2–point Markov evolutions

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Abstract

We study the Markov evolutions associated to the expected Markov processes.

Contents

1 Introduction

In classical (or expected quantum see ([9])) Markov processes, conditioning of the future on the past is equivalent to conditioning on the present. From this one deduces that these processes are canonically associated to Markov evolutions, i.e. completely positive identity preserving maps
$P(t,s)\colon \mathcal{A}_t \to \mathcal{A}_s$ satisfying $P(s,r)P(t,s) = P(t,r)$ $(r < s < t)$, where $\mathcal{A}_r$ $(r \in \mathbb{R})$ is the algebra at time r of the process. Conversely the Markov evolution $(P(t,s))_{s<t}$ allows to reconstruct all the conditional expectations of the process.
Restricted to 1 dimension, the Nelson Markov property (NMP) (see Subsection 3.4) states that if $s' < s < t < t'$, conditioning the algebra $\mathcal{A}_{(s,t)}$ on the algebra $\mathcal{A}_{(s,t)^c}$ ($(s,t)^c :=$ complement of (s,t)) (*in–out* conditioning) is the same as conditioning on $\mathcal{A}_{\{s',t'\}}$ - the algebra associated to the boundary of $(s',t')^c$). The *out–in* conditioning is the symmetric one, namely conditioning $\mathcal{A}_{(s,t)^c}$ on $\mathcal{A}_{(s,t)}$.
Since both the in–out and out–in conditioning can also be defined for usual Markov processes, it is natural to compare the result of these conditioning with the corresponding NMP. It turns out that, for the out–in conditioning, usual Markov processes satisfy the NMP *in a stronger form* (see Theorem 1). However the converse is not true, i.e. the NMP does not imply the usual Markov property.
This suggests the idea that, for processes satisfing the NMP, the Markov evolution $(P(t,s))_{s<t}$ should be replaced by a 2–**parameter Markov evolution** $P_{(t',s'),(t,s)}\colon \mathcal{A}_{(s,t)} \to \mathcal{A}_{(s',t')}$ satisfying

$$P_{(t'',s''),(t',s')}P_{(t',s'),(t,s)} = P_{(t'',s''),(t,s)} \quad , \quad s'' < s' < s < t < t' < t'' \qquad (1.1)$$

and that this evolution should allow to reconstruct the process. In Theorem 2 it is proved that this is indeed the case whenever the local σ–algebras are generated by the σ–algebras at fixed times.
This shows that the structure of the special class of Nelson Markov processes, discussed in Theorem 2, is fully analogous to the structure of usual non–time–homogeneous Markov processes with the only difference that usual Markov evolutions are replaced by 2–parameter Markov evolutions of the form (1.1). This raises the problem to characterize the generators of these evolutions. A natural conjecture is that it should be similar to the structure of Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generators, but this is an open problem.

In the case of usual (i.e. 1–parameter) evolution the condition of time–homogeneity, i.e. $P_{t+r,s+r} = P_{t,s}$ $(0 \leq s \leq t,\ r \in \mathbb{R}_+)$, reduces the 1–parameter evolution $(P_{t,s})_{s\leq t}$ becomes a 1–parameter Markov semi–group. One would expect a similar behaviour from the 2–parameter Markov processes. To formulate more precisely this conjecture, one has to define a notion of time–homogeneity for such processes. This is done in Section 3.4.2. In Section 3.5 we prove that, under the 2–parameter time–homogeneity condition, the evolution (1.1) takes the form

$$\hat{P}_{\tau_M}(-t', s')\hat{P}_{\tau_M}(-t, s) = \hat{P}_{\tau_M}(-(t'+t), -(t-s')) \tag{1.2}$$

where $\hat{P}_{\tau_M}$ is defined by (3.56) and the variables t, s, t', s' are subject to the constraints $s, t, t' > 0$, $s' > s + t$ (see (3.54), (3.55) below). It is clear that equation (1.2) is reminiscent, but definitely different from the 2–parameter Markov semi–group equation

$$P(a', b')P(a, b) = P(a' + a, b' + b) \quad ; \quad a, b, a', b' \in \mathbb{R}_+ \tag{1.3}$$

In particular, the constraint $s' > s + t$ shows that we are still in the domain of evolution equations and not of semi–group laws.
Notice that a two–parameter family $(P(a, b))$ satisfying (1.3) has the form

$$P(a, b) \overset{a'=b=0}{=} P(0 + a, b' + 0) \overset{(1.3)}{=} P(0, b')P(a, 0)$$

$$P(a', b) \overset{a=b'=0}{=} P(a' + 0, 0 + b) \overset{(1.3)}{=} P(a', 0)P(b, 0)$$

and, since a, b, a', b' are arbitrary, the above identities imply

$$P(0, b)P(a, 0) = P(a, 0)P(b, 0)$$

i.e. $P(a, b)$ is a product of two commuting one–parameter Markov semi–groups. There is no such factorization for general 2–parameter Markov evolution of the form (1.1). However, if in such evolution one chooses (as always possible) $s'' = s' = s$, (1.1) becomes

$$P_{(t'',s),(t',s)}P_{(t',s),(t,s)} = P_{(t'',s),(t,s)} \quad , \quad s < t < t' < t'' \tag{1.4}$$

which, introducing the notation

$$P_{2;s}(t', t) := P_{(t',s),(t,s)} \tag{1.5}$$

is reduced to the usual 1–parameter Markov evolution

$$P_{2;s}(t'', t')P_{2;s}(t', t) = P_{2;s}(t'', t) \quad , \quad s < t < t' < t'' \tag{1.6}$$

This suggests a condition of time shift–covariance different from the one used in this paper, which is the usual one for stochastic processes, namely: the fixed time algebras are all isomorphic to a single algebra $\mathcal{B}$ and there exists a Markov semi–group $Q^t : \mathcal{B} \to \mathcal{B}$ such that, for any choice of $s < t < t'$, $Q_2^{t-t'}$ is identified with $P_{2;s}(t', t)$. Moreover the same is required, possibly with a different semi–group $Q_1^{t-t'}$, for the evolutions $(P_{1;t}(s', s))$ obtained from (1.5) with the replacements $2 \to 1$, $s \to t$, $t \to s$, $t' \to s'$, $t'' \to s''$.
This would produce a 2–parameter Markov evolution with two naturally associated 1–parameter Markov evolutions and the problem to find the relations between the infinitesimal generators of these 2 evolutions is definitively of interest and will be discussed elsewhere. In Section 4 we show that 2–parameter Markov semi–groups arise naturally in quantum theory.

2 Projective families of Markov conditional expectations: 1–dimensional case

Definition 1 Let T be a set, $\mathcal{I}$ a family of sub–sets of T and $\mathcal{A}$ a $*$–algebra. A family $(\mathcal{A}_I)_{I\in\mathcal{I}}$ of sub–$*$–algebras of $\mathcal{A}$ satisfying

$$I \subseteq J \;\Rightarrow\; \mathcal{A}_I \subseteq \mathcal{A}_J \quad , \quad \forall I, J \in \mathcal{I}$$

is called a **localization** on $\mathcal{A}$ based on T (a W^*–localization (resp. C^*–localization) if the $\mathcal{A}_I$ are W^*–algebras (resp. C^*–algebras)). In this case, we also say that the pair $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{I}}\}$ is **a family of local algebras**.

Definition 2 A triple $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{I}}, \varphi\}$ where $\{\mathcal{A}, (\mathcal{A}_I)\}$ is a family of local algebras and φ is a state on $\mathcal{A}$ will be called a **stochastic process localized on** $\mathcal{I}$ (a **quantum stochastic process** if $\mathcal{A}$ is not abelian).

Let $\mathcal{I}$ be a family of Borel sub–sets of $\mathbb{R}$. In the following the notation $\{I^-, I', I^+\}$ means that $I^-, I', I^+ \in \mathcal{I}$ and $\sup I^- < \inf I' \le \sup I' < \inf I^+$. We will use the notations

$$E_{I'} \equiv E_{\mathbb{R},I'}$$

$$L^\infty(I) := L^\infty(\mathcal{F}_I) := L^\infty(\Omega, \mathcal{F}_I, P)$$

2.1 Translations on $\mathbb{R}$ ($\mathbb{R}_+$)

In the following we will discuss the case in which $T = \mathbb{R}$ or $\mathbb{R}_+$ and we shall assume that there is an action

$$t \in T \mapsto u_t \in \text{ End } (\mathcal{A})$$

of T on $\mathcal{A}$ by $*$–endomorphisms which satisfies:

$$u_t \mathcal{A}_I = \mathcal{A}_{I+t} \quad \text{(covariance)} \tag{2.1}$$

$$u_t \text{ has a left inverse denoted } u_t^* \quad (u_t^* \text{ is the inverse of } u_t \text{ if } T = \mathbb{R}) \tag{2.2}$$

$$u_t u_s = u_{t+s} \tag{2.3}$$

A projective family (E_I) of conditional expectations is called **covariant** if

$$u_t E_I = E_{I+t} u_t \tag{2.4}$$

this is equivalent to

$$u_t E_I u_t^* \Big|_{\mathcal{A}_{[t,+\infty)}} = E_{I+t} \Big|_{\mathcal{A}_{[t,+\infty)}} \tag{2.5}$$

