THE ENERGY-DISSIPATION PRINCIPLE FOR STOCHASTIC PARABOLIC EQUATIONS

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Abstract. The Energy-Dissipation Principle provides a variational tool for the analysis of parabolic evolution problems: solutions are characterized as so-called null-minimizers of a global functional on entire trajectories. This variational technique allows for applying the general results of the calculus of variations to the underlying differential problem and has been successfully applied in a variety of deterministic cases, ranging from doubly nonlinear flows to curves of maximal slope in metric spaces. The aim of this note is to extend the Energy-Dissipation Principle to stochastic parabolic evolution equations. Applications to stability and optimal control are also presented.

1. Introduction

This note is concerned with a global variational approach to the Cauchy problem for the abstract stochastic parabolic evolution equation

$$\mathrm{d}u + \partial\phi(u)\,\mathrm{d}t \ni F(\cdot,u)\,\mathrm{d}t + G(\cdot,u)\,\mathrm{d}W, \quad u(0) = u^0. \tag{1}$$

The trajectory $u : \Omega\times[0,T] \to H$ is defined on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ and the bounded time interval $[0,T]$ and takes values in the separable Hilbert space H. The functional ϕ is assumed to be convex and lower semicontinuous, the nonlinearities F and G are taken to be suitably smooth, and W is a cylindrical Wiener process on a second separable Hilbert space U. More precisely, solutions u of equation (1) are asked to be *Itô processes* of the form

$$u(t) = u^d(t) + \int_0^t u^s(s)\,\mathrm{d}W(s), \quad t \in [0,T], \tag{2}$$

where u^d is an absolutely continuous process and u^s is a W-stochastically integrable process. In particular, we look for solutions u of equation (1) in the space $\mathscr{U}$ consisting of all Itô processes of the form (2) with $u^d \in L^2_{\mathscr{P}}(\Omega; H^1(0,T;H))$ and $u^s \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$. Here, $\mathscr{L}^2(U,V)$ indicates the set of Hilbert-Schmidt operators from U to V, where V is a separable reflexive Banach space, densely and compactly embedded into H.

Existence, uniqueness, and continuous dependence on the initial data for stochastic evolution problems in the form (1) are addressed within the classical variational theory

2010 *Mathematics Subject Classification.* 35K55, 35R60, 60H15.
Key words and phrases. Variational principle, parabolic SPDE, generalized Itô's formula, null-minimization, stability, optimal control.

by Pardoux [44, 45] and Krylov & Rozovskiĭ,[32] in the sense of analytically weak or *martingale* solutions: we refer also to the monographs [19, 47] for a general presentation. In the context of analytically strong solutions, existence for stochastic equations in the subdifferential form (1) has been obtained by Gess [27]. Well-posedness results in a weak sense have also been obtained under more general conditions in the monograph [34] and in the papers [37, 35, 36, 50].

Following the seminal remarks by De Giorgi [1], one can variationally characterize solutions of the Cauchy problem (1) in terms of trajectories minimizing the *Energy-Dissipation-Principle (EDP)* functional $I : \mathscr{U} \to [0,\infty]$ defined as

$$\begin{aligned} I(u) = \mathbb{E}\phi(u(T)) - \mathbb{E}\phi(u(0)) + \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d\|^2 \,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T \|\partial\phi(u) - F(\cdot,u)\|^2 \,\mathrm{d}s \\ - \mathbb{E}\int_0^T (\partial_t u^d, F(\cdot,u))\,\mathrm{d}s \; - \frac{1}{2}\mathbb{E}\int_0^T \mathrm{Tr}_H\, L(u)\,\mathrm{d}s \\ + 2C_\phi \mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \mathbb{E}\|u(0)-u^0\|^2_V \end{aligned} \tag{3}$$

if $u \in C([0,T];L^2(\Omega,\mathscr{F};V))$ and $I(u) = \infty$ otherwise. Here, $L(u) := u^s(u^s)^* D_{\mathcal{G}}\partial\phi(u)$ and u^d, u^s are associated to u via the decomposition (2). The symbols $(\cdot,\cdot)$ and $\|\cdot\|$ stand for the scalar product and the norm in H, respectively, and $\partial\phi(u)$ denotes the subdifferential of ϕ, here assumed to be Gateaux-differentiable from V to V^*. The constant $C_\phi > 0$ depends on ϕ and is defined in (5) below. Within our assumption setting, we will have that $L(u) \in \mathscr{L}^1(V,V)$, where the latter is the space of *trace-class* operators from V to V. The symbol Tr_H hence denotes the *trace* of the operator with respect to an orthonormal system of H in V. The fact that I takes nonnegative values hinges on the validity of an Itô formula for ϕ, see Proposition 3.1 below.

The focus of this note is to discuss the equivalence of solutions of equation (1) and *null-minimizers* of the EDP functional I. Under general assumptions on ϕ, F, and G, our main result, Theorem 2.1, states that

$$u \;\text{ solves (1)} \quad \Leftrightarrow \quad 0 = I(u) = \min_{\mathscr{U}} I.$$

The core of this characterization resides on the nature of the EDP functional I, which in the present setting corresponds to the *squared residual* of the system

$$\partial_t u^d + \partial\phi(u) = F(\cdot,u), \quad u^s = G(\cdot,u), \quad u(0) = u^0,$$

as illustrated in Proposition 3.2 below. The approach in (3) is however more general and can be adapted in Banach spaces and doubly nonlinear problems as well, see Remark 3.3 below.

The residual nature of the EDP functional entails that the EDP variational principle $0 = I(u) = \min_{\mathscr{U}} I$ is not a mere minimization problem, for one is asked to check that the minimum is actually 0, motivating the use of the term *null-minimization*. This issue is not uncommon for global variational approaches and can be traced back to celebrated *Brezis-Ekeland-Nayroles* principle [16, 17, 42, 43]. In the current stochastic case, the existence of a unique null-minimizer follows from the well-posedness of the differential problem (1). Still, minimization cannot be tackled directly, for the functional I shows some limited semicontinuity properties, see Section 4. Apart from the case when Ω is atomic, this prevents us from providing an alternative existence theory for problem (1). On the other hand, we make use of the EDP characterization for proving stability of the Cauchy problem for equation (1) under perturbations of the data (u^0, ϕ, F, G) in Section 5 and for discussing the penalization of an optimal control problem constrained to (1) in Section 6.

Before moving on, let us mention two alternative global variational principles for SPDEs of the class (1). The mentioned Brezis-Ekeland-Nayroles principle has been indeed extended to the stochastic case. Following some specific application in [5, 6, 8], a general theory has been presented by Barbu & Röckner [7, 9, 10] in the linear multiplicative case and by Boroushaki & Ghoussoub [14] in the nonlinear multiplicative case. In our notation, the stochastic Brezis-Ekeland-Nayroles functional from [14] reads

$$u \mapsto \mathbb{E}\int_0^T \Big(\phi(u) + \phi^*(F(\cdot,u) - \partial_t u^d) - (F(\cdot,u) - \partial_t u^d, u)\Big)\,\mathrm{d}s \\ + \frac{1}{2}\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \mathbb{E}\|u(0)-u^0\|^2.$$

Here, ϕ^* stands for the Legendre conjugate of ϕ. Recall that $\phi(u) + \phi(v) \geq (v,u)$ for all u, $v \in H$ and that equality holds if and only if $v \in \partial\phi(u)$. Hence, a null-minimizer of the latter necessarily solves the Cauchy problem for (1). By resorting to the far-reaching theory of anti-self dual Lagrangians, the existence of null-minimizers of the Brezis-Ekeland-Nayroles functional has been ascertained in [14].

In the additive case, a different global variational approach to (1) is in [53], where the *Weighted-Energy-Dissipation* functional

$$u \mapsto \mathbb{E}\int_0^T e^{-s/\varepsilon}\Big(\frac{\varepsilon}{2}\|\partial_t u^d\|^2 + \phi(u) - (F,u)\Big)\,\mathrm{d}s + \mathbb{E}\int_0^T e^{-s/\varepsilon}\frac{1}{2}\|u^s - G\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s$$

is investigated. This *strictly convex* functional admits a unique minimizer u_ε over trajectories with given initial value u^0. At all levels $\varepsilon > 0$, such minimizers solve an elliptic-in-time regularization of equation (1), complemented by an extra Neumann boundary condition at the final time T. In particular, the minimization of the Weighted-Energy-Dissipation functional corresponds to a *noncausal* differential problem. As $\varepsilon \to 0$ one can prove [53] that u_ε converge to the solution to the Cauchy problem (1). In particular, causality is restored in the limit.

Compared with the Brezis-Ekeland-Nayroles approach, the null-minimization of the EDP functional is a priori not restricted to the case of a convex ϕ (although we limit ourselves to convex ϕ in this note, for simplicity) and is easily adapted to more nonlinear situations, see Remark 3.3 below. With respect to the Weighted-Energy-Dissipation approach, the null-minimization of the EDP functional does not require to take the extra limit $\varepsilon \to 0$ and is causal. It is hence better suited to discuss convergence issues.

