

# Vanishing parameter for an optimal control problem modeling tumor growth

ANDREA SIGNORI<sup>(1)</sup>

e-mail: andrea.signori02@universitadipavia.it

<sup>(1)</sup> Dipartimento di Matematica e Applicazioni, Università di Milano–Bicocca  
via Cozzi 55, 20125 Milano, Italy

## Abstract

A distributed optimal control problem for a phase field system which physical context is that of tumor growth is discussed. The system we are going to take into account consists of a Cahn-Hilliard equation for the phase variable (relative concentration of the tumor) coupled with a reaction-diffusion equation for the nutrient. The cost functional is of standard tracking-type and the control variable models the intensity at which it is possible to dispense medication. The model we deal with presents two small and positive parameters which are introduced in previous contributions as relaxation terms. Here, starting from the already investigated optimal control problem for the relaxed model, we aim at confirming the existence of optimal control and characterizing the first-order necessary optimality condition, via asymptotic schemes, when one of the two occurring parameters goes to zero.

**Key words** Asymptotic analysis, distributed optimal control, phase field model, tumor growth, cancer treatment, evolution equations, Cahn-Hilliard equation, optimal control, adjoint system, necessary optimality conditions.

**AMS (MOS) Subject Classification** 35K61, 35Q92, 49J20, 49K20, 35K86, 92C50.

# 1 Introduction

The need for investigating tumor growth from a mathematical viewpoint stems from the great impact that it may have on medical treatments. As a matter of fact, in the last years, there is increasing attention by the mathematical community toward biological and medical models (see, e.g., [15]). In particular, an open and unknown area such as the tumor field can find a useful support tool in the mathematical predictions. In fact, this latter could be able to pull out some of the main features of the evolution phenomena and, by focusing on some particular aspects, it may give some deep insights as if a given negative outcome was to be foreseen, it would be possible to prevent it. Moreover, the theoretical investigation has the huge advantage that no patient is put at risk. Furthermore, without the claim to cure the disease, the mathematical models could provide prominent a priori information as a support for the medical treatments leading to more personalized therapy. Indeed, despite the wide number of parameters involved in the disease, due to the few understanding of the tumor evolution, the corresponding clinical treatment is quite standardized, while every patient responds differently to the medications.

Among the numerous models recently proposed, we focus on the ones derived by continuum mixture and phase field theories. The evolution of a young tumor, before the development of quiescent cells, can be described as a Cahn-Hilliard equation for the phase variable (see, e.g., [36] and the huge references therein for a general, while rich, introduction to the Cahn-Hilliard equation), coupled with a reaction-diffusion for an unknown species acting as a nutrient (e.g., oxygen or glucose). The model we are going to face in this work consists of a variation of the one introduced by Hawkins-Daruud et al. in [32], where the velocity contributions are neglected (see also [14, 30, 31, 33, 46]). Several models, by interpreting the tumors and the healthy cells as inertia-less fluids, also include the contribution of the velocity field assuming a Darcy law or a Stokes-Brinkman equation. In this regards, let us refer to [16, 19, 21, 23–27, 29, 45], where further mechanisms such as active transport and chemotaxis are also taken into account. We also point out the paper [22], where a non-local model is proposed.

At first, let us point out that the symbol  $\Omega \subset \mathbb{R}^3$  is devoted to indicating the set where the evolution takes place which boundary we denote by  $\Gamma$ . Furthermore, given a final time  $T > 0$ , we set for convenience

$$\begin{aligned} Q_t &:= \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t) \quad \text{for every } t \in (0, T], \\ Q &:= Q_T, \quad \text{and} \quad \Sigma := \Sigma_T. \end{aligned} \tag{1.1}$$

In the present paper, we are going to deal with the optimal control problem consisting of minimizing the so-called objective, or tacking-type, cost functional

$$\begin{aligned} \mathcal{J}(\varphi, \sigma, u) &:= \frac{b_1}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{b_2}{2} \|\varphi(T) - \varphi_\Omega\|_{L^2(\Omega)}^2 + \frac{b_3}{2} \|\sigma - \sigma_Q\|_{L^2(Q)}^2 \\ &\quad + \frac{b_4}{2} \|\sigma(T) - \sigma_\Omega\|_{L^2(\Omega)}^2 + \frac{b_0}{2} \|u\|_{L^2(Q)}^2, \end{aligned} \tag{1.2}$$

subject to the control-box constraints

$$u \in \mathcal{U}_{\text{ad}} := \{u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q\}, \tag{1.3}$$

and under the assumption that the variables  $\varphi$  and  $\sigma$  solve the following system

$$\alpha \partial_t \mu + \partial_t \varphi - \Delta \mu = P(\varphi)(\sigma - \mu) \quad \text{in } Q \quad (1.4)$$

$$\mu = \beta \partial_t \varphi - \Delta \varphi + F'(\varphi) \quad \text{in } Q \quad (1.5)$$

$$\partial_t \sigma - \Delta \sigma = -P(\varphi)(\sigma - \mu) + u \quad \text{in } Q \quad (1.6)$$

$$\partial_n \mu = \partial_n \varphi = \partial_n \sigma = 0 \quad \text{on } \Sigma \quad (1.7)$$

$$\mu(0) = \mu_0, \varphi(0) = \varphi_0, \sigma(0) = \sigma_0 \quad \text{in } \Omega. \quad (1.8)$$

For the sake of synthesis, let us describe the physical background of the occurring variables without diving into the details. The admissible set  $\mathcal{U}_{\text{ad}}$  fix the space in which the control variable  $u$  can be chosen and it is given in terms of the bounds  $u_*$  and  $u^*$ . Moreover,  $b_0, b_1, b_2, b_3, b_4$  stand for nonnegative constants, not all zero, while  $\varphi_Q, \sigma_Q, \varphi_\Omega, \sigma_\Omega$  denote some target functions defined in  $Q$  and  $\Omega$ , respectively. The variable  $\varphi$  is an order parameter and it is designed to keep track of the evolution of the tumor in the tissue. It is a normalized relative concentration and it ranges between  $-1$  and  $+1$ , where these extremes represent the pure phases, that is the tumorous and the healthy case, respectively. Furthermore, the variable  $\mu$  stands, as usual for the Cahn-Hilliard equation, for the chemical potential for  $\varphi$ . The third unknown  $\sigma$  has the role of describing the evolution of the nutrient within the evolution process and it is normalized between 0 and 1 with the following property: the closer to one, the richer of nutrient the extra-cellular is, while the closer to zero, the poorer it is. Lastly, the variable  $u$  represents the so-called control variable and, since it appears in the nutrient equation, it can be read as a supply of a nutrient or a drug in the medical treatment. As the functions  $P$  and  $F$  are concerned, they are nonlinearities. The former models the proliferation of the tumor, while the latter is a double-well potential associated with the Cahn-Hilliard equation. A typical example of  $F$  is the regular potential which is defined as follows

$$F_{\text{reg}}(r) = \frac{1}{4}(r^2 - 1)^2 = \frac{1}{4}((r^2 - 1)^+)^2 + \frac{1}{4}((1 - r^2)^+)^2 \quad \text{for } r \in \mathbb{R}. \quad (1.9)$$

We will see that the optimal control problem we are going to deal with will demand to restrict the analysis on potentials which slightly generalize (1.9). Namely, we cannot take into account singular or non-regular potential as the well-known logarithmic double-well potential or the double-obstacle one. For different physically meaningful choices of the potentials, we refer to [1] and to the references therein, where numerical simulations and comparison with clinical data can be found as well. Further details regarding the interpretation of the model can be found in [8, 12, 13, 20].

The above system has already been investigated in [8], in the case where  $\alpha = \beta > 0$ , from the viewpoint of well-posedness and long-time behavior in terms of the omega-limit set. Further comprehension of the model has been achieved by [12, 13], where the authors show under which framework the parameters  $\alpha$  and  $\beta$  can be let to zero and also point out the existence and uniqueness of the solution to the limit problem in their natural setting. In addition, we also refer to [20] where the system formally obtained by imposing  $\alpha = \beta = 0$  is tackled. There, after providing the well-posedness, the authors focus on the long-time behavior of the solution in terms of the global attractor (see, e.g., [38] for details on the asymptotic behavior of infinite-dimensional dynamical systems). As for the long-time behavior of the same system, namely (1.4)–(1.8) with  $\alpha = \beta = 0$ , we are also aware of the recent contributions [5, 37]. Lastly, let us mention [34], where the author confirms the existence of the above problem (1.4)–(1.8) when  $\beta \searrow 0$ , extending the analysis to the case of unbounded domains by accounting for suitable approximation schemes.

As the terminology is concerned, the system that the control variable has to satisfy is referred to as the state system. Moreover, once that the well-posedness of the state system has been performed, we can introduce the control-to-state mapping as the map that assigns to a given control the associated solution, namely the function

$$\mathcal{S} : u \mapsto \mathcal{S}(u) := (\mu, \varphi, \sigma) = (\mu(u), \varphi(u), \sigma(u)).$$

Starting from this, one can interpret the cost functional  $\mathcal{J}$  as a function depending on the control variable only, giving rise to the so-called reduced cost functional reading as

$$\mathcal{J}_{red}(u) := \mathcal{J}(\mathcal{S}_2(u), \mathcal{S}_3(u), u),$$

where  $\mathcal{S}_2$  and  $\mathcal{S}_3$  denote the second and third component of the solution operator  $\mathcal{S}$ , respectively.