3 Markovianity and semi–groups

Let $\{\mathcal{A}, (\mathcal{A}_I)_{I \in \mathcal{I}}\}$ be a family of local algebras localized on $\mathbb{R}$ (or $\mathbb{R}_+$) and let be given a projective family $E_I : \mathcal{A} \to \mathcal{A}_I$ $(I \in \mathcal{I})$ of conditional expectations. To avoid circumlocutions, we adopt the convention that, if $T = \mathbb{R}_+$, the symbols

$$E_{t]} := E_{(-\infty,t]} \quad , \quad E_{[t} := E_{[t,+\infty)}, \quad \mathcal{A}_{t]} := \mathcal{A}_{(-\infty,t]}, \quad \mathcal{A}_t := \mathcal{A}_{\{t\}} \; , \; \ldots$$

stand respectively for

$$E_{[0,t]}, \quad E_{[t} \quad , \quad \mathcal{A}_{[0,t]}, \ldots$$

Definition 3 The family (E_I) is said to be Markovian if $\forall\, t \in T$

$$E_{(-\infty,t]}(\mathcal{A}_{[t,+\infty)}) \subseteq \mathcal{A}_t \tag{3.1}$$

The properties of the conditional expectations easily imply that (3.1) is equivalent to

$$E_{(-\infty,t]}(a) = E_{\{t\}}(a) \; ; \qquad \forall\, a \in \mathcal{A}_{[t,+\infty)} \tag{3.2}$$

There are many equivalent ways of formulating the Markov property. The formulation (3.1) (and its multi–dimensional analogues, see [9]) underlines the locality aspect of the Markov property and is particularly well suited for the quantum generalization.

Proposition 1 In the above notations, let (E_I) be a projective, covariant, markovian family of conditional expectations, and define

$$P^t = E_{(-\infty,0]} u_t \Big|_{\mathcal{A}_0} \quad ; \quad t \geq 0 \tag{3.3}$$

then P^t is a 1–parameter, completely positive semi–group $\mathcal{A}_0 \to \mathcal{A}_0$ such that

$$P^t(1) = 1 \ ; \qquad t \geq 0 \tag{3.4}$$

Proof. P^t is positivity preserving and $P^t(1) = 1$ since $E_{(-\infty,0]}$ and u_t have this properties; because of the Markov property

$$P^t = E_{\{0\}} u_t$$

hence

$$P^t \mathcal{A}_0 \subseteq \mathcal{A}_0$$

and

$$P^t P^s = E_{\{0\}} u_t E_{\{0\}} u_s = E_{\{0\}} E_{\{t\}} u_{t+s} = E_{\{0\}} E_{(-\infty,t]} u_{t+s} = E_{\{0\}} u_{t+s} = P^{t+s}$$

hence P^t is a semi–group. □

A completely positive 1–parameter semi–group $P^t : \mathcal{A}_0 \to \mathcal{A}_0$ such that $P^t(1) = 1$, is called a **Markov semi–group** (on $\mathcal{A}_0$). The relation (3.1) can be called the **forward Markov property** (the past conditioning the future). The **backward Markov property** (the future conditioning the past) is expressed by

$$E_{[t,+\infty)}(\mathcal{A}_{(-\infty,t]}) \subseteq \mathcal{A}_t \tag{3.5}$$

Reasoning as in the proof of Proposition 1 one verifies that, if (E_I) is backward Markovian, covariant, projective, then

$$P^t = u_t^* E_{[t,+\infty)} \Big|_{\mathcal{A}_0} ; \ t \geq 0 \tag{3.6}$$

is a Markov semi–group on $\mathcal{A}_0$.

If $T = \mathbb{R}$ and the system $\{\mathcal{A}, (\mathcal{A}_I), (E_I)(u_t)\}$ admits a reflection, then it is easy to verify that the two definitions coincide.

Remark 1. The proof of the semi–group property makes use only of covariance and projectivity, and the fact that P^t maps $\mathcal{A}_0$ into itself follows from the Markov property. Thus the construction above holds for any covariant, projective, markovian, normalized family $(E_{(-\infty,t]})$ of completely positive maps.

Remark 2. The relation (3.3) shows the deep connection between stationary (or, more generally, covariant) Markov processes and the theory of unitary dilations of semi–groups. We refer to [12] for a discussion of this topic and bibliographic references.

3.1 Semi–groups and markovianity

In section 3 we have seen that every covariant stochastic process, as defined in section 2.1, determines a Markov semi–group P^t. If the process has an initial distribution w_0 (resp. is stationary with invariant distribution w_0), then the pair $\{w_0, P^t\}$ uniquely determines the stochastic equivalence class of the process. It is important to note that the equivalence class of the process is meant here with respect to the localization given by the **finite subsets** of the index set $T \subseteq \mathbb{R}$. Without this clarification the above assertion is in general false (this is the case, for example, for Markov fields — i.e. generalized processes — on the real line, for which the natural equivalence relation is not based on the finite subsets of $\mathbb{R}$ but on the open intervals).
In the following we shall use the term **process** to imply that the localization is based on the finite subsets if the set of indices, and the term **field** for the more general situation.
There is a well known procedure which allows to associate a stochastic process (resp. stationary stochastic process) with initial (resp. stationary) distribution w_0, uniquely determined up to equivalence, to a pair $\{w_0, P^t\}$, where w_0 is a probability distribution on a measurable space $(S, \mathcal{B})$, and P^t is a Markov semi–group acting on some subspace of $L^\infty(S, \mathcal{B})$ with appropriate continuity properties (cf. [10], [11], for example). The equivalence class of the process, i.e. the joint expectations, are determined by:

$$\mu_{0,t_1,\ldots,t_n}((f_0 \circ x_0) \cdot (f_1 \circ X_{t_1}) \cdot \cdots \cdot (f_n \circ X_{t_n})) =$$
$$= w_0(f_0 \cdot [P^{t_1}[f_1 \cdot [P^{t_2-t_1} \cdot \cdots \cdot [P^{t_n-t_{n-1}} f_{t_n}]]] \ldots])$$

where $f_0, \ldots, f_n \in L^\infty(S, \mathcal{B})$, (x_t) are the random variables of the process, $0 < t_1 < \cdots < t_n$, $n \in \mathbb{R}$ and the dot denotes point–wise multiplication.
Thus all classical covariant Markov process are determined up to the initial (resp. stationary) distribution and up to stochastic equivalence, by a Markov semi–group. As shown in [2], [3], [4], the situation in the quantum case is more delicate; in particular, the extrapolation of the above assertion to the

quantum case is wrong.

3.2 The Nelson Markov property

From now on we only discuss classical stochastic processes, i.e. the algebra $\mathcal{A}$ is commutative.

Definition 4 Let $\mathcal{I}$ be a Boolean algebra of subsets of $\mathbb{R}$ containing all of intervals and their boundaries. A projective family of conditional expectations

$$E_{G,F} : \mathcal{A}_G \to \mathcal{A}_F \quad ; \quad F \subset G; \quad F, G \in \mathcal{I} \tag{3.7}$$

is said to enjoy the **in–out Nelson Markov property (in–out NMP) with respect to the interval localization**, if for any open interval (r, t), one has:

$$E_{(r,t)^c}(f_{[r,t]}) = E_{\{r,t\}}(f_{[r,t]}) = E_{\partial(r,t)}(f_{[r,t]}) \tag{3.8}$$

The family (3.7) is said to enjoy the **out–in Nelson Markov property (out–in NMP)** with respect to the same localization if, in the above notations

$$E_{[r,t]}(f_{(r,t)^c}) = E_{\{r,t\}}(f_{(r,t)^c}) = E_{\partial(r,t)}(f_{(r,t)^c}) \tag{3.9}$$

Remark. The standard formulation of the NMP is:

$$E_{S^c}(f_{\overline{S}}) = E_{\partial S}(f_{\overline{S}}) \tag{3.10}$$

where S is a bounded open set in $\mathbb{R}^d$.
In the general case there is an important difference between the NMP for bounded or unbounded sets. The term **global Markov property** was coined to denote the NMP when S is an open half–space and the extension from the Markov property to the global Markov property turned out to be a non–trivial problem. In [7] it was shown that the root of the problem was in the definition of the σ–algebras associated to an half–space and that, defining it as the intersection of the σ–algebras associated to the open sets **with bounded boundary** whose complement contains S, one can naturally deduce the global Markov property from the local one.

Remark. Even if under very general conditions, the in–out and the out–in Markov properties are equivalent, their probabilistic and physical interpretations are **quite different**.
In the **statistical mechanics** interpretation one thinks of a gas localized on the points of a 1–dimensional space, identified with $\mathbb{R}$ and interpreted

as space, and the points of the sample space are its configurations. In this case the **in–out** conditioning means that the configuration inside the finite volume $[s, t]$ (up to boundary) is conditioned by the configuration outside the same volume (on the boundary in the Markov case).
The **out–in** conditioning means that the configuration outside the volume $[s, t]$ is conditioned by the configuration inside the same volume (up to boundary).

