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Blow-up and global existence for the inhomogeneous porous medium equation with reaction

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Dedicated to the memory of Maria Assunta Pozio

Abstract. *We study finite time blow-up and global existence of solutions to the Cauchy problem for the porous medium equation with a variable density $\rho(x)$ and a power-like reaction term. We firstly consider the case that $\rho(x)$ decays at infinity like the critical case $|x|^{-2}$ divided by a positive power of the logarithm of $|x|$ and we show that for small enough initial data, solutions globally exist for any $p>1$. On the other hand, when $\rho(x)$ decays at infinity like the critical case $|x|^{-2}$ multiplied by a positive power of the logarithm of $|x|$, if the initial datum is small enough, then one has global existence of the solution for any $p>m$, while if the initial datum is large enough, then the blow-up of the solutions occurs for any $p>m$. Such results generalize those established in [27] and [28], where it is supposed that $\rho(x)$ decays at infinity like a power of $|x|$, without logarithmic terms.*

1. Introduction

We are concerned with global existence and blow-up of nonnegative solutions to the Cauchy parabolic problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}\,, \end{cases} \tag{1.1}$$

where $m>1$, $p>1$, $N\geq 3$, $\tau>0$. Furthermore, we always assume that

$$\begin{cases} \text{(i) } u_0 \in L^\infty(\mathbb{R}^N),\ u_0 \geq 0 \text{ in } \mathbb{R}^N\,; \\ \text{(ii) } \rho \in C(\mathbb{R}^N),\ \rho > 0 \text{ in } \mathbb{R}^N\,; \end{cases} \tag{1.2}$$

the function $\rho = \rho(x)$ is usually referred to as a *variable density*.

The differential equation in problem (1.1), posed in $(-1,1)$ with homogeneous Dirichlet boundary conditions, has been introduced in [21] as a mathematical model of a thermal evolution of a heated plasma.

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We refer the reader to [27, Introduction], [28, Introduction] for a comprehensive account of the literature concerning various problems related to (1.1). Here we limit ourselves to recall only some contribution of that literature. Problem (1.1) without the reaction term has been widely examined, e.g., in [3, 4, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20, 29, 30, 31, 33, 34]. Furthermore, global existence and blow-up of solutions of problem (1.1) with $m = 1$ and $\rho \equiv 1$ have been studied, e.g., in [5, 12]). If

$$p \leq 1 + \frac{2}{N},$$

then finite time blow-up occurs, for all nontrivial nonnegative data, whereas, for

$$p > 1 + \frac{2}{N},$$

global existence prevails for sufficiently small initial conditions. In addition, in [22] (see also [2]), problem (1.1) with $m = 1$ has been considered.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [13, 23, 24, 25, 32, 36] (see also [7] and [26] for the case of Riemannian manifolds); moreover, in [11] the same problem on Cartan-Hadamard manifolds has been investigated.

Global existence and blow-up of solutions for problem (1.1) with ρ satisfying

$$\frac{1}{k_1|x|^q} \leq \rho(x) \leq \frac{1}{k_2|x|^q} \quad \text{for all} \quad |x| > 1 \tag{1.3}$$

have been investigated in [27] for $q \in [0, 2)$, and in [28] for $q \geq 2$. In [27], for $q \in [0, 2)$, the following results have been established.

- ([27, Theorem 2.1]) If $p > \overline{p}$, for a certain $\overline{p} = \overline{p}(k_1, k_2, q, m, N) > m$ and the initial datum is sufficiently small, then solutions exist globally in time. Observe that
$$\overline{p} = m + \frac{2-q}{N-q} \text{ when } k_1 = k_2.$$
- ([27, Theorem 2.4]) For any $p > 1$, for all sufficiently large initial data, solutions blow-up in finite time.
- ([27, Theorem 2.6]) For $1 < p < m$, for any non trivial initial data, solutions blow-up in finite time.
- ([27, Theorem 2.7]) If $m < p < \underline{p}$, for a certain $\underline{p} = \underline{p}(k_1, k_2, q, m, N) \leq \overline{p}$, then, for any non trivial initial data, solutions blow-up in finite time, under specific extra assumptions on ρ.

