p}(T)^{-1} - S_R^{-1}(s, T) \bar{T} Q_{c,p}(T)^{-1} \end{aligned}$$

$$\begin{aligned}
 & +s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)S_L^{-1}(p, T) \\
 & +\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T)p + 2s_0S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} - 2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) \\
 = & -sS_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} + s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) \\
 & -S_R^{-1}(s, T)[\bar{T}\mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p, T)] \\
 & +\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T)p + 2s_0S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \\
 & -2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T).
 \end{aligned}$$

Finally, using another time formula (6.3) and the fact that $2s_0 - s = \bar{s}$ we obtain

$$\begin{aligned}
 & \mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}(p^2 - 2s_0p + |s|^2) \\
 & = -sS_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} + s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1}p \\
 & \quad +\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T)p + 2s_0S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \\
 & \quad -2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) \\
 = & \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \right] p \\
 & \quad -\bar{s} \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \right].
 \end{aligned}$$

It is possible to obtain formula (6.2) with similar computations. □

Remark 6.2 We can rewrite the equations obtained in Theorem 6.1 by using the left or right $*$ -products, see [19, Chap. 4], in the variables $s, p \in \rho_S(T)$ with $s \notin [p]$,

$$\begin{aligned}
 \mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} = & \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \right] \\
 & *_{s, \text{left}}(p - \bar{s})(p^2 - 2s_0p + |s|^2)^{-1}\mathcal{I},
 \end{aligned}$$

or

$$\begin{aligned}
 \mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} = & (p - \bar{s})(p^2 - 2s_0p + |s|^2)^{-1}\mathcal{I} \\
 & *_{p, \text{right}} \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p, T) - S_R^{-1}(s, T)\mathcal{Q}_{c,p}(T)^{-1} \right].
 \end{aligned}$$

Theorem 6.3 (Left and right generalized Q -resolvent equations) *Let $T \in \mathcal{BC}(X)$ with $s \in \rho_S(T)$ and set*

$$\mathcal{M}_m^L(s, T) := \sum_{i=0}^{m-1} \bar{T}^i S_L^{-1}(s, T) s^{m-i-1} \tag{6.5}$$

and

$$\mathcal{M}_m^R(s, T) := \sum_{i=0}^{m-1} s^{m-i-1} S_R^{-1}(s, T) \bar{T}^i.$$

Then for $m \geq 1$ and $s \in \rho_S(T)$, the following equations hold

$$\mathcal{Q}_{c,s}(T)^{-1}s^m - \bar{T}^m \mathcal{Q}_{c,s}(T)^{-1} = \mathcal{M}_m^L(s, T) \tag{6.6}$$

and

$$s^m \mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\bar{T}^m = \mathcal{M}_m^R(s, T).$$

Proof We prove the result by induction on m . We will prove only (6.6) since the other equality is proven with similar techniques. The case $m = 1$ is trivial because

$$\mathcal{M}_1^L(s, T) = S_L^{-1}(s, t) = \mathcal{Q}_{c,s}^{-1}(T)s - \bar{T}\mathcal{Q}_{c,s}^{-1}(T).$$

We assume that the equation holds for $m - 1$ and we will prove it for m . By inductive hypothesis, we have

$$\begin{aligned} \bar{T}^m \mathcal{Q}_{c,s}(T)^{-1} &= \bar{T}\bar{T}^{m-1} \mathcal{Q}_{c,s}(T)^{-1} = \bar{T}(\mathcal{Q}_{c,s}(T)^{-1}s^{m-1} - \mathcal{M}_{m-1}^L(s, T)) \\ &= \bar{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-1} - \bar{T}\mathcal{M}_{m-1}^L(s, T). \end{aligned}$$

Since

$$\bar{T}\mathcal{M}_{m-1}^L(s, T) = \sum_{i=0}^{m-2} \bar{T}^{i+1} S_L^{-1}(s, T)s^{m-i-2} = \sum_{i=1}^{m-1} \bar{T}^i S^{-1}(s, T)s^{m-i-1}$$

and

$$\bar{T}\mathcal{Q}_{c,s}(T)^{-1} = \mathcal{Q}_{c,s}(T)^{-1}s - S_L^{-1}(s, t),$$

we have

$$\begin{aligned} \bar{T}^m \mathcal{Q}_{c,s}(T)^{-1} &= \mathcal{Q}_{c,s}(T)^{-1}s^m - S_L^{-1}(s, T)s^{m-1} - \sum_{i=1}^{m-1} \bar{T}^i S^{-1}(s, T)s^{m-i-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1}s^m - \sum_{i=0}^{m-1} \bar{T}^i S^{-1}(s, T)s^{m-i-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1}s^m - \mathcal{M}_m^L(s, T). \end{aligned}$$

□

Now, we prove the Q -resolvent equation just in terms of the commutative pseudo S -resolvent operator.

