# Obstacle Avoidance via Hybrid Feedback

S. Berkane, *Member, IEEE,* A. Bisoffi, *Member, IEEE*, D. V. Dimarogonas, *Senior Member, IEEE*

*Abstract*—In this paper we present a hybrid feedback approach to solve the navigation problem of a point mass in the $n-$dimensional space containing an arbitrary number of ellipsoidal shape obstacles. The proposed hybrid control algorithm guarantees both global asymptotic stabilization to a reference and avoidance of the obstacles. The intuitive idea of the proposed hybrid feedback is to switch between two modes of control: stabilization and avoidance. The geometric construction of the flow and jump sets of the proposed hybrid controller, exploiting hysteresis regions, guarantees Zeno-free switching between the stabilization and the avoidance modes. Simulation results illustrate the performance of the proposed hybrid control approach for 2-dimensional and 3-dimensional scenarios.

## I. INTRODUCTION

For decades, the obstacle avoidance problem has been an active area of research in the robotics and control communities [1]. In a typical robot navigation scenario, the robot is required to reach a given goal (destination) while not colliding with a set of obstacle regions in the workspace. Since the pioneering work by Khatib [2], artificial potential fields have been widely used in the obstacle avoidance problem since they offer the possibility to combine the solution to the global find-path problem with a feedback controller for the robot, thus, allowing the high-level planner to address more abstract tasks. The idea is to generate an artificial potential field that renders the goal attractive and the obstacles repulsive. Then, by considering trajectories that navigate along the negative gradient of the artificial potential field, one can ensure that the robot will reach the desired target while avoiding to collide with the obstacles. However, artificial potential field-based algorithms suffer from 1) the presence of local minima preventing the successful navigation to the target point and 2) arbitrarily large repulsive potential near the obstacles which are in conflict with the inevitable actuator saturations.

The navigation function-based approach, which was initiated by Koditscheck and Rimon [3] for sphere worlds [3, p. 414], solves both problems. It allows to obtain artificial potential fields with the nice property that all but one of the critical points are saddles with the remaining critical point being the desired reference. Since then, the navigation function-based approach has been extended in many different directions; *e.g.,* for multi-agent systems [4]–[6], for unknown sphere words [7], and for focally admissible obstacles [8].

The major drawback of navigation functions is that they are not correct by construction. In fact, navigation functions are theoretically guaranteed to exist, but their explicit computation is not straightforward since they require an unknown tuning of a given parameter to eliminate local minima.

Recently, Loizou [9] introduced the navigation transform that diffeomorphically maps the workspace to a trivial domain called the *point world* consisting of a closed ball with a finite number of points removed. Once this transformation is found, the navigation problem is solved from almost all initial conditions without requiring any tuning. In addition, the trajectory duration is explicitly available, which provides a timed-abstraction solution to the motion-planning problem. Similarly, the recent work in [10] uses the so-called prescribed performance control to design a time-varying control law that drives the robot, in finite time, from all initial conditions to some neighborhood of the target while avoiding the obstacles. Another approach to the navigation problem is through barrier functions (see [11] and references therein), which are developed for nonlinear systems with state-space constraints and ensure safety. Model predictive control approaches have been also used for reactive robot navigation, e.g., [12], [13].

However, by using any of the approaches described above, it is not possible to ensure safety from all initial conditions in the obstacle-free state space. As pointed out in [3], the appearance of additional undesired equilibria is unavoidable when considering continuous time-invariant vector fields. This is a well known topological obstruction to global asymptotic stabilization by continuous feedback when the free state space is not diffeomorphic to an Euclidean space (see, e.g., [14, Thm. 2.2]). Furthermore, this problem is more far-reaching since, by using a continuous feedback law, it is always possible to find arbitrarily small adversarial (noise) signals acting on the vector field, such that a set of initial conditions different from the target, possibly of measure zero, can be rendered stable [15, Thm. 6.5]. To deal with such limitations, the authors in [16] proposed a hybrid state feedback controller, using Lyapunov-based hysteresis switching, to achieve robust global asymptotic regulation in $\mathbb{R}^2$ to a target while avoiding a single obstacle. This approach has been exploited in [17] to steer a planar vehicle to the source of an unknown but measurable signal while avoiding an obstacle. In [18] and [19], a hybrid control law was proposed to globally asymptotically stabilize a class of linear systems while avoiding neighbourhoods of unsafe isolated points in $\mathbb{R}^n$. Although such hybrid approaches are promising, they are still challenged by constructing the suitable hybrid feedback for higher dimensions and with more complex obstacles shapes.

In this work, we propose a hybrid control algorithm for the global asymptotic stabilization of a point mass moving in an arbitrary $n-$dimensional space while safely avoiding obstacles

that have generic ellipsoidal shapes, based on the preliminary treatment of this problem for a single spherical obstacle in [20]. The ellipsoids provide a tighter bounding volume than spheres, and in our scheme this volume can be arbitrarily flat and close to the target, which leads to a significant reduction in the level of conservatism compared, for instance, to [21, Thm. 3] (as shown in Section VI).

Our proposed hybrid algorithm employs a hysteresis-based switching between the avoidance controller and the stabilizing controller in order to guarantee forward invariance of the obstacle-free region (corresponding to safety) and global asymptotic stability of the reference position. We consider trajectories in an $n-$dimensional Euclidean space and we resort to tools from higher-dimensional geometry [22] to provide a construction of the flow and jump sets where the different modes of operation of the hybrid controller are activated. Furthermore, the hybrid control law guarantees a bounded control input, it matches the stabilizing controller in arbitrarily large subsets of the obstacle-free region by a suitable tuning of its parameters (hence qualifying as minimally invasive), it can be readily extended to a non-point mass vehicle and enjoys some level of inherent robustness to perturbations. *Structure.* Preliminaries are in Section II. The navigation problem is formulated in Section III. Our proposed hybrid control scheme is discussed in Section IV. Section V presents the main result of forward invariance of the obstacle-free space and global asymptotic stability of the target, together with other desirable complementary properties. Numerical examples are in Section VI. All the proofs are in the Appendix.

## II. PRELIMINARIES

$\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_{\geq}$ denote, respectively, the set of nonnegative integers, reals and nonnegative reals. $\mathbb{R}^n$ is the $n$-dimensional Euclidean space and $\mathbb{S}^n$ is the $n$-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$. Given the column vectors $v_1 \in \mathbb{R}^{n_1}$ and $v_2 \in \mathbb{R}^{n_2}$, $(v_1, v_2)$ denotes the stack vector $\begin{bmatrix} v_1^\top & v_2^\top \end{bmatrix}^\top$. The Euclidean norm of $x \in \mathbb{R}^n$ is defined as $\|x\| := \sqrt{x^\top x}$ and the geodesic distance between two points $x$ and $y$ on the sphere $\mathbb{S}^n$ is defined by $\mathbf{d}_{\mathbb{S}^n}(x,y) := \arccos(x^\top y)$ for all $x, y \in \mathbb{S}^n$. For an arbitrary matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_i(A)$ denotes the $i$-th eigenvalue of $A$. If $A$ is a symmetric matrix, then $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalues of $A$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, we define the distance to the set $\mathcal{A}$ by $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} \|x - y\|$. Given two sets $\mathcal{A}$ and $\mathcal{B}$, we define the distance from $\mathcal{A}$ to $\mathcal{B}$ by $\mathbf{dist}(\mathcal{A}, \mathcal{B}) := \inf\{|a - b| : a \in \mathcal{A}, b \in \mathcal{B}\}$. The closure, interior and boundary of a set $\mathcal{A} \subset \mathbb{R}^n$ are denoted as $\overline{\mathcal{A}}, \mathcal{A}^\circ$ and $\partial\mathcal{A}$, respectively. The relative complement of a set $\mathcal{B} \subset \mathbb{R}^n$ with respect to a set $\mathcal{A}$ is denoted by $\mathcal{A}\backslash\mathcal{B}$ and contains the elements of $\mathcal{A}$ which are not in $\mathcal{B}$. In particular, we use $\mathcal{A}^c$ to denote the complement of $\mathcal{A}$ in $\mathbb{R}^n$, *i.e.,* $\mathcal{A}^c = \mathbb{R}^n\backslash\mathcal{A}$. Given the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the following set identities [23] will be used

$$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C}) \tag{1a}$$

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}) \tag{1b}$$

$$(\mathcal{A} \cap \mathcal{C})\backslash(\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{C})\backslash\mathcal{B} = \mathcal{A} \cap (\mathcal{C}\backslash\mathcal{B}) \tag{1c}$$

$$\mathcal{C}\backslash(\mathcal{A}\cup\mathcal{B}) = (\mathcal{C}\backslash\mathcal{A})\cap(\mathcal{C}\backslash\mathcal{B}), \mathcal{C}\backslash(\mathcal{A}\cap\mathcal{B}) = (\mathcal{C}\backslash\mathcal{A})\cup(\mathcal{C}\backslash\mathcal{B}) \tag{1d}$$

$$(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c\cap\mathcal{B}^c, (\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c\cup\mathcal{B}^c, \mathcal{A}\backslash\mathcal{B} = \mathcal{A}\cap\mathcal{B}^c \tag{1e}$$

$$\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}, \overline{\mathcal{A} \cap \mathcal{B}} \subset \overline{\mathcal{A}} \cap \overline{\mathcal{B}}, \mathcal{A}\backslash\overline{\mathcal{A}^c} = \mathcal{A}^\circ \tag{1f}$$

$$\mathcal{A} \subset \mathcal{B} \implies \overline{\mathcal{A}} \subset \overline{\mathcal{B}}, \quad \partial\mathcal{A}^c = \partial\mathcal{A} \tag{1g}$$

$$\partial(\mathcal{A} \cup \mathcal{B}) \subset (\partial\mathcal{A}\backslash\mathcal{B}^\circ) \cup (\partial\mathcal{B}\backslash\mathcal{A}^\circ) \tag{1h}$$

$$\partial(\mathcal{A} \cap \mathcal{B}) \subset (\partial\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\partial\mathcal{B} \cap \overline{\mathcal{A}}). \tag{1i}$$

Two sets $\mathcal{A}$ and $\mathcal{B}$ are said to be disjoint if $\mathcal{A}\cap\mathcal{B} = \emptyset$. They are said to be separated if $\mathcal{A}\cap\overline{\mathcal{B}} = \emptyset = \overline{\mathcal{A}}\cap\mathcal{B}$. The notion of separated sets is stronger than mere disjointness. If two sets $\mathcal{A}$ and $\mathcal{B}$ are separated then we have [23, Exercise 1.3.A]

$$\partial(\mathcal{A} \cup \mathcal{B}) = \partial\mathcal{A} \cup \partial\mathcal{B}. \tag{2}$$

The tangent cone to a set $\mathcal{K} \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted $\mathbf{T}_\mathcal{K}(x)$, is defined as in [24, Def. 5.12 and Fig. 5.4].

### A. Projections Maps

For $z \in \mathbb{R}^n\backslash\{0\}$, we define the following projection maps:

$$\pi^\|(z) := \frac{zz^\top}{\|z\|^2}, \ \pi^\perp(z) := I_n - \frac{zz^\top}{\|z\|^2}, \ \rho(z) := I_n - 2\frac{zz^\top}{\|z\|^2} \tag{3}$$

where $I_n$ is the $n \times n$ identity matrix. The map $\pi^\|(\cdot)$ is the parallel projection map, $\pi^\perp(\cdot)$ is the orthogonal projection map [22], and $\rho(\cdot)$ is the reflector map (also called Householder transformation). Consequently, for any $x \in \mathbb{R}^n$, the vector $\pi^\|(z)x$ corresponds to the projection of $x$ onto the line generated by $z$, $\pi^\perp(z)x$ corresponds to the projection of $x$ onto the hyperplane orthogonal to $z$ and $\rho(z)x$ corresponds to the reflection of $x$ about the hyperplane orthogonal to $z$. For $z \in \mathbb{R}^n\backslash\{0\}$, some useful properties of these maps follow:

$$\pi^\|(z)z = z, \qquad \pi^\perp(z)\pi^\perp(z) = \pi^\perp(z), \tag{4a}$$

$$\pi^\perp(z)z = 0, \qquad \pi^\|(z)\pi^\|(z) = \pi^\|(z), \tag{4b}$$

$$\rho(z)z = -z, \qquad \rho(z)\rho(z) = I_n. \tag{4c}$$

We also define for $z \in \mathbb{R}^n\backslash\{0\}$ and $\theta \in \mathbb{R}$ the parametric map

$$\pi^\theta(z) := \cos^2(\theta)\pi^\perp(z) - \sin^2(\theta)\pi^\|(z), \tag{5}$$

which can also be written (thanks to $2\pi^\perp(z) - \rho(z) = 2\pi^\|(z) + \rho(z) = I_n$) as

$$\pi^\theta(z) = \frac{1}{2}\rho(z) + \frac{1}{2}\cos(2\theta)I_n. \tag{6}$$

### B. Geometric Subsets of $\mathbb{R}^n$

*1) Line:* A line is the one-dimensional subset of $\mathbb{R}^n$ described by the set

$$\mathcal{L}(c, v) := \{x \in \mathbb{R}^n : x = c + \lambda v, \lambda \in \mathbb{R}\}, \tag{7}$$

which corresponds to the line passing by the point $c \in \mathbb{R}^n$ and with direction parallel to $v \in \mathbb{R}^n\backslash\{0\}$. If in (7) $\lambda \geq 0$ (respectively $\lambda \leq 0$), then we obtain the half-line denoted by $\mathcal{L}_\geq(c, v)$ (respectively $\mathcal{L}_\leq(c, v)$).

*2) Hyperplane:* A hyperplane is the $(n-1)$-dimensional subset of $\mathbb{R}^n$ described by the set

$$\mathcal{P}(c,v) := \{x \in \mathbb{R}^n : v^\top(x-c) = 0\}, \tag{8}$$

which corresponds to the hyperplane that passes through a point $c \in \mathbb{R}^n$ and has normal vector $v \in \mathbb{R}^n \backslash \{0\}$. The hyperplane $\mathcal{P}(c,v)$ divides the Euclidean space $\mathbb{R}^n$ into two closed subsets $\mathcal{P}_\geq(c,v)$ and $\mathcal{P}_\leq(c,v)$, which are obtained by substituting the $=$ in (8) with $\geq$ and $\leq$, respectively.

*3) Sphere:* A sphere is the $(n-1)$-dimensional subset of $\mathbb{R}^n$ described by the set

$$\mathcal{S}(c,r) := \{x \in \mathbb{R}^n : \|x-c\| = r\} \tag{9}$$

where $c$ is the center of the sphere and $r \in \mathbb{R}_\geq$ is its radius. The closed interior (respectively exterior) of the sphere, also called a hyperball and denoted by $\mathcal{S}_\leq(c,r)$ (respectively $\mathcal{S}_\geq(c,r)$), is obtained from (9) by substituting the $=$ with $\leq$ (respectively $\geq$).

*4) Ellipsoid:* For a positive definite matrix $E \in \mathbb{R}^{n \times n}$, a ellipsoid is the $(n-1)$-dimensional subset of $\mathbb{R}^n$ described by the set

$$\mathcal{E}(c,E) := \{x \in \mathbb{R}^n : \|E(x-c)\|^2 = 1\} \tag{10}$$

where $c$ is the center of the ellipsoid and its $i$-th principal semi-axis is the vector $\lambda_i^{-1}(E)v_i$, with $v_i$ the unit eigenvector corresponding to the eigenvalue $\lambda_i(E)$. The closed interior (respectively exterior) of the ellipsoid, denoted by $\mathcal{E}_\leq(c,E)$ (respectively $\mathcal{E}_\geq(c,E)$), is obtained from (10) by substituting the $=$ with $\leq$ (respectively $\geq$).

**Definition 1:** *Two ellipsoids $\mathcal{E}_\leq(c_1,E_1)$ and $\mathcal{E}_\leq(c_2,E_2)$ are weakly disjoint if $\mathcal{E}_\leq(c_1,E_1) \cap \mathcal{E}_\leq(c_2,E_2) = \emptyset$.*
Explicit algebraic conditions to test weak disjointness of two ellipsoids can be found in [25, Thm. 6] for $n = 2$ and in [26, Thm. 8] for $n = 3$.

**Definition 2:** *Two ellipsoids $\mathcal{E}_\leq(c_1,E_1)$ and $\mathcal{E}_\leq(c_2,E_2)$ are strongly disjoint if $(\lambda_{\min}(E_1))^{-1} + (\lambda_{\min}(E_2))^{-1} < \|c_1-c_2\|$.* Strong disjointness means that the two smallest spherical balls containing the ellipsoids are disjoint. Strong disjointness is more conservative than weak disjointness.

*5) Cone:* For a positive definite matrix $E \in \mathbb{R}^{n \times n}$, a cone is the $(n-1)$-dimensional subset of $\mathbb{R}^n$ described by the set

$$\mathcal{C}(c,v,\theta,E) := \{x \in \mathbb{R}^n : \cos(\theta)\|Ev\|\|E(x-c)\| = v^\top E^2(x-c)\} \tag{11}$$

where $c \in \mathbb{R}^n$ is its vertex, $v \in \mathbb{R}^n \backslash \{0\}$ is its axis and $2\theta \in [0,\pi]$ is its aperture. The cone defined here is sometimes referred to as nappe or half-cone, as opposed to the double cone. The closed interior (respectively, exterior) of the cone, denoted by $\mathcal{C}_\leq(c,v,\theta,E)$ (respectively $\mathcal{C}_\geq(c,v,\theta,E)$), is obtained from (11) by substituting the $=$ with $\leq$ (respectively $\geq$). A normal vector to the cone surface $\mathcal{C}(c,v,\theta,E)$ at $x$ is

$$n(x) := E\pi^\theta(Ev)E(x-c), \tag{12}$$

and can be obtained after squaring in (11) and taking the gradient. The next fact will be used.

**Lemma 1:** *Let $v_1, v_2 \in \mathbb{S}^{n-1}$ such that $v_1^\top v_2 = \cos\theta$ for some $\theta \in (0,\pi]$. Let $\psi_1, \psi_2 \in [0,\pi/2]$ with $\psi_1 + \psi_2 < \theta$. Then for each $c \in \mathbb{R}^n$ and $E \in \mathbb{R}^{n \times n}$ positive definite,*

$$\mathcal{C}_\leq(c, E^{-1}v_1, \psi_1, E) \cap \mathcal{C}_\leq(c, E^{-1}v_2, \psi_2, E) = \{c\}.$$

### C. Hybrid Systems Framework

We consider hybrid dynamical systems of the class [24], described through constrained differential and difference inclusions for state $X \in \mathbb{R}^n$:

$$\begin{cases} \dot{X} \in \mathbf{F}(X), & X \in \mathcal{F}, \\ X^+ \in \mathbf{J}(X), & X \in \mathcal{J}, \end{cases} \tag{13}$$

where the *flow map* $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ governs the continuous evolution, the *flow set* $\mathcal{F} \subseteq \mathbb{R}^n$ dictates where continuous evolution can occur. The *jump map* $\mathbf{J} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ governs the discrete evolution, and the *jump set* $\mathcal{J} \subseteq \mathbb{R}^n$ defines where discrete evolution can occur. The hybrid system (13) is defined by its data and denoted $\mathscr{H} = (\mathcal{F}, \mathbf{F}, \mathcal{J}, \mathbf{J})$.