Even though the literature around the mathematical investigations of biological and medical models find several examples, the corresponding optimal control contributions are very few. Up to our knowledge, the first paper dealing with an optimal control problem for a system very close the one gave above, namely the case  $\alpha = \beta = 0$ , is [11]. Furthermore, we mention [41], which is our starting point. There, the author handles the optimal control problem for the classical tracking-type cost functional in the non-trivial case of the logarithmic potential, where the presence of the relaxation terms turns out to be fundamental. Moreover, in a following work, the same author proves that, accounting for an asymptotic technique known in the literature as to deep quench limit, it is also possible to generalize the assumptions for the potentials in order to take into account also singular and non-regular potentials like the double-obstacle one. In addition, we refer to [39], where it was shown that the optimal control problem for the state system (1.4)–(1.8) with  $\beta = 0$  can be solved, by letting  $\beta \searrow 0$  in the optimal control problem associated with (1.4)–(1.8). Lastly, let us address to [5], where an optimal control problem for (1.4)–(1.8), with  $\alpha = \beta = 0$ , has been discussed for a slightly more general class of cost functional which takes into account the time optimization (see also [28], where the same generalized cost functional is taken for a different state system). To conclude the overview concerning the literature, let us also point out [9], where a different kind of control problem, known as sliding mode control, is performed. As for different state systems, let us refer to the recent [17, 18], where the authors establish the existence of optimal controls and also characterize the optimality conditions for the more involved Cahn-Hilliard-Brinkman equation. Lastly, let us mention [43], where a distributed optimal control problem for the Cahn-Hilliard-Darcy system with mass source was studied.

Here, we aim to employ an asymptotic scheme similar to the one of [39], by letting  $\alpha \searrow 0$  instead of  $\beta$ , and assuming [41]. Note that the present contribution complete the picture around the optimal control problem for system (1.4)–(1.8) with the standard tracking-type cost functional. Indeed, the case  $\alpha, \beta > 0$  has been investigated in [41], the case  $\alpha > 0$  and  $\beta = 0$  has been studied in [39], whereas the case  $\alpha = \beta = 0$  has been treated in [11].

As for the interpretation of the control problem, let us just point out the following comments:

- (i) The cost functional (1.2) is designed to track the state variables during the evolution. The targets  $\varphi_Q, \sigma_Q, \varphi_\Omega, \sigma_\Omega$ , especially  $\varphi_\Omega$  and  $\sigma_\Omega$ , have to be chosen as a desirable configuration for clinical reasons, e.g., for surgery. Moreover, if some stable configuration for the system has known, it can be taken as well as a target.

- (ii) The smaller  $\|\varphi - \varphi_Q\|_{L^2(Q)}^2$  is, the closer the solution  $\varphi$  is to the target  $\varphi_Q$ , and the same goes for the other variables. On the other hand, the term  $\|u\|_{L^2(Q)}^2$  penalizes the large values of the control variable and it can be read as the side-effect that may occur if too many drugs are dispensed to the patient.
- (iii) The ratios between the constants  $b_0, b_1, b_2, b_3, b_4$  implicitly describe which targets hold the leading part in the application.

Let us anticipate that for our purpose, we have to restrict the analysis to the case in which the function  $P(\varphi)$  degenerates to a positive constant  $P$ . Thus, the system we are going to face is the following

$$\alpha \partial_t \mu_\alpha + \partial_t \varphi_\alpha - \Delta \mu_\alpha = P(\sigma_\alpha - \mu_\alpha) \quad \text{in } Q \quad (1.10)$$

$$\mu_\alpha = \beta \partial_t \varphi_\alpha - \Delta \varphi_\alpha + F'(\varphi_\alpha) \quad \text{in } Q \quad (1.11)$$

$$\partial_t \sigma_\alpha - \Delta \sigma_\alpha = -P(\sigma_\alpha - \mu_\alpha) + u_\alpha \quad \text{in } Q \quad (1.12)$$

$$\partial_n \mu_\alpha = \partial_n \varphi_\alpha = \partial_n \sigma_\alpha = 0 \quad \text{on } \Sigma \quad (1.13)$$

$$\mu_\alpha(0) = \mu_0, \varphi_\alpha(0) = \varphi_0, \sigma_\alpha(0) = \sigma_0 \quad \text{in } \Omega, \quad (1.14)$$

where we have written  $\mu_\alpha, \varphi_\alpha$  and  $\sigma_\alpha$  for the state variables to stress that they are solution to the system in which  $\alpha > 0$ . Such a state system leads to the following control problem:

**(CP) $_\alpha$**  Minimize  $\mathcal{J}(\varphi, \mu, u)$  subject to the control constraints (1.3) and under the requirement that the variables  $(\varphi, \sigma)$  solve the system (1.10)–(1.14).

On the other hand, we will denote with the symbols  $\mu, \varphi$  and  $\sigma$  their corresponding limits as  $\alpha \searrow 0$ . The asymptotic behavior of the above system, as  $\alpha \searrow 0$ , has been one of the main features of [12]. More precisely, in [12, Thm. 2.5 and Thm. 2.6] the authors discuss the passage to the limit as  $\alpha \searrow 0$  and rigorously proved in which sense system (1.10)–(1.14) converge to

$$\partial_t \varphi - \Delta \mu = P(\sigma - \mu) \quad \text{in } Q \quad (1.15)$$

$$\mu = \beta \partial_t \varphi - \Delta \varphi + F'(\varphi) \quad \text{in } Q \quad (1.16)$$

$$\partial_t \sigma - \Delta \sigma = -P(\sigma - \mu) + u \quad \text{in } Q \quad (1.17)$$

$$\partial_n \mu = \partial_n \varphi = \partial_n \sigma = 0 \quad \text{on } \Sigma \quad (1.18)$$

$$\mu(0) = \mu_0, \varphi(0) = \varphi_0, \sigma(0) = \sigma_0 \quad \text{in } \Omega, \quad (1.19)$$

having the care of showing the restrictions under which existence and uniqueness hold, respectively. Moreover, they also exhibit an error estimate between the solution to system (1.10)–(1.14) and the solution to (1.15)–(1.19), which in turn implies the uniqueness to the second. Note that to address the corresponding control problem, the uniqueness of system (1.15)–(1.19) is mandatory.

Therefore, the control problem we want to solve in this paper can be summarized as follows:

**(CP)** Minimize  $\mathcal{J}(\varphi, \mu, u)$  subject to the control constraints (1.3) and under the requirement that the variables  $(\varphi, \sigma)$  yield a solution to (1.15)–(1.19).

Here, let us sketch some strategies which are usually employed in control theory for the class of linear-quadratic cost functional, referring to, e.g., [35, 44] for a complete and

thorough presentation. The main aim of control theory is to prove the existence (eventually also uniqueness) of optimal control and provide some necessary (and eventually sufficient) conditions for optimality. Once that the well-posedness of the state system has been proved, the existence of optimal controls easily follows by combining the lower weak sequential semicontinuity of the cost functional  $\mathcal{J}$  with standard weak compactness results for reflexive Banach spaces. On the other hand, in the nonlinear constrained PDEs control theory, usually, the uniqueness is out of reach. In fact, ordinarily, one appeal to the strict convexity of the cost functional to infer uniqueness from the existence part, but, whenever the state system, and therefore the corresponding control-to-state operator, is nonlinear, one cannot hope to recover the strict convexity. The second step consists of looking for some optimality conditions. As a matter of fact, since the set of admissible controls is convex, it follows from standard results of convex analysis that the necessary condition for optimality of  $\bar{u}$  is carried out by the following variational inequality

$$D\mathcal{J}_{red}(\bar{u})(v - \bar{u}) \geq 0 \quad \text{for every } v \in \mathcal{U}_{ad}, \quad (1.20)$$

where  $D\mathcal{J}_{red}$  stands for the derivative of the reduced cost functional in a proper functional sense. Moreover, let us recall that  $\mathcal{J}_{red}$  is essentially obtained as a composition of the cost functional  $\mathcal{J}$  and the control-to-state operator  $\mathcal{S}$ . So, since  $\mathcal{J}$  is trivially Fréchet differentiable, the classical technique relies on proving the Fréchet differentiability of  $\mathcal{S}$  and then invoke the chain rule to conclude. Anyhow, the above procedure does not lead to the desired conclusion since does not provide an explicit characterization of the gradient  $\nabla J_{red}(\bar{u})$ . Hence, as in the classical constrained control theory, the Lagrange multipliers can be introduced to include the constraints in the minimization problem. This requires to solve another system, called adjoint problem and which variables are called adjoint, or co-state, variables. Finally, after solving this latter, the variational inequality (1.20) can be expressed in a more convenient way which directly allows us to represent  $\nabla J_{red}(\bar{u})$ . To conclude this overview of control theory, let us emphasize that the second-order derivative can give us meaningful information for the sufficiency. However, this is usually less investigated since it introduces some further technicalities. Just to give a simple motivation note that, formally, if a map  $\mathcal{S} : \mathcal{U} \rightarrow \mathcal{Y}$ , then it follows that  $D\mathcal{S} : \mathcal{U} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$  and  $D^2\mathcal{S} : \mathcal{U} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ . Hence, if one would like to show that  $D\mathcal{S}$  is Fréchet differentiable by checking the definition, it has to consider a double increment leading to some technical calculations. On the other hand, these issues can be overcome by employing some advanced techniques (see, e.g., [4, 44]).

Summing up, in this paper we aim to show that we can let the parameter  $\alpha$  goes to zero in  $(CP)_\alpha$  to solve  $(CP)$ . We will provide the classical results for the optimal control; namely, the existence of optimal control and the first-order necessary condition for optimality. This strategy has a huge advantage. Indeed, we will avoid the non-trivial discussion of the Fréchet differentiability of the control-to-state mapping corresponding to the state system (1.15)–(1.19). On the other hand, by adopting this approximation scheme, we need to overcome an approximation issue since it is not trivially ensured that every optimal control for  $(CP)$  can be approximated by sequences of optimal controls for  $(CP)_\alpha$ .

**Plan of the paper** We conclude the section by sketching an outline of the paper. In Section 2, we will focus the attention on two aspects; the first one is setting the framework and the notation, while the second consists in presenting the obtained results. In Section 3, we start with the corresponding proofs by checking the existence of optimal control and showing an approximation result that will be of crucial importance for the

asymptotic analysis. In Section 4, we investigate the asymptotic analysis of the adjoint system proving its well-posedness in a proper framework. Lastly, we exploit the adjoint problem and the approximation result to provide the first-order necessary condition for optimality, reading as a variational inequality.