Theorem 6.4 (The Q -resolvent equation) *Let $T \in \mathcal{BC}(X)$. Then for $s, p \in \rho_S(T)$ with $s \notin [p]$, we have the following equation*

$$\begin{aligned}
 & s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}p - s\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1} \\
 & \quad - \mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1}p + \mathcal{Q}_{c,s}(T)^{-1}\bar{T}^2\mathcal{Q}_{c,p}(T)^{-1} \\
 & = \left[(s\mathcal{Q}_{c,s}(T)^{-1} - p\mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(s\mathcal{Q}_{c,s}(T)^{-1} - p\mathcal{Q}_{c,p}(T)^{-1}) \right] \\
 & \quad (p^2 - 2s_0p + |s|^2)^{-1} \\
 & \quad + \left[(\bar{T}\mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\bar{T})p - \bar{s}(\bar{T}\mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\bar{T}) \right] \\
 & \quad (p^2 - 2s_0p + |s|^2)^{-1}. \tag{6.7}
 \end{aligned}$$

Proof Starting from the S -resolvent equation (see formula (2.6))

$$S_R^{-1}(s, T)S_L^{-1}(p, T) = \left[(S_R^{-1}(s, T) - S_L^{-1}(p, T))p - \bar{s}(S_R^{-1}(s, T) - S_L^{-1}(p, T)) \right] (p^2 - 2s_0p + |s|^2)^{-1}$$

and using the definitions of the S -resolvent operators

$$S_R^{-1}(s, T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \bar{T}),$$

and

$$S_L^{-1}(p, T) = (p\mathcal{I} - \bar{T})\mathcal{Q}_{c,p}(T)^{-1},$$

we obtain that the left hand side of the S -resolvent equation can be rewritten as

$$\begin{aligned}
 S_R^{-1}(s, T)S_L^{-1}(p, T) & = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \bar{T})(p\mathcal{I} - \bar{T})\mathcal{Q}_{c,p}(T)^{-1} \\
 & = s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}p - s\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1} \\
 & \quad - \mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1}p + \mathcal{Q}_{c,s}(T)^{-1}\bar{T}^2\mathcal{Q}_{c,p}(T)^{-1}.
 \end{aligned}$$

The right hand side can be rewritten in the following way

$$\begin{aligned}
 & \left[(S_R^{-1}(s, T) - S_L^{-1}(p, T))p - \bar{s}(S_R^{-1}(s, T) - S_L^{-1}(p, T)) \right] (p^2 - 2s_0p + |s|^2)^{-1} \\
 & = \left[(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \bar{T}) - (p\mathcal{I} - \bar{T})\mathcal{Q}_{c,p}(T)^{-1})p + \right. \\
 & \quad \left. - \bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \bar{T}) - (p\mathcal{I} - \bar{T})\mathcal{Q}_{c,p}(T)^{-1}) \right] (p^2 - 2s_0p + |s|^2)^{-1} \\
 & = \left[(s\mathcal{Q}_{c,s}(T)^{-1} - p\mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(s\mathcal{Q}_{c,s}(T)^{-1} - p\mathcal{Q}_{c,p}(T)^{-1}) \right] \\
 & \quad (p^2 - 2s_0p + |s|^2)^{-1} \\
 & \quad + \left[(\bar{T}\mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\bar{T})p - \bar{s}(\bar{T}\mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\bar{T}) \right] \\
 & \quad (p^2 - 2s_0p + |s|^2)^{-1}.
 \end{aligned}$$

By equating (6.8) and (6.8), we obtain the assertion. □

Remark 6.5 It is possible to rewrite (6.7) as

$$\begin{aligned}
 & s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}p - s\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1} + \\
 & -\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1}p + \mathcal{Q}_{c,s}(T)^{-1}\bar{T}^2\mathcal{Q}_{c,p}(T)^{-1} \\
 & = \left(s\mathcal{Q}_{c,s}(T)^{-1} - p\mathcal{Q}_{c,p}(T)^{-1} \right) *_{s,left} S_L^{-1}(p, s) \\
 & + \left(\bar{T}\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1}\bar{T} \right) *_{s,left} S_L^{-1}(p, s)
 \end{aligned} \tag{6.8}$$

Remark 6.6 Formula (6.7) or, equivalently, (6.8) can be considered the most appropriate Q -resolvent equation because

- (I) it preserves the left slice hyperholomorphicity in s and the right slice hyperholomorphicity in p ;
- (II) the product $\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}$ (multiplied by monomials or bounded operators) is written in terms of the difference $\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1}$ entangled with the left slice hyperholomorphic Cauchy kernel.