A subset $\mathbb{T} \subset \mathbb{R}_\geq \times \mathbb{N}$ is a *hybrid time domain* if it is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval (if existent) possibly of the form $[t_j, T)$ with $T$ finite or $T = +\infty$. The ordering of points on each hybrid time domain is such that $(t,j) \preceq (t',j')$ if $t < t'$, or $t = t'$ and $j \leq j'$. A *hybrid solution* is defined in [24, Def. 2.6]. A hybrid solution $\phi$ is maximal if it cannot be extended and complete if its domain $\text{dom}\,\phi$ (which is a hybrid time domain) is unbounded.

## III. PROBLEM FORMULATION

We consider a point mass vehicle moving in the $n$-dimensional Euclidean space containing $I \in \mathbb{N}$ obstacles denoted by $\mathcal{O}_1, \cdots, \mathcal{O}_I$. For each

$$i \in \{1, \cdots, I\} =: \mathbb{I},$$

the obstacle $\mathcal{O}_i$ has an ellipsoidal shape such that $\mathcal{O}_i := \mathcal{E}_\leq(c_i, E_i)$, for some center $c_i \in \mathbb{R}^n$ and some positive definite matrix $E_i \in \mathbb{R}^{n \times n}$ defining the orientation and the shape of the obstacle. The free workspace (obstacle-free region) is then defined by the closed set

$$\mathcal{W} := \bigcap_{i' \in \mathbb{I}} \mathcal{E}_\geq(c_{i'}, E_{i'}). \tag{14}$$

The vehicle is moving according to the dynamics

$$\dot{x} = u, \tag{15}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^n$ is the control input. The vehicle is required to stabilize its position to a target position while avoiding the obstacles. Without loss of generality we consider the target position to be $x = 0$ (the origin).

**Assumption 1:** $n \geq 2$.
We consider $n \geq 2$ since for $n = 1$ (i.e., the state space is a line), global asymptotic stabilization with obstacle avoidance is infeasible.

**Assumption 2:** *For all $i \in \mathbb{I}$, $\|E_i c_i\| > 1$.*
Assumption 2 requires that the target position $x = 0$ is not inside any of the obstacle regions $\mathcal{O}_i$, otherwise the considered navigation problem would be infeasible.

**Assumption 3:** *$\{\mathcal{O}_i\}_{i \in \mathbb{I}}$ are weakly pairwise disjoint.*
In Assumption 3 we impose that there is no intersection region between the obstacles. Otherwise, the union of the two intersecting obstacles forms another region which might have
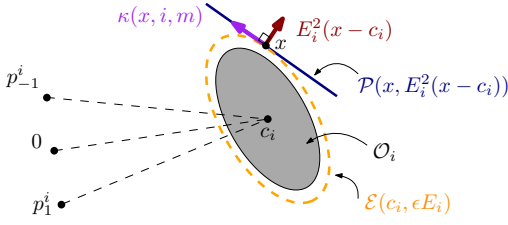
Fig. 1. Illustration of the projection-based avoidance controller. The vehicle is attracted to the auxiliary point $p_m^i$ while sliding on a neighbouring ellipsoid.

a different shape than an ellipsoid. Our *objectives* in designing a control strategy are:

  i) the obstacle-free region $\mathcal{W}$ in (14) is forward invariant,
  ii) the target $x = 0$ is globally asymptotically stable.

Objective i) guarantees that all solutions of the closed-loop system are safely avoiding the obstacles by remaining in the obstacle-free region $\mathcal{W}$ for all times while objective ii) corresponds to global stabilization of the target.

## IV. HYBRID CONTROL FOR OBSTACLE AVOIDANCE

In this section, we propose a hybrid controller that switches suitably between a *stabilizing* and an *avoidance* controller. Let us define a discrete variable

$$m \in \{-1, 0, 1\} =: \mathbb{M}.$$

The value $m = 0$ corresponds to the activation of the stabilizing controller and the values $m = -1$, $m = 1$ correspond to the activation of one of the two configurations of the avoidance controller. The avoidance controller depends also on the current obstacle $\mathcal{O}_i$, as detailed in the next sections.

### A. Control Input

In this section we propose the feedback law for the control input $u$ in (15). $u$ depends on the state $x \in \mathbb{R}^n$, the obstacle $i \in \mathbb{I}$ and the control mode $m \in \mathbb{M}$ as

$$u = \kappa(x, i, m) \tag{16}$$
$$:= \begin{cases} -k_0 x, & m = 0, \\ -k_m E_i^{-1} \pi^\perp(E_i(x - c_i)) E_i(x - p_m^i), & m \in \{-1, 1\}, \end{cases}$$

where $k_{-1}$, $k_0$, $k_1 > 0$ are the control gains for each control mode $m \in \mathbb{M}$ and the points $p_m^i \in \mathbb{R}^n$, $m \in \{-1, 1\}$ and $i \in \mathbb{I}$, are design parameters defined below. In the stabilization mode ($m = 0$), the control input in (16) steers $x$ towards the origin under a state feedback. In the avoidance mode depicted in Fig. 1, the control input minimizes the distance to the *auxiliary* attractive point $p_m^i$ *while* maintaining a constant distance to the obstacle $\mathcal{O}_i$. Indeed, the time derivative of $\|E_i(x - c_i)\|^2$ along solutions of $\dot{x} = \kappa(x, i, m)$ for $m \in \{-1, 1\}$ and $i \in \mathbb{I}$, reads

$$\frac{1}{2} \frac{d}{dt} \|E_i(x - c_i)\|^2 = (x - c_i)^\top E_i^2 \kappa(x, i, m)$$
$$= -k_m(x - c_i)^\top E_i \pi^\perp(E_i(x - c_i)) E_i(x - p_m^i) = 0 \tag{17}$$

by (4b). Then, if we activate the avoidance mode sufficiently away from the obstacle, the avoidance feedback $u = \kappa(x, i, m)$ guarantees that the vehicle does not hit the obstacle. Whereas the logic variable $i$ corresponds to obstacle $\mathcal{O}_i$, the logic

variable $m$ is selected according to a hybrid mechanism that exploits a suitable construction of the flow and jump sets as detailed in Section IV-B.

In order to clear the obstacle while approaching the desired target position at the origin, we select the points $p_1^i$ and $p_{-1}^i$ in the region between the obstacle and the origin, see Fig. 1. The motivation is that the avoidance task is equivalent (up to a linear transformation) to a stabilization problem on the unit sphere $\mathbb{S}^{n-1}$. Therefore, as pointed out for instance in [27], global asymptotic stabilization cannot be accomplished by only one continuous time-invariant controller, but it can be by a hybrid feedback with at least two configurations. For this reason, we consider two avoidance modes with $m = -1$ and $m = 1$ and, hence, the points $p_1^i$ and $p_{-1}^i$ must be distinct. More precisely, for $\theta_i > 0$ (which will be further bounded in Lemma 4), the points $p_1^i$ and $p_{-1}^i$ are selected as

$$p_1^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}, \tag{18a}$$
$$p_{-1}^i := -E_i^{-1} \rho(E_i c_i) E_i p_1^i. \tag{18b}$$

By (18), $p_{-1}^i$ opposes $p_1^i$ diametrically with respect to the cone axis (for $E_i = I_n$, $p_{-1}^i$ is obtained by an orthogonal reflection) and also belongs to $\mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}$ as shown in the next lemma.

***Lemma 2:*** $p_{-1}^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i) \backslash \{c_i\}$.

Note that the results of the paper hold for any selection of the point $p_1^i$ as long as it lies on the surface of the cone as in (18a). An explicit guided choice for those points is given in Section VI for the 2D and 3D cases. Finally, further motivation about the choice of the avoidance controller mode in (16) is detailed in Section IV-B and, in particular, in Lemma 3, which is important for the construction of flow and jump sets.

### B. Geometric Construction of the Flow and Jump sets

In this section we construct explicitly the flow and jump sets where the stabilization and avoidance controllers are activated.

*1) Safety Helmets:* Our proposed construction of flow and jump sets is based on regions that have the shape of a *helmet*, whose construction is now motivated. In the stabilization mode $m = 0$, the closed-loop system should *not* flow when: 1) $x$ is close enough to any of the obstacle regions $\mathcal{E}_\leq(c_i, E_i)$ and 2) the vector field $-k_0 x$ points inside $\mathcal{E}_\leq(c_i, E_i)$. Otherwise, the vehicle ends up hitting the obstacle $i$. Indeed, by computing the time derivative of $\|E_i(x - c_i)\|^2$ along solutions of the vector field $-k_0 x$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|E_i(x - c_i)\|^2 = -k_0 x^\top E_i^2 (x - c_i)$$
$$= k_0 c_i^\top E_i^2 c_i / 4 - k_0 (x - c_i/2)^\top E_i^2 (x - c_i/2) \tag{19}$$
$$= k_0 \|E_i \bar{c}_i\|^2 \left(1 - \|\bar{E}_i(x - \bar{c}_i)\|^2\right)$$

where $\bar{c}_i$ and $\bar{E}_i$ are defined as

$$\bar{c}_i := c_i / 2, \quad \bar{E}_i := 2 E_i / (\|E_i c_i\|). \tag{20}$$

(19) implies that the distance function $\|E_i(x - c_i)\|^2$ decreases for all $x$ in the closed set $\mathcal{E}_\geq(\bar{c}_i, \bar{E}_i)$. Consider now Fig. 2 for a sketch of the next sets and for obstacle $i$, define the *helmet-shaped* set

$$\mathcal{H}_i^* := \mathcal{E}(c_i, E_i) \cap \mathcal{E}_\geq(\bar{c}_i, \bar{E}_i). \tag{21}$$
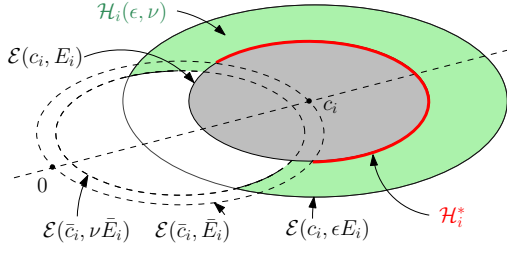
Fig. 2. The helmet $\mathcal{H}_i^*$ in (21) (red) corresponds to all boundary points where the stabilization vector field is pointing inside the obstacle (grey). The safety helmet $\mathcal{H}_i(\epsilon, \nu)$ in (22) (green) corresponds to an dilated version of $\mathcal{H}_i^*$.

$\mathcal{H}_i^*$ is the set of all points that lie on the boundary of the obstacle $\mathcal{O}_i$ and generate a vector field pointing towards the obstacle. Then, for obstacle $i$, we define the *safety helmet* as:

$$\mathcal{H}_i(\epsilon, \nu) := \mathcal{E}_{\leq}(c_i, \epsilon E_i) \cap \mathcal{E}_{\geq}(c_i, E_i) \cap \mathcal{E}_{\geq}(\bar{c}_i, \nu \bar{E}_i) \quad (22)$$

for some parameters $\epsilon, \nu > 0$. $\epsilon$ and $\nu$ determine the thickness of the safety helmet by tuning the dilation/shrinking of the ellipsoids $\mathcal{E}(c_i, E_i)$ and $\mathcal{E}(\bar{c}_i, \bar{E}_i)$, thereby generating a dilated version of $\mathcal{H}_i^*$. The safety helmet $\mathcal{H}_i(\epsilon, \nu)$ constitutes the main ingredient of our following constructions.

*2) Stabilization Mode $m = 0$:* Consider from now on Fig. 3 for a visualization of the sets we are introducing in our construction. In stabilization mode ($m = 0$), we create around each obstacle $\mathcal{O}_i$ a safety helmet $\mathcal{H}_i(\epsilon_i, \nu_i)$ which adds a safety layer to the given obstacle. The controller mode must be switched to the avoidance mode whenever the vehicle reaches this safety helmet. Specifically, we define for each $i \in \mathbb{I}$, a jump set

$$\mathcal{J}_0^i := \mathcal{H}_i(\epsilon_i, \nu_i) \cap \mathcal{W}, \quad (23)$$

where $\epsilon_i \in (0, 1)$ (dilating $\mathcal{E}_{\leq}(c_i, E_i)$ to $\mathcal{E}_{\leq}(c_i, \epsilon_i E_i)$), and $\nu_i \in (1, \infty)$ (shrinking $\mathcal{E}_{\geq}(\bar{c}_i, \bar{E}_i)$ to $\mathcal{E}_{\geq}(\bar{c}_i, \nu_i \bar{E}_i)$) and $\mathcal{W}$ is the free workspace defined in (14). We emphasize that we consider the intersection with $\mathcal{W}$ in (23) for convenience, but later we tune the parameters such that $\mathcal{H}_i(\epsilon_i, \nu_i) \subset \mathcal{W}$, which implies $\mathcal{J}_0^i$ will equal to $\mathcal{H}_i(\epsilon_i, \nu_i)$. The selection of $\mathcal{J}_0^i$ in (23) leads naturally to the following flow set of the stabilization mode (corresponding to the closed complement of $\mathcal{J}_0^i$ in the free workspace)

$$\mathcal{F}_0^i := \left( \mathcal{E}_{\geq}(c_i, \epsilon_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i, \nu_i \bar{E}_i) \right) \cap \mathcal{W}. \quad (24)$$

Finally, from (23) and (24), we take all the obstacles into account and define the flow and jump sets for the stabilization mode $m = 0$ as

$$\mathcal{F}_0 := \left( \bigcap_{i \in \mathbb{I}} \mathcal{F}_0^i \right) \times \mathbb{I}, \qquad \mathcal{J}_0 := \left( \bigcup_{i \in \mathbb{I}} \mathcal{J}_0^i \right) \times \mathbb{I}. \quad (25)$$

Indeed, the stabilization mode can be selected when the state $x$ belongs to the intersection of the sets $\mathcal{F}_0^i$ (and for *any* obstacle index $i \in \mathbb{I}$), and a jump to the avoidance mode can occur when the state $x$ belongs to the union of the sets $\mathcal{J}_0^i$ (and for *any* obstacle index $i \in \mathbb{I}$). In other words, if during the stabilization mode the vehicle reaches any one of the safety helmets, then the controller jumps to one of the avoidance modes with $m$ equal to $-1$ or $1$.
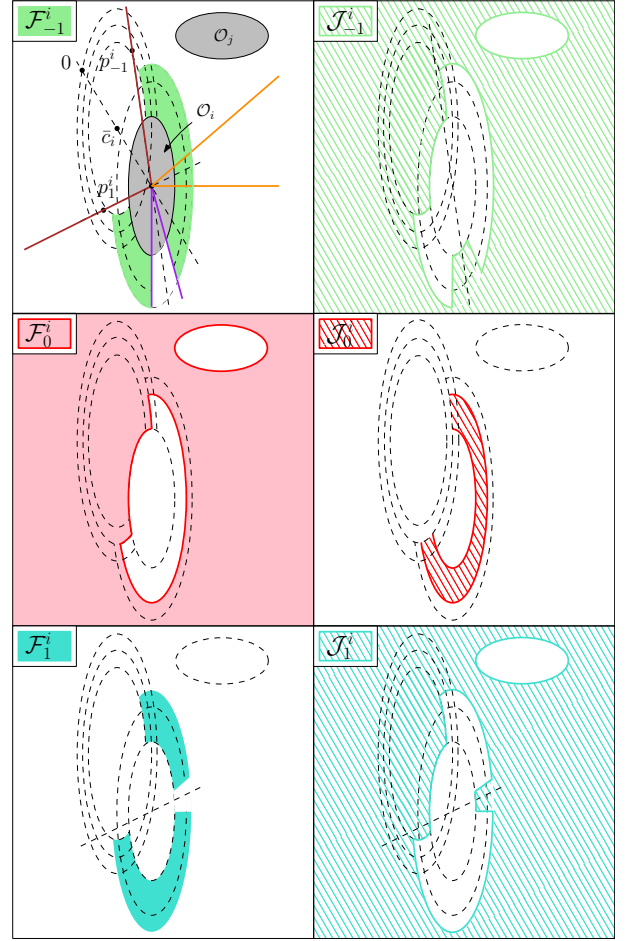


Fig. 3. 2D illustration of flow and jump sets considered in Sections IV-V corresponding to obstacle $\mathcal{O}_i$ (in the presence of a second obstacle $\mathcal{O}_j$). The stabilization-mode jump set $\mathcal{J}_0^i$ (hatched red) is constructed by using the helmet $\mathcal{H}_i(\epsilon_i, \nu_i)$, while the corresponding flow set $\mathcal{F}_0^i$ is the complement of $\mathcal{J}_0^i$ in the free workspace. For the avoidance mode we select $p_1^i$ and $p_{-1}^i$ to lie on the cone $\mathcal{C}(c_i, -c_i, \theta_i, E_i)$ (solid brown line). The avoidance flow set $\mathcal{F}_m^i$, with $m \in \{-1, 1\}$, corresponds to the helmet $\mathcal{H}_i(\delta_i, \mu_i)$ deprived of the interior of the the cone region defined by $\mathcal{C}(c_i, c_i - p_m^i, \psi_i, E_i)$ (solid purple line for $m = -1$ and solid orange line for $m = 1$). The corresponding jump set $\mathcal{J}_m^i$ is the complement of $\mathcal{F}_m^i$ in the free workspace.

*3) Avoidance Mode $m \in \{-1, 1\}$:* We consider now the construction of flow and jump sets for the avoidance modes $m \in \{-1, 1\}$ and the specific obstacle $i \in \mathbb{I}$ with the aid of Fig. 3. To highlight their motivation, we first define such flow sets and state later the corresponding jump sets (see (28)). For each $i \in \mathbb{I}$ and $m \in \{-1, 1\}$, the avoidance flow set is

$$\mathcal{F}_m^i := \mathcal{H}_i(\delta_i, \mu_i) \cap \mathcal{C}_{\geq}(c_i, c_i - p_m^i, \psi_i, E_i) \cap \mathcal{W}, \quad (26)$$

where $\delta_i \in (0, \epsilon_i)$ (dilating $\mathcal{E}_{\leq}(c_i, \epsilon_i E_i)$ to $\mathcal{E}_{\leq}(c_i, \delta_i E_i)$), $\mu_i \in (\nu_i, \infty)$ (shrinking $\mathcal{E}_{\geq}(\bar{c}_i, \nu_i \bar{E}_i)$ to $\mathcal{E}_{\geq}(\bar{c}_i, \mu_i \bar{E}_i)$), and $\psi_i \in (0, \pi/2]$. In the two configurations $m \in \{-1, 1\}$ of the avoidance of obstacle $i \in \mathbb{I}$, we want the vehicle to slide on the safety helmet $\mathcal{H}_i(\delta_i, \mu_i)$ while maintaining a constant distance to the obstacle. By selecting $\delta_i \in (0, \epsilon_i)$ and $\mu_i \in (\nu_i, \infty)$, one obtains a dilated version of $\mathcal{H}_i(\epsilon_i, \nu_i)$ used in $\mathcal{J}_0^i$ and, thus, creates a hysteresis region useful to prevent infinitely many consecutive jumps (Zeno behavior). However, the avoidance vector field $\kappa(x, i, m)$ in (16) has some undesirable equilibria,

TABLE I
SELECTION OF THE DESIGN PARAMETERS OF (30), WITH $i \in \mathbb{I}$.

| Parameter | Selection | Parameter | Selection |
|---|---|---|---|
| $\delta_i$ | $(\underline{\delta}_i, 1)$ | $\epsilon_i$ | $(\delta_i, 1)$ |
| $\mu_i$ | $(1, \bar{\mu}_i(\delta_i))$ | $\nu_i$ | $(1, \mu_i)$ |
| $\theta_i$ | $(0, \bar{\theta}_i(\delta_i, \mu_i))$ | $\bar{\psi}_i$ | $(0, \theta_i)$ |
| $k_0, k_1, k_{-1}$ | $(0, +\infty)$ | $\psi_i$ | $(0, \bar{\psi}_i)$ |
| | | $p_1^i, p_{-1}^i$ | as in (18) |

which we need to rule out from the flow sets $\mathcal{F}_1^i$ and $\mathcal{F}_{-1}^i$. These are characterized in the next lemma.