## 2 General Assumptions and Results

Let us now come to present the mathematical framework and state the main results. First of all, we recall that the set  $\Omega$  models the tissue where the evolution takes place and we assume it to be an open, bounded and regular domain in  $\mathbb{R}^3$ . Moreover, for an arbitrary Banach space  $X$ , we convey to use  $\|\cdot\|_X$  to denote its norm, the standard symbol  $X^*$  for its topological dual, and  ${}_{X^*}\langle \cdot, \cdot \rangle_X$  for the corresponding duality product between  $X^*$  and  $X$ . Likewise, for every  $p \in [1, +\infty]$ , we use the symbol  $\|\cdot\|_p$  for the usual norm in  $L^p(\Omega)$ . Since in what follows we are going to use several times some particular spaces, it turns out to be convenient to set the following conventions

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\},$$

where  $\partial_n$  stands for the outward normal derivative of  $\Gamma$ , and where these spaces are equipped with their standard norms in order to have Banach spaces. Let us remark that the canonical injections  $V \hookrightarrow H \cong H^* \hookrightarrow V^*$  are both continuous and dense. Therefore, the triplet  $(V, H, V^*)$  forms a Hilbert triplet. Indeed, we can identify, in the usual way, the duality product of  $V$  with the inner product of  $H$  as follows

$${}_{V^*}\langle u, v \rangle_V = \int_{\Omega} uv \quad \text{for every } u \in H \text{ and } v \in V.$$

Before diving into the setting we are going to use, let us underline again that our starting point is the distributed optimal control investigated in [41] which considers a quite strong framework in order to handle the tricky case of the logarithmic potential. On the other hand, in order to apply the asymptotic strategy mentioned above, we have to guarantee the well-posedness of system (1.15)–(1.19) which has been treated in [12] (see also [13]). Hence, the simplification introduced in this second work, in order to treat the asymptotic analysis, cannot be avoided. So, all the results proved in [41] hold since the following setting perfectly fits the one there considered.

As the assumptions for the above systems and the cost functional are concerned, we require that

$$\alpha, \beta > 0 \tag{2.1}$$

$$b_0, b_1, b_2, b_3, b_4 \text{ are nonnegative constants, but not all zero} \tag{2.2}$$

$$\varphi_Q, \sigma_Q \in L^2(Q), \varphi_{\Omega}, \sigma_{\Omega} \in H^1(\Omega), u_*, u^* \in L^{\infty}(Q) \text{ with } u_* \leq u^* \text{ a.e. in } Q \tag{2.3}$$

$$P \text{ is a positive constant} \tag{2.4}$$

$$\varphi_0 \in W, \mu_0 \in H^1(\Omega), \sigma_0 \in H^1(\Omega). \tag{2.5}$$

As for the control-box, we assume it to be a closed and convex set, and we also owe to the following notation

$\mathcal{U}_R \subset L^2(Q)$  be a non-empty and bounded open set such that it contains  $\mathcal{U}_{\text{ad}}$  and  $\|u\|_2 \leq R$  for all  $u \in \mathcal{U}_R$ .

Moreover, as for the nonlinear double-well potential  $F$ , we postulate that

$$F : \mathbb{R} \rightarrow [0, +\infty), \quad \text{with} \quad F := \widehat{B} + \widehat{\pi}, \quad (2.6)$$

where

$$\widehat{B} : \mathbb{R} \rightarrow [0, +\infty) \text{ is convex and lower semicontinuous, with } \widehat{B}(0) = 0 \quad (2.7)$$

$$\widehat{\pi} \in C^1(\mathbb{R}) \text{ is nonnegative, } \pi := \widehat{\pi}' \text{ is Lipschitz continuous.} \quad (2.8)$$

It follows from the above requirements that  $B := \partial\widehat{B}$  is a maximal and monotone graph  $B \subset \mathbb{R} \times \mathbb{R}$  (see, e.g., [3, Ex. 2.3.4, p. 25]) and that  $D(\widehat{B}) = \mathbb{R}$ . Furthermore, from (2.8), we also deduce that  $\widehat{\pi}$  grows at most quadratically and that its derivative  $\pi$  is linearly bounded. Unfortunately, to manage the optimal control problem introduced above, we are forced to restrict the class of admissible potentials by requiring some explicit growth assumptions. In fact, we also assume that

$$F \text{ is a } C^3 \text{ function on } \mathbb{R} \text{ satisfying } |F''(r)| \leq C_1(1 + |r|^2), \quad (2.9)$$

for a positive constant  $C_1$ . Despite such a strong framework for the potentials, the regular potential (1.9) complies with the above requirements and can be considered. We also notice that, owing to the regularity of the initial datum  $\varphi_0$  and of (2.9), we realize that  $F(\varphi_0) \in L^1(\Omega)$ . Indeed, from (2.9), we realize that  $F(r) = O(r^4)$  as  $|r| \rightarrow +\infty$ , and, owing to the Sobolev embedding, that  $\varphi_0 \in L^4(\Omega)$ .

Now, we start by recalling some already known results. First of all, we introduce the well-posedness and the asymptotic results, as  $\alpha \searrow 0$ , for system (1.10)–(1.14). In this regard, we refer to [12] and to [8, 13]. As a matter of fact, the following existence result still holds in a rather mild setting for the potential  $F$ . Namely, the requirement on the potentials can be weakened by assuming that  $\widehat{B}$  may attain also the value  $+\infty$  and that for a positive constant  $C_B$  it holds that

$$|B^\circ(r)| \leq C_B(\widehat{B}(r) + 1) \quad \text{for every } r \in \mathbb{R},$$

where  $B^\circ(r)$  denotes the element of  $B(r)$  having minimum modulus since, without assuming any regularity property,  $B$  may be multivalued. However, we reinforce the setting according to the uniqueness result [12, Thm. 2.6] unifying the description, by virtue of simplicity, as for the control problem both the results are necessary.

In view of what already pointed out, it immediately follows from [12, Thm. 2.5., Thm. 2.6] and [41] the following results.

**Theorem 2.1.** *Let (2.1)–(2.9) be fulfilled. Then, system (1.10)–(1.14) admits a unique solution  $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha)$  which satisfies*

$$\mu_\alpha, \sigma_\alpha, \varphi_\alpha \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V). \quad (2.10)$$

**Theorem 2.2.** *Suppose that (2.1)–(2.9) are fulfilled. Moreover, for given  $\alpha, \beta \in (0, 1)$  and  $u_\alpha \in \mathcal{U}_R$ , let us denote with  $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha)$  the unique solution to system (1.10)–(1.14) enjoying (2.10). Then, there exist  $\mu, \varphi, \sigma$  and a not relabeled subsequence such that, as  $\alpha \searrow 0$ ,*



we have that

$$\mu_\alpha \rightarrow \mu \text{ weakly in } L^2(0, T; V) \quad (2.11)$$

$$\begin{aligned} \varphi_\alpha \rightarrow \varphi \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \text{and strongly in } L^2(0, T; V) \cap C^0([0, T]; H) \end{aligned} \quad (2.12)$$

$$\begin{aligned} \sigma_\alpha \rightarrow \sigma \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \\ \text{and strongly in } L^2(0, T; H) \end{aligned} \quad (2.13)$$

$$\alpha\mu_\alpha \rightarrow 0 \text{ strongly in } H^1(0, T; V^*) \cap L^2(0, T; V). \quad (2.14)$$

Furthermore, there exists a positive constant  $K_1$ , independent of  $\alpha$ , such that

$$\begin{aligned} \alpha^{1/2} \|\mu_\alpha\|_{H^1(0, T; V^*)} + \|\mu_\alpha\|_{L^2(0, T; V)} + \|\varphi_\alpha\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \\ + \|\sigma_\alpha\|_{H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V)} \leq K_1. \end{aligned} \quad (2.15)$$

In addition, the limit triple  $(\mu, \varphi, \sigma)$  is the unique solution to (1.15)–(1.19) and possesses the following regularity

$$\mu \in L^2(0, T; V) \quad (2.16)$$

$$\varphi \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.17)$$

$$\sigma \in H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V). \quad (2.18)$$

Let us also point out that our assumption perfectly fits the framework of [41]. Hence, the following existence result holds.

**Lemma 2.3.** *Assume that assumptions (2.1)–(2.9) are fulfilled. Then, for every  $\alpha \in (0, 1)$ , the optimal control problem  $(CP)_\alpha$  admits, at least, a solution.*

Our next goal is to investigate the asymptotic analysis for the corresponding adjoint system which has been already studied in [41], and reads as follows

$$\beta \partial_t q_\alpha - \partial_t p_\alpha + \Delta q_\alpha - F''(\bar{\varphi}_\alpha) q_\alpha = b_1(\bar{\varphi}_\alpha - \varphi_Q) \quad \text{in } Q \quad (2.19)$$

$$q_\alpha - \alpha \partial_t p_\alpha - \Delta p_\alpha + P(p_\alpha - r_\alpha) = 0 \quad \text{in } Q \quad (2.20)$$

$$- \partial_t r_\alpha - \Delta r_\alpha + P(r_\alpha - p_\alpha) = b_3(\bar{\sigma}_\alpha - \sigma_Q) \quad \text{in } Q \quad (2.21)$$

$$\partial_n q_\alpha = \partial_n p_\alpha = \partial_n r_\alpha = 0 \quad \text{on } \Sigma \quad (2.22)$$

$$p_\alpha(T) - \beta_\alpha q(T) = b_2(\bar{\varphi}_\alpha(T) - \varphi_\Omega), \quad \alpha p_\alpha(T) = 0, \quad r_\alpha(T) = b_4(\bar{\sigma}_\alpha(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (2.23)$$

Again, as a consequence of [41, Thm. 2.8], we have at disposal the following result.

**Theorem 2.4.** *Assume that the assumptions (2.1)–(2.9) are fulfilled. Then, there exists a unique triplet  $(q_\alpha, p_\alpha, r_\alpha)$  which solves (2.19)–(2.23) and possesses the beneath regularity*

$$q_\alpha, p_\alpha, r_\alpha \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V). \quad (2.24)$$

Next, starting from this, one can obtain the first-order necessary condition for optimality (see [41, Thm. 2.9]).