We collect notation, assumptions, and the statement of the characterization, i.e., Theorem 2.1, in Section 2. Section 3 is devoted to the proof of a generalized Itô formula, which is crucial for studying the EDP functional and brings to the proof of the characterization. We discuss in Section 4 the coercivity and the restricted lower-semicontinuity of the EDP functional, as well as the fact that minimizers of I are actually null-minimizers. We then obtain a stability result with respect to data perturbations in Section 5. Eventually, in Section 6 we discuss a general penalization procedure for an optimal control problem based on (1).

2. Setting and statement

In preparation of the statement of our main result, let us collect here the assumptions on spaces and nonlinearities, which will be tacitly assumed throughout the paper.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, with right-continuous and saturated filtration $(\mathscr{F}_t)_{t\in[0,T]}$, H and U be separable Hilbert spaces, and W be a cylindrical Wiener process on U. Moreover, let V be a separable and reflexive Banach space, with $V \subset H$ densely and compactly, so that $V \subset H \subset V^*$ (dual) is a Gelfand triplet. We recall that the symbols $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in H. Moreover, $\langle\cdot,\cdot\rangle$ stands for the duality product between V^* and V. The norm in any other Banach space E will be denoted by $\|\cdot\|_E$.

In the following, we use the classical notation $\mathscr{L}(V,H)$, $\mathscr{L}^2(U,V)$, and $\mathscr{L}^1(V,V)$ to indicate the space of linear and continuous operators from V to H, the space of Hilbert-Schmidt operators from U to V, and the space of trace-class operators on V, respectively. The symbols $\mathscr{L}_s(V,H)$ and $\mathscr{L}_w(V,H)$ indicate that the space $\mathscr{L}(V,H)$ is endowed with the so-called *strong*, resp. *weak* operator topology. For all $L \in \mathscr{L}^1(V,V)$ we denote by $\mathrm{Tr}_H\, L$ the *trace* of the operator with respect to an orthonormal system $(e_k)_{k\in\mathbb{N}}$ of H contained in V, namely,

$$\mathrm{Tr}_H\, L = \sum_{i=1}^{\infty} (Le_k, e_k).$$

We denote by $\mathscr{P}$ the progressive σ-algebra on $\Omega \times [0,T]$ and write $L^s(\Omega;E)$ and $L^s(0,T;E)$ for the spaces of strongly measurable Bochner-integrable E-valued functions on Ω and $(0,T)$, for all $s \in [1,\infty]$ and a Banach space E. For $s,r \in [1,\infty)$ we use the symbol $L^s_{\mathscr{P}}(\Omega; L^r(0,T;E))$ to indicate that measurability is intended with respect to the progressive σ-algebra $\mathscr{P}$.

In the following, we will make use of the space $\mathscr{U}$ of Itô processes given by

$$\begin{aligned}\mathscr{U} = \{u \in L^2(\Omega; C([0,T];H)) \, : \, &\text{the decomposition (2) holds for} \\ &u^d \in L^2_{\mathscr{P}}(\Omega; H^1(0,T;H)),\ u^s \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))\}.\end{aligned}$$

Note that the representation $u = u^d + u^s \cdot W$ is unique and defines an isomorphism

$$\mathscr{U} \simeq L^2_{\mathscr{P}}(\Omega; H^1(0,T;H)) \times L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V))).$$

In the following, we will systematically (and tacitly) use such isomorphism by identifying $u \in \mathscr{U}$ with the corresponding pair of processes $(u^d, u^s) \in L^2_{\mathscr{P}}(\Omega; H^1(0,T;H)) \times$

$L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$. We will also make use of the the subspace $\mathscr{V} \subset \mathscr{U}$ given by

$$\mathscr{V} := \mathscr{U} \cap C([0,T]; L^2(\Omega,\mathscr{F};V)\}.$$

We ask $\phi : H \to [0,\infty]$ to be a convex and lower semicontinuous with $\phi(0) = 0$ and *essential domain* $D(\phi) = \{u \in H \ : \ \phi(u) < \infty\} = V$, and require the subdifferential $\partial\phi : D(\partial\phi) \subseteq H \to H$, where $D(\partial\phi) = \{u \in H \ : \ \partial\phi(u) \neq \emptyset\}$, to be single-valued and coercive in the following sense:

$$\exists\, c_\phi > 0 : \quad (\partial\phi(u_1) - \partial\phi(u_2), u_1 - u_2) \geq c_\phi \|u_1 - u_2\|_V^2 \quad \forall u_1,\, u_2 \in D(\partial\phi). \tag{4}$$

We moreover ask ϕ to be continuous at some point of its domain , so that the subdifferential $\partial(\phi|_V) : V \to V^*$ of the restriction $\phi|_V : V \to [0,\infty]$ is maximal monotone and coincides with $\partial\phi$ on V. We further assume $\partial\phi : V \to V^*$ to be Gâteaux-differentiable with Gâteaux-differential $D_\mathcal{G}\partial\phi \in C(V; \mathscr{L}_s(V,V^*))$ fulfilling

$$\exists\, C_\phi > 0 : \qquad \|D_\mathcal{G}\partial\phi(u)\|_{\mathscr{L}(V,V^*)} \leq C_\phi \quad \forall u \in V. \tag{5}$$

Note that the latter entails that $\partial\phi$ is linearly bounded from V to V^*. Indeed, we have that

$$\|\partial\phi(u)\|_{V^*} = \left\| \int_0^1 \langle D_\mathcal{G}\partial\phi(ru), u\rangle \,\mathrm{d}r \right\|_{V^*} \leq \int_0^1 \|D_\mathcal{G}\partial\phi(ru)\|_{\mathscr{L}(V,V^*)} \|u\|_V \,\mathrm{d}r \leq C_\phi \|u\|_V$$

and we can compute that

$$\frac{c_\phi}{2}\|u\|_V^2 \leq \phi(u) = \int_0^1 (\partial\phi(ru), u)\,\mathrm{d}r \leq \int_0^1 r C_\phi \|u\|_V^2 \,\mathrm{d}r = \frac{C_\phi}{2}\|u\|_V^2 \quad \forall u \in V. \tag{6}$$

Moreover, we have the control

$$\forall\, u \in \mathscr{U} : \qquad \mathrm{Tr}_H\, L(u) \leq C_\phi \|u^s\|^2_{\mathscr{L}^2(U,V)} \quad \text{a.e. in } \Omega \times (0,T), \tag{7}$$

where we recall that $L(u) := u^s(u^s)^* D_\mathcal{G}\partial\phi(u)$.

We require the map $F : [0,T] \times H \to H$ to be Carathéodory with $F(\cdot,0) \in L^2(0,T;H)$ and to be Lipschitz continuous, uniformly with respect to t. More precisely, we assume that

$$\begin{aligned} \exists\, c_F > 0 : \quad & \|F(t,u_1) - F(t,u_2)\| \leq c_F \|u_1 - u_2\| \\ & \forall u_1,\, u_2 \in H, \text{ for a.e. } t \in (0,T). \end{aligned} \tag{8}$$

The latter specifically implies that the process $F(\cdot,u)$ belongs to $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ for all $u \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$.

The map $G : [0,T] \times V \to \mathscr{L}^2(U,V)$ is also asked to be Carathéodory with $G(\cdot,0) \in L^2(0,T;\mathscr{L}^2(U,V))$ and uniformly Lipschitz continuous and linearly bounded with respect to t, namely,

$$\begin{aligned} \exists\, c_G > 0 : \qquad & \|G(t,u_1) - G(t,u_2)\|_{\mathscr{L}^2(U,H)} \leq c_G \|u_1 - u_2\| \\ & \forall u_1,\, u_2 \in H, \text{ for a.e. } t \in (0,T), \end{aligned} \tag{9}$$

and

$$\exists\, c_{G,2} > 0 : \qquad \|G(\cdot,u)\|_{\mathscr{L}^2(U,V)} \leq c_{G,2}\,(1 + \|u\|_V) \quad \forall u \in V. \tag{10}$$

In particular, $G(\cdot,u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$ for all $u \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;V))$.

Eventually, we prescribe the initial datum

$$u^0 \in L^2(\Omega, \mathscr{F}_0; V). \tag{11}$$

The Cauchy problem for (1) can hence be specified as

$$u(t) = u^0 + \int_0^t (F(\cdot, u) - \partial\phi(u))\,\mathrm{d}s + \int_0^t G(\cdot, u)\,\mathrm{d}W \quad \forall\, t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{12}$$

where the latter is intended as an equation in H.

Under the above assumptions, one can adapt the theory from [52] and [27] in order to prove that equation (12) admits a unique solution $u \in \mathscr{V}$ which in addition belongs to

$$L^2_{\mathscr{P}}(\Omega; L^\infty(0,T;V)) := \big\{ v : \Omega \to L^\infty(0,T;V) \text{ weakly* progressively measurable} \\ \text{with } \ \mathbb{E}\|v\|^2_{L^\infty(0,T;V)} < \infty \big\},$$

and can be obtained as limits of approximations arising from Yosida-regularizing $\partial\phi$. As such, when referring to a solution of equation (12) the regularity $u \in \mathscr{V}$ will be always assumed in the following. Note that this is not restrictive, for all strong-in-time solutions of (12), namely, $u^d \in L^2(\Omega; W^{1,1}(0,T;H))$, can a posteriori be proved to belong to $\mathscr{V}$, see Remark 3.4 below.