**Lemma 3:** Let $c \in \mathbb{R}^n$, $p \in \mathbb{R}^n \backslash \{c\}$ and $E \in \mathbb{R}^{n \times n}$ positive definite. For each $x \in \mathbb{R}^n \backslash \{c\}$, $\pi^\perp(E(x-c))E(x-p) = 0$ if and only if $x \in \mathcal{L}(c, p-c)$.

For each $m \in \{-1, 1\}$, $i \in \mathbb{I}$, we want solutions to eventually leave the set $\mathcal{F}_m^i$ of the avoidance mode, so it is necessary to select point $p_m^i$ and flow set $\mathcal{F}_m^i$ such that $\mathcal{L}(c_i, p_m^i - c_i) \cap \mathcal{F}_m^i = \emptyset$ based on Lemma 3, otherwise solutions could stay in avoidance mode indefinitely. This motivates the intersection with the cone in (26), and the next lemma.

**Lemma 4:** For each $i \in \mathbb{I}$, define the quantities

$$\underline{\delta}_i := \|E_i c_i\|^{-\frac{1}{2}} \tag{27a}$$

$$\bar{\mu}_i(\delta_i) := \left(1 - 4\underline{\delta}_i^2(1 - \underline{\delta}_i^2/\delta_i^2)\right)^{-\frac{1}{2}} \tag{27b}$$

$$\bar{\theta}_i(\delta_i, \mu_i) := \arccos\left(\frac{\underline{\delta}_i^2}{\delta_i^2} + \frac{1}{4\underline{\delta}_i^2}\left(1 - \frac{1}{\mu_i^2}\right)\right) \tag{27c}$$

and select the parameters $\delta_i, \mu_i, \theta_i, \psi_i$ as in Table I so that $\bar{\mu}_i(\delta_i)$ and $\bar{\theta}_i(\delta_i, \mu_i)$ are well-defined. Then, for each $m \in \{-1, 1\}$, $\mathcal{L}(c_i, p_m^i - c_i) \cap \mathcal{F}_m^i = \emptyset$.

From the flow set in (26), we suitably define the jump set for the avoidance mode, of an obstacle $i \in \mathbb{I}$ with configuration $m \in \{-1, 1\}$, to be the closed complement of $\mathcal{F}_m^i$ in the free workspace. For $i \in \mathbb{I}$ and $m \in \{-1, 1\}$,

$$\mathcal{J}_m^i := \Big(\mathcal{E}_\geq(c_i, \delta_i E_i) \cup \mathcal{E}_\leq(\bar{c}_i, \mu_i \bar{E}_i) \tag{28}$$

$$\cup\, \mathcal{C}_\leq(c_i, c_i - p_m^i, \psi_i, E_i)\Big) \cap \mathcal{W}.$$

Finally, the avoidance mode has overall flow and jump sets

$$\mathcal{F}_1 := \bigcup_{i \in \mathbb{I}} \left(\mathcal{F}_1^i \times \{i\}\right), \qquad \mathcal{J}_1 := \bigcup_{i \in \mathbb{I}} \left(\mathcal{J}_1^i \times \{i\}\right), \tag{29a}$$

$$\mathcal{F}_{-1} := \bigcup_{i \in \mathbb{I}} \left(\mathcal{F}_{-1}^i \times \{i\}\right), \quad \mathcal{J}_{-1} := \bigcup_{i \in \mathbb{I}} \left(\mathcal{J}_{-1}^i \times \{i\}\right), \tag{29b}$$

where $\mathcal{F}_m^i$ and $\mathcal{J}_m^i$ ($m \in \{-1, 1\}$) are defined in (26) and (28). Indeed, *each* obstacle $i$ gives rise, for the avoidance mode, to a specific flow (jump) set with two configurations $\mathcal{F}_1^i$ and $\mathcal{F}_{-1}^i$ ($\mathcal{J}_1^i$ and $\mathcal{J}_{-1}^i$), as we motivated in this section.

### C. Hybrid Mode Selection

In this section we define the hybrid switching strategy that permits a Zeno-free transition between the different control modes. The hybrid selection of the logical variables $i \in \mathbb{I}$ and $m \in \mathbb{M}$ is implemented in the hybrid system

$$\begin{cases} \dot{x} = \kappa(x, i, m) \\ \dot{i} = 0 \\ \dot{m} = 0 \end{cases} \qquad (x, i, m) \in \mathcal{F} \tag{30a}$$

$$\begin{cases} x^+ = x \\ \begin{bmatrix} i^+ \\ m^+ \end{bmatrix} \in \mathbf{L}(x, i, m) \end{cases} \qquad (x, i, m) \in \mathcal{J} \tag{30b}$$

where $\kappa(x, i, m)$ is the control input as defined in (16) and the flow and jump sets are given by

$$\mathcal{F} := \bigcup_{m \in \mathbb{M}} \left(\mathcal{F}_m \times \{m\}\right), \quad \mathcal{J} := \bigcup_{m \in \mathbb{M}} \left(\mathcal{J}_m \times \{m\}\right). \tag{30c}$$

with $\mathcal{F}_m$ and $\mathcal{J}_m$ being defined in (25) for $m = 0$ and in (29a)-(29b) for $m \in \{-1, 1\}$. We define now the (set-valued) jump map $\mathbf{L}$ in (30b). To this end, for $i \in \mathbb{I}$ and $m \in \{-1, 1\}$, define the sets $\mathcal{C}_m^i$ as

$$\mathcal{C}_m^i := \mathcal{C}_\geq(c_i, c_i - p_m^i, \bar{\psi}_i, E_i) \tag{30d}$$

which corresponds to the region outside the cone with vertex at $c_i$, axis $c_i - p_m^i$ and aperture $2\bar{\psi}_i$, where $\bar{\psi}_i$ is a design parameter selected below. The jump map $\mathbf{L}$ for $m \in \{-1, 1\}$ is then defined as

$$\mathbf{L}(x, i, -1) := \mathbf{L}(x, i, 1) := \{[\begin{smallmatrix} i \\ 0 \end{smallmatrix}]\}, \tag{30e}$$

i.e., when jumping to stabilization mode, the obstacle index $i$ is not used in the control law $\kappa$ in (16) and consequently is not updated. The jump map $\mathbf{L}$ for $m = 0$ is

$$\mathbf{L}(x, i, 0) := \left\{ [\begin{smallmatrix} i' \\ m' \end{smallmatrix}] : x \in \mathcal{J}_0^{i'}, m' \in \mathbf{M}(x, i') \right\} \tag{30f}$$

where $\mathbf{M}$ is defined, based on (30d), as

$$\mathbf{M}(x, i) := \begin{cases} \{-1\} & x \in \mathcal{C}_{-1}^i \backslash \mathcal{C}_1^i \\ \{1\} & x \in \mathcal{C}_1^i \backslash \mathcal{C}_{-1}^i \\ \{-1, 1\} & x \in \mathcal{C}_{-1}^i \cap \mathcal{C}_1^i. \end{cases} \tag{30g}$$

$\mathbf{L}(\cdot, \cdot, 0)$ captures that when jumping from the stabilization mode $m = 0$, the suitable avoidance mode of obstacle $i' \in \mathbb{I}$ with configuration $m' \in \{-1, 1\}$ is selected based on the position $x$ of the vehicle ($m'$, in particular, is selected based on whether $x$ is within the cone region $\mathcal{C}_{-1}^{i'}$ or $\mathcal{C}_1^{i'}$). A necessary condition to implement our hybrid controller is that the jump map is nonempty, for which we have the next lemma.

**Lemma 5:** Select the parameters $\bar{\psi}_i$ and $\psi_i$ as in Table I. Then, the set $\mathbf{L}(x, i, m)$ is nonempty for all $(x, i, m) \in \mathcal{J}$.

For compact notation, we write flow and jump maps as

$$(x, i, m) \mapsto \mathbf{F}(x, i, m) := (\kappa(x, i, m), 0, 0) \tag{30h}$$

$$(x, i, m) \mapsto \mathbf{J}(x, i, m) := (x, \mathbf{L}(x, i, m)), \tag{30i}$$

and the overall state of the hybrid system as

$$\xi := (x, i, m) \in \mathbb{R}^n \times \mathbb{I} \times \mathbb{M}. \tag{30j}$$

This completes the description of the hybrid controller in (30). The selections we made in this section for the parameters of (30) are summarized in Table I.

## V. MAIN RESULT

In this section, we show that the hybrid controller achieves forward invariance (Section V-A) and global asymptotic stability (Section V-B) (related to the objectives in Section III), as well as some complementary properties (Section V-C).

The mild regularity conditions satisfied by the hybrid system (30), as in the next lemma, allows us to invoke useful results on hybrid systems in the proof of our results.

**Lemma 6:** *The hybrid system with data $(\mathcal{F}, \mathbf{F}, \mathcal{J}, \mathbf{J})$ satisfies the hybrid basic conditions in [24, Assumption 6.5].*

### A. Forward Invariance

In this section, we show that all generated solutions are complete and safe. Since the state $x$ must evolve always within the free workspace $\mathcal{W}$ in (14) regardless of the logic variables $i$ and $m$, we seek forward invariance of the set $\mathcal{K}$ defined as:

$$\mathcal{K} := \bigcap_{i' \in \mathbb{I}} \mathcal{E}_{\geq}(c_{i'}, E_{i'}) \times \mathbb{I} \times \mathbb{M} = \mathcal{W} \times \mathbb{I} \times \mathbb{M}. \quad (31)$$

The next lemma shows that the union of flow and jump sets covers exactly the obstacle-free state space $\mathcal{K}$ and that solutions cannot leave $\mathcal{K}$ through jumps.

**Lemma 7:** $\mathcal{F} \cup \mathcal{J} = \mathcal{K}$ and $\mathbf{J}(\mathcal{J}) \subset \mathcal{K}$.
Forward invariance of $\mathcal{K}$ is proven in the next theorem.

**Theorem 1:** *Under Assumptions 1-3, consider the hybrid system* (30) *with parameters selected as in Table I. Assume also that the controller parameters $\delta_i$ are tuned so that the ellipsoids $\{\mathcal{E}_{\leq}(c_i, \delta_i E_i)\}_{i \in \mathbb{I}}$ are weakly pairwise disjoint. Then, the obstacle-free set $\mathcal{K}$ in* (31) *is forward invariant.*
The existence of tuning parameters $\delta_1, \ldots, \delta_I$ satisfying the weak pairwise disjointness of the sets $\{\mathcal{E}_{\leq}(c_i, \delta_i E_i)\}_{i \in \mathbb{I}}$ is guaranteed by Assumption 3, which implies that weak pairwise disjointness holds when $\delta_i = 1$ for all $i \in \mathbb{I}$. Hence, by a continuity argument, we can always tune each $\delta_i$ sufficiently close to 1 in order to guarantee the weak pairwise disjointness of the dilated obstacles $\{\mathcal{E}_{\leq}(c_i, \delta_i E_i)\}_{i \in \mathbb{I}}$. Note that algebraic tests of weak pairwise disjointness (provided in [25, Thm. 6] for $n = 2$ and in [26, Thm. 8] for $n = 3$) can be used for this tuning purpose.

### B. Global Asymptotic Stability

In this section we show that from all initial conditions in the free workspace, all solutions converge asymptotically to the origin. To this end, we define the notion of *sufficient disjointness* of a set of ellipsoids, which is slightly stronger than weak disjointness but less conservative than strong disjointness, and guarantees that each obstacle is avoided at most one time. The motivation behind the assumption of sufficient disjointness is that the ellipsoids considered here can be arbitrarily large and flat, which might lead to long detours during the avoidance mode that take the vehicle far away from the origin. In this case, specific configurations of the obstacles exist such that from a set of initial conditions, the vehicle does not converge to the origin although it remains safe. Similarly, in the Bug 0 planning algorithm [28], termination (i.e., convergence to the target) is not *always* guaranteed since the algorithm is designed to "walk toward the target whenever you can" [28]. Our hybrid feedback shares a similar philosophy since the vehicle jumps from avoidance to stabilization mode whenever the stabilization controller generates a vector field not pointing towards the obstacle (see (19)). To proceed, the next lemma characterizes the intersection of two ellipsoids of interest.
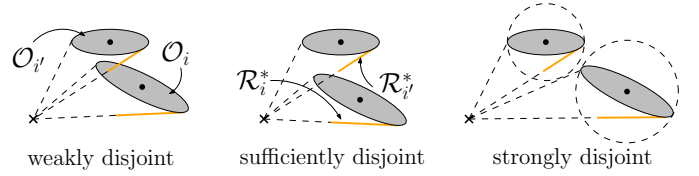


Fig. 4. Different types of disjointness introduced in the paper with set $\mathcal{R}_i^*$ (orange, see (36)). For global attractivity, sufficient disjointness is asked.

**Lemma 8:** *Consider an arbitrary $i \in \mathbb{I}$. For $\underline{\delta}_i$, $\delta \mapsto \bar{\mu}_i(\delta)$ and $(\delta, \mu) \mapsto \bar{\theta}_i(\delta, \mu)$ defined in* (27), *let $\delta \in [\underline{\delta}_i, 1]$, $\mu \in [1, \bar{\mu}_i(\delta)]$ and $\vartheta_i(\delta, \mu)$ be such that*

$$\cos(\vartheta_i(\delta, \mu)) := \frac{1 - \cos(\bar{\theta}_i(\delta, \mu))\underline{\delta}_i^2}{\sqrt{(1 + \mu^{-2})/2 - \delta^{-2}\underline{\delta}_i^4}}. \quad (32)$$

*The expression in* (32) *is well-defined and positive, and*

$$\mathcal{E}(c_i, \delta E_i) \cap \mathcal{E}(\bar{c}_i, \mu \bar{E}_i) \subset \mathcal{C}(0, c_i, \vartheta_i(\delta, \mu), E_i). \quad (33)$$

Let us consider for each obstacle $i \in \mathbb{I}$ the sphere $\mathcal{S}(0, \bar{r}_i)$ with center at the origin and radius $\bar{r}_i$ defined by the next quadratic optimization problem

$$\bar{r}_i^2 := \min \|x\|^2 \quad \text{subject to} \quad x \in \mathcal{H}_i^* \quad (34)$$

where $\mathcal{H}_i^*$ is the helmet defined in (21). The radius $\bar{r}_i$ defines the minimum distance from the helmet $\mathcal{H}_i^*$ to the origin. Let $x$ be a point belonging to the intersection of the two ellipsoids $\mathcal{E}(c_i, E_i)$ and $\mathcal{E}(\bar{c}_i, \bar{E}_i)$. Taking $\delta$ and $\mu$ equal to 1 in Lemma 8, one obtains $x \in \mathcal{C}(0, c_i, \bar{\vartheta}_i, E_i)$ with

$$\cos(\bar{\vartheta}_i) := \cos(\vartheta_i(1, 1)) = \sqrt{1 - \|E_i c_i\|^{-2}}, \quad (35)$$

from (32), (27c) and (27a). Now, let us define the set

$$\mathcal{R}_i^* := \mathcal{C}(0, c_i, \bar{\vartheta}_i, E_i) \cap \mathcal{S}_{\geq}(0, \bar{r}_i) \cap \mathcal{E}_{\geq}(c_i, E_i) \cap \mathcal{E}_{\leq}(\bar{c}_i, \bar{E}_i), \quad (36)$$

whose geometry is sketched in Fig. 4. Intuitively speaking, it is a subset of all points on the cone $\mathcal{C}(0, c_i, \bar{\vartheta}_i, E_i)$ that have a distance to the origin greater than the distance $\bar{r}_i$ of the helmet $\mathcal{H}_i^*$ to the origin. The idea is that the vehicle should not to start avoiding another obstacle while it is still in $\mathcal{R}_i^*$, otherwise there is no guarantee that the number of times the vehicle avoids the obstacles is bounded and that global attractivity holds. This motivates the next definition.

**Definition 3:** *The ellipsoids $\{\mathcal{E}(c_i, E_i)\}_{i \in \mathbb{I}}$ are sufficiently pairwise disjoint if they are weakly pairwise disjoint and*

$$\forall i, i' \in \mathbb{I} \text{ with } i \neq i', \quad \mathcal{R}_i^* \cap \mathcal{E}_{\leq}(c_{i'}, E_{i'}) = \emptyset. \quad (37)$$

Now, let us introduce the ingredients for a dilated version of $\mathcal{R}_i^*$ as in (39) below and refer to Fig. 5. First, consider the *escape annulus* cone where solutions escape from the avoidance mode by applying the stabilization vector field. This region lies between the two cones $\mathcal{C}(0, c_i, \vartheta_i(1, \mu_i), E_i)$ and $\mathcal{C}(0, c_i, \vartheta_i(\delta_i, \mu_i), E_i)$ which are related, according to Lemma 8, to the intersections $\mathcal{E}(c_i, E_i) \cap \mathcal{E}(\bar{c}_i, \mu_i \bar{E}_i)$ and $\mathcal{E}(c_i, \delta_i E_i) \cap \mathcal{E}(\bar{c}_i, \mu_i \bar{E}_i)$, respectively. Second, consider for each obstacle $i \in \mathbb{I}$ the ball $\mathcal{S}_{\geq}(0, r_i)$ where the radius $r_i$ is defined by the quadratic optimization problem

$$r_i^2 := \min \|x\|^2 \quad \text{subject to} \quad x \in \mathcal{H}_i(\delta_i, \mu_i). \quad (38)$$
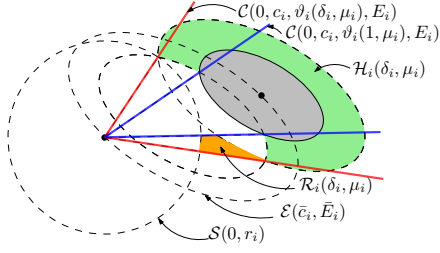
Fig. 5. Safety helmet $\mathcal{H}_i(\delta_i, \mu_i)$ (green) and the corresponding escape region $\mathcal{R}_i(\delta_i, \mu_i)$ (orange). The region $\mathcal{R}_i(\delta_i, \mu_i)$ must not intersect with any other jump set $\mathcal{J}_0^{i'}$, $i' \neq i$, to avoid starting another avoidance while the distance to target has not yet decreased.