**Theorem 2.5.** *Assume that (2.1)–(2.9) are verified. Let  $\bar{u} \in \mathcal{U}_{\text{ad}}$  be an optimal control for  $(CP)_\alpha$ , and let  $(\bar{\mu}_\alpha, \bar{\varphi}_\alpha, \bar{\sigma}_\alpha)$  and  $(p_\alpha, q_\alpha, r_\alpha)$  be the corresponding optimal state and co-state, respectively. Then, it follows that*

$$\int_Q (r_\alpha + b_0 \bar{u}_\alpha)(v - \bar{u}_\alpha) \geq 0 \quad \forall v \in \mathcal{U}_{\text{ad}}. \quad (2.25)$$

Now, let us state the novelties. We aim at showing that, as  $\alpha \searrow 0$ , the above system converge, in a proper sense, to the adjoint system corresponding to (1.15)–(1.19) which reads as

$$\beta \partial_t q - \partial_t p + \Delta q - F''(\bar{\varphi})q = b_1(\bar{\varphi} - \varphi_Q) \quad \text{in } Q \quad (2.26)$$

$$q - \Delta p + P(p - r) = 0 \quad \text{in } Q \quad (2.27)$$

$$- \partial_t r - \Delta r + P(r - p) = b_3(\bar{\sigma} - \sigma_Q) \quad \text{in } Q \quad (2.28)$$

$$\partial_n q = \partial_n p = \partial_n r = 0 \quad \text{on } \Sigma \quad (2.29)$$

$$p(T) - \beta q(T) = b_2(\bar{\varphi}(T) - \varphi_\Omega), \quad \alpha p(T) = 0, \quad r(T) = b_4(\bar{\sigma}(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (2.30)$$

We claim that, under suitable assumptions, the above system admits a unique solution in a variational sense. To avoid ambiguity, let us introduce the notion of solution we are going to employ for this latter.

**Definition 2.6.** *The triplet  $(q, p, r)$  is a solution to system (2.26)–(2.30) if it satisfies the variational formulation*

$$\begin{aligned} & -v^* \langle \partial_t(p - \beta q)(t), v \rangle_V - \int_\Omega \nabla q(t) \cdot \nabla v - \int_\Omega F''(\bar{\varphi}(t))q(t)v \\ & \quad = \int_\Omega b_1(\bar{\varphi}(t) - \varphi_Q(t))v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & \int_\Omega q(t)v + \int_\Omega \nabla p(t) \cdot \nabla v + P \int_\Omega (p(t) - r(t))v = 0 \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & -v^* \langle \partial_t r(t), v \rangle_V + \int_\Omega \nabla r(t) \cdot \nabla v + P \int_\Omega (r(t) - p(t))v \\ & \quad = \int_\Omega b_3(\bar{\sigma}(t) - \sigma_Q(t))v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T), \end{aligned}$$

and the final conditions

$$\int_\Omega (p - \beta q)(T)v = \int_\Omega b_2(\bar{\varphi}(T) - \varphi_\Omega)v \quad \text{for every } v \in V$$

and

$$\int_\Omega r(T)v = \int_\Omega b_4(\bar{\sigma}(T) - \sigma_\Omega)v \quad \text{for every } v \in V.$$

Moreover, it has to possess the following regularity

$$q \in L^2(0, T; V) \quad (2.31)$$

$$p \in L^2(0, T; W) \quad (2.32)$$

$$r \in H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \subset C^0([0, T]; H) \quad (2.33)$$

$$p - \beta q \in H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \subset C^0([0, T]; H). \quad (2.34)$$

Thus, we are in a position to introduce the results concerning the asymptotic behavior of system (2.19)–(2.23), which will be fundamental for the asymptotic investigation.

**Theorem 2.7.** *Assume that (2.1)–(2.9) are in force. Let  $(q_\alpha, p_\alpha, r_\alpha)$  be the unique solution to (2.19)–(2.23) satisfying (2.24). Then, as  $\alpha \searrow 0$ , and up to a not relabeled subsequence, we have that*

$$q_\alpha \rightarrow q \text{ weakly in } L^2(0, T; V) \quad (2.35)$$

$$p_\alpha \rightarrow p \text{ weakly in } L^2(0, T; W) \quad (2.36)$$

$$r_\alpha \rightarrow r \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \quad (2.37)$$

$$\text{and strongly in } L^2(0, T; H) \quad (2.38)$$

$$p_\alpha - \beta q_\alpha \rightarrow p - \beta q \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \quad (2.39)$$

$$\text{and strongly in } L^2(0, T; H) \quad (2.40)$$

$$\alpha p_\alpha \rightarrow 0 \text{ weakly star in } H^1(0, T; H) \quad (2.41)$$

$$\text{and strongly in } L^\infty(0, T; V) \cap L^2(0, T; W). \quad (2.42)$$

Moreover, there exists a positive constant  $K_2$ , independent of  $\alpha$ , such that

$$\begin{aligned} & \|p_\alpha - \beta q_\alpha\|_{H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V)} + \|q_\alpha\|_{L^2(0, T; V)} + \alpha^{1/2} \|p_\alpha\|_{L^\infty(0, T; V)} \\ & + \alpha \|p_\alpha\|_{H^1(0, T; H)} + \|p_\alpha\|_{L^2(0, T; W)} + \|r_\alpha\|_{H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V)} \leq K_2. \end{aligned} \quad (2.43)$$

In addition, the limit  $(q, p, r)$  is the unique solution to problem (2.26)–(2.30) in the sense of Definition 2.6.

With all these results at disposal, we can announce the results regarding the existence of optimal controls and the first-order necessary condition that every optimal control has to satisfy.

**Theorem 2.8.** *Suppose that (2.1)–(2.9) are satisfied. Then, the optimal control problem (CP) admits, at least, a solution  $\bar{u} \in \mathcal{U}_{\text{ad}}$ .*

**Theorem 2.9.** *Assume that (2.1)–(2.9) are in force and let  $\bar{u} \in \mathcal{U}_{\text{ad}}$  be an optimal control for (CP) with its corresponding optimal state  $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$ . Moreover, let us denote by  $(p, q, r)$  the associated solution to the adjoint system (2.26)–(2.30). Then, the necessary condition for optimality of  $\bar{u}$  is given by the following variational inequality*

$$\int_Q (r + b_0 \bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U}_{\text{ad}}. \quad (2.44)$$

Furthermore, whenever  $b_0 \neq 0$ , the optimal control  $\bar{u}$  is the  $L^2(0, T; H)$ –projection of  $-r/b_0$  onto the closed subspace  $\mathcal{U}_{\text{ad}}$ .

Let us emphasize a consequence which is of straightforward importance for the numerical approach. Comparing the expected theoretical condition (1.20) with the explicit (2.44), it immediately follows that we can identify, via Riesz's representation theorem, the gradient of the reduced cost functional as  $\nabla \mathcal{J}_{\text{red}}(\bar{u}) = r + b_0 \bar{u}$ . Hence, for the numerical approach, the optimal control problem can be viewed as a constrained minimization of a function,  $\mathcal{J}_{\text{red}}$ , of which we know the gradient (think of the well-known projected conjugate gradient method).

In the remainder of the section, we recall some well-known results which will be useful later on. At first, let us remind the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0.$$

In addition, we often owe to the standard Sobolev embedding

$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{which holds for every } q \in [1, 6]. \quad (2.45)$$

In the whole of the paper, let us convey to use the symbol small-case  $c$  for every constant which only depend on the structural data of the problem, that is: on the final time  $T$ , on  $\Omega$ , on  $R$ , on the shape of the nonlinearities, on the norms of the involved functions, and possibly on  $\beta$ . Differently, the capital letters are devoted to indicating some specific constant which we eventually will refer in the sequel. Moreover, since we aim to let  $\alpha \searrow 0$ , we will keep track at every step of the eventual dependence of the appearing constants by  $\alpha$ .

### 3 Existence and Approximation of Optimal Controls

#### 3.1 Existence of Optimal Controls

Here, we check the existence of optimal controls by proving Theorem 2.8.

*Proof of Theorem 2.8.* The method we are going to employ is the celebrated direct methods of calculus of variations. To begin with, let us pick an arbitrary sequence  $\{\alpha_n\}_n \subset (0, 1]$  which goes to zero as  $n \rightarrow \infty$ . Then, we take as  $\{u_n\}_n := \{u_{\alpha_n}\}_n$  a minimizing sequence for the cost functional  $\mathcal{J}$  constitutes by elements of  $\mathcal{U}_{\text{ad}}$ , which, for every  $n$ , is optimal controls for  $(CP)_{\alpha_n}$ , which exist by virtue of Lemma 2.3. Next, at every step, we introduce  $(\mu_n, \varphi_n, \sigma_n)$  as the solution associated to system (1.10)–(1.14) with  $u = u_n$ . By recalling estimate (2.15) and the boundedness of  $\mathcal{U}_{\text{ad}}$ , it straightforwardly follows from standard weak and weak-star compactness results that there exists a subsequence, which we do not relabel, some  $\bar{u} \in \mathcal{U}_{\text{ad}}$  and a triplet  $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$  such that, as  $n \rightarrow \infty$ , we have that

$$\begin{aligned} u_n &\rightarrow \bar{u} \quad \text{weakly star in } L^\infty(Q) \\ \mu_n &\rightarrow \bar{\mu} \quad \text{weakly in } L^2(0, T; V) \\ \varphi_n &\rightarrow \bar{\varphi} \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \sigma_n &\rightarrow \bar{\sigma} \quad \text{weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V). \end{aligned}$$

Moreover, compactness arguments (see, e.g., [42, Sec. 8, Cor. 4]) also yield that

$$\varphi_n \rightarrow \bar{\varphi} \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V),$$

which gives meaning to the initial condition  $\bar{\varphi}(0) = \varphi_0$ . This, along with the growth assumption (2.9), allows us to infer that

$$F'(\varphi_n) \rightarrow F'(\bar{\varphi}) \quad \text{strongly in } L^2(0, T; H).$$