The central observation of this note is the following characterization.

Theorem 2.1 (Energy Dissipation Principle)**.** *$u \in \mathscr{U}$ solves (12) if and only if $0 = I(u) = \min_{\mathscr{U}} I$.*

This characterization is proved in the next Section 3, by resorting to a generalized Itô formula for ϕ.

3. Itô formula and proof of Theorem 2.1

In the deterministic case, the Energy-Dissipation Principle hinges on the validity of the chain rule for the functional ϕ. In the stochastic case, this corresponds to a specific Itô formula, which we now present.

Proposition 3.1 (Itô formula)**.** *Let $u \in \mathscr{V}$ and assume that $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. Then,*

$$\phi(u(t)) = \phi(u(0)) + \int_0^t (\partial_t u^d, \partial\phi(u))\,\mathrm{d}s + \int_0^t (\partial\phi(u), u^s\,\mathrm{d}W) \\ + \frac{1}{2}\int_0^t \mathrm{Tr}_H\, L(u)\,\mathrm{d}s \quad \forall t \in [0,T],\ \mathbb{P}\text{-}a.s. \tag{13}$$

In particular, $t \mapsto \mathbb{E}\phi(u(t)) \in W^{1,1}(0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\phi(u(t)) = \mathbb{E}(\partial_t u^d, \partial\phi(u)) + \frac{1}{2}\mathbb{E}\,\mathrm{Tr}_H\ L(u) \quad \textit{for a.e. } t \in (0,T). \tag{14}$$

Proof. The Itô formula (13) is proved in [52, Lemma 3.2] for ϕ replaced by its Moreau-Yosida approximation ϕ_λ at level $\lambda > 0$ [15]. In particular, for all $\lambda > 0$ we have that

$$\phi_\lambda(u(t)) = \phi_\lambda(u(0)) + \int_0^t (\partial_t u^d, \partial\phi_\lambda(u))\,\mathrm{d}s + \int_0^t (\partial\phi_\lambda(u), u^s\,\mathrm{d}W) \\ + \frac{1}{2}\int_0^t \mathrm{Tr}_H(u^s(u^s)^* D_{\mathcal{G}}\partial\phi_\lambda(u))\,\mathrm{d}s \quad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{15}$$

In order to check for (13), we hence aim at taking the limit $\lambda \to 0$ in (15). The pointwise convergence of ϕ_λ to ϕ on $D(\phi)$ [15, Prop. 2.11, p. 39] ensures that $\phi_\lambda(u(t)) \to \phi(u(t))$ and $\phi_\lambda(u(0)) \to \phi(u(0))$. Moreover, from $\|\partial\phi_\lambda(u)\| \le \|\partial\phi(u)\|$ a.e. and the fact that $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ we conclude that $\partial\phi_\lambda(u) \rightharpoonup \xi$ in $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ by possibly extracting a not relabeled subsequence. On the other hand, one readily check that $\xi = \partial\phi(u)$ a.e. by passing to the limit $\lambda \to 0$ into $(\partial\phi_\lambda(u), w-u) \le \phi_\lambda(w) - \phi_\lambda(u)$ a.e. for all $w \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. Hence, extracting a subsequence was actually not needed. Eventually, the first two integrands in the right-hand side of (13) converge to the corresponding limits.

We are hence left to check the limit of the trace term. To this aim, we recall from [52, Lemma 3.1] that

$$D_{\mathcal{G}}\partial\phi_\lambda(u) = D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)$$

where we have denoted by $J_\lambda : V^* \to V$ the *resolvent* $J_\lambda := (I + \lambda\partial\phi)^{-1}$. The coercivity assumption (4) ensures that J_λ is Lipschitz continuous. Moreover, one can enhance the usual a.e. convergence $J_\lambda u \to u$ in H to

$$c_\phi \|J_\lambda u - u\|^2_{L^2_{\mathscr{P}}(\Omega;L^2(0,T;V))} \le \mathbb{E}\int_0^T (\partial\phi(J_\lambda u) - \partial\phi(u), J_\lambda u - u)\,\mathrm{d}s \to 0,$$

so that $J_\lambda u \to u$ in V a.e. Recalling that $D_{\mathcal{G}}\partial\phi \in C(V; \mathscr{L}_s(V,V^*))$ one gets that

$$u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u)) \to u^s(u^s)^* D_{\mathcal{G}}\partial\phi(u) \quad \text{in } \mathscr{L}^1_s(V,V) \text{ a.e.}$$

On the other hand, from [52, Lemma 3.1] one has that $D_{\mathcal{G}}J_\lambda(h) \to I$ in $\mathscr{L}_s(H,H)$ for all $h \in H$. In fact, under the coercivity assumption (4), the argument of [52, Lemma 3.1] can be straightforwardly extended to ensure that the convergence $D_{\mathcal{G}}J_\lambda(v) \to I$ actually holds in $\mathscr{L}_w(V,V)$ for all $v \in V$, as well. In particular, for all $k \in \mathbb{N}$ we have that

$$\big(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)e_k, e_k\big) = \big(D_{\mathcal{G}}J_\lambda(u)e_k, D_{\mathcal{G}}\partial\phi(J_\lambda(u))u^s(u^s)^*e_k\big) \\ \to \big(e_k, D_{\mathcal{G}}\partial\phi(u)u^s(u^s)^*e_k\big) = (L(u)e_k, e_k).$$

In order to use the latter and pass to the limit in

$$\mathrm{Tr}_H\,(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)) = \sum_{k=1}^\infty \big(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)e_k, e_k\big) \tag{16}$$

we now provide a bound on the series, independently of λ. We recall the invariance of the trace under permutations, namely,

$$\mathrm{Tr}_H\,(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)) = \mathrm{Tr}_U\,((u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)u^s),$$

where now Tr_U is the trace in $\mathscr{L}^1(U,U)$, related to a given (hence any) orthonormal basis $(v_k)_{k\in\mathbb{N}}\subset U$. In particular, we have that

$$\mathrm{Tr}_H\left(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)\right)=\sum_{i=1}^{\infty}\left(D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)u^sv_k,u^sv_k\right)$$

and we can argue as follows

$$\begin{aligned}&\left|\left(D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)u^sv_k,u^sv_k\right)\right|\\&\quad\le\|D_{\mathcal{G}}\partial\phi(J_\lambda(u))\|_{\mathscr{L}(V,V^*)}\|D_{\mathcal{G}}J_\lambda(u)\|_{\mathscr{L}(V,V)}\|u^sv_k\|_V^2\\&\quad\le\frac{C_\phi}{c_\phi}\|u^sv_k\|_V^2\in\ell^1.\end{aligned}$$

By the Dominated Convergence Theorem we have hence proved that

$$\mathrm{Tr}_H\left(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)\right)\to\mathrm{Tr}_H\,L(u)\quad\text{a.e.}$$

as well as

$$\left|\mathrm{Tr}_H\left(u^s(u^s)^* D_{\mathcal{G}}\partial\phi(J_\lambda(u))\,D_{\mathcal{G}}J_\lambda(u)\right)\right|\le\frac{C_\phi}{c_\phi}\|u^s\|^2_{\mathscr{L}^2(U,V)}\quad\text{a.e.}$$

As $\|u^s\|^2_{\mathscr{L}^2(U,V)}\in L^1_{\mathscr{P}}(\Omega;L^1(0,T))$ one can use again the Dominated Convergence Theorem, pass the limit in (15) as $\lambda\to0$, and get (13).

We now localize formula (13) to a subinterval $[s,t]\subset(0,T)$ in order to get that

$$\mathbb{E}\phi(u(t))=\mathbb{E}\phi(u(s))+\mathbb{E}\int_s^t(\partial_tu^d,\partial\phi(u))\,\mathrm{d}s+\frac12\mathbb{E}\int_s^t\mathrm{Tr}_H\,L(u)\,\mathrm{d}s\quad\forall\,0<s<t<T$$

This proves that $t\mapsto\mathbb{E}\phi(u(t))$ is absolutely continuous, and the differential Itô formula (14) follows by the arbitrariness of s and t. □

We now use Proposition 3.1 in order to give an equivalent formulation of the EDP functional I in terms of squared residuals.