Note the following on (38). 1) The safety helmet $\mathcal{H}_i(\delta_i, \mu_i)$ is compact and, hence, the solution to (38) exists. 2) For each $i \in \mathbb{I}$, $r_i > 0$. Indeed, for each $i \in \mathbb{I}$, $\|\delta_i E_i c_i\| = \delta_i \underline{\delta}_i^{-2} > \delta_i \underline{\delta}_i^{-1} > 1$ by Assumption 2 and the selection of $\delta_i$ in Table I, so that $0 \notin \mathcal{E}_{\leq}(c_i, \delta_i E_i)$ and in turn $0 \notin \mathcal{H}_i(\delta_i, \mu_i)$ ($\mathcal{H}_i(\delta_i, \mu_i) \subset \mathcal{E}_{\leq}(c_i, \delta_i E_i)$). Hence, since $\mathcal{H}_i(\delta_i, \mu_i)$ is compact there exists $r_i > 0$ such that $\|x\| \geq r_i$ for all $x \in \mathcal{H}_i(\delta_i, \mu_i)$. Finally, we can define the considered dilated version of $\mathcal{R}_i^*$ as

$$\mathcal{R}_i(\delta_i, \mu_i) := \mathcal{S}_{\geq}(0, r_i) \cap \mathcal{E}_{\geq}(c_i, \delta_i E_i) \cap \mathcal{E}_{\leq}(\bar{c}_i, \bar{E}_i)$$
$$\cap \, \mathcal{C}_{\geq}(0, c_i, \vartheta_i(1, \mu_i), E_i) \cap \mathcal{C}_{\leq}(0, c_i, \vartheta_i(\delta_i, \mu_i), E_i). \quad (39)$$

**Lemma 9:** *Assume that the obstacles $\{\mathcal{O}_i\}_{i \in \mathbb{I}}$ are sufficiently pairwise disjoint. Then, for each $i \in \mathbb{I}$, there exist $\delta_i^*, \mu_i^*$ such that for all $\delta_i \in (\delta_i^*, 1)$ and $\mu_i \in (1, \mu_i^*)$, we have*

$$\forall i', i'' \in \mathbb{I}, i' \neq i'', \ \mathcal{R}_{i'}(\delta_{i'}, \mu_{i'}) \cap \mathcal{E}_{\leq}(c_{i''}, \delta_{i''} E_{i''}) = \emptyset. \quad (40)$$

Property (40) of Lemma 9 will be used to show global attractivity. Intuitively, we require that after avoiding an obstacle, the distance $\|x\|$ to the target decreases before the vehicle reaches the proximity of another obstacle. Although the bounds $\delta_i^*$ and $\mu_i^*$ are not defined explicitly for generic ellipsoids, the parameters $\delta_i$ and $\mu_i$ can be tuned offline. Now, we are ready to state our main result for this section.

**Theorem 2:** *Consider the hybrid system* (30) *under the same assumptions as Theorem 1. Assume also that the obstacles $\{\mathcal{O}_i\}_{i \in \mathbb{I}}$ are sufficiently pairwise disjoint, and $\delta_i$ and $\mu_i$ are tuned such that* (40) *holds. Then, the set $\mathcal{A} := \{0\} \times \mathbb{I} \times \mathbb{M}$ is globally asymptotically stable for* (30) *and the number of jumps is bounded.*

For spherical obstacles, we show next that the extra tuning of the parameters to satisfy (40) is not needed.

**Theorem 3:** *(Spherical obstacles) Let $E_i = \lambda_i I_n$ for all $i \in \mathbb{I}$. Under the same assumptions as Theorem 1, the set $\mathcal{A} := \{0\} \times \mathbb{I} \times \mathbb{M}$ is globally asymptotically stable for* (30) *and the number of jumps is bounded.*

### C. Complementary Properties

In this section we present four relevant complementary properties of the proposed hybrid law for obstacle avoidance.

*1) Bounded Control:* First, we can show that $x$ remains always in a given ball. Indeed, let $\mathcal{S}_{\leq}(0, r_b)$, with $r_b > 0$, be the smallest ball containing all the dilated ellipsoids $\mathcal{E}(c_i, \delta_i E_i)$ (which must exist since these ellipsoids are compact). During stabilization mode the distance $\|x\|$ is decreasing and during avoidance mode the vehicle stays within the dilated ellipsoids $\mathcal{E}(c_i, \delta_i E_i)$. Then, it is guaranteed that from all $x(0,0) \in \mathcal{S}_{\leq}(0, r_b)$, $x(t,j) \in \mathcal{S}_{\leq}(0, r_b)$ for all $(t, j) \in \operatorname{dom} x$. Moreover, since the projection matrix $\pi^{\perp}(E_i(x - c_i))$ has eigenvalues in 0 and 1, it follows that we can upper bound the control input in (16) by $\|u\| \leq k\alpha(r_b + p)$ where $k = \max\{k_1, k_0, k_{-1}\}$, $\alpha = \max_{i \in \mathbb{I}}(\lambda_{\max}(E_i)/\lambda_{\min}(E_i))$ and $p = \max_{i \in \mathbb{I}} \|p_1^i\|$. The control gains can then be tuned to satisfy the inherent practical saturation of the actuators.

*2) Semiglobal Preservation:* The second property is the so-called *semiglobal preservation* property [18, §II]. This property is desirable when the original controller parameters are optimally tuned and the controller modifications imposed by the presence of the obstacles should be as minimal as possible. Such a property is also accounted for in the quadratic programming formulation of [29, III.A.]. We summarize this property for our case in the next proposition.

**Proposition 1:** *Be $\epsilon \in (0, 1)$ and $\mathcal{W}_{\epsilon} := \bigcap_{i' \in \mathbb{I}} \mathcal{E}_{\geq}(c_{i'}, \epsilon E_{i'})$. There exist controller parameters such that the control law matches, in $\mathcal{W}_{\epsilon}$, the stabilization feedback $u = -k_0 x$ ($k_0 > 0$) used in the absence of obstacles.*

*3) Non-point Mass Vehicles:* There is no loss of generality in considering a point-mass vehicle in this work. Let us rather consider that the vehicle has some volume, *e.g.,* bounded by $\mathcal{S}_{\leq}(x, r_v)$. Then, for the navigation scenario to be feasible, the radius $r_v$ of the vehicle needs to be smaller than the smallest distance between the obstacles, *i.e.,* for all $i, i' \in \mathbb{I}$ with $i \neq i'$, $r_v < \operatorname{dist}(\mathcal{E}_{\leq}(c_i, E_i), \mathcal{E}_{\leq}(c_{i'}, E_{i'}))$. For the safety of the vehicle during the stabilization mode, selecting the parameter $\epsilon_i$ as $\epsilon_i < (1 + \lambda_{\max}(E_i) r_v)^{-1}$ is sufficient (in addition to Table I) to guarantee that the vehicle starts the avoidance mode away from the obstacle. Indeed, under this condition, it is easy to show that for all $x \in \mathcal{E}_{\geq}(c_i, \epsilon_i E_i)$ (*i.e.,* the vehicle center is outside the dilated ellipsoid $\mathcal{E}(c_i, \epsilon_i E_i)$) and for all $x' \in \mathcal{S}_{\leq}(x, r_v)$, one has $x' \in \mathcal{E}_{\geq}(c_i, E_i)$, which guarantees safety of the whole volume of the vehicle.

*4) Robustness:* The constructed hybrid controller guarantees some level of robustness to perturbations (e.g., in the form of measurement noise). Hysteresis switching is one of the typical ways to ensure robustness to measurement noise, and hysteresis switching is indeed behind the designed hybrid feedback, in particular the hysteresis regions of flow and jump sets in Section IV-B and the logical selections of the jump sets in Section IV-C. More generally, fundamental results in [24, Chap. 7] guarantee structurally that global asymptotic stability of $\mathcal{A}$ in Theorem 2 is also uniform (by [24, Thm. 7.12]) and robust (by [24, Thm. 7.21]) with respect to perturbations since $\mathcal{A}$ is a compact set and the hybrid basic conditions are satisfied as in Lemma 6.

## VI. SIMULATIONS

We illustrate the effectiveness of the proposed hybrid control strategy through two simulation scenarios. The first scenario
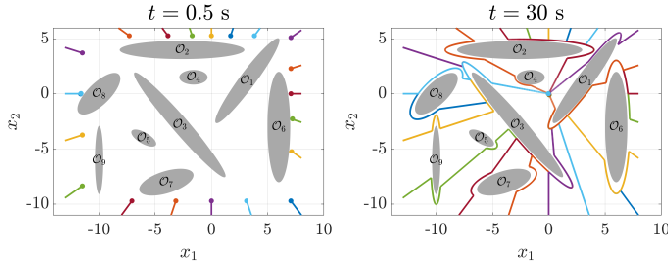
Fig. 6. Plot (at time $t = 0.5$ and $t = 30$ seconds) of the 2-dimensional trajectory of the vehicle starting at different initial conditions.

considers 9 obstacles in 2D (see Fig. 6) while the second one considers 5 obstacles in 3D (see Fig. 7). For both cases, Table I provides a suitable order to choose the parameters for each $i \in \mathbb{I}$, as follows.

1) For $\underline{\delta}_i$ in (27a), select $\delta_i$ and $\epsilon_i$ so that $\underline{\delta}_i < \delta_i < \epsilon_i < 1$;
2) For $\delta_i$ and $\bar{\mu}_i(\delta_i)$ in (27b), select $\nu_i$ and $\mu_i$ so that $1 < \nu_i < \mu_i < \bar{\mu}_i(\delta_i)$ (possibly iterating steps 1) and 2) so that $\delta_i$ and $\mu_i$ satisfy (40));
3) For $\delta_i$, $\mu_i$ and $\bar{\theta}_i(\delta_i, \mu_i)$ in (27c), select $\psi_i$, $\bar{\psi}_i$ and $\theta_i$ so that $0 < \psi_i < \bar{\psi}_i < \theta_i < \bar{\theta}_i(\delta_i, \mu_i)$.

Any parameter selection according to this guideline guarantees our results, and can be carried out keeping in mind the physical interpretation illustrated in Section IV-B for these parameters. The gains are $k_0 = k_1 = k_{-1} = 1/4$ and determine the speed of convergence of the scheme. By (18a), the point $p_1^i$ can be selected arbitrarily as long as it is on $\mathcal{C}(c_i, -c_i, \theta_i, E_i) \setminus \{c_i\}$. A suitable choice is given by

$$p_1^i = \pi^\perp (E_i^{-1} \mathbf{R}(\theta_i) E_i c_i) c_i \tag{41}$$

where $\mathbf{R}(\theta_i)$ is the standard $2 \times 2$ rotation matrix with angle $\theta_i$ or the standard $3 \times 3$ axis-angle rotation matrix with angle $\theta_i$ and an arbitrary vector of $\mathbb{S}^2$ as axis. The idea behind (41) is to project $c_i$ on the plane orthogonal to a rotated version of $c_i$, in order to obtain the point lying on the cone and closest to the origin. Having all points $p_m^i$ close enough to the origin is an effective way so that $k_0$, $k_1$, $k_{-1}$ can take the same values and yield comparable speeds for avoidance and stabilization, independently of the obstacles.

Fig. 6 (Fig. 7, respectively) shows that the solution generated by the closed loop hybrid system avoids the 2D obstacles (3D obstacles, respectively) and Fig. 8 shows the convergence of solutions to the origin. Complete simulation videos for the 2D and 3D cases can be found at https://youtu.be/CnXJlhzlzd8, https://youtu.be/4mzTXPR6D9Y.

Finally, we note that for the very obstacle configuration of the 2D scenario, the state-of-the-art approach of navigation functions [3], [21] cannot be applied since the condition [21, Thm. 3, Eq. (23)] is violated for all obstacles except obstacle $\mathcal{O}_5$, where [21, Eq. (23)] intuitively corresponds to the fact that obstacles are not too flat and not too close to the target position. ([21, Eq. (23)] is violated for all obstacles of the 3D scenario.) Moreover, navigation function approaches require tuning a parameter sufficiently large ($k$ in [21, Eq. (17) and Remark 5]), which may conflict with actuator limitations. Instead, our approach provides a clear tuning guideline for
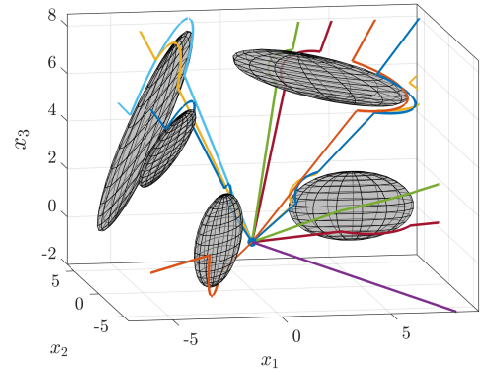


Fig. 7. Plot (at time $t = 30$ seconds) of the 3-dimensional trajectory of the vehicle starting at different initial conditions.
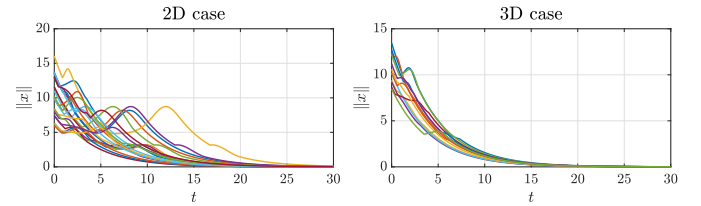


Fig. 8. Plot of the position norm $\|x\|$ versus time showing the convergence of the solutions to the origin from the considered initial conditions.

all parameters (given in this section) and actuator limitations can be taken into account (see Section V-C1).

## VII. Conclusions

We proposed a novel hybrid feedback on $\mathbb{R}^n$ to solve the obstacle avoidance problem for arbitrarily flat ellipsoidal obstacles. Our control strategy ensures global asymptotic stabilization to the target and safety (thus, successful navigation from all initial conditions) while guaranteeing a Zeno-free switching between the avoidance and stabilization modes. Moreover, the control input remains bounded (in particular, arbitrarily close to any obstacle) and matches semi-globally in the free-state space the nominal feedback used in the absence of obstacles.

Future work will be devoted to considering more complex vehicle dynamics (*e.g.*, under-actuated and second-order dynamics) and more generic obstacle shapes (*e.g.*, convex obstacles). Furthermore, although our scheme considers static obstacles to obtain formal guarantees for global asymptotic stability and safety, extending this approach to deal with unknown environments is an interesting research direction that we aim at pursuing in the future.

## Appendix

In the appendix, an equation number over an (in)equality indicates which equation has been used to obtain the (in)equality.

*1) Proof of Theorem 1:* Define $\mathbf{S}_{\mathscr{H}}(\mathcal{K})$ as the set of all maximal solutions $\phi$ to $\mathscr{H}$ with $\phi(0,0) \in \mathcal{K}$. Each $\phi \in \mathbf{S}_{\mathscr{H}}(\mathcal{K})$ has range $\mathrm{rge}\,\phi \subset \mathcal{K} = \mathcal{F} \cup \mathcal{J}$ by Lemma 7 and the definition of hybrid solution [24, p. 124], so $\mathcal{K}$ is forward pre-invariant [30, Def. 3.3]. The set $\mathcal{K}$ is in fact forward invariant [30, Def. 3.3] if for each $\xi \in \mathcal{K}$ there exists

one solution and each $\phi \in \mathbf{S}_{\mathscr{H}}(\mathcal{K})$ is complete, which we show in the rest of the proof through [24, Prop. 6.10]. In the rest of the proof, let

$$\mathcal{F}_0^* := \bigcap_{i \in \mathbb{I}} \mathcal{F}_0^i, \quad \mathcal{J}_0^* := \bigcup_{i \in \mathbb{I}} \mathcal{J}_0^i. \tag{42}$$

**Lemma 10:** *Under the assumptions of Theorem 1, we have for each $i \in \mathbb{I}$ and $m \in \{-1, 1\}$*

$$\mathcal{J}_0^i = \mathcal{H}_i(\epsilon_i, \nu_i), \tag{43a}$$

$$\mathcal{F}_m^i = \mathcal{H}_i(\delta_i, \mu_i) \cap \mathcal{C}_{\geq}(c_i, c_i - p_m^i, \psi_i, E_i), \tag{43b}$$

$$\partial \mathcal{F}_0^* \backslash \mathcal{J}_0^* \subset \bigcup_{i \in \mathbb{I}} \left( \mathcal{E}(c_i, E_i) \backslash \mathcal{E}_{\geq}(\bar{c}_i, \bar{E}_i) \right), \tag{43c}$$

$$\begin{aligned}\partial \mathcal{F}_m^i \backslash \mathcal{J}_m^i \subset &\mathcal{E}(c_i, E_i) \backslash (\mathcal{E}_{\leq}(\bar{c}_i, \mu_i \bar{E}_i) \\ &\cup \mathcal{C}_{\leq}(c_i, c_i - p_m^i, \psi_i, E_i)).\end{aligned} \tag{43d}$$

First, let us show that the viability condition

$$\mathbf{F}(x, i, m) \cap \mathbf{T}_{\mathcal{F}}(x, i, m) \neq \emptyset \tag{44}$$

holds for all $(x, i, m) \in \mathcal{F} \backslash \mathcal{J}$. Let $(x, i, m) \in \mathcal{F} \backslash \mathcal{J}$, which implies by (30c) that $(x, i) \in \mathcal{F}_m \backslash \mathcal{J}_m$ for some $m \in \mathbb{M}$, and divide into the cases $m = 0$ and $m \in \{-1, 1\}$. When $m = 0$, from (25) there exists $i \in \mathbb{I}$ such that $x \in \mathcal{F}_0^* \backslash \mathcal{J}_0^*$. If $x \in (\mathcal{F}_0^*)^\circ \backslash \mathcal{J}_0^*$ (hence, $x$ is in the interior of $\mathcal{F}_0^*$), then $\mathbf{T}_{\mathcal{F}_0^*}(x) = \mathbb{R}^n$, so that $\mathbf{T}_{\mathcal{F}}(\xi) = \mathbb{R}^n \times \{0\} \times \{0\}$ and (44) holds. If $x \in \partial \mathcal{F}_0^* \backslash \mathcal{J}_0^*$, which satisfies the set inclusion (43c), the weak pairwise disjointness of $\{\mathcal{E}(c_i, E_i)\}_{i \in \mathbb{I}}$ yields:

$$\begin{aligned}x &\in \mathcal{E}(c_i, E_i), \ i \in \mathbb{I} \\ \mathbf{T}_{\mathcal{F}}(x, i, 0) &= \mathcal{P}_{\geq}(0, E_i^2(x - c_i)) \times \{0\} \times \{0\}.\end{aligned} \tag{45}$$

By (19) and $x \notin \mathcal{E}_{\geq}(\bar{c}_i, \bar{E}_i)$ by (43c), we obtain

$$-k_0 x^\top E_i^2 (x - c_i) = k_0 \|E_i \bar{c}_i\|^2 \left(1 - \|\bar{E}_i(x - \bar{c}_i)\|^2\right) > 0, \tag{46}$$

hence, $\kappa(x, i, 0) \in \mathcal{P}_{\geq}(0, E_i^2(x - c_i))$ in (45), and (44) holds for $m = 0$. When $m \in \{-1, 1\}$, we have $i \in \mathbb{I}$ and $x \in \partial \mathcal{F}_m^i \backslash \mathcal{J}_m^i$, which satisfies the set inclusion (43d), and so

$$\mathbf{T}_{\mathcal{F}}(x, i, m) = \mathcal{P}_{\geq}(0, E_i^2(x - c_i)) \times \{0\} \times \{0\}. \tag{47}$$

$\kappa(x, i, m) \in \mathcal{P}_{\geq}(0, E_i^2(x - c_i))$ in (47) because

$$-k_m(x - p_m^i)^\top E_i \pi^\perp(E_i(x - c_i)) E_i(x - c_i) = 0, \tag{48}$$

so the viability condition (44) holds for $m \in \{-1, 1\}$ as well.