For more information of the properties of the resolvent equations in hyperholomorphic spectral theories see the paper [16].

7 The Riesz Projectors for Harmonic Functional Calculus

We now take advantage of the Q -resolvent equation in Theorem 6.4 to study the Riesz projectors for the harmonic functional calculus. In the sequel we need the crucial result originally proved in [1, Lemma 3.23].

Lemma 7.1 (See [20]) *Let $B \in \mathcal{B}(X)$. Let G be an axially symmetric domain and assume $f \in N(G)$. Then for $p \in G$, we have*

$$\frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (\bar{s}B - Bp)(p^2 - 2s_0p + |s|^2)^{-1} = Bf(p).$$

Theorem 7.2 (The Riesz projectors) *Let $T = T_1e_1 + T_2e_2 + T_3e_3$ and assume that the operators $T_l, l = 1, 2, 3$, have real spectrum. Let $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with $\text{dist}(\sigma_1, \sigma_2) > 0$.*

Let $G_1, G_2 \subset \mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1 \subset G_1, \bar{G}_1 \subset G_2$ and $\text{dist}(G_2, \sigma_2) > 0$. Then the operator

$$\tilde{P}(T) := \frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} = \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p$$

is a projection, i.e.,

$$\tilde{P}^2 = \tilde{P}.$$

Moreover, the operator \tilde{P} commutes with T , i.e. we have

$$T\tilde{P} = \tilde{P}T. \tag{7.1}$$

Proof First, we multiply equation (6.7) by ds_J on the left and we integrate it on $\partial(G_2 \cap \mathbb{C}_J)$ with respect to ds_J , and then we multiply it by dp_J on the right and we integrate it on $\partial(G_1 \cap \mathbb{C}_J)$ with respect to dp_J . We obtain

$$\begin{aligned} & \int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \mathcal{Q}_{c,s}(T)^{-1} T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \mathcal{Q}_{c,s}(T)^{-1} T^2 \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J \\ & = \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[(s \mathcal{Q}_{c,s}(T)^{-1} - p \mathcal{Q}_{c,p}(T)^{-1}) p \right. \\ & \quad \left. - \bar{s} (s \mathcal{Q}_{c,s}(T)^{-1} - p \mathcal{Q}_{c,p}(T)^{-1}) \right] \mathcal{Q}_s(p)^{-1} dp_J \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[(\bar{T} \mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1} \bar{T}) p \right. \\ & \quad \left. - \bar{s} (\bar{T} \mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1} \bar{T}) \right] \mathcal{Q}_s(p)^{-1} dp_J, \end{aligned} \tag{7.2}$$

where we set $\mathcal{Q}_s(p) = p^2 - 2s_0p + |s|^2$. By Lemma 5.12 the expression on the left-hand side of (7.2) simplifies to

$$\int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p.$$

Now, we focus on the right hand-side of (7.2). We start by rewriting it as

$$\begin{aligned} & \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} s [\mathcal{Q}_{c,s}(T)^{-1} p - \bar{s} \mathcal{Q}_{c,s}(T)^{-1}] \mathcal{Q}_s(p)^{-1} dp_J \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [-\mathcal{Q}_{c,s}(T)^{-1} \bar{T} p + \bar{s} \mathcal{Q}_{c,s}(T)^{-1} \bar{T}] \mathcal{Q}_s(p)^{-1} dp_J \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\bar{s} \mathcal{Q}_{c,p}(T)^{-1} p - p \mathcal{Q}_{c,p}(T)^{-1} p] \mathcal{Q}_s(p)^{-1} dp_J \\ & + \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [-\bar{s} \bar{T} \mathcal{Q}_{c,p}(T)^{-1} + \bar{T} \mathcal{Q}_{c,p}(T)^{-1} p] \mathcal{Q}_s(p)^{-1} dp_J. \end{aligned} \tag{7.3}$$

Now, since $\overline{G}_1 \subset G_2$, for any $s \in \partial(G_2 \cap \mathbb{C}_J)$ the functions

$$p \rightarrow p\mathcal{Q}_s(p)^{-1}$$

and

$$p \rightarrow \mathcal{Q}_s(p)^{-1}$$

are intrinsic slice hyperholomorphic on \overline{G}_1 . By the Cauchy integral formula we have

$$\int_{\partial(G_1 \cap \mathbb{C}_J)} p\mathcal{Q}_s(p)^{-1} dp_J = 0, \quad \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_s(p)^{-1} dp_J = 0.$$