Proposition 3.2 (Equivalent formulation). *For $u\in\mathscr{V}$ with $\partial\phi(u)\in L^2_{\mathscr{P}}(\Omega;L^2(0,T;H))$ one has*

$$\begin{aligned}I(u)=\frac12\mathbb{E}\int_0^T\|\partial_tu^d+\partial\phi(u)-F(\cdot,u)\|^2\,\mathrm{d}s+2C_\phi\mathbb{E}\int_0^T\|u^s-G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s\\+\mathbb{E}\|u(0)-u^0\|_V^2.\end{aligned}\tag{17}$$

Proof. Under the assumptions $u \in \mathscr{V}$ and $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ the Itô formula (13) holds and we can compute

$$\begin{aligned}
I(u) &= \mathbb{E}\phi(u(T)) - \mathbb{E}\phi(u(0)) + \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d\|^2 \,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T \|F(\cdot,u) - \partial\phi(u)\|^2 \,\mathrm{d}s \\
&\quad - \frac{1}{2}\mathbb{E}\int_0^T \mathrm{Tr}_H\, L(u)\,\mathrm{d}s - \mathbb{E}\int_0^T (\partial_t u^d, F(\cdot,u))\,\mathrm{d}s \\
&\quad + 2C_\phi \mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,H)}\,\mathrm{d}s + \mathbb{E}\|u(0) - u^0\|_V^2 \\
&\overset{(13)}{=} \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d\|^2\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T \|F(\cdot,u) - \partial\phi(u)\|^2\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^T (\partial_t u^d, \partial\phi(u) - F(\cdot,u))\,\mathrm{d}s + 2C_\phi\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \\
&\quad + \mathbb{E}\|u(0) - u^0\|_V^2 \\
&= \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d + \partial\phi(u) - F(\cdot,u)\|^2\,\mathrm{d}s + 2C_\phi\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \\
&\quad + \mathbb{E}\|u(0) - u^0\|_V^2. \qquad \square
\end{aligned}$$

Owing to the equivalence from Proposition 3.2 we are now in the position of checking the characterization Theorem 2.1.

Proof of Theorem 2.1. Let $u \in \mathscr{U}$ be such that $I(u) = 0$. The boundedness of I in particular entails that $u \in \mathscr{V}$ and that the difference $\partial\phi(u) - F(\cdot,u)$ belongs to $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. As $u \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ one has $F(\cdot,u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ owing to (8). This implies that $\partial\phi(u)$ is in $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. We can hence use equation (17) and obtain that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d + \partial\phi(u) - F(\cdot,u)\|^2\,\mathrm{d}s + 2C_\phi\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \\
&\quad + \mathbb{E}\|u(0) - u^0\|_V^2 \overset{(17)}{=} I(u) = 0.
\end{aligned}$$

This proves that $\partial_t u^d + \partial\phi(u) = F(\cdot,u)$ and $u^s = G(\cdot,u)$ a.e. in $\Omega\times(0,T)$ and $u(0) = u^0$ $\mathbb{P}$-a.s. Hence, u solves equation (12).

Let now $u \in \mathscr{V}$ solve equation (12). In particular, we have that $\partial_t u^d + \partial\phi(u) = F(\cdot,u)$ and $u^s = G(\cdot,u)$ a.e. in $\Omega\times(0,T)$ and $u(0) = u^0$ $\mathbb{P}$-a.s. As $\partial_t u^d \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ and $F(\cdot,u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ from (8), we have that $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ as well. Again, the equivalence (17) holds and we have that

$$\begin{aligned}
I(u) &\overset{(17)}{=} \frac{1}{2}E\int_0^T \|\partial_t u^d + \partial\phi(u) - F(\cdot,u)\|^2\,\mathrm{d}s + 2C_\phi\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \\
&\quad + \mathbb{E}\|u(0) - u^0\|_V^2 = 0.
\end{aligned}$$

This concludes the proof of the theorem. □

Remark 3.3. The equivalence (17) reveals that the EDP functional is indeed nothing but the square residual of the system

$$\partial_t u^d + \partial\phi(u) = F(\cdot, u), \quad u^s = G(\cdot, u^s), \quad u(0) = u^0.$$

In particular, the expression in (17) could have been used as alternative and possibly more informative definition for I. On the other hand, definition (3) is the direct stochastic extension of the classical one [1, 20] and has the advantage of making sense also out of the purely Hilbertian setting. Without going into the greatest generality, which would call for considering stochastic integrals in Banach spaces, let us mention that the present results (in particular, the validity of Theorem 2.1) could be extended to the EDP functional

$$\begin{aligned} I(u) = {} & \mathbb{E}\phi(u(T)) - \mathbb{E}\phi(u(0)) + \mathbb{E}\int_0^T \psi_A(\partial_t u^d, -\partial\phi(u) + F(\cdot, u))\,\mathrm{d}s \\ & - \mathbb{E}\int_0^T (\partial_t u^d, F(\cdot, u))\,\mathrm{d}s - \mathbb{E}\int_0^T \mathrm{Tr}_H\, L(u)\,\mathrm{d}s \\ & + 2C_\phi \mathbb{E}\int_0^T \|u^s - G(\cdot, u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \mathbb{E}\|u(0) - u^0\|^2_V. \end{aligned} \tag{18}$$

Here, $\psi_A : H \times H \to (-\infty, \infty]$ is a convex function *representing* the maximal monotone operator $A : H \to H$ in the sense of the Fitzpatrick theory [23], see also [56, 57] for additional material and details. In particular,

$$\psi_A(v, w) \geq (v, w) \quad \forall v,\, w \in H, \tag{19}$$

$$\psi_A(v, w) = (v, w) \quad \Leftrightarrow \quad w \in A(v). \tag{20}$$

An example for such ψ_A is the so-called *Fitzpatrick function*

$$\psi_A(v, w) = \sup\{(\hat{v}, w) + (v, \hat{w}) - (\hat{v}, \hat{w}) \,:\, \hat{v},\, \hat{w} \in H,\, \hat{w} \in A(\hat{v})\}.$$

If A is cyclic, namely $A = \partial\eta$ for some $\eta : H \to (-\infty, \infty]$ convex, proper, and lower semicontinuous, a second example for ψ_A is the *Fenchel function*

$$\psi_A(v, w) = \eta(v) + \eta^*(w)$$

where η^* is the Legendre conjugate of η.

By using (19)-(20) one can prove that null-minimizers of I from (18) solve the *doubly nonlinear* equation

$$A(\partial_t u^d)\,\mathrm{d}t + u^s\,\mathrm{d}W + \partial\phi(u)\,\mathrm{d}t = F(\cdot, u)\,\mathrm{d}t + G(\cdot, u)\,\mathrm{d}W$$

where $A : H \to H$ is maximal monotone, nondegenerate, and linearly bounded but not necessarily cyclic. The latter, under suitable assumptions, has been proved to admit martingale solutions in [52].

Remark 3.4. By adapting the argument of Proposition 3.1 one can check that strong-in-time solutions with $u^d \in L^2(\Omega; W^{1,1}(0,T;H))$ of equation (12) are actually in $\mathscr{V}$, so that assuming $u \in \mathscr{V}$ is actually not restrictive. Indeed, given $u^d \in L^2(\Omega; W^{1,1}(0,T;H))$ and taking (8)–(9) into account one has that $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^1(0,T;H))$, $F(\cdot, u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$, and $G(\cdot, u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$. In order to conclude for $u \in \mathscr{U}$ it hence suffices to prove that indeed $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. At some approximation level (for instance, that of Yosida approximations u_λ from the proof of Proposition 3.1), the Itô formula holds for $\partial\phi(u)$ in $L^2_{\mathscr{P}}(\Omega; L^1(0,T;H))$, as well. On solutions of the equation (12) one can hence replace $\partial_t u^d$ in the Itô formula (14) by $F(\cdot,u) - \partial\phi(u)$ and easily check that indeed $\partial\phi(u) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$. In fact, by a standard application of the Burkholder-Davis-Gundy inequality one additionally obtains that solutions of (12) belong to $L^2_{\mathscr{P}}(\Omega; L^\infty(0,T;V))$, as well.

4. Some properties of the EDP functional

As mentioned above, under the assumptions of Section 2 equation (12) admits a unique solution u. In particular, the null-minimization problem for I is uniquely solvable.

In this section, we comment on the possibility of tackling the null-minimization problem for I directly. We prove that I is coercive in $\mathscr{V}$ (Proposition 4.1) and that minimizers are actually null-minimizers (Proposition 4.3). Moreover, we check that I is lower semicontinuous, up to possibly changing the underlying stochastic basis (Proposition 4.2). Unfortunately, this lower semicontinuity property is too weak to allow for an application of the Direct Method, preventing us from obtaining a complete alternative existence proof for (12).

The case of an atomic Ω is special. Here, no change in the stochastic basis is needed for lower semicontinuity and the null-minimization of I can be directly carried out, bringing to a fully variational existence proof for (12).