Second, we apply [24, Prop. 6.10]. By it and (44), there exists a nontrivial solution to $\mathscr{H}$ from each initial condition in $\mathcal{K}$. Finite escape times can only occur through flow. They can neither occur for $x$ in the set $\mathcal{F}_{-1}^i \cup \mathcal{F}_1^i$ ($\mathcal{F}_{-1}^i$ and $\mathcal{F}_1^i$ are bounded by their definitions in (26)) nor for $x$ in the set $\mathcal{F}_0^*$ because they would make $x^\top x$ grow unbounded, and this would contradict that $\frac{d}{dt}(x^\top x) \leq 0$ by the definition of $\kappa(x, i, 0)$ and by (30a). So, all maximal solutions do not have finite escape times. By Lemma 7, $\mathbf{J}(\mathcal{J}) \subset \mathcal{K} = \mathcal{F} \cup \mathcal{J}$. Hence, by [24, Prop. 6.10], all maximal solutions are complete.

*2) Proof of Theorem 2:* We prove global asymptotic stability of $\mathcal{A}$ by [24, Def. 7.1]. For each $i \in \mathbb{I}$, $\|\delta_i E_i c_i\| = \delta_i \underline{\delta}_i^{-2} > \delta_i \underline{\delta}_i^{-1} > 1$ by Assumption 2 and the selection of $\delta_i$ in Table I, so $0 \notin \mathcal{E}_{\leq}(c_i, \delta_i E_i)$. As a consequence, there exists $\varepsilon^* > 0$ such that the ball $\mathcal{S}_{\leq}(0, \varepsilon^*)$ does not intersect with any of the dilated obstacles $\mathcal{E}_{\leq}(c_i, \delta_i E_i)$. It can be shown easily that for each $\varepsilon \in [0, \varepsilon^*]$, the set $\mathcal{S} := \mathcal{S}_{\leq}(0, \varepsilon) \times \mathbb{I} \times \mathbb{M}$ is forward invariant because $\mathcal{S}_{\leq}(0, \varepsilon)$ is disjoint from $\mathcal{J}_0^*$ and the component $x$ of solutions evolves, after at most one jump, with the stabilization mode $\dot{x} = -k_0 x$. Thanks to forward invariance of $\mathcal{S}$, stability of $\mathcal{A}$ for (30) is immediate from [24, Def. 7.1]. Let us prove global attractivity of $\mathcal{A}$. Before that, we need the next intermediate result.

*Lemma 11: There exists $\sigma > 0$ such that for all solutions $\xi = (x, i, m)$ with $\xi(t, j) \in \mathcal{F}_l \times \{l\}$ for some $l \in \{-1, 1\}$ and $(t, j) \in \operatorname{dom} \xi$, there exists $(s, \ell) \in \operatorname{dom} \xi$ such that $(s, \ell) \succeq (t, j)$ and*

$$\|x(s, \ell)\| \leq \|x(t, j)\| - \sigma. \tag{49}$$

Now, for each solution $\xi$ to (30), there exists a finite time $(T, J) \succeq (0, 0)$ after which the solution does not evolve with the avoidance controller any longer, *i.e.*, $m(t, j) = 0$ for all $(t, j) \succeq (T, J)$. Otherwise, there would exist a sequence of hybrid times $\{(t_k, j_k)\}_{k=0}^\infty$ such that $\xi(t_k, j_k) \in \mathcal{F}_{l_k} \times \{l_k\}$ with $l_k \in \{-1, 1\}$ and this would imply by Lemma 11 that $\|x(t_{k+1}, j_{k+1})\| \leq \|x(t_k, j_k)\| - \sigma$ for all $k \in \mathbb{N}$. This is indeed a contradiction as it would lead to $\|x(\cdot, \cdot)\|$ becoming negative. Then, the solution $\xi$ enters the stabilizing mode $m = 0$ after $(T, J)$ and its flow map $\dot{x} = -k_0 x$ guarantees in turn global attractivity. Moreover, $J$ is the maximum number of jumps of the hybrid system since any extra jump will cause $m$ to take values in $\{-1, 1\}$, which is not possible after $(T, J)$.

*3) Proof of Theorem 3:* To prove the theorem, it is sufficient to show that for spherical obstacles the result of Lemma 11 holds. The proof of Lemma 11 under the assumptions of Theorem 3 is the same up to (85). From (85) we have $x(t', j+1) = x(t', j) \in \tilde{\mathcal{P}}_{m(t', j), 3}^{i(t', j)}(\delta, \psi)$, $i(t', j+1) = i(t', j) =: \iota$ and $m(t', j) = 0$. However, since $x(t, j)$ and $x(t', j+1)$ both belong to $\mathcal{E}(c_\iota, \delta E_\iota)$, we can write $\|x(t, j) - c_\iota\|^2 = \|x(t', j+1) - c_\iota\|^2$ (since $E_\iota = \lambda_\iota I_n$) and hence

$$\|x(t, j)\|^2 - 2c_\iota^\top x(t, j) = \|x(t', j+1)\|^2 - 2c_\iota^\top x(t', j+1). \tag{50}$$

$x(t', j+1) \in \tilde{\mathcal{P}}_{0,3}^\iota(\delta, \psi) \subset \mathcal{E}(\bar{c}_\iota, \mu_\iota \bar{E}_\iota)$ implies also that

$$c_\iota^\top x(t', j+1) = \|x(t', j+1)\|^2 + (1 - \mu_\iota^{-2})\|\bar{c}_\iota\|^2,$$

thus, with (50), we have

$$c_\iota^\top x(t, j) = \frac{\|x(t, j)\|^2 + \|x(t', j+1)\|^2}{2} + (1 - \mu_\iota^{-2})\|\bar{c}_\iota\|^2.$$

However, since $x(t, j) \in \mathcal{E}_{\geq}(\bar{c}_\iota, \mu_\iota \bar{E}_\iota)$, we have

$$c_\iota^\top x(t, j) \leq \|x(t, j)\|^2 + (1 - \mu_\iota^{-2})\|\bar{c}_\iota\|^2,$$

and, hence, $\|x(t', j+1)\|^2 \leq \|x(t, j)\|^2$ must hold. Also, by (85), $x(t', j+1) \in \mathcal{E}(c_\iota, \delta E_\iota) \cap \mathcal{E}(\bar{c}_\iota, \mu_\iota \bar{E}_\iota)$, and, by Lemma 8, $x(t', j+1) \in \mathcal{C}(0, c_\iota, \vartheta_\iota(\delta, \mu_\iota), E_\iota)$. In view of Step 2 of the proof of Lemma 11, both the sets $\mathcal{E}_{\leq}(\bar{c}_\iota, \bar{E}_\iota)$

and $\mathcal{C}(0, c_\iota, \vartheta_\iota(\delta, \mu_\iota), E_\iota)$ are forward invariant under the stabilization flow map for $x$, i.e., $-k_0 x$. Since the obstacles are weakly disjoint, the solution then flows in $\mathcal{E}_\leq(\bar{c}_\iota, \bar{E}_\iota) \cap \mathcal{C}(0, c_\iota, \vartheta_\iota(\delta, \mu_\iota), E_\iota)$ until it reaches the set $\mathcal{E}(c_\iota, \delta_\iota E_\iota)$ at $(t'', j + 1)$. Since flow with stabilization mode decreases the distance to the origin we have

$$\|x(t'', j + 1)\|^2 \leq \|x(t', j + 1)\|^2 \leq \|x(t, j)\|^2$$

Also, the solution must flow after $(t'', j + 1)$ up to some $(s, j + 1)$ with the stabilization mode (since obstacles are weakly disjoint) such that at least $\sigma_1$ in (86) is traversed, i.e.,

$$\|x(s, j + 1)\|^2 + \sigma_1 \leq \|x(t'', j + 1)\|^2 \leq \|x(t, j)\|^2.$$

This proves Lemma 11 and in turn Theorem 2.

*4) Proof of Proposition 1:* Note preliminarily that thanks to $\epsilon < 1$, $\mathcal{W}_\epsilon \subset \mathcal{W}$ in (14). It is sufficient to show that the closed loop system under the proposed hybrid feedback cannot flow except with stabilization mode $m = 0$ when $x \in \mathcal{W}_\epsilon$. Indeed, if in Table I we further constrain $\delta_i$ as $\delta_i \in (\max(\underline{\delta}_i, \epsilon), 1)$ for all $i \in \mathbb{I}$, then $\mathcal{F}_m^i \subset \mathcal{H}(\delta_i, \mu_i) \subset \mathcal{E}_\leq(c_i, \delta_i E_i) \subset \mathcal{E}_\leq(c_i, \epsilon E_i)$ and $\mathcal{E}_\leq(c_i, \delta_i E_i) \neq \mathcal{E}_\leq(c_i, \epsilon E_i)$. Therefore, we have $\mathcal{F}_m^i \cap \mathcal{W}_\epsilon = \emptyset$ for all $i \in \mathbb{I}$ and $m \in \{-1, 1\}$. This implies that solutions cannot flow with the avoidance mode when $x$ belongs to $\mathcal{W}_\epsilon$ and must then flow with the stabilization mode.

*5) Proof of Lemma 1:* Let $x_l \in \mathcal{C}_\leq(c, E^{-1}v_l, \psi_l, E)\backslash\{c\}$, $l = 1, 2$, and be otherwise arbitrary. Define then $z_l := E(x_l - c)/\|E(x_l - c)\| \in \mathbb{S}^{n-1}$ for $l = 1, 2$. Hence, $z_l \in \mathcal{S}_l := \mathcal{C}_\leq(0, v_l, \psi_l, I_n) \cap \mathbb{S}^{n-1}$, $l = 1, 2$. For $l = 1, 2$, $z_l$ satisfies, by (11), $\cos(\psi_l) \leq z_l^\top v_l$, and consequently $\mathbf{d}_{\mathbb{S}^{n-1}}(v_l, z_l) \leq \psi_l$. It follows from the triangle inequality that $\theta = \mathbf{d}_{\mathbb{S}^{n-1}}(v_1, v_2) \leq \mathbf{d}_{\mathbb{S}^{n-1}}(v_1, z_1) + \mathbf{d}_{\mathbb{S}^{n-1}}(z_1, z_2) + \mathbf{d}_{\mathbb{S}^{n-1}}(v_2, z_2) \leq \mathbf{d}_{\mathbb{S}^{n-1}}(z_1, z_2) + \psi_1 + \psi_2$. Hence, in view of the condition $\psi_1 + \psi_2 < \theta$, $\mathbf{d}_{\mathbb{S}^{n-1}}(z_1, z_2) > 0$. This fact implies that the compact sets $\mathcal{S}_1$ and $\mathcal{S}_2$ (and in turn $\mathcal{C}_\leq(c, E^{-1}v_l, \psi_l, E)\backslash\{c\}$, $l = 1, 2$) are disjoint.

*6) Proof of Lemma 2:* First, by (18b) and (3) we have

$$p_{-1}^i - c_i = -E_i^{-1}\rho(E_i c_i)E_i(p_1^i - c_i), \text{ hence} \tag{51}$$

$$\|E_i(p_{-1}^i - c_i)\|^2 \overset{(51)}{=} (p_1^i - c_i)^\top E_i \rho(E_i c_i)\rho(E_i c_i)E_i(p_1^i - c_i)$$

$$\overset{(4c)}{=} (p_1^i - c_i)^\top E_i^2(p_1^i - c_i) = \|E_i(p_1^i - c_i)\|^2 \tag{52}$$

(so that $p_1^i \neq c_i$ implies $p_{-1}^i \neq c_i$). Based on (11) for $\mathcal{C}(c_i, -c_i, \theta_i, E_i)$, one has

$$-c_i^\top E_i^2(p_{-1}^i - c_i) \overset{(51)}{=} c_i^\top E_i \rho(E_i c_i)E_i(p_1^i - c_i)$$

$$\overset{(4c)}{=} -c_i^\top E_i^2(p_1^i - c_i) \overset{(18a)}{=} \cos(\theta_i)\|E_i(-c_i)\|\|E_i(p_1^i - c_i)\|$$

$$\overset{(52)}{=} \cos(\theta_i)\|E_i(-c_i)\|\|E_i(p_{-1}^i - c_i)\|.$$

This concludes by (11) that $p_{-1}^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i)\backslash\{c_i\}$.

*7) Proof of Lemma 3:* As for the $\Longrightarrow$ implication, let $x \in \mathbb{R}^n\backslash\{c\}$ be such that $\pi^\perp(E(x - c))E(x - p) = 0$, which is equivalent to $\pi^\perp(E(x - c))E(p - c) = 0$. By substituting the definition of $\pi^\perp(\cdot)$ in (3), one obtains $\|E(x - c)\|^2(p - c) = ((p - c)^\top E^2(x - c))(x - c)$. This very equation excludes that $(p - c)^\top E^2(x - c) = 0$ since $E$ is positive definite, $x \neq c$, and $p \neq c$ by assumption. So, letting $\lambda = \|E(x - c)\|^2/((p - c)^\top E^2(x - c))$ in (7), one deduces that $x \in \mathcal{L}(c, p - c)$. The $\Longleftarrow$ implication is straightforward.

*8) Proof of Lemma 4:* The quantities in (27b)-(27c) are well-defined. Indeed, we have for (27b) that $1 - 4\underline{\delta}_i^2(1 - \underline{\delta}_i^2/\delta_i^2) = (2\underline{\delta}_i^2 - 1)^2 + 4\underline{\delta}_i^4(\delta_i^{-2} - 1) > 0$ thanks to $\delta_i \in (\underline{\delta}_i, 1)$. Moreover, by $\mu_i \in (1, \bar{\mu}_i(\delta_i))$, the argument of the arccos in (27c) belongs to $(0, 1)$, so $\bar{\theta}_i(\delta_i, \mu_i)$ is also well-defined. Now, define

$$\hat{\mathcal{F}}_m^i := \mathcal{E}_\leq(c_i, \delta_i E_i) \cap \mathcal{E}_\geq(\bar{c}_i, \mu_i \bar{E}_i) \cap$$
$$\mathcal{C}_\geq(c_i, c_i - p_m^i, \psi_i, E_i) \cap \mathcal{E}_\geq(c_i, E_i) \supset \mathcal{F}_m^i. \tag{53}$$

By proving that for each $i \in \mathbb{I}$ and $m \in \{-1, 1\}$

$$\mathcal{L}(c_i, p_m^i - c_i) \cap \hat{\mathcal{F}}_m^i = \emptyset, \tag{54}$$

the claim of the lemma is also proven. We prove then (54) for an arbitrary $i \in \mathbb{I}$ and an arbitrary $m \in \{-1, 1\}$. For this proof, select the following angle $\psi_i'$ as *any* angle $\psi_i' \in (0, \psi_i)$. First, let us show that the following set inclusions hold

$$\mathcal{L}_\leq(c_i, p_m^i - c_i) \subset \mathcal{C}_\leq(c_i, c_i - p_m^i, \psi_i', E_i), \tag{55a}$$

$$\mathcal{L}_\geq(c_i, p_m^i - c_i) \subset \mathcal{C}(c_i, -c_i, \theta_i, E_i). \tag{55b}$$

Let $x \in \mathcal{L}_\leq(c_i, p_m^i - c_i)$. Then there exists $\lambda \leq 0$ such that $x - c_i = \lambda(p_m^i - c_i)$. Such $x - c_i$ verifies the condition $\cos(\psi_i')\|E_i(c_i - p_m)\|\|E_i(x - c_i)\| \leq (c_i - p_m^i)^\top E^2(x - c_i)$ corresponding to $\mathcal{C}_\leq(c_i, c_i - p_m, \psi_i', E_i)$ by simple computations for any $0 < \psi_i' < \psi_i$ (since $\cos(\psi_i') \leq 1$). This proves (55a). Now, let $x \in \mathcal{L}_\geq(c_i, p_m^i - c_i)$. Then there exists $\lambda \geq 0$ such that $x - c_i = \lambda(p_m^i - c_i)$. Such $x - c_i$ verifies the condition $\cos(\theta_i)\|E_i(-c_i)\|\|E_i(x - c_i)\| = -c_i^\top E_i^2(x - c_i)$ corresponding to $\mathcal{C}(c_i, -c_i, \theta_i, E_i)$ by simple computations using that $\cos(\theta_i)\|E_i(-c_i)\|\|E_i(p_m^i - c_i)\| = -c_i^\top E_i^2(p_m^i - c_i)$ (corresponding to $p_m^i \in \mathcal{C}(c_i, -c_i, \theta_i, E_i)$ from (18a) and Lemma 2). This proves (55b). Second, from (55) one has

$$\hat{\mathcal{F}}_m^i \cap \mathcal{L}(c_i, p_m^i - c_i)$$
$$= (\hat{\mathcal{F}}_m^i \cap \mathcal{L}_\leq(c_i, p_m^i - c_i)) \cup (\hat{\mathcal{F}}_m^i \cap \mathcal{L}_\geq(c_i, p_m^i - c_i))$$
$$\subset (\hat{\mathcal{F}}_m^i \cap \mathcal{C}_\leq(c_i, c_i - p_m^i, \psi_i', E_i)) \cup (\hat{\mathcal{F}}_m^i \cap \mathcal{C}(c_i, -c_i, \theta_i, E_i)) \tag{56}$$

and we prove that the two intersections in (56) are empty. Since $0 < \psi_i' < \psi_i$, one obtains readily from the definition of the cone in (11) that

$$\mathcal{C}_\geq(c_i, c_i - p_m^i, \psi_i, E_i) \cap \mathcal{C}_\leq(c_i, c_i - p_m^i, \psi_i', E_i) = \{c_i\}.$$

This relationship and the definition of $\hat{\mathcal{F}}_m^i$ in (53) imply that the first intersection in (56) is empty. We show now that the second intersection in (56) is also empty. Let $x \in \hat{\mathcal{F}}_m^i \cap \mathcal{C}(c_i, -c_i, \theta_i, E_i)$. So,

$$c_i^\top E_i^2(x - c_i) \overset{(20)}{=} -(x - c_i)^\top E_i^2(x - c_i)$$
$$+ \tfrac{1}{4}c_i^\top E_i^2 c_i (x - \bar{c}_i)^\top \bar{E}_i^2(x - \bar{c}_i) - \tfrac{1}{4}c_i^\top E_i^2 c_i$$
$$= -\|E_i(x - c_i)\|^2 + \tfrac{1}{4}\|E_i c_i\|^2\|\bar{E}_i(x - \bar{c}_i)\|^2 - \tfrac{1}{4}\|E_i c_i\|^2$$
$$\geq -\frac{1}{\delta_i^2} - \frac{1}{4}\|E_i c_i\|^2\left(1 - \frac{1}{\mu_i^2}\right) \tag{57}$$

where the bound holds since $x \in \hat{\mathcal{F}}_m^i$ implies $x \in \mathcal{E}_\leq(c_i, \delta_i E_i)$ and $x \in \mathcal{E}_\geq(\bar{c}_i, \mu_i \bar{E}_i)$. We continue (57) as

$$c_i^\top E_i^2(x - c_i) \geq -\frac{1}{\delta_i^2} - \frac{1}{4}\|E_i c_i\|^2\left(1 - \frac{1}{\mu_i^2}\right)$$

$$\stackrel{(27c),(27a)}{=} -\cos(\bar{\theta}_i(\delta_i,\mu_i))/\underline{\delta}_i^2 \stackrel{(27a)}{=} -\cos(\bar{\theta}_i(\delta_i,\mu_i))\|E_ic_i\|$$
$$\geq -\cos(\bar{\theta}_i(\delta_i,\mu_i))\|E_ic_i\|\|E_i(x-c_i)\| \qquad (58)$$

since $\cos(\bar{\theta}_i(\delta_i,\mu_i)) \geq 0$ and $\|E_i(x-c_i)\| \geq 1$ ($x \in \hat{\mathcal{F}}_m^i$ implies $x \in \mathcal{E}_\geq(c_i,E_i)$). It is also $x \in \mathcal{C}(c_i,-c_i,\theta_i,E_i)$. So,

$$c_i^\top E_i^2(x-c_i) = -\cos(\theta_i)\|E_ic_i\|\|E_i(x-c_i)\| \qquad (59)$$
$$< -\cos(\bar{\theta}_i(\delta_i,\mu_i))\|E_ic_i\|\|E_i(x-c_i)\|$$

from the bound on $\theta_i$ in Table I. (58) and (59) contradict each other, so the second intersection in (56) is *also* empty. Then, (54) is proven.