In the **probabilistic** (or open system) interpretation one thinks of the evolution of a system in time, identified with a one–sided lattice, typically $\mathbb{R}_+^* \equiv (0, +\infty)$ or a subset of it and the points of the sample space are its trajectories. The point 0 is excluded because $\mathcal{A}_0$ is usually introduced as an algebra algebraically independent of $\mathcal{A}_{(0,+\infty)}$ in the sense that $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_{(0,+\infty)}$ and $\mathcal{A}_0$ interpreted as algebra of observables of the system in the open system interpretation and as as algebra of the initial data in the probabilistic interpretation. In this case the finite volume $(0, t]$ is the **past** and it conditions the future (the outside). Its boundary $\{t\}$ is the **present**. This justifies the term **in–out** conditioning which becomes **backward** in the probabilistic interpretation. The term **forward** used for the time reversed process. In this case the algebra $\mathcal{A}_0$ is often taken to be 1–dimensional, $\mathcal{A}_0 = \mathbb{C} \equiv \mathbb{C}\chi_{\{x\}}$, corresponding to the fact that the state of the system at time 0 is the δ–function at $x \in \mathbb{R}$.

3.3 Deduction of the out–in Nelson Markov property from standard Markovianity in the 1–dimensional case

We have seen that the general Nelson Markov property (3.10) in the 1–dimensional case and with the choice $S = [r, t]$ (so that $\partial S = \{r, t\}$), becomes (3.9). In this section we deal with the 1–dimensional case and prove that the **usual Markov property implies a stronger version of the out–in Nelson Markov property**. The idea of the proof is to use the fact that a classical processes is forward Markov if and only if it is backward Markov (see [9]), to project forward the functions localized in the **past** of a given set S and backward those localized in the **future** of S.
The in–out NMP is more delicate and will be discussed in section 3.4.
The **usual backward Markov property** is defined, for the projective family $(E_{t]}) \equiv (E_{(-\infty,t]})$, by

$$E_{t]}(f_{[t}) = E_{\{t\}}(f_{[t}) \in \mathcal{A}_{\{t\}} \tag{3.11}$$

and the **usual forward Markov property** is defined, for the projective family $(E_{[t})\equiv(E_{[t,+\infty)})$, by

$$E_{[t}(f_{t]})=E_{\{t\}}(f_{t]})\in\mathcal{A}_{\{t\}}\tag{3.12}$$

Lemma 1 Suppose that the families $(E_{t]})$ and $(E_{[s})$ satisfy conditions (3.12) and (3.11) respectively and that $\mathcal{A}$ is linearly generated by the products of the form

$$f_{s]}f_{[s,t]}f_{[t}\tag{3.13}$$

where $f_{s]}\in\mathcal{A}_{s]}$, $f_{[s,t]}\in\mathcal{A}_{[s,t]}$, $f_{[t}\in\mathcal{A}_{[t}$. Then for any $f_{s]},f_{[s,t]},f_{[t}$ as before, one has:

$$E_{[s,t]}(f_{s]}f_{[s,t]}f_{[t})=E_{\{s\}}(f_{s]})f_{[s,t]}E_{\{t\}}(f_{[t})\tag{3.14}$$

In particular

$$E_{[s,t]}(f_{s]}f_{[t})=E_{\{s\}}(f_{s]})E_{\{t\}}(f_{[t})\tag{3.15}$$

$$E_{[s,t]}(\mathcal{A}_{(s,t)^c})\subset\mathcal{A}_{\{s,t\}}=\mathcal{A}_{\partial[s,t]}\tag{3.16}$$

Proof. In the notations of the statement, one has

$$\begin{aligned}
&E_{[s,t]}(f_{s]}f_{[s,t]}f_{[t})\overset{projectivity}{=}E_{t]}(f_{s]}f_{[s,t]}f_{[t})=E_{[s,t]}(f_{s]}f_{[s,t]}E_{t]}(f_{[t}))\\
&\overset{(3.11)}{=}E_{[s,t]}(f_{s]}f_{[s,t]}E_{\{t\}}(f_{[t}))=E_{[s,t]}(f_{s]}f_{[s,t]})E_{\{t\}}(f_{[t})\\
&\overset{projectivity}{=}E_{[s,t]}E_{[s}(f_{s]}f_{[s,t]})E_{\{t\}}(f_{[t})=E_{[s,t]}(E_{[s}(f_{s]})f_{[s,t]})E_{\{t\}}(f_{[t})\\
&=E_{[s,t]}(E_{\{s\}}(f_{s]})f_{[s,t]})E_{\{t\}}(f_{[t})\overset{(3.12)}{=}E_{\{s\}}(f_{s]})E_{[s,t]}(f_{[s,t]})E_{\{t\}}(f_{[t})\\
&=E_{\{s\}}(f_{s]})f_{[s,t]}E_{\{t\}}(f_{[t})
\end{aligned}$$

□

Remark. If algebras localized on disjoint sets commute, (3.14) and (3.15) are equivalent.

Remark. Note that (3.15) is **strictly stronger** than (3.16) (i.e. (3.9)) even in the case one assumes that $\mathcal{A}_{\{s,t\}}$ is generated by $\mathcal{A}_{\{s\}}$ and $\mathcal{A}_{\{t\}}$. In fact, even in this case, an element of $\mathcal{A}_{\{s,t\}}$ will be a sum of products of the form $g_{\{s\}}g_{\{t\}}$ with $g_{\{s\}}\in\mathcal{A}_{\{s\}}$, $g_{\{t\}}\in\mathcal{A}_{\{t\}}$ or a limit thereof while the right hand side of (3.15) involves **a single product**. A similar totality argument can be repeated for the products of the form (3.13).

Corollary 1 Any classical Markov process satisfies the out–in NMP in the stronger form (3.14).

Proof. We know that, for a classical process $(E_{t]})$–Markovianity and $(E_{[s})$–Markovianity are equivalent and that linear combinations of products of the form (3.13) are dense in $\mathcal{A}=L^\infty(\Omega,\mathcal{F},P)$. The thesis then follows from Lemma 1. □

3.3.1 The out–in Markov property for finite point localizations

Let $\mathcal{I}_{fin}$ be the family of finite sub–sets of $\mathbb{R}$ with their natural order. We write equivalently

$$F = \{t_1, \dots, t_n\} = \{t_1 < \dots < t_n\} \in \mathcal{I}_{fin}$$

In many stochastic processes (but definitively not all) the main objects associated to the process (measure, conditional expectations, ...) are completely determined by their restrictions on the elements of the form $f_{t_1} \dots f_{t_n}$ with $f_{t_i} \in \mathcal{A}_{t_i}$ $(i = 1, \dots, n)$, $t_1 < \dots < t_n$, i.e. on the algebras localized on finite sets. We will use the notation

$$\mathcal{A}_{\{t_1,\dots,t_n\}} := \mathcal{A}_{t_1} \vee \dots \vee \mathcal{A}_{t_n} \tag{3.17}$$

Theorem 1 Let $(E_{G,F})$ be a family of projective Markovian conditional expectation localized on finite sets. Then, the two families of operators respectively defined for $s < t$ by:

$$E^-_{t,s} := E_{s]}\Big|_{\mathcal{A}_t} \ : \ \mathcal{A}_t \to \mathcal{A}_s \quad ; \quad s < t \tag{3.18}$$

$$E^+_{s,t} := E_{[t}\Big|_{\mathcal{A}_s} \ : \ \mathcal{A}_s \to \mathcal{A}_t \quad ; \quad s < t \tag{3.19}$$

satisfy the following relations:

$$E^-_{t,s} \text{ and } E^+_{s,t} \text{ are completely positive maps} \tag{3.20}$$

$$E^-_{t,s}(1) = E^+_{s,t}(1) = 1 \tag{3.21}$$

$$E^-_{s,r}E^-_{t,s} = E^-_{t,r} \qquad ; \qquad E^+_{s,t}E^+_{r,s} = E^+_{r,t} \ , \ r < s < t \tag{3.22}$$

Conversely, **if the local algebras are of tensor product type**, two families of completely positive operators $(E^-_{t,s})_{s<t}$, $(E^+_{s,t})_{s<t}$, satisfying (3.20), (3.21), (3.22), determine a unique family of projective conditional expectations $(E_{G,F})$ localized on finite sets through the equality:

$$E_{\{t_1,t_2,\dots,t_{k-1},t_k,\dots,t_{k+m},t_{k+m+1},\dots t_n\},\{t_k,\dots,t_{k+m}\}}\left(f_{t_1}\cdots f_{t_{k-1}}f_{t_k}\cdots f_{t_{k+m}}f_{t_{k+m+1}}\cdots f_{t_n}\right) \tag{3.23}$$

$$= E^+_{t_{k-1},t_k}\left(E^+_{t_{k-2},t_{k-1}}\left(\cdots E^+_{t_2,t_3}(E^+_{t_1,t_2}(f_{t_1})f_{t_2})f_{t_3}\cdots\right)f_{t_{k-1}}\right)f_{t_k}\cdots f_{t_{k+m}}\cdot$$

$$\cdot E^-_{t_{k+m+1},t_{k+m}}\left(f_{t_{k+m+1}}E^-_{t_{k+m+2},t_{k+m+1}}\cdots f_{t_{n-3}}E^-_{t_{n-2},t_{n-1}}(f_{t_{n-2}}E^-_{t_{n-1},t_n}(f_{t_n})\cdots)\right)$$

for any $t_1 < t_2 < \dots < t_{k-1} < t_k < \dots < t_{k+m} < t_{k+m+1} < \dots t_n$ and $f_{t_j} \in \mathcal{A}_{t_j}$ $(j = 1, \dots, n)$. Moreover the family constructed in this way satisfies the stronger version of the out–in NMP (3.15).