Proposition 4.1 (Coercivity)**.** *The sublevels of I are bounded in $\mathscr{V}$.*

Proof. Assume $I(v) < \infty$. Then $v \in \mathscr{V} \subset L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ and we have $F(\cdot,v) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ and $G(\cdot,v) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U;V)))$ from (8)-(9). As $I(v)$ is finite, we deduce that $\partial\phi(v) \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ as well. Applying the Itô formula (14) and integrating on the interval $[0,t]$ for $t \in [0,T]$ we deduce that

$$\begin{aligned}
&\mathbb{E}\phi(v(t)) - \mathbb{E}\phi(v(0)) + \frac{1}{2}\mathbb{E}\int_0^t \|\partial_t v^d\|^2 \,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^t \|\partial\phi(v) - F(\cdot,v)\|^2 \,\mathrm{d}s \\
&\quad - \mathbb{E}\int_0^t (F(\cdot,v), \partial_t v^d) \,\mathrm{d}s - \frac{1}{2}\mathbb{E}\int_0^t \mathrm{Tr}_H\, L(v) \,\mathrm{d}s + 2C_\phi \mathbb{E}\int_0^T \|v^s - G(\cdot,v)\|^2_{\mathscr{L}^2(U,V)} \,\mathrm{d}s \\
&\quad = \frac{1}{2}\mathbb{E}\int_0^t \|\partial_t v^d + \partial\phi(v) - F(\cdot,v)\|^2 \,\mathrm{d}s + 2C_\phi \mathbb{E}\int_0^t \|v^s - G(\cdot,v)\|^2_{\mathscr{L}^2(U,V)} \,\mathrm{d}s \\
&\quad \le I(v).
\end{aligned} \tag{21}$$

We now use the coercivity (6) in order to get that

$$
\begin{aligned}
&\frac{c_\phi}{2}\mathbb{E}\|v(t)\|_V^p + \frac14\mathbb{E}\int_0^t \|\partial_t v^d\|^2\,\mathrm{d}s + \frac12\mathbb{E}\int_0^t \|\partial\phi(v) - F(\cdot,v)\|^2\,\mathrm{d}s\\
&\quad + \frac{3C_\phi}{2}\mathbb{E}\int_0^T \|v^s\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s\\
&\le \mathbb{E}\phi(v(0)) + \mathbb{E}\int_0^t \|F(\cdot,v)\|^2\,\mathrm{d}s + \frac12\mathbb{E}\int_0^t \mathrm{Tr}_H\, L(v)\,\mathrm{d}s\\
&\quad + 8C_\phi\mathbb{E}\int_0^t \|G(\cdot,v)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + I(v).
\end{aligned}
$$

By the Lipschitz continuity of F and the linear boundedness of G from (8)–(10) and the bounds (6)-(7) we get

$$
\begin{aligned}
&\frac{c_\phi}{2}\mathbb{E}\|v(t)\|_V^2 + \frac14\mathbb{E}\int_0^t \|\partial_t v^d\|^2\,\mathrm{d}s + \frac12\mathbb{E}\int_0^t \|\partial\phi(v) - F(\cdot,v)\|^2\,\mathrm{d}s\\
&\quad + \frac{3C_\phi}{2}\mathbb{E}\int_0^T \|v^s\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s\\
&\le \frac{C_\phi}{2}\mathbb{E}\|v(0)\|_V^2 + C\int_0^t \mathbb{E}\|v\|_V^2\,\mathrm{d}s + \frac{C_\phi}{2}\int_0^t \mathbb{E}\|v^s\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + C + I(v).
\end{aligned}
$$

for some positive constant C, depending on the data c_ϕ, C_ϕ, c_F, c_G, $c_{G,2}$, $\|u^0\|_V$, and $\|F(\cdot,0)\|_{L^2(0,T;H)}$, but independent of v. An application of the Gronwall Lemma ensures that

$$
\max_{[0,T]}\mathbb{E}\|v\|_V^2 + \mathbb{E}\int_0^T \|\partial_t v^d\|^2\,\mathrm{d}s + \mathbb{E}\int_0^T \|v^s\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \le C(1 + I(v)),
$$

possibly by updating the constant. The assertion follows. □

In order to discuss lower limits, we make the notation for the EDP functional more precise by explicitly indicating the background stochastic structure and the given initial value. When needed in the following, we use the extended notation

$$
u \mapsto \hat I(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P},W,u^0,u)
$$

instead of $u \mapsto I(u)$. Correspondingly, we specify the dependence on the stochastic basis of the space of Itô processes by using the notation $\hat{\mathscr{U}}(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P},W)$. Our lower-semicontinuity result reads as follows.

Proposition 4.2 (lim inf tool)**.** *For all $u_\varepsilon \stackrel{*}{\rightharpoonup} u$ in $\mathscr{V}$ one can find a stochastic basis $(\hat\Omega,\hat{\mathscr{F}},(\hat{\mathscr{F}}_t)_{t\in[0,T]},\hat{\mathbb{P}})$, a not relabeled sequence of measurable maps $\eta_\varepsilon : (\hat\Omega,\hat{\mathscr{F}}) \to (\Omega,\mathscr{F})$ with $\mathbb{P}\circ\eta_\varepsilon = \hat{\mathbb{P}}$, a cylindrical Wiener process $\hat W$ on U, a process*

$$
\hat u \in \hat{\mathscr{U}}(\hat\Omega,\hat{\mathscr{F}},(\hat{\mathscr{F}}_t)_{t\in[0,T]},\hat{\mathbb{P}},\hat W) \cap C([0,T];L^2(\hat\Omega,\hat{\mathscr{F}};V)),
$$

and an initial value $\hat u^0 \in L^2(\hat\Omega,\hat{\mathscr{F}}_0;V)$ such that $u_\varepsilon\circ\eta_\varepsilon \to \hat u$ in $C([0,T];H)$ a.e. in $\hat\Omega$ and

$$
\hat I(\hat\Omega,\hat{\mathscr{F}},(\hat{\mathscr{F}}_t)_{t\in[0,T]},\hat{\mathbb{P}},\hat W,\hat u^0,\hat u) \le \liminf_{\varepsilon\to 0} I(u_\varepsilon). \tag{22}
$$

Proof. As $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathscr{V}$, the classical result [24, Lemma 2.1] ensures that

$$\iota_\varepsilon := \int_0^\cdot u_\varepsilon^s \, \mathrm{d}W$$

is uniformly bounded in $L^2_{\mathscr{P}}(\Omega; H^\mu(0,T;V))$ for some $\mu \in (0,1/2)$. Since V is compact in H, the Aubin-Lions Lemma [55] ensures that

$$\begin{aligned}
&H^\mu(0,T;V) \subset\subset L^2(0,T;H),\\
&H^1(0,T;H) \subset\subset C([0,T];V^*),\\
&L^2(0,T;V) \cap \big(H^1(0,T;H) + H^\mu(0,T;V)\big) \subset\subset L^2(0,T;H).
\end{aligned}$$

This entails that the laws of $(u_\varepsilon, u^0, u_\varepsilon^d, \iota_\varepsilon, W)$ are tight in

$$L^2(0,T;H) \times V \times C([0,T];V^*) \times L^2(0,T;H) \times C([0,T];U_1),$$

where U_1 is a separable Hilbert space such that the inclusion $U \hookrightarrow U_1$ is Hilbert-Schmidt. By the Skorohod Theorem [31, Thm. 2.7] one can hence find another probability space $(\hat\Omega, \hat{\mathscr{F}}, \hat{\mathbb{P}})$, a sequence of measurable maps $\eta_\varepsilon : (\hat\Omega, \hat{\mathscr{F}}) \to (\Omega, \mathscr{F})$ with $\mathbb{P} \circ \eta_\varepsilon = \hat{\mathbb{P}}$ for all $\varepsilon > 0$, and some measurable

$$(\hat u, \hat u^0, \hat u^d, \hat\iota, \hat W) : (\hat\Omega, \hat{\mathscr{F}}) \to L^2(0,T;H) \times V \times C([0,T];V^*) \times L^2(0,T;H) \times C([0,T];U_1),$$

such that, letting $\hat u_\varepsilon := u_\varepsilon \circ \eta_\varepsilon$, $\hat u_\varepsilon^d := u_\varepsilon^d \circ \eta_\varepsilon$, $\hat\iota_\varepsilon := \iota_\varepsilon \circ \eta_\varepsilon$, and $\hat W_\varepsilon := W_\varepsilon \circ \eta_\varepsilon$,

$$\hat u_\varepsilon \to \hat u \quad \text{in } L^2(0,T;H), \ \hat{\mathbb{P}}\text{-a.s.,} \tag{23}$$
$$\hat u_\varepsilon(0) \to \hat u^0 \quad \text{in } V, \ \hat{\mathbb{P}}\text{-a.s.,} \tag{24}$$
$$\hat u_\varepsilon^d \to \hat u^d \quad \text{in } C([0,T];V^*), \ \hat{\mathbb{P}}\text{-a.s.,} \tag{25}$$
$$\hat\iota_\varepsilon \to \hat\iota \quad \text{in } L^2(0,T;H), \ \hat{\mathbb{P}}\text{-a.s.,} \tag{26}$$
$$\hat W_\varepsilon \to \hat W \quad \text{in } C([0,T];U_1), \ \hat{\mathbb{P}}\text{-a.s.} \tag{27}$$

In fact, as η_ε preserves the laws, we also have, setting $\hat u_\varepsilon^s := u_\varepsilon^s \circ \eta_\varepsilon$, that

$$\partial_t \hat u_\varepsilon^d \rightharpoonup \partial_t \hat u^d \quad \text{in } L^2(\hat\Omega; L^2(0,T;H)), \tag{28}$$
$$\hat u_\varepsilon^s \rightharpoonup \hat u^s \quad \text{in } L^2(\hat\Omega; L^2(0,T;\mathscr{L}^2(U,V)), \tag{29}$$
$$\hat u_\varepsilon(0) \rightharpoonup \hat u^0 \quad \text{in } L^2(\hat\Omega, \hat{\mathscr{F}}_0; V), \tag{30}$$
$$\partial\phi(\hat u_\varepsilon) \rightharpoonup \hat\xi \quad \text{in } L^2(\hat\Omega; L^2(0,T;H)), \tag{31}$$