Then, it suffices to take into account the variational formulation of system (1.10)–(1.14), written for  $(\mu_n, \varphi_n, \sigma_n)$ , and pass to the limit as  $n \rightarrow \infty$ . From this passage, we infer that  $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$  is admissible for  $(CP)$ , that is  $\bar{\mu}$  and  $\bar{\varphi}$  are the unique solutions to (1.15)–(1.19) associated with  $\bar{u}$ . Lastly, the weak sequential lower semicontinuity of the cost functional leads to conclude that  $\bar{u}$  is a minimizer for  $(CP)$ .  $\square$

### 3.2 Approximation of Optimal Controls

After existence has been shown, we would like to infer some information on the behavior of the optimal controls, pointing out some necessary conditions for optimality. We would like to achieve this goal by letting  $\alpha \searrow 0$  in the necessary condition for  $(CP)_\alpha$  expressed by the variational inequality (2.25). Although from a formal perspective it could seem reasonable, we cannot directly proceed this way. In fact, if we want to let  $\alpha \searrow 0$  without any restriction, we have to ensure that every optimal control for  $(CP)$  can be approximated by a sequence of optimal controls for  $(CP)_\alpha$ . Unfortunately, we are unable to prove such a strong global approximation result. Anyhow, a partial one can be stated localizing the problem by following the idea firstly introduced by Barbu in [2]. Let us refer the interested reader, among others, to the contributions [6, 7, 10, 40], where an application of such a technique can be found. The key ingredient relies on a local perturbation of the cost functional  $\mathcal{J}$ . Then, instead of looking for approximating sequence made up by optimal controls for  $(CP)_\alpha$ , we seek for a sequence of optimal controls for a modified optimization problem. Namely, we still consider the same state system, whereas we are going to minimize the so-called adapted cost functional which, for every optimal control  $\bar{u}$  for  $(CP)$ , is defined by

$$\tilde{\mathcal{J}}(\varphi, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2. \quad (3.1)$$

Due to the fact that the state system is the same, it is straightforward to deduce that this slight modification of  $\mathcal{J}$  do not change the corresponding adjoint system. Hence, it is natural to consider the following new minimization problem:

$(\widetilde{CP})_\alpha$  Minimize  $\tilde{\mathcal{J}}(\varphi, \mu, u)$  subject to the control constraints (1.3) and under the requirement that the variables  $(\varphi, \sigma)$  yield a solution to (1.10)–(1.14).

It is worth emphasizing that  $\tilde{\mathcal{J}}$  reduces to  $\mathcal{J}$  whenever it is restricted to act on optimal controls for  $(CP)$ . Moreover, the above control problem perfectly complies with the framework of [41] and therefore, we also have the following lemma at disposal.

**Lemma 3.1.** *Assume that assumptions (2.1)–(2.9) are satisfied. Then, for every  $\alpha \in (0, 1)$ , there exists at least an optimal control for  $(\widetilde{CP})_\alpha$ .*

In a similar fashion as above, it also follows from [41] how the first-order necessary condition for optimality can be outlined:

**Theorem 3.2.** *Assume that (2.1)–(2.9) are in force. Let  $\bar{u}_\alpha \in \mathcal{U}_{\text{ad}}$  be an optimal control for  $(\widetilde{CP})_\alpha$ , and  $(\bar{\mu}_\alpha, \bar{\varphi}_\alpha, \bar{\sigma}_\alpha)$  and  $(p_\alpha, q_\alpha, r_\alpha)$  be the corresponding state and co-state, respectively. Then, the first-order necessary condition for optimality is characterized by the variational formulation*

$$\int_Q (r_\alpha + b_0 \bar{u}_\alpha + (\bar{u}_\alpha - \bar{u})) (v - \bar{u}_\alpha) \geq 0 \quad \forall v \in \mathcal{U}_{\text{ad}}. \quad (3.2)$$

With all these ingredients, we are finally in a position to introduce the aforementioned approximation result.

**Theorem 3.3.** *Assume that (2.1)–(2.9) are fulfilled. Let us denote  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$  an optimal triplet for  $(CP)$  and let  $\{\alpha_n\}_n \subset (0, 1]$  be a sequence which goes to zero as  $n \rightarrow \infty$ . Then, there exists an approximating optimal sequence, namely a sequence that, for every  $n$ , consists of an optimal triplet  $(\bar{\varphi}_{\alpha_n}, \bar{\sigma}_{\alpha_n}, \bar{u}_{\alpha_n})$  for  $(\widetilde{CP})_{\alpha_n}$ , such that the following convergences are satisfied*

$$\bar{u}_n := \bar{u}_{\alpha_n} \rightarrow \bar{u} \text{ strongly in } L^2(Q) \quad (3.3)$$

$$\bar{\varphi}_n := \bar{\varphi}_{\alpha_n} \rightarrow \bar{\varphi} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (3.4)$$

$$\bar{\sigma}_n := \bar{\sigma}_{\alpha_n} \rightarrow \bar{\sigma} \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V) \quad (3.5)$$

$$\tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \rightarrow \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) \quad (3.6)$$

as  $n \rightarrow \infty$ , and up to a not relabeled subsequence.

*Proof.* By virtue of Lemma 3.1, for every  $n \in \mathbb{N}$ , we can take an optimal triplet  $(\bar{\varphi}_{\alpha_n}, \bar{\sigma}_{\alpha_n}, \bar{u}_{\alpha_n})$  for  $(\widetilde{CP})_{\alpha_n}$  that, for convenience, we will denote by  $(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n)$ . From the bound pointed out by estimate (2.15), together with the boundedness of the control-box, after extraction of a subsequence, we easily get that

$$\begin{aligned} \bar{u}_n &\rightarrow u \text{ weakly star in } L^\infty(Q) \\ \bar{\varphi}_n &\rightarrow \varphi \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \bar{\sigma}_n &\rightarrow \sigma \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H) \cap L^2(0, T; V). \end{aligned}$$

On the other hand, the continuity of the control-to-state mapping entails that the limit triplet  $(\mu, \varphi, u)$  is admissible for the control problem  $(CP)$ , that is  $\varphi$  and  $\sigma$  are the solution to (1.15)–(1.19) corresponding to  $u$ . Thus, our purpose is now checking that the limit  $u$  is not only admissible, but it is actually optimal which, in turn, will imply that  $\varphi$  and  $\sigma$  are the corresponding optimal states. In this direction, we rely on monotonicity arguments. Firstly, the optimality of  $(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n)$  for  $(\widetilde{CP})_{\alpha_n}$  yields that

$$\tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) \quad \text{for every } n \in \mathbb{N}$$

and passing to the superior limit to both sides and exploiting the definition of the adapted cost functional  $\tilde{\mathcal{J}}$ , we realize that

$$\limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}). \quad (3.7)$$

Moreover, it is straightforward to see that also  $\tilde{\mathcal{J}}$  is weak sequential lower semicontinuous, which implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) &\geq \tilde{\mathcal{J}}(\varphi, \sigma, u) = \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 \\ &\geq \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2, \end{aligned} \quad (3.8)$$

where the optimality of  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$  for  $(CP)$  and the definition of the adapted cost functional have been invoked. By combining (3.7) with (3.8), we get the first convergence we are looking for since it follows that we have arrived at the identity

$$\frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 = 0, \quad (3.9)$$

so that  $\bar{u}_n$  weakly star converges to  $\bar{u}$ . These limits, also lead us to infer that the triplet  $(\varphi, \sigma, u)$  is nothing but  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$ . As (3.6) is concerned, it suffices to remember that  $\mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u})$  and the fact that the above inferior and superior limits coincide. In fact, we realize that the following chain of equality has been shown:

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}).$$

Thus, we are reduced to prove (3.3). Using the above estimates, we infer that

$$\mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \lim_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) + \frac{1}{2} \|\bar{u}_n - \bar{u}\|_{L^2(Q)}^2. \quad (3.10)$$

On the other hand, the lower semicontinuity of the cost functional, along with the above estimates, entails that

$$\begin{aligned} \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) &\leq \liminf_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \limsup_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \\ &\leq \limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}), \end{aligned}$$

so that

$$\mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \lim_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n)$$

is verified. Therefore, by combining the above property with (3.10) we deduce that

$$\frac{1}{2} \|\bar{u}_n - \bar{u}\|_{L^2(Q)}^2 \rightarrow 0,$$

which conclude the proof.  $\square$

## 4 Optimality Conditions

Next, we establish the necessary condition that an optimal control has to verify. As explained above, in order to pass to the limit in the variational inequality (2.25), we have to deal with the asymptotic of system (2.19)–(2.23) and with the approximation issue presented above. Since the approximating system has been already investigated in the above section, only the asymptotic analysis of the adjoint system (2.19)–(2.23) has been left unanswered.

### 4.1 The Adjoint System

Below, we proceed formally by only providing some a priori estimates. The justification can be carried out within a Faedo–Galerkin scheme as already made in [41, Sec. 4.4]. Let us just point out that, in the approximation, the duality product is replaced by the  $L^2$ –inner product and that the final conditions are replaced by the corresponding  $L^2$ –orthogonal projection onto the finite space spanned by the element of the Galerkin basis.

*Proof of Theorem 2.7.* The estimates we are going to perform in a while are twofold. Firstly, within a proper approximation scheme, they will be the key argument to prove the existence of a solution. Secondly, since we will keep track at every step of the dependence of the appearing constants by  $\alpha$ , they will also be the starting point to let  $\alpha \searrow 0$  to handle the asymptotic analysis of system (2.19)–(2.23).