Proof. Using (3.14) with $[s,t] \to [t_k, t_{k+m}]$, $f_{s]} \to f_{t_1} \cdots f_{t_{k-1}}$, $f_{[s,t]} \to f_{t_k} \cdots f_{t_{k+m}}$ and $f_{[t} \to f_{t_{k+m+1}} \cdots f_{t_n}$ one obtains

$$\begin{aligned} & E_{[t_k,t_{k+m}]}(f_{t_1} \cdots f_{t_{k-1}} f_{t_k} \cdots f_{t_{k+m}} f_{t_{k+m+1}} \cdots f_{t_n}) \\ & = E_{\{t_k\}}(f_{t_1} \cdots f_{t_{k-1}}) f_{t_k} \cdots f_{t_{k+m}} E_{\{t_{k+m}\}}(f_{t_{k+m+1}} \cdots f_{t_n}) \end{aligned} \tag{3.24}$$

and, by the forward and backward Markov property the right hand side of (3.24) is equal to

$$E_{[t_k}(f_{t_1} \cdots f_{t_{k-1}}) f_{t_k} \cdots f_{t_{k+m}} E_{t_{k+m}]}(f_{t_{k+m+1}} \cdots f_{t_n})$$

$$= E_{[t_k} E_{[t_2}(f_{t_1} \cdots f_{t_{k-1}})\ f_{t_k} \cdots f_{t_{k+m}} E_{t_{k+m}]} E_{t_{n-1}]}(f_{t_{k+m+1}} \cdots f_{t_n})$$

$$= E_{[t_k}\left(E_{[t_2}(f_{t_1}) f_{t_2} \cdots f_{t_{k-1}}\right)\ f_{t_k} \cdots f_{t_{k+m}} E_{t_{k+m}]}\left(f_{t_{k+m+1}} \cdots f_{t_{n-1}} E_{t_{n-1}]}(f_{t_n})\right)$$

$$\begin{aligned} = E_{[t_k}\left(E_{[t_3}(E_{[t_2}(f_{t_1}) f_{t_2}) f_{t_3} \cdots f_{t_{k-1}}\right)\ f_{t_k} \cdots f_{t_{k+m}} \\ E_{t_{k+m}]}\left(f_{t_{k+m+1}} \cdots f_{t_{n-3}} E_{t_{n-2}]}(f_{t_{n-2}} E_{t_{n-1}]}(f_{t_n}))\right) \end{aligned}$$

$$\begin{aligned} = E_{[t_k}\left(E_{[t_{k-1}}\left(\cdots E_{[t_3}(E_{[t_2}(f_{t_1}) f_{t_2}) f_{t_3} \cdots\right) f_{t_{k-1}}\right)\ f_{t_k} \cdots f_{t_{k+m}} \\ E_{t_{k+m}]}\left(f_{t_{k+m+1}} E_{t_{k+m+2}]}\left(\cdots f_{t_{n-3}} E_{t_{n-2}]}(f_{t_{n-2}} E_{t_{n-1}]}(f_{t_n})) \cdots\right)\right) \end{aligned}$$

$$= E^+_{t_{k-1},t_k}\left(E^+_{t_{k-2},t_{k-1}}\left(\cdots E^+_{t_2,t_3}(E^+_{t_1,t_2}(f_{t_1}) f_{t_2}) f_{t_3} \cdots\right) f_{t_{k-1}}\right) f_{t_k} \cdots f_{t_{k+m}} \cdot$$

$$\cdot E^-_{t_{k+m+1},t_{k+m}}\left(f_{t_{k+m+1}} E^-_{t_{k+m+2},t_{k+m+1}}\left(\cdots f_{t_{n-3}} E^-_{t_{n-2},t_{n-1}}(f_{t_{n-2}} E^-_{t_{n-1},t_n}(f_{t_n})) \cdots\right)\right)$$

where the families $(E^-_{t,s})$ and $(E^+_{s,t})$ are defined respectively by (3.18) and (3.19). Finally conditon (3.19) follows from Markovianity, (3.21) is normalization of conditional expectations and (3.22) follows from projectivity.

Conversely, suppose that algebras localized on disjoint sets are of tensor product type and let be given two families of operators $(E^-_{t,s})_{s<t}$, $(E^+_{s,t})_{s<t}$, satisfying (3.20), (3.21), (3.22). Define

$$E_{\{S^-,S,S^+\},S} := E_{\{t_1,t_2,\ldots,t_{k-1},t_k,\ldots,t_{k+m},t_{k+m+1},\ldots t_n\},\{t_k,\ldots,t_{k+m}\}}$$

through the right hand side of (3.23).
Because of (3.20), (3.21) the families $(E^-_{t,s})_{s<t}$, $(E^+_{s,t})_{s<t}$ are conditional expectations, hence completely positive. Therefore, since algebras localized on disjoint sets commute and their intersection is $\mathbb{C} \cdot 1$, the right hand side of (3.23) defines a completely positive norm 1 projection onto $\mathcal{A}_S$, hence by Tomiyama's Theorem, a conditional expectation that, by construction, satisfies the out–in NMP. The projectivity of the family constructed in this way follows from (3.21) and (3.22). □

3.4 The in–out NMP: 1–dimensional case

We suppose that (3.13) is satisfied, i.e. that $\mathcal{A}$ is linearly generated by the products of the form

$$f_{s]}f_{[s,t]}f_{[t} \tag{3.25}$$

where $f_{s]} \in \mathcal{A}_{s]}$, $f_{[s,t]} \in \mathcal{A}_{[s,t]}$, $f_{[t} \in \mathcal{A}_{[t}$. This implies that, if $I_1, I_2, I_3 \subseteq \mathbb{R}$ are mutually disjoint intervals such that

$$\overline{I}_1 \cup \overline{I}_2 \cup \overline{I}_3 = \mathbb{R}$$

then

$$\mathcal{A}_{\overline{I}_1} \vee \mathcal{A}_{I_2} \vee \mathcal{A}_{I_3} = \mathcal{A}$$

Let $(E_{(s,t)^c})$ be a projective family of Umegaki conditional expectations satisfying the NMP. Then

$$E_{(s,t)^c}(f_{s]}f_{(s,t)}f_{[t)}) = f_{s]}E_{\{s,t\}}(f_{(s,t)})f_{[t}$$

Thus $E_{(s,t)^c}$ is uniquely determined by the expectation values of the form $E_{\{s,t\}}(f_{(s,t)})$.

3.4.1 The in–out NMP for finite point localizations

Suppose that

$$\mathcal{A} = \bigvee_{t\in\mathbb{R}} \mathcal{A}_t \tag{3.26}$$

In this case condition (3.25) is replaced by the requirement that $\mathcal{A}$ is linearly generated by the products of the form

$$f_{s_1}f_{s_2}\cdots f_{s_{n-1}}f_{s_n} \quad , \quad s < s_1 < \cdots < s_n < t \subset \mathbb{R} \ , \ f_{s_j} \in \mathcal{A}_{s_j} \ , \ \forall j \tag{3.27}$$

Theorem 2 Let $(E_I)_{I\in\mathcal{I}}$ be a projective family of Umegaki conditional expectations satisfying the in–out NMP. With the natural identification $\{s', s'\} \equiv \{s'\}$, denote, for $s < s' \leq t' < t$

$$\tilde{P}_{(s',t'),(s,t)} := E_{\{s,t\}}\Big|_{\mathcal{A}_{\{s',t'\}}} = E_{(s,t)^c}\Big|_{\mathcal{A}_{\{s',t'\}}} \ : \ \mathcal{A}_{\{s',t'\}} \to \mathcal{A}_{\{s,t\}} \tag{3.28}$$

The family $(\tilde{P}_{(s',t'),(s,t)})$ uniquely determines the projective family $(E_{(s,t)^c})$ through the identity

$$E_{(s,t)^c}(f_{s_1}f_{s_2}\cdots f_{s_{n-1}}f_{s_n}) = \tilde{P}_{(s_1,s_n),(s,t)}(f_{s_1}\tilde{P}_{(s_2,s_{n-1}),(s_1,s_n)}(f_{s_2}\cdot$$

$$\cdot\tilde{P}_{(s_3,s_{n-2}),(s_2,s_{n-1})}(\cdots\tilde{P}_{(s_p,s_{p+1}),(s_{p-1},s_{p+2})}(f_{s_p}f_{s_{p+1}})\cdots)f_{s_{n-1}})f_{s_n}) \tag{3.29}$$

if $n = 2p$ and, if $n = 2p+1$:

$$E_{(s,t)^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n}) = \tilde{P}_{(s_1,s_n),(s,t)}(f_{s_1} \tilde{P}_{(s_2,s_{n-1})(s_1,s_n)}(f_{s_2} \cdot$$

$$\cdot \tilde{P}_{(s_3,s_{n-2}),(s_2,s_{n-1})}(\cdots \tilde{P}_{(s_{p+1},s_{p+1}),(s_p,s_{p+2})}(f_{s_{p+1}}) \cdots) f_{s_{n-1}}) f_{s_n}) \tag{3.30}$$

for any $n \in \mathbb{N}$, $s < s_1 < \cdots < s_n < t$ and $f_{s_j} \in \mathcal{A}_{s_j}$ for $j \in \{1, \ldots, n\}$. Moreover:

(i) For each $s < s' < t' < t$, $\tilde{P}_{(s',t'),(s,t)} : \mathcal{A}_{\{s',t'\}} \to \mathcal{A}_{\{s,t\}}$ is a completely positive linear map.