The combination of convergences (25) and (31) entail that $\hat{\xi} = \partial\phi(\hat{u})$ a.e. Moreover, the Lipschitz continuity of F and G gives

$$F(\cdot, \hat{u}_\varepsilon) \to F(\cdot, \hat{u}) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;H)), \tag{32}$$

$$G(\cdot, \hat{u}_\varepsilon) \to G(\cdot, \hat{u}) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;\mathscr{L}^2(U,H)), \tag{33}$$

$$G(\cdot, \hat{u}_\varepsilon) \rightharpoonup G(\cdot, \hat{u}) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;\mathscr{L}^2(U,V)). \tag{34}$$

Setting $(\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}$ as the filtration generated by $(\hat{u}_\varepsilon, \hat{u}^d_\varepsilon, \hat{\imath}_\varepsilon, \hat{W}_\varepsilon)$, using again the fact that η_ε preserves laws, one has that

$$I(u_\varepsilon) = \hat{I}(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}_\varepsilon, \hat{u}^0_\varepsilon, \hat{u}_\varepsilon). \tag{35}$$

Moreover, setting $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$ as the filtration generated by $(\hat{u}, \hat{u}^d, \hat{\imath}, \hat{W})$, a classical argument (see [51, 52]) ensures that $\hat{W}$ is a U-cylindrical Wiener process, $\hat{\imath} = \hat{u}^s \cdot \hat{W}$, and

$$\hat{u} = \hat{u}^0 + \int_0^\cdot \partial_t \hat{u}^d(s)\,\mathrm{d}s + \int_0^\cdot \hat{u}^s(s)\,\mathrm{d}\hat{W}(s).$$

Now, since we have that $\hat{u} \in \hat{\mathscr{U}}(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}) \cap C([0,T]; L^2(\hat{\Omega}, \hat{\mathscr{F}}; V))$, $\partial\phi(\hat{u}) \in L^2_{\hat{\mathscr{P}}}(\hat{\Omega}; L^2(0,T;H))$, as well as $\hat{u}_\varepsilon \in \hat{\mathscr{U}}(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}, \hat{\mathbb{P}}) \cap C([0,T]; L^2(\hat{\Omega}, \hat{\mathscr{F}}; V))$, $\partial\phi(\hat{u}_\varepsilon) \in L^2_{\hat{\mathscr{P}}_\varepsilon}(\hat{\Omega}; L^2(0,T;H))$, we can apply the equivalence (17) and pass to the lim inf owing to convergences (28)-(33) getting

$$\begin{aligned}
&\hat{I}(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}, \hat{u}^0, \hat{u}) \\
&\quad \overset{(17)}{=} \frac{1}{2}\hat{\mathbb{E}}\int_0^T \|\partial_t \hat{u}^d + \partial\phi(\hat{u}) - F(\cdot,\hat{u})\|^2\,\mathrm{d}s + 2C_\phi \hat{\mathbb{E}}\int_0^T \|\hat{u}^s - G(\cdot,\hat{u})\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s \\
&\qquad + \hat{\mathbb{E}}\|\hat{u}(0) - \hat{u}^0\|^2_V \\
&\quad \le \liminf_{\varepsilon\to 0}\left(\frac{1}{2}\hat{\mathbb{E}}\int_0^T \|\partial_t \hat{u}^d_\varepsilon + \partial\phi(\hat{u}_\varepsilon) - F(\cdot,\hat{u}_\varepsilon)\|^2\,\mathrm{d}s\right. \\
&\qquad \left. + 2C_\phi \hat{\mathbb{E}}\int_0^T \|\hat{u}^s_\varepsilon - G(\cdot,\hat{u}_\varepsilon)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \hat{\mathbb{E}}\|\hat{u}_\varepsilon(0) - \hat{u}^0\|^2_V\right) \\
&\quad \overset{(17)}{=} \liminf_{\varepsilon\to 0} \hat{I}(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}_\varepsilon, \hat{u}^0_\varepsilon, \hat{u}_\varepsilon) \overset{(35)}{=} \liminf_{\varepsilon\to 0} I(u_\varepsilon). \qquad \square
\end{aligned}$$

The combination of Propositions 4.1 and 4.2 still does not allow to prove the existence of minimizers of the EDP functional I, for the stochastic basis is changed in the limit. In the special case of an atomic Ω, however, no change in the basis is actually required and one can find a minimizer of I via the Direct Method.

We conclude this section by proving that minimizers u of I are actually null-minimizers ($I(u) = 0$), hence solve (12). The reader is referred to [3, 29, 48] for some similar argument, although in different variational settings.

Proposition 4.3 (Minimizers are null-minimizers)**.** *Assume that $F(t,\cdot)$ and $G(t,\cdot)$ are Gateaux-differentiable for all $t \in [0,T]$ and let $u \in \mathscr{U}$ minimize I. Then, $I(u) = 0$.*

Proof. In order to check that $I(u) = 0$ let us start by considering the linear problem

$$\mathrm{d}v + D_{\mathcal{G}}\partial\phi(u)v\,\mathrm{d}t = \big(D_{\mathcal{G}}F(\cdot,u)v - f\big)\,\mathrm{d}t + \big(D_{\mathcal{G}}G(\cdot,u)v - g\big)\,\mathrm{d}W, \quad v(0) = z, \tag{36}$$

where $f \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$, $g \in L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$, and $z \in L^2(\Omega,\mathscr{F}_0;V)$ are given. Owing to our assumptions, we have that the latter is uniquely solvable, for the time-dependent positive linear operator $D_{\mathcal{G}}\partial\phi(u)$ is coercive, uniformly with respect to time.

We now use the equivalence of Proposition 3.2 in order to rewrite

$$\begin{aligned}\hat{I}(u) &= \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d + \partial\phi(u) - F(\cdot,u)\|^2\,\mathrm{d}s\\ &\quad + 2C_\phi\mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \mathbb{E}\|u(0) - u^0\|^2_V.\end{aligned} \tag{37}$$

Let now v be the solution of (36) and compute the variation of I at u in direction v by letting $0 = g'(0)$ for $g(t) = I(u + tv)$. Owing to the Gateaux differentiability of $\partial\phi$, $F(t,\cdot)$, and $G(t,\cdot)$ we obtain that

$$\begin{aligned}0 &= \mathbb{E}\int_0^T \big(\partial_t u^d + \partial\phi(u) - F(\cdot,u), \partial_t v^d + D_{\mathcal{G}}\partial\phi(u)v - D_{\mathcal{G}}F(\cdot,u)v\big)\,\mathrm{d}s\\ &\quad + \mathbb{E}\int_0^T \big(u^s - G(\cdot,u), v^s - D_{\mathcal{G}}G(\cdot,u)v\big)_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + 2\mathbb{E}(u(0) - u^0, v(0))_V\\ &= \mathbb{E}\int_0^T \big(\partial_t u^d + \partial\phi(u) - F(\cdot,u), f\big)\,\mathrm{d}s\\ &\quad + \mathbb{E}\int_0^T \big(u^s - G(\cdot,u), g\big)_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + 2\mathbb{E}(u(0) - u^0, z)_V.\end{aligned}$$

Since f, g, and z are arbitrary we have proved that $\hat{u}$ solves $\partial_t u^d + \partial\phi(u) = F(\cdot,u)$, $\hat{u}^s = G(\cdot,u)$, and $u(0) = u^0$ a.e. Hence, $u$solves (12). In particular, $\hat{I}(u) = 0$. □

5. Application to stability

Let us now give an application of Theorem 2.1 to the analysis of the stability of problem (12) with respect to data perturbations. In the deterministic case, such stability results have to be traced back to Attouch [2]. See also [58] for some recent developments. In the stochastic regime, the reader is referred to Gess & Tölle [28], where the case $\phi_n \to \phi$ is discussed.

Assume to be given a sequence $(u^0_\varepsilon, \phi_\varepsilon, F_\varepsilon, G_\varepsilon)_{\varepsilon>0}$ of data, as well as a limiting data set $(u^0_0, \phi_0, F_0, G_0)$, all fulfilling the assumptions of Section 2, uniformly with respect to $\varepsilon \in [0,1)$. We are interested in qualifying the convergences $u^0_\varepsilon \to u^0_0$, $\phi_\varepsilon \to \phi_0$, $F_\varepsilon \to F_0$, and $G_\varepsilon \to G_0$ in such a way that solutions u_ε of equation (12) with data $(u^0_\varepsilon, \phi_\varepsilon, F_\varepsilon, G_\varepsilon)$, namely,

$$u_\varepsilon(t) = u^0_\varepsilon + \int_0^t (F_\varepsilon(\cdot,u_\varepsilon) - \partial\phi_\varepsilon(u_\varepsilon))\,\mathrm{d}s + \int_0^t G_\varepsilon(\cdot,u_\varepsilon)\,\mathrm{d}W \quad \forall t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{38}$$

converge to solutions u_0 of equation (12) with data $(u_0^0, \phi_0, F_0, G_0)$, that is

$$u_0(t) = u_0^0 + \int_0^t (F_0(\cdot, u_0) - \partial\phi_0(u_0))\, \mathrm{d}s + \int_0^t G_0(\cdot, u_0)\, \mathrm{d}W \quad \forall t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{39}$$

The reformulation of these problems in terms of null-minimization of EDP functionals allows to readily treat the stability question. In the case of gradient flows, the approach dates back to Sandier & Serfaty [49, 54]. Recently, this variational treatment of limiting processes has been applied to different kind of parameter-dependent nonlinear dissipative evolution problems and has been originating the concept of *EDP convergence* [20, 25, 39, 40]. To the best of our knowledge, we present here the first application of this technique in the stochastic setting.