Therefore

$$\int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} s[\mathcal{Q}_{c,s}(T)^{-1}p - \overline{s}\mathcal{Q}_{c,s}(T)^{-1}]\mathcal{Q}_s(p)^{-1} dp_J = 0$$

and

$$\int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [-\mathcal{Q}_{c,s}(T)^{-1}\overline{T}p + \overline{s}\mathcal{Q}_{c,s}(T)^{-1}\overline{T}]\mathcal{Q}_s(p)^{-1} dp_J = 0.$$

Thus the right hand side in (7.3) is just

$$\begin{aligned} & \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\overline{s}\mathcal{Q}_{c,p}(T)^{-1}p - p\mathcal{Q}_{c,p}(T)^{-1}p]\mathcal{Q}_s(p)^{-1} dp_J \\ & - \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\overline{s}\overline{T}\mathcal{Q}_{c,p}(T)^{-1} - \overline{T}\mathcal{Q}_{c,p}(T)^{-1}p]\mathcal{Q}_s(p)^{-1} dp_J. \end{aligned}$$

We can further simplify the previous expression by applying Lemma 7.1 twice: in the first integral for

$$B := p\mathcal{Q}_{c,p}(T)^{-1}$$

and in the second integral for

$$B := \overline{T}\mathcal{Q}_{c,p}(T)^{-1}.$$

Thus we obtain

$$\begin{aligned} & \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\overline{s}\mathcal{Q}_{c,p}(T)^{-1}p - p\mathcal{Q}_{c,p}(T)^{-1}p]\mathcal{Q}_s(p)^{-1} dp_J \\ & - \int_{\partial(G_2 \cap \mathbb{C}_J)} ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\overline{s}\overline{T}\mathcal{Q}_{c,s}(T)^{-1} - \overline{T}\mathcal{Q}_{c,s}(T)^{-1}p]\mathcal{Q}_s(p)^{-1} dp_J \end{aligned}$$

$$\begin{aligned}
 &= 2\pi \int_{\partial(G_1 \cap \mathbb{C}_J)} p \mathcal{Q}_{c,p}(T)^{-1} dp_J - 2\pi \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{T} \mathcal{Q}_{c,p}(T)^{-1} dp_J \\
 &= 2\pi \int_{\partial(G_1 \cap \mathbb{C}_J)} p \mathcal{Q}_{c,p}(T)^{-1} dp_J.
 \end{aligned}$$

In the last equation we have used Lemma 5.12. In conclusion Eq. (7.2) reduces to

$$\begin{aligned}
 &\int_{\partial(G_2 \cap \mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p \\
 &= 2\pi \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p
 \end{aligned}$$

and, by the definition of the operator \tilde{P} given in the statement, the previous equality means

$$\tilde{P}^2 = \tilde{P}.$$

Now, we prove (7.1). Since $T_0 = 0$ and

$$\bar{T} \mathcal{Q}_{c,p}(T)^{-1} = \mathcal{Q}_{c,p}(T)^{-1} p - S_L^{-1}(p, T),$$

we get

$$\begin{aligned}
 T \tilde{P} &= -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{T} \mathcal{Q}_{c,p}(T)^{-1} dp_J p \\
 &= -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} (\mathcal{Q}_{c,p}(T)^{-1} p - S_L^{-1}(p, T)) dp_J p \\
 &= -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p^2 + \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J p.
 \end{aligned}$$

Thus we get

$$T \tilde{P} = -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p^2 + \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J p. \tag{7.4}$$

On the other side, since

$$\mathcal{Q}_{c,p}(T)^{-1} \bar{T} = \mathcal{Q}_{c,p}(T)^{-1} p - S_R^{-1}(p, T),$$

we get

$$\begin{aligned} \tilde{P}T &= -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} pdp_J \mathcal{Q}_{c,p}(T)^{-1} \bar{T} \\ &= -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p^2 + \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} pdp_J S_R^{-1}(p, T). \end{aligned}$$

From the fact that $p\chi_{G_1}(p)$ is intrinsic slice hyperholomorphic in G_1 , it follows by [20, Thm. 3.2.11] that

$$\tilde{P}T = -\frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p^2 + \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) pdp_J. \tag{7.5}$$

Since (7.4) and (7.5) are equal we get the statement. □

Remark 7.3 The Q -resolvent equation stated in Theorem 6.1 preserves the slice hyperholomorphicity, however it is not useful to prove Theorem 7.2.