(ii) Each $\tilde{P}_{(s',t'),(s,t)}$ is identity preserving:

$$\tilde{P}_{(s',t'),(s,t)}(1_{\{s',t'\}}) = 1_{\{s,t\}} \quad ; \quad \tilde{P}_{(s',s'),(s,t)}(1_{\{s'\}}) = 1_{\{s,t\}}$$

(iii) The family $(\tilde{P}_{(s',t'),(s,t)})$ is a 2–parameter evolution, i.e., for $s'' < s < s' < t' < t < t''$, one has

$$\tilde{P}_{(s,t),(s'',t'')} \tilde{P}_{(s',t'),(s,t)} = \tilde{P}_{(s',t'),(s'',t'')} \tag{3.31}$$

Conversely, **if the local algebras are of tensor product type**, a 2–parameter family $(\tilde{P}_{(s',t'),(s,t)})_{s<s'\leq t'<t}$ satisfying conditions (i), (ii), (iii) above uniquely defines a projective family of Umegaki conditional expectations

$$E_{(s,t)^c} : \mathcal{A} \equiv \mathcal{A}_{s]} \otimes \mathcal{A}_{(s,t)} \otimes \mathcal{A}_{[t} \to \mathcal{A}_{(s,t)^c} \equiv \mathcal{A}_{s]} \otimes 1_{(s,t)} \otimes \mathcal{A}_{[t}$$

which satisfies the in–out NMP.

Proof. For $s < s_1 < \cdots < s_n < t$, consider the expectation values of the form

$$\begin{aligned} & E_{(s_1,s_2,\ldots s_{n-1},s_n)^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n}) \\ & := E_{(s,t)^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n}) \\ & = E_{(s,t)^c}(f_{s_1} E_{(s_1,s_n)^c}(f_{s_2} \cdots f_{s_{n-1}}) f_{s_n}) \\ & = E_{(s,t)^c}(f_{s_1} E_{(s_1,s_n)^c}(f_{s_2} E_{(s_2,s_{n-1})^c}(f_{s_3} \cdots f_{s_{n-2}})) f_{s_{n-1}}) f_{s_n}) \\ & = E_{\{s,t\}}(f_{s_1} E_{\{s_1,s_n\}}(f_{s_2} E_{(s_2,s_{n-1})^c}(f_{s_3} \cdots f_{s_{n-2}})) f_{s_{n-1}}) f_{s_n}) \end{aligned} \tag{3.32}$$

Iterating this procedure one finds, if $n = 2p$:

$$E_{(s,t)^c}(f_{s_1} E_{(s_1,s_n)^c}(f_{s_2} E_{(s_2,s_{n-1})^c}(\cdots E_{(s_{p-1},s_{p+2})^c}(f_{s_p} f_{s_{p+1}}) \cdots) f_{s_{n-1}}) f_{s_n})$$

$$= E_{\{s,t\}}(f_{s_1} E_{\{s_1,s_n\}}(f_{s_2} E_{\{s_2,s_{n-1}\}}(\cdots E_{\{s_{p-1},s_{p+2}\}}(f_{s_p} f_{s_{p+1}}) \cdots) f_{s_{n-1}}) f_{s_n}) \tag{3.33}$$

If $n = 2p + 1$, writing

$$\begin{aligned} E_{(s,t)^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n}) &= E_{(s,t)^c}(1_s f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n}) \\ &= E_{(s,t)^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_n} 1_t) \end{aligned}$$

one can always suppose that, in (3.32), n is even. In the following we always consider the case of even n.

$$\begin{aligned} & E_{(s_1,s_2,\dots s_{n-1},s_{2n})^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_{2n}}) \\ &= E_{\{s,t\}}(f_{s_1} E_{\{s_1,s_{2n}\}}(f_{s_2} E_{\{s_2,s_{n-1}\}}(\cdots \\ &\qquad \cdots f_{s_{n-1}} E_{\{s_{n-1},s_{n+2}\}}(f_{s_n} f_{s_{n+1}}) f_{s_{n+2}} \cdots) f_{s_{2n-1}}) f_{s_{2n}}) \end{aligned} \tag{3.34}$$

Since $f_{s_{n-1}} E_{\{s_{n-1},s_{n+2}\}}(f_{s_n} f_{s_{n+1}}) f_{s_{n+2}} \in \mathcal{A}_{s_{n-1}} \vee \mathcal{A}_{s_{n+2}}$, the expression $E_{\{s_{n-2},s_{n+3}\}}(f_{s_{n-1}} E_{\{s_{n-1},s_{n+2}\}}(f_{s_n} f_{s_{n+1}}) f_{s_{n+2}})$ is well defined and one has

$$\begin{aligned} & E_{(s_1,s_2,\dots s_{n-1},s_{2n})^c}(f_{s_1} f_{s_2} \cdots f_{s_{n-1}} f_{s_{2n}}) \\ &= \tilde{P}_{(s_1,s_{2n}),(s,t)}(f_{s_1} \tilde{P}_{(s_2,s_{2n-1}),(s_1,s_{2n})}(f_{s_2}(\cdots \\ &\quad \cdots f_{s_{n-1}} \tilde{P}_{(s_n,s_{n+1}),(s_{n-1},s_{n+2})}(f_{s_n} f_{s_{n+1}}) f_{s_{n+2}} \cdots) f_{s_{2n-1}}) f_{s_{2n}}) \end{aligned} \tag{3.35}$$

where $\tilde{P}_{(s',t'),(s,t)}$ is defined by (3.28). Then (i) and (ii) hold because, due to (3.28), $\tilde{P}_{(s',t'),(s,t)}$ is obtained from $E_{\{s,t\}}$ by restriction to a sub-algebra with identity and any such restriction of a completely positive identity preserving map enjoys these properties. Moreover the NMP and projectivity imply that, for each $g_{s',t'} \in \mathcal{A}_{\{s',t'\}}$, one has

$$\begin{aligned} \tilde{P}_{(s,t),(s'',t'')} \tilde{P}_{(s',t'),(s,t)}(g_{s',t'}) &= E_{\{s'',t''\}} E_{\{s,t\}}(g_{s',t'}) \\ &= E_{(s'',t'')^c} E_{(s,t)^c}(g_{s',t'}) \\ &= E_{(s'',t'')^c}(g_{s',t'}) \\ &= E_{\{s'',t''\}}(g_{s',t'}) \\ &= \tilde{P}_{(s',t'),(s'',t'')}(g_{s',t'}) \end{aligned}$$

i.e. (iii) holds.

Conversely let be given a 2–parameter evolution

$$\tilde{P}_{(s',t'),(s,t)} : \mathcal{A}_{\{s',t'\}} \to \mathcal{A}_{\{s,t\}} \quad ; \quad s < s' \le t' < t$$

of completely positive identity preserving maps. By assumption

$$\mathcal{A}_{s_1,\dots,s_{2n}} \sim \mathcal{A}_{s_1} \otimes \cdots \otimes \mathcal{A}_{s_{2n}}$$

$$\sim (\mathcal{A}_{s_1} \otimes \mathcal{A}_{s_n}) \otimes (\mathcal{A}_{s_2} \otimes \mathcal{A}_{s_{n-1}}) \otimes (\mathcal{A}_{s_3} \otimes \mathcal{A}_{s_{n-2}}) \otimes \cdots \otimes (\mathcal{A}_{s_n} \otimes \mathcal{A}_{s_{n+1}})$$

$$\sim \bigotimes_{j=0}^{n} (\mathcal{A}_{s_j} \otimes \mathcal{A}_{s_{n-j+1}})$$