Let I_ε and I_0 indicate the EDP functionals (3) defined with data $(u_\varepsilon^0, \phi_\varepsilon, F_\varepsilon, G_\varepsilon)_{\varepsilon>0}$ and $(u_0^0, \phi_0, F_0, G_0)$, respectively. In order to prove that u_0 solves (39) one has to check that $I_0(u_0) = 0$. Since I_0 is nonnegative, this would follow from

$$I_0(u_0) \le \liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) = 0. \tag{40}$$

This is nothing but a Γ-$\liminf$ inequality for I_ε [18], which we check below, by extending the argument of Proposition 4.2. In fact, the EDP functional approach is flexible enough to deliver convergence also for *approximate* minimizers v_ε of I_ε, namely for $I_\varepsilon(v_\varepsilon) \to 0$. The main result of this section is the following.

Theorem 5.1 (Stability)**.** *Let $(u_\varepsilon^0, \phi_\varepsilon, F_\varepsilon, G_\varepsilon)_{\varepsilon>0}$ and $(u_0^0, \phi_0, F_0, G_0)$ fulfill the assumptions of Section 2, uniformly with respect to ε. Moreover, assume that, as $\varepsilon \to 0$,*

$$u_\varepsilon^0 \to u_0^0 \quad \text{in } L^2(\Omega, \mathscr{F}_0; V), \tag{41}$$

$$\phi_\varepsilon \to \phi \quad \text{in the Mosco sense in } H, \tag{42}$$

and that for all $w_\varepsilon \overset{*}{\rightharpoonup} w_0$ in $\mathscr{V}$ the following convergences hold

$$F_\varepsilon(\cdot, w_\varepsilon) \rightharpoonup F_0(\cdot, w_0) \quad \text{in } L^2_{\mathscr{P}}(\Omega; L^2(0,T;H)), \tag{43}$$

$$G_\varepsilon(\cdot, w_\varepsilon) \rightharpoonup G_0(\cdot, w_0) \quad \text{in } L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U;V))). \tag{44}$$

If $I_\varepsilon(v_\varepsilon) \to 0$ than $v_\varepsilon \overset{*}{\rightharpoonup} u_0$ in $\mathscr{V}$, where u_0 solves (39)*.*

Proof. As $I_\varepsilon(v_\varepsilon) \to 0$, the sequence $(v_\varepsilon)_\varepsilon$ is bounded in $\mathscr{V}$ by Proposition 4.1. This implies that $\partial_t v_\varepsilon^d$ and $F(\cdot, v_\varepsilon)$ are bounded in $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ and v_ε^s and $G(\cdot, v_\varepsilon^s)$ are bounded in $L^2_{\mathscr{P}}(\Omega; L^2(0,T;\mathscr{L}^2(U,V)))$. Moreover, since $I_\varepsilon(v_\varepsilon)$ are bounded we have that $\partial\phi_\varepsilon(v_\varepsilon)$ are bounded in $L^2_{\mathscr{P}}(\Omega; L^2(0,T;H))$ as well.

Define $\iota_\varepsilon := \int_0^\cdot v_\varepsilon^s\, \mathrm{d}W$. By adapting the argument of Proposition (4.2), possibly passing to a not relabeled subsequence we find a probability space $(\hat\Omega, \hat{\mathscr{F}}, \hat{\mathbb{P}})$, a sequence of measurable maps $\eta_\varepsilon : (\hat\Omega, \hat{\mathscr{F}}) \to (\Omega, \mathscr{F})$ with $\mathbb{P} \circ \eta_\varepsilon = \hat{\mathbb{P}}$ for all $\varepsilon > 0$, and some measurable

$$(\hat u_0, \hat u_0^0, \hat u_0^d, \hat\iota, \hat W) : (\hat\Omega, \hat{\mathscr{F}}) \to L^2(0,T;H) \times V \times C([0,T];V^*) \times L^2(0,T;H) \times C([0,T];U_1),$$

such that, letting $\hat{v}_\varepsilon := v_\varepsilon \circ \eta_\varepsilon$, $\hat{\iota}_\varepsilon := \iota_\varepsilon \circ \eta_\varepsilon$, and $\hat{W}_\varepsilon := W_\varepsilon \circ \eta_\varepsilon$, and the following convergences hold

$$
\begin{aligned}
\hat{v}_\varepsilon &\stackrel{*}{\rightharpoonup} \hat{u}_0 && \text{in } C([0,T]; L^2(\hat{\Omega}, \hat{\mathscr{F}}; V)), && (45)\\
\hat{v}_\varepsilon &\to \hat{u}_0 && \text{in } L^2(0,T;H),\ \hat{\mathbb{P}}\text{-a.s.}, && (46)\\
\hat{v}^d_\varepsilon &\to \hat{u}^d_0 && \text{in } C([0,T];V^*),\ \hat{\mathbb{P}}\text{-a.s.}, && (47)\\
\partial_t \hat{v}^d_\varepsilon &\rightharpoonup \partial_t \hat{u}^d_0 && \text{in } L^2(\hat{\Omega}; L^2(0,T;H)), && (48)\\
\hat{v}^s_\varepsilon &\rightharpoonup \hat{u}^s_0 && \text{in } L^2(\hat{\Omega}; L^2(0,T;\mathscr{L}^2(U,V)), && (49)\\
\hat{v}_\varepsilon(0) &\rightharpoonup \hat{u}^0_0 && \text{in } L^2(\hat{\Omega}, \hat{\mathscr{F}}_0; V), && (50)\\
\partial\phi_\varepsilon(\hat{v}_\varepsilon) &\rightharpoonup \hat{\xi}_0 && \text{in } L^2(\hat{\Omega}; L^2(0,T;H)). && (51)
\end{aligned}
$$

The Mosco convergence (42) together with convergences (46) and (51) ensures that $\hat{\xi}_0 = \partial\phi_0(\hat{u}_0)$ a.e., hence

$$
\partial\phi_\varepsilon(\hat{v}_\varepsilon) \rightharpoonup \partial\phi_0(\hat{u}_0) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;H)). \tag{52}
$$

Eventually, the weak-continuous-convergence properties (43)-(44) entail that

$$
\begin{aligned}
F_\varepsilon(\cdot, \hat{v}_\varepsilon) &\rightharpoonup F_0(\cdot, \hat{u}_0) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;H)), && (53)\\
G_\varepsilon(\cdot, \hat{v}_\varepsilon) &\rightharpoonup G_0(\cdot, \hat{u}_0) \quad \text{in } L^2(\hat{\Omega}; L^2(0,T;\mathscr{L}^2(U;V))). && (54)
\end{aligned}
$$

We set now $(\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}$ as the filtration generated by $(\hat{v}_\varepsilon, \hat{v}^d_\varepsilon, \hat{\iota}_\varepsilon, \hat{W}_\varepsilon)$, and $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$ as the filtration generated by $(\hat{u}_0, \hat{u}^d_0, \hat{\iota}, \hat{W})$. As in the previous section, a classical argument (see [51, 52]) ensures again that $\hat{W}$ is a U-cylindrical Wiener process, $\hat{\iota} = \hat{u}^s \cdot \hat{W}$ and

$$
\hat{u}_0 = \hat{u}^0_0 + \int_0^\cdot \partial_t \hat{u}^d_0(s)\,\mathrm{d}s + \int_0^\cdot \hat{u}^s_0(s)\,\mathrm{d}\hat{W}(s).
$$

Using the equivalence (17), the convergence of the initial data (41), and convergences (48)-(50) and (53)-(54) we can pass to the liminf in $I_\varepsilon(v_\varepsilon) = I_\varepsilon(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}, \hat{v}_\varepsilon)$

and check that

$$\begin{aligned}
&\hat{I}_0(\hat{\Omega},\hat{\mathscr{F}},(\hat{\mathscr{F}}_t)_{t\in[0,T]},\hat{\mathbb{P}},\hat{W},\hat{u}_0)\\
&\stackrel{(17)}{=} \frac{1}{2}\hat{\mathbb{E}}\int_0^T \|\partial_t \hat{u}_0^d + \partial\phi_0(\hat{u}_0) - F_0(\cdot,\hat{u}_0)\|^2\,\mathrm{d}s\\
&\quad + 2C_\phi\hat{\mathbb{E}}\int_0^T \|\hat{u}_0^s - G_0(\cdot,\hat{u}_0)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \hat{\mathbb{E}}\|\hat{u}_0(0)-\hat{u}_0^0\|_V^2\\
&\le \liminf_{\varepsilon\to 0}\frac{1}{2}\hat{\mathbb{E}}\int_0^T \|\partial_t\hat{v}_\varepsilon^d + \partial\phi_\varepsilon(\hat{v}_\varepsilon) - F_\varepsilon(\cdot,\hat{v}_\varepsilon)\|^2\,\mathrm{d}s\\
&\quad + 2C_\phi\hat{\mathbb{E}}\int_0^T \|\hat{v}_\varepsilon^s - G_\varepsilon(\cdot,\hat{v}_\varepsilon)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \hat{\mathbb{E}}\|\hat{v}_\varepsilon(0)-\hat{u}_\varepsilon^0\|_V^2\\
&\stackrel{(17)}{=} \liminf_{\varepsilon\to 0}\hat{I}_\varepsilon(\hat{\Omega},\hat{\mathscr{F}},(\hat{\mathscr{F}}_{\varepsilon,t})_{t\in[0,T]},\hat{\mathbb{P}},\hat{W}_\varepsilon,\hat{v}_\varepsilon) = 0.
\end{aligned}$$