Such results extend those stated in [35] for problem (1.1) with $\rho \equiv 1$, $m > 1$, $p > 1$ (see also [6]).

Furthermore, assume that (1.3) holds with $q \geq 2$. In [28] the following results have been showed.

- ([28, Theorem 2.1]) If $q = 2$ and $p > m$, then, for sufficiently small initial data, solutions exist globally in time.
- ([28, Theorem 2.2]) If $q = 2$ and $p > m$, then, for sufficiently large initial data, solutions blow-up in finite time.
- ([28, Theorem 2.3]) If $q > 2$, then, for any $p > 1$, for sufficiently small initial data, solutions exist globally in time.

Finally, in [7], (1.1) is addressed, when $p < m$. It is assumed that (1.2) is satisfied, and that the weighted Poincaré inequality with weight ρ holds. Moreover, in view of the assumption on ρ also the weighted Sobolev inequality is fulfilled. By using such functional inequalities, it is showed that global existence for L^m data occurs, as well as a smoothing effect for the solution, i.e. solutions corresponding to such data are bounded for any positive time. In addition, a quantitative bound on the L^∞ norm of the solution is given.

In what follows, we always consider two types of density functions ρ. To be more specific, we always make one of the following two assumptions:

$$\begin{gathered}\text{there exist } k \in (0, +\infty) \text{ and } \alpha > 1 \text{ such that} \\ \frac{1}{\rho(x)} \geq k\,(\log|x|)^{\alpha}\,|x|^2 \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,;\end{gathered} \tag{H_1}$$

$$\begin{gathered}\text{there exist } k_1, k_2 \in (0, +\infty) \text{ with } k_1 \leq k_2 \text{ and } \alpha > 1 \text{ such that} \\ k_1 \frac{|x|^2}{(\log|x|)^{\alpha}} \leq \frac{1}{\rho(x)} \leq k_2 \frac{|x|^2}{(\log|x|)^{\alpha}} \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,.\end{gathered} \tag{H_2}$$

Assume (H_1). For $1 < p < m$ and for suitable initial data $u_0 \in L^\infty(\mathbb{R}^N)$, we show the existence of global solutions belonging to $L^\infty(\mathbb{R}^N \times (0, \tau))$ for each $\tau > 0$. Indeed, in this case, the global existence follows from the results in [7] for $u_0 \in L^m_\rho(\mathbb{R}^N)$. However, now we consider a different class of initial data u_0. In fact, $u_0 \in L^\infty(\mathbb{R}^N)$ and satisfies a decaying condition as $|x| \to +\infty$; however, u_0 not necessarily belongs to $L^m_\rho(\mathbb{R}^N)$.

On the other hand, for $p > m > 1$, if u_0 satisfies a suitable decaying condition as $|x| \to +\infty$, then problem (1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$.

Now, assume (H_2). For any $p > m$, if u_0 is sufficiently large, then the solutions to problem (1.1) blow-up in finite time. Moreover, if $p > m$, u_0 has compact support and is small enough, then, under suitable assumptions on k_1 and k_2, there exist global in time solutions to problem (1.1), which belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$.

The proofs mainly relies on suitable comparison principles and properly constructed sub- and supersolutions, which crucially depend on the behavior at infinity of the density function $\rho(x)$. More precisely, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{(\log(|x| + r_0))^q}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{1.4}$$

for any $(x,t) \in \left[\mathbb{R}^N \setminus B_e(0)\right] \times [0,T)$, for suitable functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0, r_0 > 0$ and $q > 1$. The paper is organized as follows. In Section 2 we state our main results, in Section 3 we give the precise definitions of solutions and we recall some auxiliary results. In Section 4 we prove Theorem 2.1. The blow-up result (that is, Theorem 2.2) is proved in Section 5. Finally, in Section 6 Theorem 2.3 is proved .