To begin with, it is convenient to rewrite the system (2.19)–(2.23) in a different form. Let us formally motivate this statement; by considering the vanishing of  $\alpha$ , it is straightforward to realize that the final condition  $\alpha p_\alpha = 0$  disappears. Moreover, by comparing equation (2.19) with the corresponding final condition, it turns out that the variable  $p_\alpha - \beta q_\alpha$  has to be considered as a single variable, since only for such a linear combination the final condition is available. Moving from this consideration, let us set the following notation

$$w_\alpha := p_\alpha - \beta q_\alpha, \quad (4.1)$$

which, in turn, implies

$$q_\alpha = \frac{p_\alpha - w_\alpha}{\beta} \quad \text{and} \quad p_\alpha = w_\alpha + \beta q_\alpha.$$

According to the above definitions, we rewrite the above system in terms of the variable  $w_\alpha$  to obtain the new system

$$-\partial_t w_\alpha + \frac{1}{\beta} \Delta p_\alpha - \frac{1}{\beta} \Delta w_\alpha - \frac{1}{\beta} F''(\bar{\varphi}_\alpha) p_\alpha + \frac{1}{\beta} F''(\bar{\varphi}_\alpha) w_\alpha = b_1(\bar{\varphi}_\alpha - \varphi_Q) \quad \text{in } Q \quad (4.2)$$

$$\frac{1}{\beta} p_\alpha - \frac{1}{\beta} w_\alpha - \alpha \partial_t p_\alpha - \Delta p_\alpha + P(p_\alpha - r_\alpha) = 0 \quad \text{in } Q \quad (4.3)$$

$$-\partial_t r_\alpha - \Delta r_\alpha + P(r_\alpha - p_\alpha) = b_3(\bar{\sigma}_\alpha - \sigma_Q) \quad \text{in } Q \quad (4.4)$$

$$\partial_n w_\alpha = \partial_n p_\alpha = \partial_n r_\alpha = 0 \quad \text{on } \Sigma \quad (4.5)$$

$$w_\alpha(T) = b_2(\bar{\varphi}_\alpha(T) - \varphi_\Omega), \quad \alpha p_\alpha(T) = 0, \quad r_\alpha(T) = b_4(\bar{\sigma}_\alpha(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (4.6)$$

Now, we start presenting the estimates.

**First estimate** In the first place, we multiply equation (4.2) by  $w_\alpha$ , (4.3) by  $p_\alpha - \Delta p_\alpha$ , (4.4) by  $r_\alpha$  and integrate over  $Q_t^T$  and by parts to obtain, upon rearranging the terms, that

$$\begin{aligned} & \frac{1}{2} \int_\Omega |w_\alpha(t)|^2 + \frac{1}{\beta} \int_{Q_t^T} |\nabla w_\alpha|^2 + \frac{\alpha}{2} \int_\Omega (|p_\alpha(t)|^2 + |\nabla p_\alpha(t)|^2) + \left( \frac{1}{\beta} + P \right) \int_{Q_t^T} |p_\alpha|^2 \\ & + \left( \frac{1}{\beta} + P + 1 \right) \int_{Q_t^T} |\nabla p_\alpha|^2 + \int_{Q_t^T} |\Delta p_\alpha|^2 + \frac{1}{2} \int_\Omega |r_\alpha(t)|^2 + \int_{Q_t^T} |\nabla r_\alpha|^2 + P \int_{Q_t^T} |r_\alpha|^2 \\ & = \frac{1}{2} \int_\Omega |b_2(\bar{\varphi}_\alpha(T) - \varphi_\Omega)|^2 + \frac{1}{2} \int_\Omega |b_4(\bar{\sigma}_\alpha(T) - \sigma_\Omega)|^2 + \int_{Q_t^T} b_1(\bar{\varphi}_\alpha - \varphi_Q) w_\alpha \\ & + \int_{Q_t^T} b_3(\bar{\sigma}_\alpha - \sigma_Q) r_\alpha + \frac{1}{\beta} \int_{Q_t^T} F''(\bar{\varphi}_\alpha) p_\alpha w_\alpha - \frac{1}{\beta} \int_{Q_t^T} F''(\bar{\varphi}_\alpha) w_\alpha^2 - \frac{2}{\beta} \int_{Q_t^T} \Delta p_\alpha w_\alpha \\ & + P \int_{Q_t^T} r_\alpha (p_\alpha - \Delta p_\alpha) + \frac{1}{\beta} \int_{Q_t^T} w_\alpha p_\alpha + P \int_{Q_t^T} p_\alpha r_\alpha, \end{aligned}$$

where we denote the terms on the right-hand side by  $I_1, \dots, I_{10}$ , in this order. Moreover, the integrals on the left-hand side are nonnegative, whereas the ones on the right-hand



side can be bounded as follows. Using the final conditions (4.6), assumptions (2.2), (2.3), and (2.10), we infer by the Young inequality that

$$|I_1| + |I_2| + |I_3| + |I_4| \leq c \int_{Q_t^T} (|w_\alpha|^2 + |r_\alpha|^2) + c.$$

As for  $I_5$  and  $I_6$ , we recall the growth assumption (2.9) and the fact that  $\bar{\varphi}_\alpha$ , as a solution to (1.10)–(1.14), verifies estimate (2.15). Thus, along with the Hölder and Young inequalities, and the standard embedding (2.45), we get that

$$\begin{aligned} |I_5| + |I_6| &\leq \frac{C_1}{\beta} \int_{Q_t^T} (1 + |\bar{\varphi}_\alpha^2|) p_\alpha w_\alpha + \frac{C_1}{\beta} \int_{Q_t^T} (1 + |\bar{\varphi}_\alpha^2|) w_\alpha^2 \\ &\leq c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_3) \|p_\alpha\|_6 \|w_\alpha\|_2 + c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_3) \|w_\alpha\|_6 \|w_\alpha\|_2 \\ &\leq c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_6^2) \|p_\alpha\|_6 \|w_\alpha\|_2 + c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_6^2) \|w_\alpha\|_6 \|w_\alpha\|_2 \\ &\leq c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_V^2) \|p_\alpha\|_V \|w_\alpha\|_H + c \int_t^T (1 + \|\bar{\varphi}_\alpha\|_V^2) \|w_\alpha\|_V \|w_\alpha\|_H \\ &\leq \delta \int_t^T \|p_\alpha\|_V^2 + \delta \int_{Q_t^T} |\nabla w_\alpha|^2 + c_\delta \int_{Q_t^T} |w_\alpha|^2, \end{aligned}$$

for a positive constant  $\delta$  yet to be determined. Next, invoking once more the Young inequality, we argue that

$$|I_7| \leq \frac{1}{4} \int_{Q_t^T} |\Delta p_\alpha|^2 + \frac{2}{\beta} \int_{Q_t^T} |w_\alpha|^2,$$

and also that

$$|I_8| + |I_9| + |I_{10}| \leq 3\delta \int_{Q_t^T} |p_\alpha|^2 + \frac{1}{4} \int_{Q_t^T} |\Delta p_\alpha|^2 + c_\delta \int_{Q_t^T} (|w_\alpha|^2 + |r_\alpha|^2).$$

Hence, upon collecting all these terms, we realize that it suffices to fix  $\delta$  small enough. Namely, we pick  $\delta = \widehat{\delta}$  such that

$$\widehat{\delta} < \min \left\{ \frac{1}{\beta}, \frac{1}{4} \left( \frac{1}{\beta} + P \right), \frac{1}{\beta} + P + 1 \right\}.$$

Therefore, a Gronwall argument yields that

$$\|w_\alpha\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \alpha^{1/2} \|p_\alpha\|_{L^\infty(0,T;V)} + \|p_\alpha\|_{L^2(0,T;W)} + \|r_\alpha\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c,$$

for a suitable positive constant  $c$  independent of  $\alpha$ . Moreover, let us note that

$$\|\alpha p_\alpha\|_{L^\infty(0,T;V)} \leq c \alpha^{1/2}.$$

**Second estimate** Multiplying (4.2) by an arbitrary  $v \in L^2(0, T; V)$ , integrating over  $Q$  and by parts, and making use of the above bounds, we infer that

$$\begin{aligned} \left| \int_Q \partial_t w_\alpha v \right| &\leq c \|\nabla p_\alpha\|_{L^2(0,T;H)} \|\nabla v\|_{L^2(0,T;H)} + c \|\nabla w_\alpha\|_{L^2(0,T;H)} \|\nabla v\|_{L^2(0,T;H)} \\ &\quad + c \|p_\alpha\|_{L^2(0,T;H)} \|v\|_{L^2(0,T;V)} + c \|w_\alpha\|_{L^2(0,T;H)} \|v\|_{L^2(0,T;V)} + c \|v\|_{L^2(0,T;H)} \\ &\leq c \|v\|_{L^2(0,T;V)}. \end{aligned}$$

Then, dividing both sides by  $\|v\|_{L^2(0,T;V)}$  and passing to the superior limit leads to conclude that

$$\|\partial_t w_\alpha\|_{L^2(0,T;V^*)} \leq c.$$

**Third estimate** By the same token, we employ the above estimates to obtain that

$$\|\partial_t r_\alpha\|_{L^2(0,T;V^*)} \leq c.$$

**Fourth estimate** Lastly, comparison in equation (4.3), along with the above estimates, produces

$$\|\alpha \partial_t p_\alpha\|_{L^2(0,T;H)} \leq c.$$

It is now a standard matter to show that the above estimates will be sufficient, withing a Galerkin scheme, to provide the existence of a solution to (2.26)–(2.30) which also satisfies (2.31)–(2.34). Furthermore, the existence, together with the linearity of the system, also implies its uniqueness.

**Passage to the limit** Here, we draw some consequences from the aforementioned estimates checking that, in a proper sense, system (2.19)–(2.23) converges to (2.26)–(2.30). Owing to standard weak compactness arguments it turns out that, up to a not relabeled subsequence, the following convergences hold

$$\begin{aligned} w_\alpha &\rightarrow w \text{ weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V) \\ p_\alpha &\rightarrow p \text{ weakly in } L^2(0,T;W) \\ r_\alpha &\rightarrow r \text{ weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V). \end{aligned}$$

Moreover, the compact embedding of  $H^1(0,T;V^*) \cap L^2(0,T;V)$  into  $C^0([0,T];H)$  guarantees that the final data are meaningful and that

$$\begin{aligned} w_\alpha &\rightarrow w \text{ strongly in } L^2(0,T;H) \\ r_\alpha &\rightarrow r \text{ strongly in } L^2(0,T;H) \\ \alpha p_\alpha &\rightarrow 0 \text{ weakly star in } H^1(0,T;H) \text{ and strongly in } L^\infty(0,T;V) \cap L^2(0,T;W). \end{aligned}$$

Hence, by combining the above first and second convergences with the definition of the auxiliary variable  $w_\alpha$  given by (4.1), we realize that

$$q_\alpha \rightarrow q \text{ weakly in } L^2(0,T;V). \quad (4.7)$$

Therefore, the above convergences implies that the weak limit of  $w_\alpha$  can be identified with  $w = p - \beta q$ . So, in what follows, we are legitimate to conveniently interchange the variables  $w$  and  $q$  as convenience.