As $\hat{I}_0(\hat{\Omega},\hat{\mathscr{F}},(\hat{\mathscr{F}}_t)_{t\in[0,T]},\hat{\mathbb{P}},\hat{W},\hat{u}_0) = 0$, Theorem 2.1 guarantees that $\hat{u}_0$ is a martingale solution of (12). As already commented, the pathwise uniqueness of martingale solutions of (12) ensures that all the limits above hold in the original stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$, as well, without the need of passing to a different basis. In particular, the weak$*$ limit u_0 of v_ε fulfills $I_0(u_0) = 0$ and solves (12). Eventually, since solutions of (12) are unique, no extraction of subsequences is actually needed. □

6. Application to optimal control

Consider now the equation

$$\mathrm{d}u + \partial\phi(u)\,\mathrm{d}t \ni f\,\mathrm{d}t + G(\cdot,u)\,\mathrm{d}W. \tag{55}$$

This corresponds to equation to (1), where the nonlinearity $F(\cdot,u)$ is replaced by $f \in L^2(0,T;H)$. The datum f is interpreted as a *control*, which for simplicity we assume to be deterministic. Given the initial value u^0, the Cauchy problem for equation (55) corresponds to find a process $u \in \mathscr{U}$ such that

$$u(t) = u^0 + \int_0^t (f - \partial\phi(u))\,\mathrm{d}s + \int_0^t G(\cdot,u)\,\mathrm{d}W \quad \forall\, t\in[0,T],\ \mathbb{P}\text{-a.s.} \tag{56}$$

Under our assumptions, for all $f \in L^2(0,T;H)$ there exists a unique solution $u \in \mathscr{U}\cap L^2_{\mathscr{P}}(\Omega;L^\infty(0,T;V))$ of (56). This defines the *solution operator*

$$S : L^2(0,T;H) \to \mathscr{U}\cap L^2_{\mathscr{P}}(\Omega;L^\infty(0,T;V)), \quad S(f) := u.$$

We are interested in the following optimal control problem

$$\min_{f\in\mathscr{A}}\{J(f,u) \,:\, u = S(f)\}. \tag{57}$$

Here, $\mathscr{A}$ represent the set of *admissible* controls, which we assume to be nonempty and weakly compact in $L^2(0,T;H)$, and $J : L^2(0,T;H)\times\mathscr{V}\to[0,\infty)$ is an abstract *target functional*, here considered to be lower semicontinuous with respect to the weak$*$ topology in $L^2(0,T;H)\times\mathscr{V}$. An f^* solving (57) is called *optimal control*, the corresponding

$u^* = S(f)$ is an *optimal state*, and the pair (f^*, u^*) is an *optimal pair*. Let us start from the following.

Proposition 6.1 (Existence)**.** *There exists an optimal pair (f^*, u^*) for problem* (57)*.*

Proof. Let $f_n \in \mathscr{A}$ be a infimizing sequence for problem (57). As $\mathscr{A}$ is weakly compact in $L^2(0,T;H)$ we can extract a not relabeled subsequence such that $f_n \rightharpoonup f^*$ in $L^2(0,T;H)$. By letting $u_n := S(f_n)$ and using Theorem 5.1 we have that $u_n \overset{*}{\rightharpoonup} u^*$ in $\mathscr{V}$, where $u^* \in S(f^*)$. Owing to the lower semicontinuity of J we get

$$J(f^*, u^*) \leq \liminf_{n\to\infty} J(f_n, u_n) = \inf_{f\in A} \{J(f,u) \,:\, u = S(f)\}$$

so that (f^*, u^*) is an optimal pair. □

By the characterization of Theorem 2.1, one readily finds that

$$u = S(f) \quad \Leftrightarrow \quad I(f,u) = 0,$$

where the *controlled EDP functional* $I : L^2(0,T;H) \times \mathscr{U} \to [0,\infty]$ is defined as

$$\begin{aligned} I(f,u) = {} & \mathbb{E}\phi(u(T)) - \mathbb{E}\phi(u(0)) + \frac{1}{2}\mathbb{E}\int_0^T \|\partial_t u^d\|^2 \,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T \|\partial\phi(u) - f\|^2 \,\mathrm{d}s \\ & - \mathbb{E}\int_0^T (\partial_t u^d, f)\,\mathrm{d}s - \frac{1}{2}\mathbb{E}\int_0^T \mathrm{Tr}_H\, L(u)\,\mathrm{d}s \\ & + 2C_\phi \mathbb{E}\int_0^T \|u^s - G(\cdot,u)\|^2_{\mathscr{L}^2(U,V)}\,\mathrm{d}s + \mathbb{E}\|u(0) - u^0\|_V^2 \end{aligned} \tag{58}$$

if $u \in C([0,T]; L^2(\Omega, \mathscr{F}; V))$ and $I(u) = \infty$ otherwise. The controlled EDP functional can be used to penalize the SPDE constraint $u = S(f)$ in problem (57). We consider the *penalized optimal control* problems

$$\min_{\mathscr{A}\times\mathscr{U}} F_\delta \quad \text{with} \quad F_\delta(f,u) := J(f,u) + \frac{1}{\delta} I(f,u) \tag{59}$$

where $\delta > 0$ is the penalization parameter. Let us mention that the penalization of optimal control problems via weighted residuals is classical and can be traced back to Lions [33]. Indeed, it has already been applied to different stationary and evolutive situations, see [11, 12, 13, 26, 30, 41] for a collection of results. In the deterministic setting, this penalization method via EDP functionals has been discussed in [46].

By combining the coercivity and the lower-limit tool from Propositions 4.1-4.2 one can find a minimizer of F_δ for each fixed $\delta > 0$, at the price of possibly changing the underlying stochastic basis.

In the limit $\delta \to 0$ one recovers optimal pairs for the original problem (57) as limit of *approximate optimal pairs* at level δ, without redefining the stochastic basis. The main result of this section is the following.

Theorem 6.2 (Limit $\delta \to 0$)**.** *Let $(f_\delta, u_\delta)_{\delta>0} \in \mathscr{A} \times \mathscr{U}$ be such that*

$$\liminf_{\delta\to 0} \left(F_\delta(f_\delta, u_\delta) - \inf_{\mathscr{A}\times\mathscr{U}} F_\delta \right) = 0. \tag{60}$$

Then, up to a not relabeled subsequence we have that $(f_\delta, u_\delta) \stackrel{*}{\rightharpoonup} (f^*, u^*)$ *in* $L^2(0,T;H) \times \mathscr{V}$ *where* (f^*, u^*) *is an optimal pair for* (57).

Proof. Choose $f_0 \in \mathscr{A}$ and let $u_0 = S(f_0)$, so that $I(f_0, u_0) = 0$. Owing to the weak compactness of $\mathscr{A}$ into $L^2(0,T;H)$ we can extract without relabeling so that $f_\delta \rightharpoonup f$ in $L^2(0,T;H)$. From (60) we get that

$$\liminf_{\delta\to 0} \frac{1}{\delta} I(f_\delta, u_\delta) \le \liminf_{\delta\to 0} F_\delta(f_\delta, u_\delta) \le \limsup_{\delta\to 0} \inf_{\mathscr{A}\times\mathscr{U}} F_\delta \le J(f_0, u_0).$$

Again by extracting some not relabeled subsequence, this entails that

$$\frac{1}{\delta} I(f_\delta, u_\delta) \le 1 + J(f_0, u_0).$$

As $\delta \to 0$ one has that $I(f_\delta, u_\delta) \to 0$ and we are in the setting of Theorem 5.1. In particular, $u_\delta \stackrel{*}{\rightharpoonup} u^*$ in $\mathscr{V}$ and $u^* = S(f^*)$. Moreover, owing to the lower semicontinuity of J, for any $u \in S(f)$ we find

$$J(f^*, u^*) \le \liminf_{\delta\to 0} J(f_\delta, u_\delta) \le \liminf_{\delta\to 0} F_\delta(f_\delta, u_\delta) \le \limsup_{\delta\to 0} \inf_{\mathscr{A}\times\mathscr{U}} F_\delta \le J(f, u)$$

which proves that the (f^*, u^*) is an optimal pair for problem (57). □

Acknowledgement

LS is partially supported by the Austrian Science Fund (FWF) through the Lise-Meitner project M 2876. US is partially supported by the Austrian Science Fund (FWF) through projects F 65, W 1245, I 4354, I 5149, and P 32788, and by the OeAD-WTZ project CZ 01/2021.

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