2. Statements of the main results

For any $x_0 \in \mathbb{R}^N$ and $R > 0$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^N : \|x - x_0\| < R\}. \tag{2.1}$$

When $x_0 = 0$, we write $B_R \equiv B_R(0)$.

2.1. Density ρ satisfying (H_1)

The first result concerns the global existence of solutions to problem (1.1) for any $p > 1$ and $m > 1$, $p \neq m$. We introduce the parameter $b \in \mathbb{R}$ such that

$$0 < b < \alpha - 1. \tag{2.2}$$

Moreover, since $N \geq 3$, we can choose $\varepsilon > 0$ so that

$$N - 2 - \varepsilon(b+1) > 0, \tag{2.3}$$

and $r_0 > e$ so that

$$\frac{1}{\log(|x| + r_0)} < \varepsilon \quad \text{for any } x \in \mathbb{R}^N. \tag{2.4}$$

Finally, we can find $\bar{c} > 0$ such that

$$[\log(|x| + r_0)]^{-\frac{\bar{b}p}{m}} \leq \bar{c} \quad \text{for any } x \in \mathbb{R}^N\,. \tag{2.5}$$

Observe that, thanks to (1.2)-(i) and (H_1), we can say that there exists $k_0 > 0$ such that

$$\frac{1}{\rho(x)} \geq k_0 \left[\log(|x| + r_0)\right]^{\alpha} (|x| + r_0)^2 \quad \text{for any} \;\; x \in \mathbb{R}^N\,. \tag{2.6}$$

Theorem 2.1. *Let assumptions (1.2), (H_1) be satisfied. Suppose that*

$$p > 1, \quad p \neq m,$$

and that u_0 is small enough. Then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0,\tau))$ for any $\tau > 0$. More precisely, we have the following cases.

(a) *Let $1<p<m$. If $C>0$ is big enough, $T>1$, $\beta>0$,*

$$u_0(x) \le CT^{\beta}\left(\log(|x|+r_0)\right)^{-\frac{b}{m}} \quad \text{for any } x\in\mathbb{R}^N\,, \tag{2.7}$$

with b,ε,r_0 as in (2.2), (2.3) and (2.4), then problem (1.1) admits a global solution u, which satisfies the bound from above

$$u(x,t) \le C(T+t)^{\beta}\left(\log(|x|+r_0)\right)^{-\frac{b}{m}} \text{ for any } (x,t)\in\mathbb{R}^N\times(0,+\infty)\,. \tag{2.8}$$

(b) *Let $p>m>1$. If $C>0$ is small enough, $T>0$ and (2.7) holds with $\beta=0$, then problem (1.1) admits a global solution $u\in L^\infty(\mathbb{R}^N\times(0,+\infty))$, which satisfies the bound from above (2.8) with $\beta=0$.*

2.2. Density ρ satisfying (H_2)

The next result concerns the blow-up of solutions in finite time, for every $p>m>1$, provided that the initial datum is sufficiently large. We assume that hypotheses (1.2) and (H_2) hold. In view of (1.2)-(i), there exist $\rho_1,\rho_2\in(0,+\infty)$ with $\rho_1\le\rho_2$ such that

$$\rho_1 \le \frac{1}{\rho(x)} \le \rho_2 \quad \text{for all } x\in\overline{B_e(0)}. \tag{2.9}$$

Let

$$\underline{b} := \alpha+1, \tag{2.10}$$

and

$$\mathfrak{s}(x) := \begin{cases} (\log|x|)^{\underline{b}} & \text{if } \ x\in\mathbb{R}^N\setminus B_e, \\ \dfrac{\underline{b}\,|x|^2}{2e^2}+1-\dfrac{\underline{b}}{2} & \text{if } \ x\in B_e\,. \end{cases} \tag{2.11}$$