*9) Proof of Lemma 5:* Given (30e)-(30f), we just need to show that $\mathbf{L}(x,i,0) \neq \emptyset$ for all $(x,i) \in \mathcal{J}_0$. This holds if we show, as we do in the rest of the proof, that for each $x \in \mathbb{R}^n$ and $i \in \mathbb{I}$, $\mathbf{M}(x,i) \neq \emptyset$. First, we show that $\cap_{m=-1,1}\mathcal{C}_\leq(c_i,c_i-p_m^i,\bar{\psi}_i,E_i) = \{c_i\}$ for each $i \in \mathbb{I}$. To this end, note that $p_1^i \in \mathcal{C}(c_i,-c_i,\theta_i,E_i)$ by (18a), and this implies $\cos^2(\theta_i)\|E_ic_i\|^2\|E_i(p_1^i-c_i)\|^2 = (c_i^\top E_i^2(p_1^i-c_i))^2$ or, equivalently,

$$(p_1^i-c_i)^\top E_i\pi^{\theta_i}(E_ic_i)E_i(p_1^i-c_i) = 0. \qquad (60)$$

Introduce then $v_m^i := E_i(c_i-p_m^i)/\|E_i(c_i-p_m^i)\|$ for $m \in \{-1,1\}$, and compute

$$(v_1^i)^\top v_{-1}^i = \frac{(p_1^i-c_i)^\top E_i^2(p_{-1}^i-c_i)}{\|E_i(p_1^i-c_i)\|\|E_i(p_{-1}^i-c_i)\|}$$
$$\stackrel{(52)}{=} -\frac{(p_1^i-c_i)^\top E_i\rho(E_ic_i)E_i(p_1^i-c_i)}{\|E_i(p_1^i-c_i)\|^2}$$
$$\stackrel{(6)}{=} -\frac{(p_1^i-c_i)^\top E_i(2\pi^{\theta_i}(E_ic_i)-\cos(2\theta_i)I_n)E_i(p_1^i-c_i)}{\|E_i(p_1^i-c_i)\|^2}$$
$$\stackrel{(60)}{=} \frac{\cos(2\theta_i)(p_1^i-c_i)^\top E_i^2(p_1^i-c_i)}{\|E_i(p_1^i-c_i)\|^2} = \cos(2\theta_i)$$

Then, by Lemma 1 and $2\bar{\psi}_i < 2\theta_i$, $\cap_{m=-1,1}\mathcal{C}_\leq(c_i,c_i-p_m^i,\bar{\psi}_i,E_i) = \{c_i\}$. Second, note that

$$\cup_{m=-1,1}\mathcal{C}_>(c_i,c_i-p_m^i,\bar{\psi}_i,E_i) =$$
$$\left(\cap_{m=-1,1}\mathcal{C}_\leq(c_i,c_i-p_m^i,\bar{\psi}_i,E_i)\right)^\mathsf{c} = \mathbb{R}^n\backslash\{c_i\}. \qquad (61)$$

Therefore, we have

$$\cup_{m=-1,1}\mathcal{C}_m^i \stackrel{(30d)}{=} \cup_{m=-1,1}\mathcal{C}_\geq(c_i,c_i-p_m^i,\bar{\psi}_i,E_i) = \mathbb{R}^n \qquad (62)$$

since $\cup_{m=-1,1}\mathcal{C}_m^i$ is a superset of the set in (61) and contains $c_i$. So, for each $x \in \mathbb{R}^n$ and $i \in \mathbb{I}$, $\mathbf{M}(x,i) \neq \emptyset$ in (30g).

*10) Proof of Lemma 6:* $\mathcal{F}$ and $\mathcal{J}$ are closed subsets of $\mathbb{R}^n\times\{-1,0,1\}$. $\mathbf{F}$ is continuous on $\mathcal{F}$. $\mathbf{J}(x,i,m) \neq \emptyset$ for each $(x,i,m) \in \mathcal{J}$ thanks to Lemma 5 and $\mathbf{J}$ has a closed graph relative to $\mathcal{J}$ because, in particular, the construction in (30g) allows $\mathbf{M}$ to be set-valued whenever $x \in \cap_{m=-1,1}\mathcal{C}_m^i$. Then, $\mathbf{J}$ is outer semicontinuous and locally bounded relative to $\mathcal{J}$.

*11) Proof of Lemma 7:* If we prove that $\forall i \in \mathbb{I}$, $m \in \{-1,1\}$

$$\left(\bigcap_{i'\in\mathbb{I}}\mathcal{F}_0^{i'}\right)\cup\left(\bigcup_{i'\in\mathbb{I}}\mathcal{J}_0^{i'}\right) = \mathcal{W} = \mathcal{F}_m^i\cup\mathcal{J}_m^i \qquad (63)$$

then (25), (29a), (29b) imply straightforwardly $\mathcal{F}_0 \cup \mathcal{J}_0 = \mathcal{W}\times\mathbb{I} = \mathcal{F}_1\cup\mathcal{J}_1 = \mathcal{F}_{-1}\cup\mathcal{J}_{-1}$ and, in turn, (30c) implies

$\mathcal{F}\cup\mathcal{J} = \mathcal{W}\times\mathbb{I}\times\mathbb{M} =: \mathcal{K}$. Therefore, we just need to prove (63) in the remainder. For each $i \in \mathbb{I}$ and $m \in \{-1,1\}$,

$$\mathcal{F}_m^i \cup \mathcal{J}_m^i \stackrel{(26),(28)}{=} \left(\mathcal{E}_\leq(c_i,\delta_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\mu_i\bar{E}_i)\right.$$
$$\left.\cap\mathcal{C}_\geq(c_i,c_i-p_m^i,\psi_i,E_i)\cap\mathcal{W}\right)\cup\left(\left(\mathcal{E}_\geq(c_i,\delta_iE_i)\cup\mathcal{E}_\leq(\bar{c}_i,\mu_i\bar{E}_i)\right.\right.$$
$$\left.\left.\cup\mathcal{C}_\leq(c_i,c_i-p_m^i,\psi_i,E_i)\right)\cap\mathcal{W}\right) = \mathcal{W}. \qquad (64)$$

We are left with proving $\left(\bigcap_{i'\in\mathbb{I}}\mathcal{F}_0^{i'}\right)\cup\left(\bigcup_{i'\in\mathbb{I}}\mathcal{J}_0^{i'}\right) = \mathcal{W}$ in (63). First, note that for each $i'$, $\mathcal{H}_{i'}(\epsilon_{i'},\nu_{i'})\cap\mathcal{W} = \mathcal{E}_\leq(c_{i'},\epsilon_{i'}E_{i'})\cap\mathcal{E}_\geq(\bar{c}_{i'},\nu_{i'}\bar{E}_{i'})\cap\mathcal{W}$ and, hence,

$$\left(\bigcap_{i'\in\mathbb{I}}\mathcal{F}_0^{i'}\right)\cup\left(\bigcup_{i'\in\mathbb{I}}\mathcal{J}_0^{i'}\right) = \bigcap_{i'\in\mathbb{I}}\left(\left(\mathcal{E}_\geq(c_{i'},\epsilon_{i'}E_{i'})\right.\right.$$
$$\left.\left.\cup\mathcal{E}_\leq(\bar{c}_{i'},\nu_{i'}\bar{E}_{i'})\right)\cap\mathcal{W}\right)\cup\bigcup_{i'\in\mathbb{I}}\left(\mathcal{H}_{i'}(\epsilon_{i'},\nu_{i'})\cap\mathcal{W}\right)$$
$$= \left(\bigcap_{i'\in\mathbb{I}}\left(\mathcal{E}_\geq(c_{i'},\epsilon_{i'}E_{i'})\cup\mathcal{E}_\leq(\bar{c}_{i'},\nu_{i'}\bar{E}_{i'})\right)\right.$$
$$\left.\cup\bigcup_{i'\in\mathbb{I}}\left(\mathcal{E}_\leq(c_{i'},\epsilon_{i'}E_{i'})\cap\mathcal{E}_\geq(\bar{c}_{i'},\nu_{i'}\bar{E}_{i'})\right)\right)\cap\mathcal{W} = \mathbb{R}^n\cap\mathcal{W} = \mathcal{W}.$$

From $\mathcal{F}\cup\mathcal{J} = \mathcal{K}$, the definition of $\mathcal{K}$ in (31) and $x^+ = x$ in the jump map $\mathbf{J}$, it follows immediately that $\mathbf{J}(\mathcal{J}) \subset \mathcal{K}$.

*12) Proof of Lemma 8:* The intersection of $\mathcal{E}(c_i,\delta E_i)$ and $\mathcal{E}(\bar{c}_i,\mu\bar{E}_i)$ corresponds to the two quadratic equations

$$\begin{cases} \delta^2\|E_i(x-c_i)\|^2 = 1 \\ \mu^2\|\bar{E}_i(x-\bar{c}_i)\|^2 = 1 \end{cases}. \qquad (65)$$

By expanding squares and using (20), (65) is equivalent to

$$\begin{cases} \|E_ix\|^2 - 2c_i^\top E_i^2x + \|E_ic_i\|^2 - \delta^{-2} = 0 \\ \|E_ix\|^2 - c_i^\top E_i^2x + \|E_ic_i\|^2(1-\mu^{-2})/4 = 0 \end{cases}.$$

Solving for $\|E_ix\|^2$ and $c_i^\top E_i^2x$, we obtain using (27a)

$$\begin{cases} \|E_ix\|^2 = \|E_ic_i\|^2\left((1+\mu^{-2})/2 - \delta^{-2}\underline{\delta}_i^4\right) \\ c_i^\top E_i^2x = \|E_ic_i\|^2\left((3+\mu^{-2})/4 - \delta^{-2}\underline{\delta}_i^4\right) \end{cases} \qquad (66)$$

and both right-hand sides of (66) are positive because

$$(1+\mu^{-2})/2 - \delta^{-2}\underline{\delta}_i^4$$
$$\geq (1+\bar{\mu}_i(\delta)^{-2})/2 - \delta^{-2}\underline{\delta}_i^4 \stackrel{(27b)}{=} 1 - 2\underline{\delta}_i^2 + \delta^{-2}\underline{\delta}_i^4 \quad (67a)$$
$$= (\underline{\delta}_i^2-1)^2 + \underline{\delta}_i^4(\delta^{-2}-1) \geq (\underline{\delta}_i^2-1)^2 > 0$$
$$(3+\mu^{-2})/4 - \delta^{-2}\underline{\delta}_i^4 \geq (3+\bar{\mu}_i(\delta)^{-2})/4 - \delta^{-2}\underline{\delta}_i^4 \qquad (67b)$$
$$= 1 - \underline{\delta}_i^2 > 0, \text{ by Assumption 2.}$$

From (66), one obtains with some computations

$$\frac{c_i^\top E_i^2x}{\|E_ic_i\|\|E_ix\|} \stackrel{(66)}{=} \frac{\|E_ic_i\|^2\left((3+\mu^{-2})/4-\delta^{-2}\underline{\delta}_i^4\right)}{\|E_ic_i\|\|E_ic_i\|\sqrt{(1+\mu^{-2})/2-\delta^{-2}\underline{\delta}_i^4}}$$
$$\stackrel{(27c)}{=} \frac{1-\cos(\bar{\theta}_i(\delta,\mu))\underline{\delta}_i^2}{\sqrt{(1+\mu^{-2})/2-\delta^{-2}\underline{\delta}_i^4}}. \qquad (68)$$

For $\mu \in [1, \left(1-4\underline{\delta}_i^2(1-\underline{\delta}_i^2/\delta^2)\right)^{-\frac{1}{2}}]$ and $\delta < 1$, we can prove

$$\frac{1-\cos(\bar{\theta}_i(\delta,\mu))\underline{\delta}_i^2}{\sqrt{(1+\mu^{-2})/2-\delta^{-2}\underline{\delta}_i^4}} = \frac{(3+\mu^{-2})/4-\delta^{-2}\underline{\delta}_i^4}{\sqrt{(1+\mu^{-2})/2-\delta^{-2}\underline{\delta}_i^4}} < 1 \qquad (69)$$

(*e.g.*, set $\chi := (1 + \mu^{-2})/2$, obtain the bounds of $\chi$ from the bounds of $\mu$, substitute $\chi$ in (69), and note that the obtained quadratic inequality holds true for such bounds of $\chi$ due to $\delta < 1$). Because of (67a), (67b) and (69), the expression in (32) is well-defined and positive. Since (68) yields

$$\frac{c_i^\top E_i^2 x}{\|E_i c_i\| \|E_i x\|} = \frac{1 - \cos(\bar{\theta}_i(\delta,\mu))\underline{\delta}_i^2}{\sqrt{(1+\mu^{-2})/2 - \delta^{-2}\underline{\delta}_i^4}} \overset{(32)}{=} \cos(\vartheta_i(\delta,\mu)),$$

(33) holds as well by the cone definition in (11).

*13) Proof of Lemma 9:* The obstacles $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ are sufficiently pairwise disjoint, so (37) holds. Moreover, the sets $\mathcal{R}_{i'}(\delta_{i'},\mu_{i'})$ in (39) and $\mathcal{E}_{\leq}(c_{i''},\delta_{i''}E_{i''})$ are bounded, and for $\delta_{i'},\delta_{i''},\mu_{i'} \to 1$, $\mathcal{R}_{i'}(\delta_{i'},\mu_{i'})$ and $\mathcal{E}_{\leq}(c_{i''},\delta_{i''}E_{i''})$ reduce respectively to $\mathcal{R}_{i'}^*$ and $\mathcal{E}_{\leq}(c_{i''},E_{i''})$. By a continuity argument, there exist parameters $\delta_i^*$ and $\mu_i^*$ such that the lemma holds.

*14) Proof of Lemma 10:* We prove the claim for arbitrary $i \in \mathbb{I}$ and $m \in \{-1,1\}$. Let us prove (43a)-(43b). Thanks to the weak pairwise disjointness of $\{\mathcal{E}_{\leq}(c_i,\delta_i E_i)\}_{i\in\mathbb{I}}$ we have $\mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_{i'},\delta_{i'}E_{i'}) = \mathcal{E}_{\leq}(c_i,\delta_i E_i)$ for all $i' \neq i$. Then, $\mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{W} = \mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_i,E_i)$, which implies by (26) that $\mathcal{F}_m^i$ satisfies (43b). By a similar argument, $\mathcal{J}_0^i$ satisfies (43a) as well.

Let us prove (43c). Write the complement of $\mathcal{F}_0^*$ as

$$(\mathcal{F}_0^*)^{\mathsf{c}} \overset{(42)}{=} \left(\bigcap_{i\in\mathbb{I}}\mathcal{F}_0^i\right)^{\mathsf{c}} \overset{(1)}{=} \bigcup_{i\in\mathbb{I}}(\mathcal{F}_0^i)^{\mathsf{c}}$$

$$\overset{(24)}{=} \bigcup_{i\in\mathbb{I}}\left(\left(\mathcal{E}_{\geq}(c_i,\epsilon_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\nu_i\bar{E}_i)\right) \cap \mathcal{W}\right)^{\mathsf{c}}$$

$$\overset{(1)}{=} \bigcup_{i\in\mathbb{I}}\left(\mathcal{E}_{\geq}(c_i,\epsilon_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\nu_i\bar{E}_i)\right)^{\mathsf{c}} \cup \mathcal{W}^{\mathsf{c}}$$

$$\overset{(14)}{=} \bigcup_{i\in\mathbb{I}}\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap \mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \bigcup_{i\in\mathbb{I}}\mathcal{E}_{<}(c_i,E_i) \quad (70)$$

$$= \bigcup_{i\in\mathbb{I}}\left(\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap \mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i)\right) \cup \mathcal{E}_{<}(c_i,E_i)$$

$$= \bigcup_{i\in\mathbb{I}}(\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cup \mathcal{E}_{<}(c_i,E_i)) \cap (\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i))$$

$$= \bigcup_{i\in\mathbb{I}}\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap (\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i))$$

because $\epsilon_i < 1$. Thanks to the weak pairwise disjointness of $\{\mathcal{E}(c_i,\delta_i E_i)\}_{i\in\mathbb{I}}$ and $\delta_i < \epsilon_i$, the sets $\{\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap (\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i))\}_{i\in\mathbb{I}}$ can actually be proven to be pairwise separated. Then, we can use (2) to obtain the boundary of the set $\mathcal{F}_0^*$ as

$$\partial\mathcal{F}_0^* = \partial\left((\mathcal{F}_0^*)^{\mathsf{c}}\right)$$

$$\overset{(70)}{=} \partial\left(\bigcup_{i\in\mathbb{I}}\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap (\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i))\right)$$

$$\overset{(2)}{=} \bigcup_{i\in\mathbb{I}}\partial\left(\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap (\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i))\right)$$

$$\overset{(1)}{\subset} \bigcup_{i\in\mathbb{I}}\left(\partial\mathcal{E}_{<}(c_i,\epsilon_i E_i) \cap \overline{\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i)}\right.$$

$$\left.\cup \partial\left(\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{<}(c_i,E_i)\right) \cap \overline{\mathcal{E}_{<}(c_i,\epsilon_i E_i)}\right)$$

$$\overset{(1)}{\subset} \bigcup_{i\in\mathbb{I}}\left(\mathcal{E}(c_i,\epsilon_i E_i) \cap \left(\mathcal{E}_{\geq}(\bar{c}_i,\nu_i\bar{E}_i) \cup \mathcal{E}_{\leq}(c_i,E_i)\right)\right.$$

$$\cup \left(\left(\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_{<}(c_i,E_i)\right) \cup \left(\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i)\right)\right)$$

$$\left.\cap \mathcal{E}_{\leq}(c_i,\epsilon_i E_i)\right)$$

$$= \bigcup_{i\in\mathbb{I}}\left(\left(\mathcal{E}(c_i,\epsilon_i E_i) \cap \mathcal{E}_{\geq}(\bar{c}_i,\nu_i\bar{E}_i)\right) \cup \left(\left(\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_{<}(c_i,E_i)\right)\right.\right.$$

$$\left.\left.\cap \mathcal{E}_{\leq}(c_i,\epsilon_i E_i)\right) \cup \left(\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_{>}(\bar{c}_i,\nu_i\bar{E}_i)\right)\right) =: \bigcup_{i\in\mathbb{I}}\mathcal{P}_0^i$$

(43c) is finally proven in (74), for which we note that: *(a)* $\mathcal{J}_0^*$ is simplified into $\cup_{i\in\mathbb{I}}\mathcal{H}_i(\epsilon_i,\nu_i)$ thanks to (43a); *(b)* $\mathcal{P}_0^i \cap \mathcal{H}_{i'}(\epsilon_{i'},1) = \emptyset$ for all $i \neq i'$; *(c)* most of the sets in the next-to-last expression are empty, so the last expression follows.