Now, let us take into account the variational formulation of system (2.19)–(2.23). It

consists of seeking for a triplet  $(w_\alpha, p_\alpha, r_\alpha)$  such that satisfies the following problem

$$\begin{aligned}
& - {}_V^* \langle \partial_t w_\alpha(t), v \rangle_V - \int_\Omega \nabla q_\alpha(t) \cdot \nabla v - \int_\Omega F''(\bar{\varphi}_\alpha(t)) q_\alpha(t) v \\
& \quad = \int_\Omega b_1(\bar{\varphi}_\alpha(t) - \varphi_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\
& \int_\Omega q_\alpha v - \alpha \int_\Omega \partial_t p_\alpha(t) v + \int_\Omega \nabla p_\alpha(t) \cdot \nabla v + P \int_\Omega (p_\alpha(t) - r_\alpha(t)) v \\
& \quad = 0 \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\
& - {}_V^* \langle \partial_t r_\alpha(t), v \rangle_V + \int_\Omega \nabla r_\alpha(t) \cdot \nabla v + P \int_\Omega (r_\alpha(t) - p_\alpha(t)) v \\
& \quad = \int_\Omega b_3(\bar{\sigma}_\alpha(t) - \sigma_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T).
\end{aligned}$$

Moreover, owing to the final conditions (4.6), it has also to verify the final conditions

$$\int_\Omega w_\alpha(T) v = \int_\Omega b_2(\bar{\varphi}_\alpha(T) - \varphi_\Omega) v \quad \text{for every } v \in V$$

and

$$\int_\Omega r_\alpha(T) v = \int_\Omega b_4(\bar{\sigma}_\alpha(T) - \sigma_\Omega) v \quad \text{for every } v \in V.$$

By virtue of the above discussion, along with the convergences (2.11)–(2.14), we would conclude that, as  $\alpha \searrow 0$ , the above system converges to the following problem:

$$\begin{aligned}
& - {}_V^* \langle \partial_t (p - \beta q)(t), v \rangle_V - \int_\Omega \nabla q(t) \cdot \nabla v - \int_\Omega F''(\bar{\varphi}(t)) q(t) v \\
& \quad = \int_\Omega b_1(\bar{\varphi}(t) - \varphi_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\
& \int_\Omega q(t) v + \int_\Omega \nabla p(t) \cdot \nabla v + P \int_\Omega (p(t) - r(t)) v \\
& \quad = 0 \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\
& - {}_V^* \langle \partial_t r(t), v \rangle_V + \int_\Omega \nabla r(t) \cdot \nabla v + P \int_\Omega (r(t) - p(t)) v \\
& \quad = \int_\Omega b_3(\bar{\sigma}(t) - \sigma_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T)
\end{aligned}$$

with the corresponding final conditions

$$\int_\Omega (p - \beta q)(T) v = \int_\Omega b_2(\bar{\varphi}(T) - \varphi_\Omega) v \quad \text{for every } v \in V$$

and

$$\int_\Omega r(T) v = \int_\Omega b_4(\bar{\sigma}(T) - \sigma_\Omega) v \quad \text{for every } v \in V.$$

To do that, we multiply the first system by a regular function  $\delta \in C_c^\infty(0, T)$ , integrate over  $(0, T)$ , and then pass to the limit accounting for the above estimates. Thus, since the

obtained limit system holds for every  $\delta \in C_c^\infty(0, T)$ , one finally recover the last system. Anyhow, to prove such a passage, we need to handle the asymptotics of the nonlinear term  $F''(\bar{\varphi}_\alpha)q_\alpha$ . We claim that, by combining the growth assumptions on the potential (2.9) with the strong convergence (2.12), it follows that

$$F''(\bar{\varphi}_\alpha) \rightarrow F''(\bar{\varphi}) \quad \text{strongly in } L^2(0, T; H). \quad (4.8)$$

Therefore, by combining (4.7) with (4.8), the nonlinear term can be handled since we have

$$F''(\bar{\varphi}_\alpha)q_\alpha \rightarrow F''(\bar{\varphi})q \quad \text{weakly in } L^2(0, T; H),$$

and this conclude the proof.  $\square$

## 4.2 First-order Necessary Condition

In this last section, we are going to prove Theorem 2.9 which gives us the first-order necessary condition for optimality.

*Proof of Theorem 2.9.* As already mention, we try to recover the first-order necessary condition for the control problem (CP) via asymptotic techniques by letting  $\alpha \searrow 0$ , in a suitable sense, in the variational inequality (2.25). The main issue has been already introduced above and consists in the fact that we have to guarantee that every optimal control for (CP) can be found as a limit of a sequence made up by optimal controls for  $(CP)_\alpha$ . This can be overcome by invoking the investigated approximation result. In fact, we consider a sequence  $\{\alpha_n\} \subset (0, 1]$  which goes to zero as  $n \rightarrow \infty$ , and introduce the sequence  $\bar{u}_n := \bar{u}_{\alpha_n}$  of optimal controls for  $(\widehat{CP})_{\alpha_n}$  introduced in Theorem 3.3. After further extraction of a subsequence  $\{\alpha_{n_k}\}$ , the convergence pointed out by (2.35)–(2.41) and (3.3)–(3.6) allow us to pass to the limit, as  $k \rightarrow \infty$ , in (3.2) to achieve the necessary condition we are looking for.

Finally, the last sentence follows from an application of the well-known Hilbert projection theorem, since  $\mathcal{U}_{\text{ad}}$  is a non-empty, closed and convex subset of  $L^2(0, T; H)$ .  $\square$

Finally, due to the structure of the control-box  $\mathcal{U}_{\text{ad}}$ , in the case of  $b_0 > 0$ , we can provide an equivalent implicit characterization of the optimal control (see, e.g., [44]).

**Corollary 4.1.** *Suppose that (2.1)–(2.9) and that  $b_0 > 0$ . Then, the optimal control  $\bar{u}$  for (CP) satisfies*

$$\bar{u}(x, t) = \max\left\{u_*(x, t), \min\left\{u^*(x, t), -\frac{1}{b_0}r(x, t)\right\}\right\} \quad \text{for a.a. } (x, t) \in Q.$$

**Remark 4.2.** From a little investigation of Theorems 2.2 and 2.7, one realize that the requirements  $\varphi_\Omega, \sigma_\Omega \in H^1(\Omega)$  and  $\varphi_0 \in W$ , and  $\mu_0, \sigma_0 \in V$  turn out to be superabundant. In fact, for the limit optimal control problem (CP), to be meaningful, it suffices that

$$\varphi_\Omega, \sigma_\Omega \in H \quad \text{and} \quad \varphi_0 \in V, \mu_0, \sigma_0 \in H. \quad (4.9)$$

The framework we have introduced was motivated by the fact that, in order to manage  $(CP)_\alpha$ , we directly rely on the results of [41]. Thus, it has been chosen by comparing the framework of [41] with the additional assumptions introduced in [12] to deal with

the asymptotic analysis of system (1.10)–(1.14). Indeed, whenever  $\alpha > 0$ , both the requirements (2.3) and (2.5) have to be fulfilled.

In such a perspective, one may wonder if the given requirements can be somehow weakened. One possible way to proceed can be to assume (4.9) and define some regularizing sequences

$$\begin{aligned} \{\varphi_\Omega^\alpha\}_\alpha, \{\sigma_\Omega^\alpha\}_\alpha &\in H^1(\Omega) \\ \{\varphi_0^\alpha\}_\alpha &\in W, \{\mu_0^\alpha\}_\alpha \in V, \{\sigma_0^\alpha\}_\alpha \in V \end{aligned}$$

which satisfy, as  $\alpha \searrow 0$ , the following strong convergences

$$\begin{aligned} \varphi_\Omega^\alpha &\rightarrow \varphi_\Omega, \sigma_\Omega^\alpha \rightarrow \sigma_\Omega \text{ strongly in } H \\ \varphi_0^\alpha &\rightarrow \varphi_0 \text{ strongly in } V, \text{ and } \mu_0^\alpha \rightarrow \mu_0, \sigma_0^\alpha \rightarrow \sigma_0 \text{ strongly in } H. \end{aligned}$$

Then, for every  $\alpha \in (0, 1)$ , the initial conditions in the state system (1.14) has to be replaced with the approximated version

$$\mu_\alpha(0) = \mu_0^\alpha, \varphi_\alpha(0) = \varphi_0^\alpha, \sigma_\alpha(0) = \sigma_0^\alpha \quad \text{in } \Omega.$$

Moreover, the cost functional  $\mathcal{J}$  has to be substituted by

$$\begin{aligned} \mathcal{J}^\alpha(\varphi, \sigma, u) &:= \frac{b_1}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{b_2}{2} \|\varphi(T) - \varphi_\Omega^\alpha\|_{L^2(\Omega)}^2 + \frac{b_3}{2} \|\sigma - \sigma_Q\|_{L^2(Q)}^2 \\ &\quad + \frac{b_4}{2} \|\sigma(T) - \sigma_\Omega^\alpha\|_{L^2(\Omega)}^2 + \frac{b_0}{2} \|u\|_{L^2(Q)}^2, \end{aligned}$$

and the adapted cost with  $\tilde{\mathcal{J}}^\alpha$ , defined according to (3.1). Thus, the new  $(CP)_\alpha$  consists of minimizing the cost functional  $\mathcal{J}^\alpha$  subject to the control-box constraints  $\mathcal{U}_{\text{ad}}$ , and under the assumption that  $\varphi, \sigma$  are solution to this new approximated state system, namely system (1.10)–(1.13) coupled with the above initial data. It is immediately clear that the corresponding investigation will became more technical and it is not clear if such an effort is worth to be pursued.