Theorem 2.2. *Let assumptions (1.2), (H_2). Let*

$$p>m,$$

$T>0$; suppose that, for $C>0$ and $a>0$ large enough, the initial datum satisfies

$$u_0(x) \ge CT^{-\frac{1}{p-1}}\left[1-\frac{\mathfrak{s}(x)}{a}\,T^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \text{for any } x\in\mathbb{R}^N\,, \tag{2.12}$$

with $\underline{b}$ and $\mathfrak{s}(x)$ as in (2.10) and (2.11), then there exists $S\in(0,T]$ such that the solution u of problem (1.1) blows-up at time S, in the sense that

$$\|u(t)\|_\infty \to \infty \text{ as } t\to S^-\,. \tag{2.13}$$

Moreover, the solution u satisfies the bound from below

$$u(x,t) \ge C(T-t)^{-\frac{1}{p-1}}\left[1-\frac{\mathfrak{s}(x)}{a}\,(T-t)^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} \text{ for any } (x,t)\in\mathbb{R}^N\times(0,S)\,. \tag{2.14}$$

Observe that if u_0 satisfies (2.12), then

$$\operatorname{supp} u_0 \supseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < aT^{\frac{p-m}{p-1}}\}\,.$$

From (2.14) we can infer that

$$\operatorname{supp} u(\cdot, t) \supseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < a(T-t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t \in [0, S)\,. \tag{2.15}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 5.2.

The next result concerns the global existence of solutions to problem (1.1) for $p > m$. We assume that ρ satisfies a stronger condition than (H_2). Indeed, we suppose that

$$k_1 \frac{(|x| + r_0)^2}{(\log(|x| + r_0))^\alpha} \leq \frac{1}{\rho(x)} \leq k_2 \frac{(|x| + r_0)^2}{(\log(|x| + r_0))^\alpha} \quad \text{for all } x \in \mathbb{R}^N\,, \tag{2.16}$$

where

$$r_0 > e, \quad \frac{k_2}{k_1} < m + (N-3)\left(\frac{m-1}{\bar{b}}\right)\,, \tag{2.17}$$

and

$$\bar{b} := \alpha + 2. \tag{2.18}$$

Theorem 2.3. *Assume (1.2), (2.16), (2.17). Suppose that*

$$p > m\,,$$

and that u_0 is small enough and has compact support. Then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.

More precisely, if $C > 0$ is small enough, $a > 0$ is so that

$$0 < \omega_0 \leq \frac{C^{m-1}}{a} \leq \omega_1$$

for suitable $0 < \omega_0 < \omega_1$, $T > 0$,

$$u_0(x) \leq CT^{-\frac{1}{p-1}} \left[1 - \frac{(\log(|x| + r_0))^{\bar{b}}}{a} T^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \textit{for any } x \in \mathbb{R}^N\,, \tag{2.19}$$

with $\bar{b}$ as in (2.18), then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$. Moreover,

$$u(x,t) \leq C(T+t)^{-\frac{1}{p-1}} \left[1 - \frac{(\log(|x| + r_0))^{\bar{b}}}{a} (T+t)^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \tag{2.20}$$

for any $(x,t) \in \mathbb{R}^N \times (0, +\infty)$.

Observe that if u_0 satisfies (2.19), then

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N \,:\, (\log(|x|+r_0))^{\overline{b}} \leq aT^{\frac{p-m}{p-1}}\}\,.$$

From (2.20) we can infer that

$$\operatorname{supp} u(\cdot,t) \subseteq \{x \in \mathbb{R}^N \,:\, (\log(|x|+r_0))^{\overline{b}} \leq a(T+t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t>0\,. \tag{2.21}$$

The choice of the parameters $C>0, T>0$ and $a>0$ is discussed in Remark 6.2.

3. Preliminaries

In this section we give the precise definitions of solutions of all problems we address. Moreover, we recall some auxiliary results. The proofs can be found in [27, Section 3].

Throughout the paper we deal with *very weak* solutions to problem (1.1) and to the same problem set in different domains, according to the following definitions.