By condition (i) the map

$$E_{\{s<s_1<\cdots<s_n<t\},\{s,t\}} : \mathcal{A}_{s)} \otimes \mathcal{A}_{\{s<s_1<\cdots<s_n<t\}} \otimes \mathcal{A}_{(t} \to \mathcal{A}_{(s,t)^c}$$

defined by

$$\begin{aligned} &E_{\{s<s_1<\cdots<s_n<t\},\{s,t\}}(f_{s]} \otimes f_{s_1} \otimes f_{s_2} \otimes \cdots \otimes f_{s_{n-1}} \otimes f_{s_n} \otimes f_{[t)}) \\ &:= f_{s]} \tilde{P}_{(s_1,s_n),(s,t)}(f_{s_1} \tilde{P}_{(s_2,s_{n-1}),(s_1,s_n)}(f_{s_2} \cdot \\ &\cdot \tilde{P}_{(s_3,s_{n-2}),(s_2,s_{n-1})}(\cdots \tilde{P}_{(s_p,s_{p+1}),(s_{p-1},s_{p+2})}(f_{s_p} f_{s_{p+1}}) \cdots) f_{s_{n-1}}) f_{s_n}) f_{[t} \end{aligned}$$

if $n = 2p$ and, if $n = 2p + 1$:

$$\begin{aligned} &E_{\{s<s_1<\cdots<s_n<t\},\{s,t\}}(f_{s]} \otimes f_{s_1} \otimes f_{s_2} \otimes \cdots \otimes f_{s_{n-1}} \otimes f_{s_n} \otimes f_{[t)}) \\ &:= f_{s]} \tilde{P}_{(s_1,s_n),(s,t)}(f_{s_1} \tilde{P}_{(s_2,s_{n-1})(s_1,s_n)}(f_{s_2} \cdot \\ &\cdot \tilde{P}_{(s_3,s_{n-2}),(s_2,s_{n-1})}(\cdots \tilde{P}_{(s_{p+1},s'_{p+1}),(s_p,s_{p+2})}(f_{s_{p+1}}) \cdots) f_{s_{n-1}}) f_{s_n}) f_{[t} \end{aligned}$$

is completely positive.

For any $s < t$, the family $\mathcal{F}_{(s,t),fin,even}$, of finite subsets of (s,t) of the form $\{s < s_1 < \cdots < s_{2n} < t\}$ is an increasing net for the partial order defined by:
$\{s < s_1 < \cdots < s_{2n} < t\} \prec \{s < s'_1 < \cdots < s'_{2p} < t\}$ if and only if
(i) $S := \{s_1, \ldots, s_{2n}\} \subseteq S' := \{s'_1, \ldots, s'_{2p}\}$,
(ii) for each $j \in \{0, \ldots, 2n+1\}$, denoting $s_0 := s$ and $s_{2n+1} := t$, there is an even number of elements of $S' \setminus S$ between s_j and s_{j+1}.
By definition of local algebras, one has

$$\mathcal{A}_{(s,t)} = \bigcup_{S \in \mathcal{F}_{(s,t)}} \mathcal{A}_S$$

Using conditions (ii) and (iii) one verifies that the family of maps defined above is projective in the sense that, if $\{s < s_1 < \cdots < s_{2n} < t\} \prec$ $\{s < s'_1 < \cdots < s'_{2p} < t\}$, then one has

$$E_{\{s<s'_1<\cdots s'_{2p}<t\}}(f_{s]} \otimes f_{s'_1} \otimes \cdots \otimes \cdots \otimes f_{s'_{2p}} \otimes f_{[t)})$$

$$= E_{\{s<s_1<\cdots<s_n<t\}}(f_{s]} \otimes f_{s_1} \otimes \cdots \otimes f_{s_n} \otimes f_{[t)})$$

Therefore condition (3.26) and projectivity implies that the limit

$$\lim_{\{s<s_1<\cdots<s_j<\cdots<s_{2n}<t\}} E_{\{s<s_1<\cdots<s_j<\cdots<s_n<t\},\{s,t\}} =: E_{(s,t)^c}$$

exists in the strongly finite sense on $\mathcal{A}$ and defines an Umegaki conditional expectation

$$E_{(s,t)^c} : \mathcal{A} \equiv \mathcal{A}_{s]} \otimes \mathcal{A}_{(s,t)} \otimes \mathcal{A}_{[t} \to \mathcal{A}_{(s,t)^c}$$

which, by construction satisfies the in–out NMP. The projectivity of the family $(E_{\{s<s_1<\cdots<s_j<\cdots<s_n<t\},\{s,t\}})$ implies the projectivity of the family $(E_{(s,t)^c})$. $\square$

Definition 5 A family $(\tilde{P}_{(s',t'),(s,t)})_{s<s'<t'<t}$ satisfying conditions (i), (ii), (iii) of Theorem 2 is called a 2–**parameter Markov evolution**.

3.4.2 The homogeneous case

Definition 6 The family of local algebras $(\mathcal{A}_t)$ is called:

(i) **homogeneous** if there exists a $*$–algebra $\mathcal{B}$ and injective embeddings

$$j_t : \mathcal{B} \to \mathcal{A} \qquad , \qquad t \in \mathbb{R}$$

such that for each $t \in \mathbb{R}$

$$j_t(\mathcal{B}) = \mathcal{A}_t$$

(ii) **of tensor product type** if

$$\mathcal{A}_{\{s,t\}} := \mathcal{A}_{\{s\}} \vee \mathcal{A}_{\{t\}} \sim \mathcal{A}_s \otimes \mathcal{A}_t \sim \mathcal{B} \otimes \mathcal{B}$$

Remark. In the tensor product case the existence of the family of $*$–isomorphisms $j_t : \mathcal{B} \to \mathcal{A}$ can be easily proved. However there other interesting cases is which such a family exists (e.g. the Fermi case). In this case, since by assumption each j_t has a left inverse, denoted j_t^{-1} , the same is true for each $j_s \otimes j_t := j_{s,t}$ $(s < t)$ and

$$(j_s \otimes j_t)^{-1} = j_s^{-1} \otimes j_t^{-1} : \mathcal{A}_{\{s,t\}} := \mathcal{A}_{\{s\}} \vee \mathcal{A}_{\{t\}} \to \mathcal{B} \otimes \mathcal{B}$$

is a $*$–isomorphism. Therefore the family of maps

$$P^0_{(s',t'),(s,t)} := j_{s,t}^{-1} \tilde{P}_{(s',t'),(s,t)} j_{s',t'} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{B} \tag{3.36}$$

with $s < s' \le t' < t$, is a 2–parameter evolution on $\mathcal{B} \otimes \mathcal{B}$, i.e. if:
$s'' < s < s' \le t' < t < t''$, then

$$P^0_{(s,t),(s'',t'')} P^0_{(s',t'),(s,t)} = P^0_{(s',t'),(s'',t'')} \tag{3.37}$$

Definition 7 Suppose that the family of local algebras $(\mathcal{A}_t)$ is knematically homogeneous and admits a 2–parameter family of $*$–isomorphisms as in (3.36). A 2–parameter evolution

$$\tilde{P}_{(s',t'),(s,t)} \ : \ \mathcal{A}_{\{s',t'\}} \to \mathcal{A}_{\{s,t\}} \quad ; \quad s < s' \leq t' < t \tag{3.38}$$

is called **translation covariant** if

$$u_r \tilde{P}_{(s',t'),(s,t)} = \tilde{P}_{(s'+r,t'+r),(s+r,t+r)} u_r \quad , \quad \forall r \in \mathbb{R} \tag{3.39}$$

where u_r satisfies (2.1).

Lemma 2 The identity (3.39) is equivalent to

$$P^0_{(s',t'),(s,t)} = P^0_{(u',v'),(u,v)} \tag{3.40}$$

whenever $s < s' \leq t' < t$ and $u < u' \leq v' < v$ are such that:

$$s' - s = u' - u \quad ; \quad t - t' = v - v' \quad ; \quad t - s = v - u \tag{3.41}$$

Proof. For $s < s' \leq t' < t$ and $u < u' \leq v' < v$ satisfying condition (3.41), define

$$0 < \tau_L := s' - s = u' - u \quad ; \quad 0 < \tau_R := t - t' = v - v' \tag{3.42}$$

$$0 < \tau_M := t' - s' = v' - u' \tag{3.43}$$

$$\tau := u - s \in \mathbb{R} \tag{3.44}$$

and notice that, in the notations above,

$$t - s = (t - t') + (t' - s') + (s' - s) = v - u = (v - v') + (v' - u') + (u' - u)$$

$$= \tau_R + (t' - s') + \tau_L = \tau_R + (v' - u') + \tau_L \iff t' - s' = v' - u'$$

Therefore τ_M is well defined by (3.43) and from these one obtains

$$u' \overset{(3.42)}{=} u + \tau_L \overset{(3.44)}{=} s + \tau + \tau_L \overset{(3.42)}{=} s' + \tau$$

$$\begin{aligned} v' &\overset{(3.43)}{=} u' + \tau_M = u' - \tau_L + \tau_L + \tau_M \overset{(3.42)}{=} u + \tau_L + \tau_M \\ &\overset{(3.42)}{=} u + s' - s + \tau_M \overset{(3.44)}{=} \tau + s' + \tau_M \overset{(3.43)}{=} \tau + t' \end{aligned}$$

$$u \overset{(3.44)}{=} s + \tau$$

$$v \overset{(3.42)}{=} v' + \tau_R = t' + \tau + \tau_R \overset{(3.42)}{=} t + \tau$$

Using these identities, (3.40) becomes

$$P^0_{(s',t'),(s,t)} = P^0_{(s'+\tau,t'+\tau),(s+\tau,t+\tau)} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{B} \quad , \quad \forall \tau \in \mathbb{R} \tag{3.45}$$

Now, identity (3.39) is equivalent to

$$u_\tau \tilde{P}_{(s',t'),(s,t)} u_\tau^{-1} = \tilde{P}_{(s'+\tau,t'+\tau),(s+\tau,t+\tau)} \quad , \quad \forall \tau \in \mathbb{R}$$

which is equivalent to

$$\begin{aligned} j_{s,t}^{-1} \tilde{P}_{(s',t'),(s,t)} j_{s',t'} &= j_{s,t}^{-1} u_\tau^{-1} \tilde{P}_{(s'+\tau,t'+\tau),(s+\tau,t+\tau)} u_\tau j_{s',t'} \\ &= j_{s+\tau,t+\tau}^{-1} \tilde{P}_{(s'+\tau,t'+\tau),(s+\tau,t+\tau)} j_{s'+\tau,t'+\tau} \end{aligned}$$

which is (3.45) $\square$

The following proposition justifies the terminology introduced in Definition 7.