Let us prove (43d). First, note that by (43b) and (22)

$$\mathcal{F}_m^i = \mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_i,E_i)$$
$$\cap \mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i) \cap \mathcal{C}_{\geq}(c_i,c_i - p_m^i,\psi_i,E_i),$$

which is an intersection of four closed sets. By successive applications of (1i) and (1f), the boundary of $\mathcal{F}_m^i$ satisfies

$$\partial\mathcal{F}_m^i \subset \bigcup_{k\in\{1,2,3,4\}} \mathcal{P}_{m,k}^i \qquad (71)$$

with the following definitions

$$\mathcal{P}_{m,1}^i := \mathcal{E}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_i,E_i) \cap \mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap \mathcal{C}_{\geq}(c_i,c_i - p_m^i,\psi_i,E_i) \quad (72\text{a})$$

$$\mathcal{P}_{m,2}^i := \mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}(c_i,E_i) \cap \mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap \mathcal{C}_{\geq}(c_i,c_i - p_m^i,\psi_i,E_i) \quad (72\text{b})$$

$$\mathcal{P}_{m,3}^i := \mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_i,E_i) \cap \mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap \mathcal{C}_{\geq}(c_i,c_i - p_m^i,\psi_i,E_i) \quad (72\text{c})$$

$$\mathcal{P}_{m,4}^i := \mathcal{E}_{\leq}(c_i,\delta_i E_i) \cap \mathcal{E}_{\geq}(c_i,E_i) \cap \mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap \mathcal{C}(c_i,c_i - p_m^i,\psi_i,E_i). \quad (72\text{d})$$

Second, note that since $\mathcal{F}_m^i$ is closed, $\partial\mathcal{F}_m^i \subset \mathcal{F}_m^i \subset \mathcal{W}$, and hence $\partial\mathcal{F}_m^i\backslash\mathcal{W} \subset \mathcal{W}\backslash\mathcal{W} = \emptyset$. By this fact, we can write that

$$\partial\mathcal{F}_m^i\backslash\mathcal{J}_m^i \overset{(28)}{=} \partial\mathcal{F}_m^i\backslash\left(\left(\mathcal{E}_{\geq}(c_i,\delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\right.\right.$$

$$\left.\left.\cup\, \mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right) \cap \mathcal{W}\right)$$

$$\overset{(1)}{=} \partial\mathcal{F}_m^i\backslash\mathcal{W} \cup \partial\mathcal{F}_m^i\backslash\left(\mathcal{E}_{\geq}(c_i,\delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\right.$$

$$\left.\cup\, \mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right)$$

$$= \partial\mathcal{F}_m^i\backslash\left(\mathcal{E}_{\geq}(c_i,\delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i) \cup \mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right)$$

$$\overset{(71)}{\subset} \left(\cup_{k\in\{1,2,3,4\}}\mathcal{P}_{m,k}^i\right)\backslash\left(\mathcal{E}_{\geq}(c_i,\delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\right.$$

$$\left.\cup\, \mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right)$$

$$\overset{(1)}{=} \cup_{k\in\{1,2,3,4\}}\left(\mathcal{P}_{m,k}^i\backslash\left(\mathcal{E}_{\geq}(c_i,\delta_i E_i) \cup \mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\right.\right.$$

$$\left.\left.\cup\, \mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right)\right)$$

$$\overset{(1)}{\subset} \cup_{k\in\{1,2,3,4\}}\left(\left(\mathcal{P}_{m,k}^i\backslash\mathcal{E}_{\geq}(c_i,\delta_i E_i)\right)\right.$$

$$\left.\cap \left(\mathcal{P}_{m,k}^i\backslash\mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\right) \cap \left(\mathcal{P}_{m,k}^i\backslash\mathcal{C}_{\leq}(c_i,c_i - p_m^i,\psi_i,E_i)\right)\right).$$

Finally, we simplify this expression through the following facts

$$\mathcal{P}_{m,1}^i\backslash\mathcal{E}_{\geq}(c_i,\delta_iE_i)=\emptyset,\quad \mathcal{P}_{m,3}^i\backslash\mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)=\emptyset$$
$$\mathcal{P}_{m,4}^i\backslash\mathcal{C}_{\leq}(c_i,c_i-p_m^i,\psi_i,E_i)=\emptyset,\tag{73}$$

which are an immediate consequence of (72) and yield

$$\partial\mathcal{F}_m^i\backslash\mathcal{J}_m^i \overset{(73)}{\subset} \mathcal{P}_{m,2}^i\backslash\Big(\mathcal{E}_{\geq}(c_i,\delta_iE_i)\cup\mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cup\,\mathcal{C}_{\leq}(c_i,c_i-p_m^i,\psi_i,E_i)\Big)$$
$$\overset{(1)}{=}\mathcal{P}_{m,2}^i\cap\Big(\mathcal{E}_{\geq}(c_i,\delta_iE_i)\cup\mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\cup\mathcal{C}_{\leq}(c_i,c_i-p_m^i,\psi_i,E_i)\Big)^c$$
$$\overset{(1)}{=}\mathcal{P}_{m,2}^i\cap\mathcal{E}_{<}(c_i,\delta_iE_i)\cap\mathcal{E}_{>}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_{>}(c_i,c_i-p_m^i,\psi_i,E_i)$$
$$=\mathcal{E}_{<}(c_i,\delta_iE_i)\cap\mathcal{E}(c_i,E_i)\cap\mathcal{E}_{>}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_{>}(c_i,c_i-p_m^i,\psi_i,E_i)$$
$$=\mathcal{E}(c_i,E_i)\cap\mathcal{E}_{>}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_{>}(c_i,c_i-p_m^i,\psi_i,E_i)$$
$$\overset{(1)}{=}\mathcal{E}(c_i,E_i)\backslash(\mathcal{E}_{\leq}(\bar{c}_i,\mu_i\bar{E}_i)\cup\mathcal{C}_{\leq}(c_i,c_i-p_m^i,\psi_i,E_i)).$$

*15) Proof of Lemma 11:* We divide the proof into steps. *Step 1:* For each $i\in\mathbb{I}$ and $m\in\{-1,1\}$, $\delta\in[\delta_i,1]$ and $\psi\in[\psi_i,\bar{\psi}_i)$, consider the sets

$$\tilde{\mathcal{F}}_m^i(\delta,\psi):=\mathcal{E}_{\leq}(c_i,\delta E_i)\cap\mathcal{E}_{\geq}(c_i,E_i)\cap\mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap\,\mathcal{C}_{\geq}(c_i,c_i-p_m^i,\psi,E_i).\tag{75a}$$
$$\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi):=\mathcal{E}_{\leq}(c_i,\delta E_i)\cap\mathcal{E}_{\geq}(c_i,E_i)$$
$$\cap\,\mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_{\geq}(c_i,c_i-p_m^i,\psi,E_i)\tag{75b}$$

*Each maximal solution $x$ to the flow-only hybrid system*

$$\dot{x}=\kappa(x,i,m)=:\mathfrak{u}(x),\quad x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)\tag{76}$$

*has $T=\sup_t\mathrm{dom}\,x<+\infty$ and $x(T)\in\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$.* We first prove that $T$ is finite. Consider the following nonnegative function $\mathbf{V}(x):=\frac{1}{2}\|E_i(x-p_m^i)\|^2$. Simple computations, (16), and (4a) yield that for all $x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)$

$$\langle\nabla\mathbf{V}(x),\kappa(x,i,m)\rangle$$
$$=-k_m(x-p_m^i)^\top E_i\pi^\perp(E_i(x-c_i))E_i(x-p_m^i)$$
$$=-k_m\|\pi^\perp(E_i(x-c_i))E_i(x-p_m^i)\|<-\mathfrak{e}<0,$$

where $\mathfrak{e}>0$ follows from $\pi^\perp(E_i(x-c_i))E_i(x-p_m^i)$ vanishing only for $x\in\mathcal{L}(c_i,p_m^i-c_i)$ (Lemma 3), and $\mathcal{L}(c_i,p_m^i-c_i)$ is separated by a positive distance from $\tilde{\mathcal{F}}_m^i(\delta,\psi)$ (Lemma 4). Then, $T$ is finite, otherwise $\mathbf{V}$ evaluated along solutions would become negative. In order to show $x(T)\in\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$, we resort to a viability argument based on tangent cones. To this end, we need the next lemma.

**Lemma 12:** *For all $x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)\backslash\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$,*

$$\mathfrak{u}(x)\in\mathbf{T}_{\mathcal{E}_{\leq}(c_i,\delta E_i)}(x)\tag{77a}$$
$$\mathfrak{u}(x)\in\mathbf{T}_{\mathcal{E}_{\geq}(c_i,E_i)}(x)\tag{77b}$$
$$\mathfrak{u}(x)\in\mathbf{T}_{\mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)}(x)\tag{77c}$$
$$\mathfrak{u}(x)\in\mathbf{T}_{\mathcal{C}_{\geq}(c_i,c_i-p_m^i,\psi,E_i)}(x).\tag{77d}$$

$\tilde{\mathcal{F}}_m^i(\delta,\psi)$ in (75a) is the intersection of four closed sets: if

$$\mathfrak{u}(x)\in\mathbf{T}_{\mathcal{E}_{\leq}(c_i,\delta E_i)}(x)\cap\mathbf{T}_{\mathcal{E}_{\geq}(c_i,E_i)}(x)$$
$$\cap\,\mathbf{T}_{\mathcal{E}_{\geq}(\bar{c}_i,\mu_i\bar{E}_i)}(x)\cap\mathbf{T}_{\mathcal{C}_{\geq}(c_i,c_i-p_m^i,\psi,E_i)}(x),\tag{78}$$

then $\mathfrak{u}(x)\in\mathbf{T}_{\tilde{\mathcal{F}}_m^i(\delta,\psi)}(x)$ by the next fact, which is an immediate corollary of [31, Thm. 5].

*Fact 1 ([31, Thm. 5]):* Let $v\in\mathcal{A}\cap\mathcal{B}$ with $\mathcal{A}$, $\mathcal{B}$ closed subsets of $\mathbb{R}^n$. Suppose $\mathbf{T}_{\mathcal{A}}(v)\cap(\mathbf{T}_{\mathcal{B}}(v)^\circ)\neq\emptyset$. Then, $\mathbf{T}_{\mathcal{A}\cap\mathcal{B}}(v)\supset\mathbf{T}_{\mathcal{A}}(v)\cap\mathbf{T}_{\mathcal{B}}(v)$.

The condition (78) has been checked in Lemma 12 for each $x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)\backslash\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$, hence

$$\mathfrak{u}(x)\in\mathbf{T}_{\tilde{\mathcal{F}}_m^i(\delta,\psi)}(x)\quad\forall x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)\backslash\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi).\tag{79}$$

Then, it can only be $x(T)\in\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$, otherwise the solution could be further extended by viability results such as [24, Lemma 5.26(b)].

*Step 2:* For each $i\in\mathbb{I}$, both the sets $\mathcal{E}_{\leq}(\bar{c}_i,\bar{E}_i)$ and $\mathcal{C}(0,c_i,\vartheta_i(\delta,\mu_i),E_i)$ $(\delta\in[\delta_i,1])$ are forward invariant under the vector field $-k_0x$. For $x\in\mathcal{E}(\bar{c}_i,\bar{E}_i)$, we have

$$-k_0x^\top\bar{E}_i^2(x-\bar{c}_i)\overset{(20)}{=}-\frac{k_0}{2}\|\bar{E}_ix\|^2-\frac{k_0}{2}x^\top\bar{E}_i^2(x-c_i)$$
$$\overset{(19)}{=}-\frac{k_0}{2}\|\bar{E}_ix\|^2+\frac{k_0}{2}\left(1-\|\bar{E}_i(x-\bar{c}_i)\|^2\right)\tag{80}$$
$$=-\frac{k_0}{2}\|\bar{E}_ix\|^2\leq 0,$$

where the last equality follows from $x\in\mathcal{E}(\bar{c}_i,\bar{E}_i)$. Therefore, $\{-k_0x\}\in\mathcal{P}_{\leq}(0,\bar{E}_i^2(x-\bar{c}_i))=\mathbf{T}_{\mathcal{E}(\bar{c}_i,\bar{E}_i)}(x)$. For $x\in\mathcal{C}(0,c_i,\vartheta_i(\delta,\mu_i),E_i)$, we have

$$-k_0x^\top E_i\pi^{\vartheta_i(\delta,\mu_i)}(E_ic_i)E_ix=0.\tag{81}$$

Therefore, $\{-k_0x\}\in\mathcal{P}(0,E_i\pi^{\vartheta_i(\delta,\mu_i)}(E_ic_i)E_ix)=\mathbf{T}_{\mathcal{C}(0,c_i,\vartheta_i(\delta,\mu_i),E_i)}(x)$ (cf. (12)). Forward invariance follows then from the classical Nagumo's theorem.

*Step 3:* *Proof of (49).* Let $\xi(t,j)=(x(t,j),i(t,j),m(t,j))\in\mathcal{F}_l\times\{l\}$ with $l\in\{-1,1\}$ and $(t,j)\in\mathrm{dom}\,\xi$. Hence, by (38),

$$\|x(t,j)\|\geq r_{i(t,j)}>0\tag{82}$$

thanks to the discussion below (38). We further divide into mutually exclusive subcases.

*Step 3a:* $\xi(t,j)\in(\mathcal{F}_l\cap\mathcal{J}_l)\times\{l\}$ *and the solution jumps.* By (30b) and (30e), $\xi(t,j+1)=(x(t,j),i(t,j),0)$. Depending on $x(t,j)$, either the solution never jumps again or reaches the set $\mathcal{J}_0^*$ in (42) at some $(t',j+1)$. Consider the former case. The flow map for $x$ in (30a) ensures $x(\tau,j+1)=\exp(-k_0(\tau-t))x(t,j)$ for all $\tau\geq t$. By (82), $s\geq t$ exists such that $\|x(s,j+1)\|=\min_{i'\in\mathbb{I}}r_{i'}/2=:\sigma_0>0$. So,

$$\|x(s,j+1)\|+\sigma_0=\min_{i'\in\mathbb{I}}r_{i'}\leq r_{i(t,j)}\leq\|x(t,j)\|\tag{83}$$

and the claim of the proposition is proven. Consider the latter case, i.e., there exist $t'\geq t$ and $i'\in\mathbb{I}$ such that $x(t',j+1)\in\mathcal{J}_0^{i'}\subset\mathcal{J}_0^*$. $\mathcal{J}_0^{i'}$ is a shrinking of the set $\mathcal{F}_1^{i'}\cup\mathcal{F}_{-1}^{i'}$ by construction of the flow and jump sets (cf. (23) and (26), where the union of the cones from (26) gives $\mathbb{R}^n$ from the same arguments yielding (62)). Then, by Lemma 5, there exist $l'$ such that $\xi(t',j+2)\in(\mathcal{F}_{l'}\backslash\mathcal{J}_{l'})\times\{l'\}$ and the solution is forced to flow as in Step 3b below.

*Step 3b:* $\xi(t,j)\in(\mathcal{F}_l\backslash\mathcal{J}_l)\times\{l\}$ *and the solution flows.* If $x(t,j)\in\mathcal{F}_l^i\backslash\mathcal{J}_l^i$, there exists $\delta\in(\delta_i,1]$ and $\psi\in(\psi_i,\bar{\psi}_i)$ such that $x(t,j)\in\tilde{\mathcal{F}}_l^i(\delta,\psi)$ as defined in (75a). From the facts

$$\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{E}_{\geq}(c_i,\delta_iE_i)=\emptyset\tag{84a}$$
$$\mathcal{C}_{\geq}(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{C}_{\leq}(c_i,c_i-p_m^i,\psi_i,E_i)$$
$$=\{c_i\}\notin\mathcal{E}_{\geq}(c_i,E_i),\tag{84b}$$

$$\partial\mathcal{F}_0^*\backslash\mathcal{J}_0^*\subset\Big(\bigcup_{i\in\mathbb{I}}\mathcal{P}_0^i\Big)\backslash\Big(\bigcup_{i\in\mathbb{I}}\mathcal{H}_i(\epsilon_i,\nu_i)\Big)=\bigcup_{i\in\mathbb{I}}\Big(\mathcal{P}_0^i\backslash\bigcup_{i'\in\mathbb{I}}(\mathcal{H}_{i'}(\epsilon_{i'},\nu_{i'}))\Big)=\bigcup_{i\in\mathbb{I}}\Big(\bigcap_{i'\in\mathbb{I}}(\mathcal{P}_0^i\backslash\mathcal{H}_{i'}(\epsilon_{i'},\nu_{i'}))\Big)=\bigcup_{i\in\mathbb{I}}\Big(\mathcal{P}_0^i\backslash\mathcal{H}_i(\epsilon_i,\nu_i)\cap\bigcap_{i'\in\mathbb{I},i'\neq i}\mathcal{P}_0^i\Big)$$

$$=\bigcup_{i\in\mathbb{I}}(\mathcal{P}_0^i\backslash\mathcal{H}_i(\epsilon_i,\nu_i))=\bigcup_{i\in\mathbb{I}}\Big((\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i))\cup((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i))\cup(\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\Big)\backslash\mathcal{H}_i(\epsilon_i,\nu_i)$$

$$=\bigcup_{i\in\mathbb{I}}\Big((\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i))\cup((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i))\cup(\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\Big)\cap\mathcal{H}_i(\epsilon_i,\nu_i)^c$$

$$=\bigcup_{i\in\mathbb{I}}\Big((\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i))\cup((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i))\cup(\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\Big)\cap\big(\mathcal{E}_>(c_i,\epsilon_iE_i)\cup\mathcal{E}_<(\bar{c}_i,\nu_i\bar{E}_i)\cup\mathcal{E}_<(c_i,E_i)\big)$$