Definition 3.1. Let $u_0 \in L^\infty(\mathbb{R}^N)$ with $u_0 \geq 0$. Let $\tau>0$, $p>1, m>1$. We say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0,S))$ for any $S<\tau$ is a solution of problem (1.1) if

$$\begin{aligned} -\int_{\mathbb{R}^N}\int_0^\tau \rho(x) u \varphi_t \,dt\,dx = & \int_{\mathbb{R}^N} \rho(x) u_0(x)\varphi(x,0)\,dx \\ & + \int_{\mathbb{R}^N}\int_0^\tau u^m \Delta\varphi\,dt\,dx \\ & + \int_{\mathbb{R}^N}\int_0^\tau \rho(x) u^p \varphi\,dt\,dx \end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0,\tau)), \varphi \geq 0$. Moreover, we say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0,S))$ for any $S<\tau$ is a subsolution (supersolution) if it satisfies (3.1) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ " with $\varphi \geq 0$.

Proposition 3.2. *Let hypotheses (1.2) be satisfied. Then there exists a solution u to problem (1.1) with*

$$\tau \geq \tau_0 := \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}\,.$$

Moreover, u is the minimal solution*, in the sense that for any solution v to problem (1.1) there holds*

$$u \leq v \quad \text{in} \quad \mathbb{R}^N \times (0,\tau)\,.$$

We state the following two comparison results, which will be used in the sequel.

Proposition 3.3. *Let hypothesis (1.2) be satisfied. Let $\bar u$ be a supersolution to problem (1.1). Then, if u is the minimal solution to problem (1.1) given by Proposition 3.2, then*

$$u \leq \bar u \quad a.e. \ in \ \mathbb{R}^N \times (0,\tau)\,.$$

In particular, if $\bar u$ exists until time τ, then also u exists at least until time τ.

Proposition 3.4. *Let hypothesis (1.2) be satisfied. Let u be a solution to problem (1.1) for some time $\tau = \tau_1 > 0$ and $\underline{u}$ a subsolution to problem (1.1) for some time $\tau = \tau_2 > 0$. Suppose also that*

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N\times[0,S]} \ \textit{is compact for every} \ S \in (0,\tau_2)\,.$$

Then

$$u \geq \underline{u} \quad in \ \ \mathbb{R}^N \times (0, \min\{\tau_1,\tau_2\})\,.$$

In what follows we also consider solutions of equations of the form

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in} \ \ \Omega\times(0,\tau), \tag{3.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open subset. Solutions are meant in the following sense.

Definition 3.5. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\Omega\times(0,S))$ for any $S < \tau$ is a solution of equation (3.1) if

$$\begin{aligned} -\int_\Omega\int_0^\tau \rho(x) u\,\varphi_t\,dt\,dx = &\int_\Omega\int_0^\tau u^m\Delta\varphi\,dt\,dx \\ &+ \int_\Omega\int_0^\tau \rho(x)u^p\varphi\,dt\,dx \end{aligned} \tag{3.2}$$

for any $\varphi \in C_c^\infty(\overline{\Omega}\times[0,\tau))$ with $\varphi|_{\partial\Omega} = 0$ for all $t\in[0,\tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(\Omega\times(0,S))$ for any $S<\tau$ is a subsolution (supersolution) if it satisfies (3.2) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ ", with $\varphi \geq 0$.

Finally, let us recall the following well-known criterion, that will be used in the sequel. Let $\Omega\subseteq\mathbb{R}^N$ be an open set. Suppose that $\Omega = \Omega_1\cup\Omega_2$ with $\Omega_1\cap\Omega_2=\emptyset$, and that $\Sigma := \partial\Omega_1\cap\partial\Omega_2$ is of class C^1.