Remark 7.4 Theorem 7.2 can be proved using directly the F -functional calculus. Indeed, in the same hypothesis of the theorem, it is proved in [20, Thm. 7.4.2] that

$$\check{P}_1^2 = \check{P}_1 \quad \text{and} \quad T\check{P}_1 = \check{P}_1 T$$

for

$$\check{P}_1 := -\frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J p^2.$$

Now, using Theorem 2.29 and [20, Lemma 7.4.1], we have

$$\begin{aligned} \tilde{P} &= \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p \\ &= -\frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) p - T F_L(p, T) dp_J p \\ &= -\frac{1}{8\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J p^2 = \check{P}. \end{aligned}$$

8 Further Properties of the Harmonic Functional Calculus

In this section we prove other important properties of the harmonic functional calculus.

Theorem 8.1 *Let $T \in \mathcal{BC}(X)$. Let $m \in \mathbb{N}_0$, and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$. For every $J \in \mathbb{S}$ we have*

$$H_m(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1}, \tag{8.1}$$

where

$$H_m(T) := \sum_{k=0}^m T^{m-k} \bar{T}^k.$$

Proof We start by considering U to be the ball $B_r(0)$ with $\|T\| < r$. We know that

$$\mathcal{Q}_{c,s}(T)^{-1} = \sum_{n=1}^{+\infty} \sum_{k=1}^n T^{n-k} \bar{T}^{k-1} s^{-1-n}$$

for every $s \in \partial B_r(0)$. By Proposition 5.3 we know that the series converges on $\partial B_r(0)$. Thus we have

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} &= \frac{1}{2\pi} \sum_{n=1}^{+\infty} \sum_{k=1}^n T^{n-k} \bar{T}^{k-1} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-n+m} ds_J \\ &= \sum_{k=1}^{m+1} T^{m+1-k} \bar{T}^{k-1} = \sum_{k=0}^m T^{m-k} \bar{T}^k = H_m(T) \end{aligned}$$

since

$$\int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-n+m} ds_J = \begin{cases} 0 & \text{if } n \neq m + 1 \\ 2\pi & \text{if } n = m + 1. \end{cases}$$

This proves the result for the case $U = B_r(0)$. Now we get the result for an arbitrary bounded Cauchy domain U that contains $\sigma_S(T)$. Then there exists a radius r such that $\bar{U} \subset B_r(0)$. The operator $\mathcal{Q}_{c,s}(T)^{-1}$ is right slice hyperholomorphic and the monomial s^{m+1} is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0) \setminus U$. By the Cauchy’s integral formula we get

$$\begin{aligned} &\frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} - \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} \\ &= \frac{1}{2\pi} \int_{\partial((B_r(0) \setminus U) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} = 0. \end{aligned}$$

Finally we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} = \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J s^{m+1} = H_m(T),$$

and this concludes the proof. □

Remark 8.2 Unlike what happens in the S -functional calculus (see [20, Thm. 3.2.2]) we do not have a left slice hyperholomorphic polynomial on the left hand side of equality (8.1), but we have harmonic polynomials. Another difference with respect to [20, Thm. 3.2.2] is that in Theorem 8.1 we do not have a difference between right and left part, because by Proposition 5.3

$$\sum_{m=1}^{\infty} \sum_{k=1}^m T^{m-k} \bar{T}^{k-1} s^{-1-m} = \sum_{m=1}^{\infty} \sum_{k=1}^m s^{-1-m} T^{m-k} \bar{T}^{k-1} = \mathcal{Q}_{c,s}(T)^{-1}.$$

For the intrinsic functions we have the following result, see [20, Theorem 3.2.11].

Lemma 8.3 *Let $T \in \mathcal{BC}(X)$. If $f \in N(\sigma_S(T))$ and U is a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\bar{U} \subset \text{dom}(f)$, then we have*

$$\tilde{f}(T) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(T)^{-1}.$$

Proof It follows by the definitions of intrinsic functions, of the Q -functional calculus and Runge’s theorem (see [20, Theorem 2.1.37]). □

For the \mathcal{Q} -functional calculus it is possible to prove a generalized product rule.

Theorem 8.4 *Let $T \in \mathcal{BC}(X)$ and assume $f \in N(\sigma_S(T))$ and $g \in SH_L(\sigma_S(T))$ then*

$$2[\mathcal{D}((\cdot)f g)(T) - \bar{T} \mathcal{D}(f g)(T)] = f(T) \mathcal{D}((\cdot)g)(T) - f(T) \bar{T} \mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T) g(T) - \mathcal{D}(f)(T) \bar{T} g(T). \tag{8.2}$$