(74)

$$=\bigcup_{i\in\mathbb{I}}\Big(\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i)\cap\mathcal{E}_>(c_i,\epsilon_iE_i)\Big)\cup\Big(\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i)\cap\mathcal{E}_<(\bar{c}_i,\nu_i\bar{E}_i)\Big)\cup\Big(\mathcal{E}(c_i,\epsilon_iE_i)\cap\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i)\cap\mathcal{E}_<(c_i,E_i)\Big)$$

$$\cup\Big((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i)\cap\mathcal{E}_>(c_i,\epsilon_iE_i)\Big)\cup\Big((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i)\cap\mathcal{E}_<(\bar{c}_i,\nu_i\bar{E}_i)\Big)$$

$$\cup\Big((\mathcal{E}(\bar{c}_i,\nu_i\bar{E}_i)\backslash\mathcal{E}_<(c_i,E_i))\cap\mathcal{E}_\leq(c_i,\epsilon_iE_i)\cap\mathcal{E}_<(c_i,E_i)\Big)\cup\Big((\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\cap\mathcal{E}_>(c_i,\epsilon_iE_i)\Big)$$

$$\cup\Big((\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\cap\mathcal{E}_<(\bar{c}_i,\nu_i\bar{E}_i)\Big)\cup\Big((\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_>(\bar{c}_i,\nu_i\bar{E}_i))\cap\mathcal{E}_<(c_i,E_i)\Big)=\bigcup_{i\in\mathbb{I}}\mathcal{E}(c_i,E_i)\backslash\mathcal{E}_\geq(\bar{c}_i,\nu_i\bar{E}_i).$$

$\tilde{\mathcal{F}}_l^i(\delta,\psi)\cap\mathcal{J}_l^i$ is a subset of $\tilde{\mathcal{P}}_{l,3}^i(\delta,\psi)$ because

$$\tilde{\mathcal{F}}_l^i(\delta,\psi)\cap\mathcal{J}_l^i=\mathcal{E}_\leq(c_i,\delta E_i)\cap\mathcal{E}_\geq(c_i,E_i)$$
$$\cap\mathcal{E}_\geq(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{J}_l^i$$
$$\subset\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)$$
$$\cap\big(\mathcal{E}_\geq(c_i,\delta_iE_i)\cup\mathcal{E}_\geq(\bar{c}_i,\mu_i\bar{E}_i)\cup\mathcal{C}_<(c_i,c_i-p_l^i,\psi_i,E_i)\big)$$
$$=\big(\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{E}_\geq(c_i,\delta_iE_i)\big)$$
$$\cup\big(\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{E}_\leq(\bar{c}_i,\mu_i\bar{E}_i)\big)$$
$$\cup\big(\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{C}_\leq(c_i,c_i-p_l^i,\psi_i,E_i))\big)$$
$$\overset{(84)}{=}\mathcal{H}_i(\delta,\mu_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)\cap\mathcal{E}_\leq(\bar{c}_i,\mu_i\bar{E}_i)$$
$$=\mathcal{E}_\leq(c_i,\delta E_i)\cap\mathcal{E}_\geq(c_i,E_i)\cap\mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_\geq(c_i,c_i-p_l^i,\psi,E_i)$$
$$\overset{(75b)}{=}\tilde{\mathcal{P}}_{l,3}^i(\delta,\psi).\qquad(85)$$

This fact combined with Step 1, shows that the solution leaves the set $\mathcal{F}_l^i$ in finite time through the set $\tilde{\mathcal{P}}_{l,3}^i(\delta,\psi)$, where it jumps. Then, we have $x(t',j+1)=x(t',j)\in\tilde{\mathcal{P}}_{m(t',j),3}^{i(t',j)}(\delta,\psi)$, $i(t',j+1)=i(t',j)=:\iota$ and $m(t',j)=0$. Then, by (85), $x(t',j+1)\in\mathcal{E}(c_\iota,\delta E_\iota)\cap\mathcal{E}(\bar{c}_\iota,\mu_\iota\bar{E}_\iota)$, and, by Lemma 8, $x(t',j+1)\in\mathcal{C}(0,c_\iota,\vartheta_\iota(\delta,\mu_\iota),E_\iota)$. We have shown in Step 2 that both the sets $\mathcal{E}_\leq(\bar{c}_\iota,\bar{E}_\iota)$ and $\mathcal{C}(0,c_\iota,\vartheta_\iota(\delta,\mu_\iota),E_\iota)$ are forward invariant under the stabilization flow map for $x$, i.e., $-k_0x$. Since the obstacles are weakly disjoint, the solution then flows in $\mathcal{E}_\leq(\bar{c}_\iota,\bar{E}_\iota)\cap\mathcal{C}(0,c_\iota,\vartheta_\iota(\delta,\mu_\iota),E_\iota)$ until it reaches the set $\mathcal{E}(c_\iota,\delta_\iota E_\iota)$ at $(t'',j+1)$. We either have $\|x(t'',j+1)\|<r_\iota$ or $\|x(t'',j+1)\|\geq r_\iota$. Consider the former case. Define

$$\sigma_1:=\min_{i,i'\in\mathbb{I},\,i\neq i'}\mathbf{dist}(\mathcal{E}_\leq(c_i,\delta_iE_i),\mathcal{E}_\leq(c_{i'},\delta_{i'}E_{i'}))>0,\quad(86)$$

which is positive because obstacle are compact, pairwise disjoint sets. Since $x(t'',j+1)\in\mathcal{E}(c_\iota,\delta_\iota E_\iota)$ and the obstacles are weakly pairwise disjoint, the solution can only flow up to the time $(s,j+1)$ such that $\sigma_1$ is traversed, i.e.,

$$\|x(s,j+1)\|+\sigma_1\leq\|x(t'',j+1)\|<r_\iota\leq\|x(t,j)\|,\quad(87)$$

and the claim of the proposition is proven. Consider the latter case, i.e., $\|x(t'',j+1)\|\geq r_\iota$. Then, the definition of the set $\mathcal{R}_\iota(\delta_\iota,\mu_\iota)$ in (39) implies that $x(t'',j+1)\in\mathcal{R}_\iota(\delta_\iota,\mu_\iota)$.

However, thanks to (40), the solution can only flow with stabilization mode while in $\mathcal{R}_\iota(\delta_\iota,\mu_\iota)$, so that $(t''',j+1)$ exists such that $\|x(t''',j+1)\|=r_\iota$. Define

$$\sigma_2:=\min_{i,i'\in\mathbb{I},i\neq i'}\mathbf{dist}(\mathcal{R}_i(\delta_i,\mu_i),\mathcal{E}_\leq(c_{i'},\delta_{i'}E_{i'}))>0,\quad(88)$$

which is positive because the considered sets are compact and pairwise sufficiently disjoint. Before a jump to avoidance mode at $(s,j+1)$ can occur, we have

$$\|x(s,j+1)\|+\sigma_2\leq\|x(t''',j+1)\|=r_\iota\leq\|x(t,j)\|.\quad(89)$$

*Step 3c: $\xi(t,j)\in(\mathcal{F}_l\cap\mathcal{J}_l)\times\{l\}$ and the solution flows.* The solution cannot flow forever, as established in Step 1. If it flows until its component $x$ reaches the set $\mathcal{P}_{l,3}^{i(t,j)}(\delta_{i(t,j)},\psi_{i(t,j)})$, the second part of the argument of Step 3b still applies, in particular (87) or (89). If it jumps beforehand, Step 3a applies. Then we do not have circularity. By combining (83), (87) and (89), (49) is proven with $\sigma:=\min\{\sigma_0,\sigma_1,\sigma_2\}>0$.

*16) Proof of Lemma 12:* As for (77a), we have that $\mathbf{T}_{\mathcal{E}_\leq(c_i,\delta E_i)}(x)$ is either $\mathbb{R}^n$ for $x\in\mathcal{E}_<(c_i,\delta E_i)$ or $\mathcal{P}_\leq(0,E_i^2(x-c_i))$ for $x\in\mathcal{E}(c_i,\delta E_i)$. For all $x\in\mathcal{E}_\leq(c_i,\delta E_i)\supset\tilde{\mathcal{F}}_m^i(\delta,\psi)$, $\mathfrak{u}(x)\in\mathcal{P}(0,E_i^2(x-c_i))$ (see (48)), so (77a) is proven. A similar argument yields (77b). As for (77c), we note that

$$\tilde{\mathcal{F}}_m^i(\delta,\psi)\backslash\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)=\mathcal{E}_\leq(c_i,\delta E_i)\cap\mathcal{E}_\geq(c_i,E_i)$$
$$\cap\mathcal{E}_\geq(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_\geq(c_i,c_i-p_m^i,\psi,E_i)\backslash\Big(\mathcal{E}_\leq(c_i,\delta E_i)$$
$$\cap\mathcal{E}_\geq(c_i,E_i)\cap\mathcal{E}(\bar{c}_i,\mu_i\bar{E}_i)\cap\mathcal{C}_\geq(c_i,c_i-p_m^i,\psi,E_i)\Big)\quad(90)$$
$$\overset{(1c)}{=}\mathcal{E}_\leq(c_i,\delta E_i)\cap\mathcal{E}_\geq(c_i,E_i)\cap\mathcal{E}_>(\bar{c}_i,\mu_i\bar{E}_i)$$
$$\cap\mathcal{C}_\geq(c_i,c_i-p_m^i,\psi,E_i).$$

Then, for all $x\in\tilde{\mathcal{F}}_m^i(\delta,\psi)\backslash\tilde{\mathcal{P}}_{m,3}^i(\delta,\psi)$, $\mathbf{T}_{\mathcal{E}_\geq(\bar{c}_i,\mu_i\bar{E}_i)}(x)=\mathbb{R}^n$ thanks to (90), and this proves (77c). As for (77d), we have that $\mathbf{T}_{\mathcal{C}_\geq(c_i,c_i-p_m^i,\psi,E_i)}(x)$ is either $\mathbb{R}^n$ for $x\in\mathcal{C}_>(c_i,c_i-p_m^i,\psi,\bar{E}_i)$ or $\mathcal{P}_\geq(0,n_m^i(x))$ for $x\in\mathcal{C}(c_i,c_i-p_m^i,\psi,E_i)$ with $n_m^i(x):=E_i\pi^{\psi}(E_i(c_i-p_m^i))E_i(x-c_i)$ by (12). If we prove

that $\mathfrak{u}(x)^\top n_m^i(x) \geq 0$ for all $x \in \mathcal{C}(c_i, c_i - p_m^i, \psi, E_i) \supset \tilde{\mathcal{F}}_m^i(\delta, \psi)$, then (77d) holds. Indeed, this last step follows from

$$
\begin{aligned}
\mathfrak{u}(x)^\top \cdot n_m^i(x) = & -k_m(x - p_m^i)^\top E_i \pi^\perp(E_i(x - c_i)) E_i^{-1} \\
& \cdot E_i \pi^\psi(E_i(c_i - p_m^i)) E_i(x - c_i) \\
= & \ k_m(p_m^i - c_i)^\top E_i \pi^\perp(E_i(x - c_i)) \\
& \cdot \pi^\psi(E_i(c_i - p_m^i)) E_i(x - c_i) \\
\overset{(5),(3)}{=} & \ k_m(p_m^i - c_i)^\top E_i \pi^\perp(E_i(x - c_i)) \\
& \cdot \big( \cos^2(\psi) I_n - \pi^\|(E_i(c_i - p_m^i)) \big) E_i(x - c_i) \\
= & -k_m(p_m^i - c_i)^\top E_i \pi^\perp(E_i(x - c_i)) \\
& \cdot \pi^\|(E_i(c_i - p_m^i)) E_i(x - c_i) \\
\overset{(3)}{=} & \ k_m(c_i - p_m^i)^\top E_i \pi^\perp(E_i(x - c_i)) E_i(c_i - p_m^i) \\
& \cdot (c_i - p_m^i)^\top E_i^2(x - c_i) \| E_i(c_i - p_m^i) \|^{-2} \geq 0
\end{aligned}
\tag{91}
$$

because for all $x \in \mathcal{C}(c_i, c_i - p_m^i, \psi, E_i)$, one has $x \in \mathcal{P}_{\geq}(c_i, E_i^2(c_i - p_m^i))$.

## REFERENCES

[1] M. Hoy, A. S. Matveev, and A. V. Savkin, "Algorithms for collision-free navigation of mobile robots in complex cluttered environments: a survey," *Robotica*, vol. 33, no. 3, pp. 463–497, 2015.

[2] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," in *Autonomous robot vehicles*. Springer, 1986, pp. 396–404.

[3] D. E. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Advances in applied mathematics*, vol. 11, no. 4, pp. 412–442, 1990.

[4] H. G. Tanner and A. Kumar, "Formation stabilization of multiple agents using decentralized navigation functions." in *Robotics: Science and systems*, vol. 1, 2005, pp. 49–56.

[5] D. V. Dimarogonas, S. G. Loizou, K. J. Kyriakopoulos, and M. M. Zavlanos, "A feedback stabilization and collision avoidance scheme for multiple independent non-point agents," *Automatica*, vol. 42, no. 2, pp. 229–243, 2006.

[6] G. Roussos and K. J. Kyriakopoulos, "Decentralized and prioritized navigation and collision avoidance for multiple mobile robots," in *Distributed Autonomous Robotic Systems*. Springer, 2013, pp. 189–202.

[7] G. Lionis, X. Papageorgiou, and K. J. Kyriakopoulos, "Locally computable navigation functions for sphere worlds," in *IEEE International Conference on Robotics and Automation*, 2007, pp. 1998–2003.

[8] I. Filippidis and K. J. Kyriakopoulos, "Navigation functions for focally admissible surfaces," in *Amer. Control Conference*, 2013, pp. 994–999.

[9] S. G. Loizou, "The navigation transformation," *IEEE Trans. on Robotics*, vol. 33, no. 6, pp. 1516–1523, 2017.

[10] C. Vrohidis, P. Vlantis, C. P. Bechlioulis, and K. J. Kyriakopoulos, "Prescribed time scale robot navigation," *IEEE Robotics and Automation Letters*, vol. 3, no. 2, pp. 1191–1198, 2018.

[11] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans. Automat. Contr.*, vol. 62, no. 8, pp. 3861–3876, 2017.

[12] P. Falcone, F. Borrelli, J. Asgari, H. E. Tseng, and D. Hrovat, "A real-time model predictive control approach for autonomous active steering," *Nonlinear Model Predictive Control for Fast Systems, Grenoble, France*, p. 7, 2006.

[13] M. Defoort, A. Kokosy, T. Floquet, W. Perruquetti, and J. Palos, "Motion planning for cooperative unicycle-type mobile robots with limited sensing ranges: A distributed receding horizon approach," *Robotics and autonomous systems*, vol. 57, no. 11, pp. 1094–1106, 2009.

[14] F. Wilson Jr, "The structure of the level surfaces of a Lyapunov function," *Journal of Differential Equations*, 1967.

[15] R. G. Sanfelice, *Robust hybrid control systems*. University of California, Santa Barbara, 2007.

[16] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel, "Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points," in *Amer. Control Conference*, 2006, pp. 3352–3357.

[17] J. I. Poveda, M. Benosman, A. R. Teel, and R. G. Sanfelice, "A hybrid adaptive feedback law for robust obstacle avoidance and coordination in multiple vehicle systems," in *Amer. Control Conference*, 2018, pp. 616–621.

[18] P. Braun, C. M. Kellett, and L. Zaccarian, "Unsafe point avoidance in linear state feedback," *IEEE Conference on Decision and Control*, 2018.

[19] ——, "Explicit construction of robust avoidance controllers for linear systems," 2018, available from https://hal.archives-ouvertes.fr/hal-01893027.

[20] S. Berkane, A. Bisoffi, and D. V. Dimarogonas, "A hybrid controller for obstacle avoidance in an $n$-dimensional euclidean space," in *Europ. Control Conference*, 2019, pp. 764–769.

[21] S. Paternain, D. E. Koditschek, and A. Ribeiro, "Navigation functions for convex potentials in a space with convex obstacles," *IEEE Trans. Automat. Contr.*, vol. 63, no. 9, pp. 2944–2959, 2018.

[22] C. D. Meyer, *Matrix analysis and applied linear algebra*. SIAM, 2000.

[23] R. Engelking, *General topology*. Heldermann, 1989.

[24] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[25] Y.-K. Choi, W. Wang, Y. Liu, and M.-S. Kim, "Continuous collision detection for two moving elliptic disks," *IEEE Trans. on Robotics*, vol. 22, no. 2, pp. 213–224, 2006.

[26] W. Wang, J. Wang, and M.-S. Kim, "An algebraic condition for the separation of two ellipsoids," *Computer aided geometric design*, vol. 18, no. 6, pp. 531–539, 2001.

[27] C. G. Mayhew and A. R. Teel, "Global stabilization of spherical orientation by synergistic hybrid feedback with application to reduced-attitude tracking for rigid bodies," *Automatica*, vol. 49, no. 7, pp. 1945–1957, 2013.

[28] V. J. Lumelsky and A. Stepanov, "Dynamic path planning for a mobile automaton with limited information on the environment," *IEEE Trans. Automat. Contr.*, vol. 31, no. 11, pp. 1058–1063, 1986.

[29] L. Wang, A. D. Ames, and M. Egerstedt, "Safety barrier certificates for collisions-free multirobot systems," *IEEE Trans. on Robotics*, vol. 33, no. 3, pp. 661–674, 2017.

[30] J. Chai and R. G. Sanfelice, "Forward invariance of sets for hybrid dynamical systems (Part I)," *IEEE Trans. Automat. Contr.*, 2019.

[31] R. T. Rockafellar, "Clarke's tangent cones and the boundaries of closed sets in $\mathbf{R}^n$," *Nonlinear analysis: Theory, Methods & Applications*, vol. 3, pp. 145–154, 1979.

**Soulaimane Berkane** received his Engineering and M.Sc. degrees in Automatic Control from Ecole Nationale Polytechnique, Algeria, in 2013, and his PhD in Electrical Engineering from the University of Western Ontario, Canada, in 2017. He held postdoctoral positions at the University of Western Ontario, Canada, and at KTH Royal Institute of Technology, Sweden, between 2018 and 2019. He is currently an assistant professor at the Department of Computer Science and Engineering, University of Quebec in Outaouais, Canada. His research interests are in the area of nonlinear control theory with applications to robotic and autonomous systems.

**Andrea Bisoffi** received the M.Sc. degree in Automatic Control Engineering from Politecnico di Milano, Italy, in 2013 and the Ph.D. degree in Mechatronics from the University of Trento, Italy, in 2017. In 2015—2016 he was a visiting scholar in the Control Group at the University of Cambridge, UK. He was a postdoctoral researcher at KTH Royal Institute of Technology, Sweden in 2017-2019 and is currently one at the University of Groningen, The Netherlands. His current research interests include hybrid and nonlinear control systems, with applications to mechanical, and robotic systems.

**Dimos V. Dimarogonas** received the Diploma in Electrical and Computer Engineering in 2001 and the Ph.D. in Mechanical Engineering in 2007, both from the National Technical University of Athens (NTUA), Greece. From 2007 to 2010, he held postdoctoral positions at KTH Royal Institute of Technology and Massachusetts Institute of Technology. He is currently a Professor in Automatic Control, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology. His current research interests include multi-agent and hybrid systems with applications to autonomous systems. He serves on the Editorial Board of Automatica and the IEEE Transactions on Control of Network Systems. Dr. Dimarogonas is a recipient of an ERC Starting Grant in 2014, an ERC Consolidator Grant in 2019, and a Wallenberg Academy Fellowship in 2015. He is a member of the Technical Chamber of Greece and a Senior Member of the IEEE.