Let n be the unit outwards normal to Ω_1 at Σ. Let

$$u = \begin{cases} u_1 & \text{in} \ \ \Omega_1\times[0,T), \\ u_2 & \text{in} \ \ \Omega_2\times[0,T)\,, \end{cases} \tag{3.3}$$

where $\partial_t u \in C(\Omega_1\times(0,T)), u_1^m \in C^2(\Omega_1\times(0,T))\cap C^1(\overline{\Omega}_1\times(0,T)), \partial_t u_2 \in C(\Omega_2\times(0,T)), u_2^m \in C^2(\Omega_2\times(0,T))\cap C^1(\overline{\Omega}_2\times(0,T))$.

Lemma 3.6. *Let assumption (1.2) be satisfied.*

(i) Suppose that

$$\begin{aligned} \partial_t u_1 &\geq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \textit{for any } (x,t) \in \Omega_1 \times (0,T), \\ \partial_t u_2 &\geq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \textit{for any } (x,t) \in \Omega_2 \times (0,T), \end{aligned} \tag{3.4}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \geq \frac{\partial u_2^m}{\partial n} \quad \textit{for any } (x,t) \in \Sigma \times (0,T)\,. \tag{3.5}$$

Then u, defined in (3.3), is a supersolution to equation (3.1), in the sense of Definition 3.5.

(ii) Suppose that

$$\begin{aligned} \partial_t u_1 &\leq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \textit{for any } (x,t) \in \Omega_1 \times (0,T), \\ \partial_t u_2 &\leq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \textit{for any } (x,t) \in \Omega_2 \times (0,T), \end{aligned}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \leq \frac{\partial u_2^m}{\partial n} \quad \textit{for any } (x,t) \in \Sigma \times (0,T)\,.$$

Then u, defined in (3.3), is a subsolution to equation (3.1), in the sense of Definition 3.5.

4. Proof of Theorem 2.1

In what follows we set $r \equiv |x|$. We assume (1.2), (H_1), (2.2) and (2.3). We want to construct a suitable family of supersolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{ in } \mathbb{R}^N \times (0,+\infty). \tag{4.1}$$

In order to do this, we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar{u}(x,t) \equiv \bar{u}(r(x),t) := C\zeta(t)\,(\log(r+r_0))^{-\frac{b}{m}}\,; \tag{4.2}$$

where $\zeta \in C^1([0,+\infty);[0,+\infty))$, $C>0$ and $r_0 > e$ such that (2.4) is verified.

Proposition 4.1. *Let $\zeta \in C^1([0,+\infty);[0,+\infty)), \zeta' \geq 0$. Assume (1.2), ($H_1$), (2.2), (2.3), (2.4), (2.5), (2.6) and that*

$$k_0 b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\,C^p\zeta^p \geq 0. \tag{4.3}$$

Then $\bar{u}$ defined in (4.2) is a supersolution of equation (4.1).

Proof of Proposition 4.1. In view of (2.3) and (2.4), for any $(x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)$,

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\
&\geq C\zeta' \left(\log(r+r_0)\right)^{-\frac{b}{m}} + \frac{1}{\rho}\left\{N-2-\varepsilon(b+1)\right\} C^m \zeta^m b \frac{\left(\log(r+r_0)\right)^{-b-1}}{(r+r_0)^2} \\
&- C^p\zeta^p \left(\log(r+r_0)\right)^{-\frac{bp}{m}} .
\end{aligned} \tag{4.4}$$

Thanks to hypotheses (2.2), (2.5) and (2.6), we have

$$\begin{aligned}
&\frac{1}{\rho}\frac{\left(\log(r+r_0)\right)^{-\bar{b}-1}}{(r+r_0)^2} \geq k_0 \,\frac{\left(\log(r+r_0)\right)^{\alpha-\bar{b}-1}}{(r+r_0)^2}(r+r_0)^2 \geq k_0 \,, \\
&-\left(\log(r+r_0)\right)^{-\frac{bp}{m}} \geq -\bar{c} \,.
\end{aligned} \tag{4.5}$$

Since $\zeta' \geq 0$, from (4.5) we get

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq k_0 \, b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\, C^p\zeta^p \,. \tag{4.6}$$