Proposition 2 The following statements are equivalent:

(i) The family $\{E_{\{s,t\}}\}_{s<t}$, constructed from the 2–parameter evolution $(\tilde{P}_{(s',t'),(s,t)})$ as in the second part of Theorem 2, is translation covariant in the sense that

$$u_r E_{\{s,t\}} = E_{\{s+r,t+r\}} u_r \tag{3.46}$$

where u_r satisfies (2.1).

(ii) The 2–parameter evolution $\tilde{P}_{(s',t'),(s,t)}$ $(s < s' \leq t' < t)$, defined in Theorem 2, is translation covariant in the sense of Definition 7.

Proof. Suppose that (ii) holds. Denote $(E_{(s,t)^c})$ the projective family of Markov conditional expectations associated to the 2–parameter evolution $(\tilde{P}_{(s',t'),(s,t)})$ as in Theorem 2. Identity (3.28) implies that, for $s < s' \leq t' < t$,

$$\tilde{P}_{(s',t'),(s,t)}(j_{s'} \otimes j_{t'}) = E_{\{s,t\}}\Big|_{\mathcal{A}_{\{s',t'\}}}(j_{s'} \otimes j_{t'}) = E_{\{s,t\}}(j_{s'} \otimes j_{t'}) \tag{3.47}$$

It follows that from Lemma 2 and (3.47)

$$\begin{aligned} u_r E_{\{s,t\}}(j_{s'} \otimes j_{t'}) &= u_r \tilde{P}_{(s',t'),(s,t)}(j_{s'} \otimes j_{t'}) \\ &= \tilde{P}_{(s'+r,t'+r),(s+r,t+r)}(j_{s+r} \otimes j_{t+r}) \\ &= E_{\{s+r,t+r\}}(j_{s'+r} \otimes j_{t'+r}) \\ &= E_{\{s+r,t+r\}} u_r (j_{s'} \otimes j_{t'}) \end{aligned}$$

Thus (i) holds.
Conversely, suppose that (i) holds. We have

$$\begin{aligned} \tilde{P}_{(s'+r,t'+r),(s+r,t+r)} j_{s'+r} \otimes j_{t'+r} &= E_{\{s+r,t+r\}} j_{s'+r} \otimes j_{t'+r} \\ &= E_{\{s+r,t+r\}} u_r (j_{s'} \otimes j_{t'}) \\ &= u_r E_{\{s,t\}} (j_{s'} \otimes j_{t'}) \\ &\overset{(3.47)}{=} u_r \tilde{P}_{(s',t'),(s,t)} (j_{s'} \otimes j_{t'}) \\ &= u_r \tilde{P}_{(s',t'),(s,t)} u_r^{-1} j_{s'+r} \otimes j_{t'+r} \end{aligned}$$

which proves the statement (ii). □

3.5 Standard forms

Every 1–parameter translation invariant evolution $(P_{s,t})_{s<t}$ on $\mathcal{B}$, defines in a standard way a unique semi–group $(P_{0,t})_{0<t}$, using the identities:

$$P_{s,t} = P_{s+\tau,t+\tau} \overset{\tau=-s}{=} P_{0,t-s}$$

$$P_{t,t''}P_{s,t} = P_{0,t''-t}P_{0,t-s} = P_{s,t''} = P_{0,t''-s}$$

In this section, we adapt this strategy in the case of 2–parameter translation invariant evolutions $(P^0_{(s',t'),(s,t)})_{(s',t')\subset(s,t)}$ on $\mathcal{B}\otimes\mathcal{B}$.
To this goal, we exploit the translation invariance condition (3.45) which, in the notations (3.42), (3.43), (3.44), gives

$$\begin{aligned} P^0_{(s',t'),(s,t)} &= P^0_{(s'+\tau,t'+\tau),(s+\tau,t+\tau)} \\ &\overset{\tau=-s'}{=} P^0_{(0,t'-s'),(s-s',t-s')} = P^0_{(0,\tau_M),(-\tau_L,\tau_R+\tau_M)} \\ &\overset{\tau=-s}{=} P^0_{(\tau_L,\tau_L+\tau_M),(0,\tau_L+\tau_M+\tau_R)} \end{aligned} \tag{3.48}$$

We will use the following notations:

$P^0_{(0,\tau_M),(-\tau_L,\tau_R+\tau_M)}$ =: standard form of first kind of $P^0_{(s',t'),(s,t)}$

$P^0_{(\tau_L,\tau_L+\tau_M),(0,\tau_L+\tau_M+\tau_R)}$ =: standard form of second kind of $P^0_{(s',t'),(s,t)}$

In conclusion: up to translation invariance, the pair $((s',t'),(s,t))$ is equivalent to the pair $((0,t'-s'),(s-s',t-s'))$ and this latter pair, in terms of the τ–parameters, can be represented as $((0,\tau_M),(-\tau_L,\tau_R+\tau_M))$.

Lemma 3 In terms of standard forms of the first kind and for $s''<s<s'<t'<t<t''$, the evolution equation

$$P^0_{(s,t),(s'',t'')}P^0_{(s',t'),(s,t)} = P^0_{(s',t'),(s'',t'')} \tag{3.49}$$

on $\mathcal{B}\otimes\mathcal{B}$ becomes

$$P^0_{(0,\tau_M),(-\tau''_L,\tau_M+\tau_R+\tau_L+\tau''_R)}P^0_{(0,\tau_M),(-\tau_L,\tau_M+\tau_R)} = P^0_{(0,\tau_M),(-\tau''_L-\tau_L,\tau_M+\tau_R+\tau''_R)} \tag{3.50}$$

where

$$\tau''_L = s-s'' \geq 0 \qquad ; \qquad \tau''_R = t''-t \geq 0$$

Proof. The standard form of the first kind of $P^0_{(s',t'),(s'',t'')}$ is:

$$\begin{aligned} P^0_{(s',t'),(s'',t'')} = P^0_{(s'+\tau,t'+\tau),(s''+\tau,t''+\tau)} &\overset{\tau=-s'}{=} P^0_{(0,t'-s'),(s''-s',t''-s')} \\ &\overset{(3.43)}{=} P^0_{(0,\tau_M),(-\tau''_L-\tau_L,\tau_M+\tau_R+\tau''_R)} \end{aligned}$$

The standard form of the first kind of $P^0_{(s,t),(s'',t'')}$ is:

$$\begin{aligned} P^0_{(s,t),(s'',t'')} &= P^0_{(s+\tau,t+\tau),(s''+\tau,t''+\tau)} \\ &\overset{\tau=-s}{=} P^0_{(0,t-s),(s''-s,t''-s)} \\ &\overset{(3.42)}{=} P^0_{(0,\tau_M),(-\tau''_L,\tau_L+\tau_R+\tau''_R+\tau_M)} \end{aligned} \tag{3.51}$$

Thus, using the standard form of first kind of $P^0_{(s',t'),(s,t)}$ given by (3.48), one obtains

$$P^0_{(0,\tau_M),(-\tau''_L,\tau_L+\tau_R+\tau''_R+\tau_M)}P^0_{(0,\tau_M),(-\tau_L,\tau_R+\tau_M)} = P^0_{(0,\tau_M),(-\tau''_L-\tau_L,\tau_M+\tau_R+\tau''_R)} \quad (3.52)$$

which is (3.50). $\square$

Lemma 4 The evolution equation

$$P^0_{(s,t),(s'',t'')}P^0_{(s',t'),(s,t)} = P^0_{(s',t'),(s'',t'')} \quad (3.53)$$

on $\mathcal{B} \otimes \mathcal{B}$ can be equivalently written in the form

$$\hat{P}_{\tau_M}(-t',s')\hat{P}_{\tau_M}(-t,s) = \hat{P}_{\tau_M}(-(t'+t),-(t-s')) \quad (3.54)$$

where the variables t, s, t', s' are subject to the constraints

$$s > 0 \quad ; \quad t > 0 \quad ; \quad s' > s+t \quad ; \quad t' > 0 \quad (3.55)$$

Proof. Introducing the variable

$$\sigma := \tau_M + \tau_R > 0$$

and the notation

$$\hat{P}_{\tau_M}(x,y) := P^0_{(0,\tau_M),(x,y)} \quad ; \quad x \le y \quad (3.56)$$