Proof Let G_1 and G_2 be two bounded slice Cauchy domains such that contain $\sigma_S(T)$ and $\bar{G}_1 \subset G_2$ and $\bar{G}_2 \subset \text{dom}(f) \cap \text{dom}(g)$. We choose $p \in \partial(G_1 \cap \mathbb{C}_J)$ and $s \in \partial(G_2 \cap \mathbb{C}_J)$. By Definitions 2.23 and 5.7 for $J \in \mathbb{S}$ we have

$$\begin{aligned} & f(T) \mathcal{D}((\cdot)g)(T) - f(T) \bar{T} \mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T) g(T) - \mathcal{D}(f)(T) \bar{T} g(T) \\ &= -\frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, T) \frac{1}{\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} p dp_J g(p) \\ &+ \frac{1}{2\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, T) \frac{\bar{T}}{\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J g(p) \\ &- \frac{1}{\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} s ds_J f(s) \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p) \\ &+ \frac{1}{\pi} \int_{\partial(G_2 \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s) \frac{\bar{T}}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p). \end{aligned}$$

Since the function f is intrinsic by Lemma 8.3 we get

$$\begin{aligned}
 & f(T)\mathcal{D}((\cdot)g)(T) - f(T)\bar{T}\mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T)g(T) - \mathcal{D}(f)(T)\bar{T}g(T) \\
 &= -\frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} p dp_J g(p) \\
 &\quad + \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J S_R^{-1}(s, T)\bar{T} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J g(p) \\
 &\quad - \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J s \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p) \\
 &\quad + \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-1} \bar{T} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p) \\
 &= \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[-S_R^{-1}(s, T) \mathcal{Q}_{c,p}(T)^{-1} p \right. \\
 &\quad \left. + S_R^{-1}(s, T)\bar{T} \mathcal{Q}_{c,p}(T)^{-1} \right. \\
 &\quad \left. - s \mathcal{Q}_{c,s}(T)^{-1} S_L^{-1}(p, T) + \mathcal{Q}_{c,s}(T)^{-1} \bar{T} S_L^{-1}(p, T) \right] dp_J g(p) \\
 &= \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[-S_R^{-1}(s, T)(p\mathcal{I} - \bar{T}) \mathcal{Q}_{c,p}(T)^{-1} \right. \\
 &\quad \left. + \mathcal{Q}_{c,s}(T)^{-1}(\bar{T} - s\mathcal{I}) S_L^{-1}(p, T) \right] dp_J g(p) \\
 &= \frac{1}{2\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[-S_R^{-1}(s, T) S_L^{-1}(p, T) \right. \\
 &\quad \left. - S_R^{-1}(s, T) S_L^{-1}(p, T) \right] dp_J g(p) \\
 &= -\frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[S_R^{-1}(s, T) S_L^{-1}(p, T) \right] dp_J g(p).
 \end{aligned}$$

By the S -resolvent equation (see (2.6)) and by setting $\mathcal{Q}_s(p) := p^2 - 2s_0p + |s|^2$, we get

$$\begin{aligned}
 & f(T)\mathcal{D}((\cdot)g)(T) - f(T)\bar{T}\mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T)g(T) - \mathcal{D}(f)(T)\bar{T}g(T) \\
 &= -\frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[(S_R^{-1}(s, T) - S_L^{-1}(p, T))p - \bar{s}(S_R^{-1}(s, T) \right. \\
 &\quad \left. - S_L^{-1}(p, T)) \right] \cdot \mathcal{Q}_s(p)^{-1} dp_J g(p) \\
 &= -\frac{1}{\pi^2} \left[\int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} S_R^{-1}(s, T)p \mathcal{Q}_s(p)^{-1} dp_J g(p) + \right. \\
 &\quad \left. - \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T)p \mathcal{Q}_s(p)^{-1} dp_J g(p) \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} S_R^{-1}(s, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) \\
 & + \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} S_L^{-1}(p, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) \Big].
 \end{aligned}$$

From the Cauchy formula we obtain

$$\begin{aligned}
 & \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} S_R^{-1}(s, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) = 0, \\
 & \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} S_R^{-1}(s, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) = 0.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & f(T) \mathcal{D}((\cdot)g)(T) - f(T) \bar{T} \mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T)g(T) - \mathcal{D}(f)(T) \bar{T} g(T) \\
 & = -\frac{1}{\pi^2} \left[- \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) \right. \\
 & \quad \left. + \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} S_L^{-1}(p, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) \right] \\
 & = -\frac{1}{\pi^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \left[\bar{s} S_L^{-1}(p, T) - S_L^{-1}(p, T) p \right] \\
 & \quad \mathcal{Q}_s(p)^{-1} dp_J g(p).
 \end{aligned}$$

Using Lemma 7.1 with $B := S_L^{-1}(p, T)$ we get

$$\begin{aligned}
 & f(T) \mathcal{D}((\cdot)g)(T) - f(T) \bar{T} \mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T)g(T) - \mathcal{D}(f)(T) \bar{T} g(T) \\
 & = -\frac{2}{\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J f(p)g(p).
 \end{aligned}$$