## Acknowledgments

The author would like to thank especially one of the referees for the careful reading and the precious suggestions which have improved the manuscript.

## References

- [1] A. Agosti, P.F. Antonietti, P. Ciarletta, M. Grasselli and M. Verani, A Cahn-Hilliard-type equation with application to tumor growth dynamics, *Math. Methods Appl. Sci.*, **40** (2017), 7598-7626.
- [2] V. Barbu, Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities, *J. Math. Anal. Appl.* **80** (1981), 566-597.
- [3] H. Brezis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

- [4] E. Casas, J.C. de los Reyes and F. Tröltzsch, Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints, *SIAM J. Optim.* **19**(2), (2008), 616-643.
- [5] C. Cavaterra, E. Rocca and H. Wu, Long-time Dynamics and Optimal Control of a Diffuse Interface Model for Tumor Growth, *Appl. Math. Optim.* (2019), <https://doi.org/10.1007/s00245-019-09562-5>.
- [6] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi and J. Sprekels, Optimal boundary control of a viscous Cahn-Hilliard system with dynamic boundary condition and double obstacle potentials, *SIAM J. Control Optim.* **53** (2015), 2696-2721.
- [7] P. Colli, M.H. Farshbaf-Shaker and J. Sprekels, A deep quench approach to the optimal control of an Allen-Cahn equation with dynamic boundary conditions and double obstacles, *Appl. Math. Optim.* **71** (2015), 1-24.
- [8] P. Colli, G. Gilardi and D. Hilhorst, On a Cahn-Hilliard type phase field system related to tumor growth, *Discrete Contin. Dyn. Syst.* **35** (2015), 2423-2442.
- [9] P. Colli, G. Gilardi, G. Marinoschi and E. Rocca, Sliding mode control for a phase field system related to tumor growth, *Appl. Math. Optim.* to appear (2019), doi:10.1007/s00245-017-9451-z.
- [10] P. Colli, G. Gilardi and J. Sprekels, Optimal velocity control of a convective Cahn-Hilliard system with double obstacles and dynamic boundary conditions: a ‘deep quench’ approach. *J. Convex Anal.*, to appear (2018).
- [11] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth, *Nonlinearity* **30** (2017), 2518-2546.
- [12] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Asymptotic analyses and error estimates for a Cahn-Hilliard type phase field system modeling tumor growth, *Discrete Contin. Dyn. Syst. Ser. S* **10** (2017), 37-54.
- [13] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Vanishing viscosities and error estimate for a Cahn-Hilliard type phase field system related to tumor growth, *Nonlinear Anal. Real World Appl.* **26** (2015), 93-108.
- [14] V. Cristini, X. Li, J.S. Lowengrub, S.M. Wise, Nonlinear simulations of solid tumor growth using a mixture model: invasion and branching. *J. Math. Biol.* **58** (2009), 723-763.
- [15] V. Cristini, J. Lowengrub, Multiscale Modeling of Cancer: An Integrated Experimental and Mathematical Modeling Approach. *Cambridge University Press*, Leiden (2010).
- [16] M. Dai, E. Feireisl, E. Rocca, G. Schimperna, M. Schonbek, Analysis of a diffuse interface model of multispecies tumor growth, *Nonlinearity* **30** (2017), 1639.
- [17] M. Ebenbeck and P. Knopf, Optimal control theory and advanced optimality conditions for a diffuse interface model of tumor growth *preprint arXiv:1903.00333 [math.OC]*, (2019), 1-34.
- [18] M. Ebenbeck and P. Knopf, Optimal medication for tumors modeled by a Cahn-Hilliard-Brinkman equation, *preprint arXiv:1811.07783 [math.AP]*, (2018), 1-26.
- [19] M. Ebenbeck and H. Garcke, Analysis of a Cahn-Hilliard-Brinkman model for tumour growth with chemotaxis. *J. Differential Equations*, (2018) <https://doi.org/10.1016/j.jde.2018.10.045>.
- [20] S. Frigeri, M. Grasselli, E. Rocca, On a diffuse interface model of tumor growth, *European J. Appl. Math.* **26** (2015), 215-243.
- [21] S. Frigeri, K.F. Lam, E. Rocca, G. Schimperna, On a multi-species Cahn-Hilliard-Darcy tumor growth model with singular potentials, *Comm. in Math. Sci.* **(16)(3)** (2018), 821-856.

- [22] S. Frigeri, K.F. Lam and E. Rocca, On a diffuse interface model for tumour growth with non-local interactions and degenerate mobilities, In *Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs*, P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels (ed.), *Springer INdAM Series*, **22**, Springer, Cham, 2017.
- [23] H. Garcke and K. F. Lam, Well-posedness of a Cahn-Hilliard-Darcy system modelling tumour growth with chemotaxis and active transport, *European. J. Appl. Math.* **28** (2) (2017), 284-316.
- [24] H. Garcke and K. F. Lam, Analysis of a Cahn-Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis, *Discrete Contin. Dyn. Syst.* **37** (8) (2017), 4277-4308.
- [25] H. Garcke and K. F. Lam, Global weak solutions and asymptotic limits of a Cahn-Hilliard-Darcy system modelling tumour growth, *AIMS Mathematics* **1** (3) (2016), 318-360.
- [26] H. Garcke and K. F. Lam, On a Cahn-Hilliard-Darcy system for tumour growth with solution dependent source terms, in *Trends on Applications of Mathematics to Mechanics*, E. Rocca, U. Stefanelli, L. Truskinovski, A. Visintin (ed.), *Springer INdAM Series* **27**, Springer, Cham, 2018, 243-264.
- [27] H. Garcke, K. F. Lam, R. Nürnberg and E. Sitka, A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis, *Mathematical Models and Methods in Applied Sciences* **28** (3) (2018), 525-577.
- [28] H. Garcke, K. F. Lam and E. Rocca, Optimal control of treatment time in a diffuse interface model of tumor growth, *Appl. Math. Optim.* **78**(3) (2018), 495-544.
- [29] H. Garcke, K.F. Lam, E. Sitka, V. Styles, A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport. *Math. Models Methods Appl. Sci.* **26**(6) (2016), 1095-1148.
- [30] A. Hawkins, J.T Oden, S. Prudhomme, General diffuse-interface theories and an approach to predictive tumor growth modeling. *Math. Models Methods Appl. Sci.* **58** (2010), 723-763.
- [31] A. Hawkins-Daarud, S. Prudhomme, K.G. van der Zee, J.T. Oden, Bayesian calibration, validation, and uncertainty quantification of diffuse interface models of tumor growth. *J. Math. Biol.* **67** (2013), 1457-1485.
- [32] A. Hawkins-Daruud, K. G. van der Zee and J. T. Oden, Numerical simulation of a thermodynamically consistent four-species tumor growth model, *Int. J. Numer. Math. Biomed. Engng.* **28** (2011), 3-24.
- [33] D. Hilhorst, J. Kampmann, T. N. Nguyen and K. G. van der Zee, Formal asymptotic limit of a diffuse-interface tumor-growth model, *Math. Models Methods Appl. Sci.* **25** (2015), 1011-1043.
- [34] S. Kurima, Asymptotic analysis for Cahn-Hilliard type phase field systems related to tumor growth in general domains, *Math. Methods in the Appl. Sci.* (2019), <https://doi.org/10.1002/mma.5520>.
- [35] J. L. Lions, Contrôle optimal de systèmes gouvernés par des equations aux dérivées partielles, Dunod, Paris, 1968.
- [36] A. Miranville, The Cahn-Hilliard equation and some of its variants, *AIMS Mathematics*, **2** (2017), 479-544.
- [37] A. Miranville, E. Rocca, and G. Schimperna, On the long time behavior of a tumor growth model, *Commun. Pure Appl. Anal.* **8** (2009) 881-912.
- [38] A. Miranville, S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains, in *Handbook of Differential Equations: Evolutionary Equations*, Vol. IV (eds. C.M. Dafermos and M. Pokorný), Elsevier/North-Holland, 103200, 2008.
- [39] A. Signori, Optimal treatment for a phase field system of Cahn-Hilliard type modeling tumor growth by asymptotic scheme. *Preprint: arXiv:1902.01079 [math.AP]* (2019), 1-28.

- [40] A. Signori, Optimality conditions for an extended tumor growth model with double obstacle potential via deep quench approach. *Preprint: arXiv:1811.08626 [math.AP]* (2018), 1-25.
- [41] A. Signori, Optimal distributed control of an extended model of tumor growth with logarithmic potential. *Appl. Math. Optim.* (2018), <https://doi.org/10.1007/s00245-018-9538-1>.
- [42] J. Simon, Compact sets in the space  $L^p(0, T; B)$ , *Ann. Mat. Pura Appl.* **146** (4) (1987) 65-96.
- [43] J. Sprekels and H. Wu, Optimal Distributed Control of a Cahn-Hilliard-Darcy System with Mass Sources, *Appl. Math. Optim.* (2019), <https://doi.org/10.1007/s00245-019-09555-4>.
- [44] F. Tröltzsch, Optimal Control of Partial Differential Equations. Theory, Methods and Applications, *Grad. Stud. in Math.*, Vol. **112**, AMS, Providence, RI, 2010.
- [45] S.M. Wise, J.S. Lowengrub, H.B. Frieboes, V. Cristini, Three-dimensional multispecies nonlinear tumor growthI: model and numerical method. *J. Theor. Biol.* **253**(3) (2008), 524-543.
- [46] X. Wu, G.J. van Zwieten and K.G. van der Zee, Stabilized second-order splitting schemes for Cahn-Hilliard models with applications to diffuse-interface tumor-growth models, *Int. J. Numer. Meth. Biomed. Engng.* **30** (2014), 180-203.