(3.50) becomes

$$\hat{P}_{\tau_M}(-\tau''_L,\sigma+\tau_L+\tau''_R)\hat{P}_{\tau_M}(-\tau_L,\sigma) = \hat{P}_{\tau_M}(-\tau''_L-\tau_L,\sigma+\tau''_R) \quad (3.57)$$

Introducing the new variables

$$s := \sigma > 0 \quad ; \quad t := \tau_L > 0 \quad ; \quad t' := \tau''_L > 0 \quad ; \quad u := \tau''_R > 0 \quad (3.58)$$

(3.50) becomes

$$\hat{P}_{\tau_M}(-t',s+t+u)\hat{P}_{\tau_M}(-t,s) = \hat{P}_{\tau_M}(-t'-t,s+u) \quad (3.59)$$
$$= \hat{P}_{\tau_M}(-t'-t,(s+t+u)-t)$$

Therefore, introducing the new variable

$$s+t+u := s' \ge s+t \quad (3.60)$$

(3.50) becomes

$$\hat{P}_{\tau_M}(-t',s')\hat{P}_{\tau_M}(-t,s) = \hat{P}_{\tau_M}(-t'-t,s'-t) \quad (3.61)$$

The constraints on the new variables are:

$$s \overset{(3.58)}{>} 0 \quad ; \quad t \overset{(3.58)}{>} 0 \quad ; \quad s' \overset{(3.60)}{\ge} s+t \quad ; \quad t' \overset{(3.58)}{>} 0 \quad (3.62)$$

and this proves (3.55) $\square$

4 2–parameter Markov semi–groups in quantum theory

The previous discussion of 2–parameter Markov semi–groups concerned the classical case. Here we outline the construction of examples of such semi–groups in the quantum case.

Definition 8 Let $\mathcal{I}$ be a localization on $\mathbb{R}$. A family $(\mathcal{A}_I)_{I\in\mathcal{I}}$ of local $*$–algebras is called **factorizable** if, for any $I \in \mathcal{I}$ and any partition $I = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{I}$, there is a $*$–isomorphism

$$u_{I;I_1,I_2} : \mathcal{A}_I \to \mathcal{A}_{I_1} \otimes \mathcal{A}_{I_2} \tag{4.1}$$

where $\otimes$ denotes **algebraic tensor product**.

Classical examples of factorizable families of local W^*–algebras are provided by the local algebras of independent increment classical real valued stochastic process X indexed by $\mathbb{R}$. In this case $\mathcal{I}$ is the family of all intervals in $\mathbb{R}$,

$$\mathcal{A} := L^\infty_{\mathbb{C}}(X) := \text{algebra of bounded measurable functions of the process } X$$

and, for $I \in \mathcal{I}$, $\mathcal{A}_I$ is the sub–algebra of $\mathcal{A}$ generated by the functions of the increments $X_t - X_s$ with $(s,t) \subseteq I$. A non–commutative example is provided by the algebra of bounded operators on $\Gamma(L^2(\mathbb{R};\mathcal{H}))$, the Fock space over $L^2(\mathbb{R};\mathcal{H})$ (square integrable $\mathcal{H}$–valued functions, $\mathcal{H}$ Hilbert space) and, for $I \in \mathcal{I}$, $\mathcal{A}_I$ is the sub–algebra of $\mathcal{A}$ generated by the Weyl operators with test functions having support in I. In the former case the expectation value with respect to the constant function equal to 1 gives a state on $\mathcal{A}$, in the latter a state is given by the vacuum expectation value. In both cases we denote φ the state and Φ the Hilbert space vector implementing it.
Both states are factorizable in the sense that, denoting φ_I the restriction of φ to $\mathcal{A}_I$, one has

$$\varphi_I \circ u^{-1}_{I;I_1,I_2} = \varphi_{I_1} \otimes \varphi_{I_2} \tag{4.2}$$

This implies that in both cases for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{A} \to \mathcal{A}_I$ characterized by the property

$$E_I(a_I a_{I^c}) = a_I \varphi(a_{I^c}) \qquad ; \qquad a_I \in \mathcal{A}_I \ , \ a_{I^c} \in \mathcal{A}_{I^c}$$

and that the family (E_I) of conditional expectations is factorizable in the sense that, for any pair of disjoint sub–sets $I_1, I_2 \in \mathcal{I}$ one has

$$E_{I_1\cup I_2} = E_{I_1}E_{I_2} = E_{I_2}E_{I_1}$$

In particular

$$E_{(s,t)^c} = E_{s]} E_{[t} = E_{[t} E_{s]} \tag{4.3}$$

Using this and Markov cocycles constructed with stochastic calculus, one can construct examples of multi–parameter Markov semigroups as described in the following.

Recall the random walk scheme, where one couples tensorially an algebra $\mathcal{B}_0$ (initial or system algebra) to a factorizable family $(\mathcal{A}, (\mathcal{A}_I))$ of local algebras thus producing a new family of local algebras

$$\tilde{\mathcal{A}} := \mathcal{B}_0 \otimes \mathcal{A}$$

$$\tilde{\mathcal{A}}_{[-s,t]} := \mathcal{B}_0 \otimes \mathcal{A}_{[-s,t]} \quad ; \quad \tilde{\mathcal{A}}_{[-s,t]^c} := 1_{\mathcal{B}_0} \otimes \mathcal{A}_{-s)} \otimes \mathcal{A}_{(t} \quad ; \quad s, t \geq 0$$

One also introduces a new family of conditional expectations

$$\tilde{E}_{(-s,t)^c} := id_{\mathcal{B}_0} \otimes E_{(-s,t)^c} \overset{(4.3)}{=} id_{\mathcal{B}_0} \otimes (E_{-s)} E_{(t)}) = id_{\mathcal{B}_0} \otimes (E_{(t} E_{-s)})$$

Let (;) be a shift–covariant localized right multiplicative functional (Markov cocycle)

$$U_{[s,t]} U_{[r,s]} = U_{[r,t]} \quad ; \quad r < s < t \quad ; \quad U_{[s,t]} \in \mathcal{A}_{[s,t]}$$

It is known that in the factorizable case one can associate to it various Markov semi–groups

$$P^t_-(x_0) := E_0(U^*_{[0,t]} x_0 U_{[0,t]}) = E_{0]}(U^*_{[0,t]} x_0 U_{[0,t]})$$

$$P^s_+(x_0) := E_0(U^*_{[-s,0]} x_0 U_{[-s,0]}) = E_{0]}(U^*_{[-s,0]} x_0 U_{[-s,0]})$$

$$Q^t_- := E_0(U_{[0,t]}) = E_{0]}(U_{[0,t]}) \qquad ; \qquad Q^s_+ := E_0(U_{[-s,0]}) = E_{[0}(U_{[-s,0]})$$

Example 1: If $-s < -s' < 0 < t' < t$, one has

$$E_{(s,t)^c}(U^*_{[-s',0]} x_0 U_{[0,t']}) = E_{s]} E_{[t}(U^*_{[-s',0]} x_0 U_{[0,t']}) = E_{s]}(E_{[t}(U^*_{[-s',0]}) x_0 U_{[0,t']}))$$

$$= E_{s]}(E_{[t}(U_{[-s',0]})^* x_0 U_{[0,t']})) = E_{s]}(E_{[0}(U_{[-s',0]})^* x_0 U_{[0,t']})) = E_{s]}((Q^{s'}_+)^* x_0 U_{[0,t']})$$

$$= E_{s]}((Q^{s'}_+)^* x_0 U_{[0,t']}) = (Q^{s'}_+)^* x_0 E_{s]}(U_{[0,t']}) = (Q^{s'}_+)^* x_0 Q^{t'}_-$$

which is a 2–parameter semi-group. In this example there is no dependence on (s,t) unless $(s,t) = (s',t')$.

Example 2: If $-s < -s' < 0 < t' < t$, one has

$$P^0_{(s',t'),(s,t)}(x_0) := E_{(-s,t)^c}(U^*_{[-s',t']} x_0 U_{[-s',t']}) = E_{(-s,t)^c}(u^0_{s'}(U_{[0,t'-s']})^* x_0 u^0_{s'}(U_{[0,t'-s']}))$$

$$= E_{-s]} E_{[t}(u^0_{s'}(U^*_{[0,t'-s']} x_0 U_{[0,t'-s']})) = E_{-s]} u^0_{s'}(E_{[t-s'}(U^*_{[0,t'-s']} x_0 U_{[0,t'-s']}))$$

$$= E_0(U^*_{[0,t'-s']} x_0 U_{[0,t'-s']}) = P_-^{t'-s'}(x_0)$$

and the evolution becomes

$$P^0_{(s,t),(s'',t'')} P^0_{(s',t'),(s,t)}(x_0) = P^0_{(s,t),(s'',t'')}(P_-^{t'-s'}(x_0)) = P_-^{t-s}(P_-^{t'-s'}(x_0))$$

$$= (P_-^{(t-s)+(t'-s')}(x_0))$$

Thus in this case

$$P^0_{(s,t),(0,0)} = P_-^{t-s}$$

and the 2–parameter evolution becomes equivalent to a single Markov semi–group. Example 2 is the usual semi–group deduced from the quantum Feynman–Kac formula [5].

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