Finally by definition of the S -resolvent operator we obtain

$$\begin{aligned}
 & f(T) \mathcal{D}((\cdot)g)(T) - f(T) \bar{T} \mathcal{D}(g)(T) + \mathcal{D}(f(\cdot))(T)g(T) - \mathcal{D}(f)(T) \bar{T} g(T) \\
 & = -\frac{2}{\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} (p\bar{I} - \bar{T}) \mathcal{Q}_{c,p}(T)^{-1} dp_J f(p)g(p) \\
 & = -\frac{2}{\pi} \left(\int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p f(p)g(p) - \bar{T} \right. \\
 & \quad \left. \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J f(p)g(p) \right) \\
 & = -2 \left[\bar{T} \mathcal{D}(fg)(T) - \mathcal{D}((\cdot)fg)(T) \right].
 \end{aligned}$$

□

9 The Q -functional Calculus and New Properties of the F -functional Calculus

The introduction of the Q -functional calculus is essential to prove a product formula for the F -functional calculus. Before to go through this, we prove the following property of the F -functional calculus.

Theorem 9.1 *Let us consider $T \in \mathcal{BC}(X)$ and $m \in \mathbb{N}_0$. Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$. For every $J \in \mathbb{S}$, we have*

$$Q_m(T, \bar{T}) = -\frac{1}{4\pi(m+1)(m+2)} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \tag{9.1}$$

and

$$Q_m(T, \bar{T}) = -\frac{1}{4\pi(m+1)(m+2)} \int_{\partial(U \cap \mathbb{C}_J)} s^{m+2} ds_J F_R(s, T), \tag{9.2}$$

where

$$Q_m(T, \bar{T}) = \sum_{j=0}^m \frac{2(m-j+1)}{(m+1)(m+2)} T^{m-j} \bar{T}^j.$$

Proof We start by considering the set U as the ball $B_r(0)$ with $\|T\| < r$. Then, by [16, Thm. 3.9] we know that $F_L(s, T) = \sum_{n=2}^{\infty} -2(n-1)nQ_{n-2}(T, \bar{T})s^{-1-n}$ for every $s \in \partial B_r(0)$. This series converges uniformly on $\partial B_r(0)$. Thus we have

$$\begin{aligned} & -\frac{1}{4\pi(m+1)(m+2)} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \\ &= \frac{1}{2\pi(m+1)(m+2)} \sum_{n=0}^{+\infty} (n+1)(n+2)Q_n(T, \bar{T}) \int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-1-n+m} ds_J. \end{aligned}$$

Due to the fact that

$$\int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-1-n+m} ds_J = \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m, \end{cases}$$

we obtain

$$-\frac{1}{4\pi(m+1)(m+2)} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} = Q_m(T, \bar{T}).$$

Now, we consider U an arbitrary bounded slice Cauchy domain that contains $\sigma_S(T)$. We suppose that there exists a radius r such that $\bar{U} \subset B_r(0)$. The left F -resolvent operator $F_L(s, T)$ is right slice hyperholomorphic in the variable s and the monomial

s^{m+2} is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0) \setminus U$. By the Cauchy’s integral theorem (see Theorem 2.13) we get

$$\begin{aligned} & - \frac{1}{4\pi(m+1)(m+2)} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \\ & + \frac{1}{4\pi(m+1)(m+2)} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \\ & = - \frac{1}{4\pi(m+1)(m+2)} \int_{\partial((B_r(0) \setminus U) \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} = 0. \end{aligned}$$

This implies that

$$\begin{aligned} & - \frac{1}{4\pi(m+1)(m+2)} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \\ & = - \frac{1}{4\pi(m+1)(m+2)} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} F_L(s, T) ds_J s^{m+2} \\ & = \mathcal{Q}_m(T, \bar{T}). \end{aligned}$$

□

Remark 9.2 Theorem 9.1 is consistent with respect to the definition of the F -functional calculus. Indeed, the results of the integrals (9.1) and (9.2) are Fueter regular polynomials in T . That kind of polynomials were introduced in [9, 10].

Now, we prove a product rule for the F -function calculus.

Theorem 9.3 *Let $T \in \mathcal{BC}(X)$ and assume $f \in N(\sigma_S(T))$ and $g \in SH_L(\sigma_S(T))$ then we have*

$$\Delta(fg)(T) = \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T). \tag{9.3}$$

where \mathcal{D} is the Fueter operator.

Proof Let G_1 and G_2 be two bounded slice Cauchy domains such that contain $\sigma_S(T)$ and $\bar{G}_1 \subset G_2$, with $\bar{G}_2 \subset \text{dom}(f) \cap \text{dom}(g)$. We choose $p \in \partial(G_1 \cap \mathbb{C}_J)$ and $s \in \partial(G_2 \cap \mathbb{C}_J)$. For every $J \in \mathbb{S}$, from the definitions of F -functional calculus, S -functional calculus and Q - functional calculus, we get

$$\begin{aligned} & \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\ & = \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J F_R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p) \\ & + \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J g(p) \\ & - \frac{1}{(\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s) \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J g(p). \end{aligned}$$

Since the function f is intrinsic by Lemma 8.3 we have

$$\begin{aligned} &\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\ &= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J F_R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p, T) dp_J g(p) \\ &\quad + \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J g(p) \\ &\quad - \frac{1}{(\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J g(p). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\ &= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} \int_{\partial(G_1 \cap \mathbb{C}_J)} f(s) ds_J [F_R(s, T)S_L^{-1}(p, T) + \\ &\quad + S_R^{-1}(s, T)F_L(p, T) - 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}] dp_J g(p). \end{aligned}$$

By the following equation (see [20, Lemma 7.3.2])

$$\begin{aligned} &F_R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)F_L(p, T) - 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} \\ &= [(F_R(s, T) - F_L(p, T))p - \bar{s}(F_R(s, T) - F_L(p, T))] \mathcal{Q}_s(p)^{-1}, \end{aligned}$$

where $\mathcal{Q}_s(p) = p^2 - 2s_0p + |s|^2$, we obtain

$$\begin{aligned} &\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\ &= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} \int_{\partial(G_1 \cap \mathbb{C}_J)} f(s) [(F_R(s, T) - F_L(p, T))p \\ &\quad - \bar{s}(F_R(s, T) - F_L(p, T))] \mathcal{Q}_s(p)^{-1} dp_J g(p). \end{aligned}$$

By the linearity of the integrals follows that

$$\begin{aligned} &\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\ &= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} F_R(s, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) \\ &\quad - \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) \\ &\quad - \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} F_R(s, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) \\ &\quad + \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} F_L(p, T) \mathcal{Q}_s(p)^{-1} dp_J g(p). \end{aligned}$$

Since the functions $p \mapsto p\mathcal{Q}_s(p)^{-1}, p \mapsto \mathcal{Q}_s(p)^{-1}$ are intrinsic slice hyperholomorphic on \bar{G}_1 , by the Cauchy integral formula, see Theorem 2.13 we have

$$\begin{aligned} \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} F_R(s, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) &= 0, \\ \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} F_R(s, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) &= 0. \end{aligned}$$

Thus, we get

$$\begin{aligned} \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) &= -\frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) p \mathcal{Q}_s(p)^{-1} dp_J g(p) + \\ &+ \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \bar{s} F_L(p, T) \mathcal{Q}_s(p)^{-1} dp_J g(p) \\ &= \frac{1}{(2\pi)^2} \int_{\partial(G_2 \cap \mathbb{C}_J)} f(s) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} [\bar{s} F_L(p, T) \\ &- F_L(p, T) p] \mathcal{Q}_s(p)^{-1} dp_J g(p). \end{aligned}$$

By applying Lemma 7.1 with $B := F_L(p, T)$ and by the definition of the F -functional calculus we obtain

$$\begin{aligned} \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) &= \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J f(p)g(p) \\ &= \frac{1}{2\pi} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_L(p, T) dp_J (fg)(p) = \Delta(fg)(T). \end{aligned}$$

□

Corollary 9.4 *Let $T \in \mathcal{BC}(X)$ and assume $g \in N(\sigma_S(T))$ and $f \in SH_R(\sigma_S(T))$ then we have*

$$\Delta(fg)(T) = \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T). \tag{9.4}$$

Remark 9.5 The classical formula

$$\Delta(fg) = \Delta(f) \cdot g + f \cdot \Delta(g) + 2\langle \nabla f, \nabla g \rangle, \tag{9.5}$$

is true for any \mathbb{C}^2 quaternionic valued functions and it inspires formula (9.3). However, formula (9.3) is true only for slice hyperholomorphic functions. Indeed its proof relies heavily on the slice hyperholomorphic Cauchy integral representation formula. Thus (9.3) is not applicable in the case when f and g are real valued functions and, in this sense, it is not a generalization of the formula (9.5).

Remark 9.6 The product fg in Theorem 9.3 and Corollary 9.4 is respectively slice hyperholomorphic left or right slice hyperholomorphic.

Remark 9.7 Formula (9.3) is a general case of the well-known formula $\Delta(qg(q)) = q\Delta(g(q)) + 2\mathcal{D}(g(q))$. Indeed, it is enough to replace the operator T by q and to take $f(q) := q$ in formula (9.3).

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