Hence (4.6) is nonnegative if

$$k_0 \, b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\, C^p\zeta^p \geq 0 \,, \tag{4.7}$$

which is guaranteed by (2.3) and (4.3). So, we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty) \,.$$

Now observe that

$$\begin{aligned}
&\bar{u} \in C(\mathbb{R}^N \times [0,+\infty)) \,, \\
&\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0,+\infty)) \,, \\
&\bar{u}^m_r(0,t) \leq 0 \,.
\end{aligned}$$

Hence, thanks to a Kato-type inequality we can infer that $\bar{u}$ is a supersolution to equation (4.1) in the sense of Definition 3.5. □

Remark 4.2. Let assumption (H_1) be satisfied. In Theorem 2.1 the precise hypotheses on parameters β, $C > 0$, $T > 0$ are as follows.

(a) Let $p < m$. We require that

$$\beta > 0, \tag{4.8}$$

$$k_0 \, b(N-2-\varepsilon(b+1))C^m - \bar{c}\, C^p \geq 0 \,. \tag{4.9}$$

(b) Let $p > m$. We require that

$$\beta = 0, \tag{4.10}$$

$$k_0 \, b(N-2-\varepsilon(b+1))C^m - \bar{c}\, C^p \geq 0 \,. \tag{4.11}$$

Lemma 4.3. *All the conditions in Remark 4.2 can hold simultaneously.*

Proof. (a) We observe that, due to (2.3),

$$N - 2 - \varepsilon(b+1) > 0.$$

Therefore, we can select $C > 0$ sufficiently large to guarantee (4.9).
(b) We choose $C > 0$ sufficiently small to guarantee (4.11). □

Proof of Theorem 2.1. We now prove Theorem 2.1 in view of Proposition 4.1. In view of Lemma 4.3 we can assume that all conditions in Remark 4.2 are fulfilled. Set

$$\zeta(t) = (T+t)^{\beta}, \quad \text{for all} \quad t \geq 0\,.$$

Let $p < m$. Inequality (4.3) reads

$$k_0\, b(N-2-\varepsilon(b+1))C^m(T+t)^{m\beta} - \bar{c}\, C^p(T+t)^{p\beta} \geq 0 \quad \text{for all} \quad t > 0\,.$$

This follows from (4.8) and (4.9), for $T > 1$. Hence, by Propositions 4.1 and 3.2 the thesis follows in this case.

Let $p > m$. Conditions (4.10) and (4.11) are equivalent to (4.3). Hence, by Propositions 4.1 and 3.2 the thesis follows in this case too. The proof is complete. □

5. Proof of Theorem 2.2

We construct a suitable family of subsolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,T). \tag{5.1}$$

We assume (1.2) and (H_2). Let

$$\underline{w}(x,t) \equiv \underline{w}(r(x),t) := \begin{cases} \underline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_e(0)] \times [0,T), \\ \underline{v}(x,t) & \text{in } B_e(0) \times [0,T), \end{cases} \tag{5.2}$$

where

$$\underline{u}(x,t) \equiv \underline{u}(r(x),t) := C\zeta(t)\left[1 - \frac{(\log r)^{\frac{b}{2}}}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \tag{5.3}$$

and

$$\underline{v}(x,t) \equiv \underline{v}(r(x),t) := C\zeta(t)\left[1 - \left(\frac{br^2}{2e^2} + 1 - \frac{b}{2}\right)\frac{\eta}{a}\right]_+^{\frac{1}{m-1}}. \tag{5.4}$$

Let

$$F(r,t) := 1 - \frac{(\log r)^{\frac{b}{2}}}{a}\eta(t)\,,$$

and

$$G(r,t) := 1 - \left(\frac{br^2}{2e^2} + 1 - \frac{b}{2}\right)\frac{\eta}{a}\,.$$

Observe that for any $(x,t) \in [\mathbb{R}^N \setminus B_e(0)] \times (0,T)$, we have:

$$\underline{u}_t = C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}. \tag{5.5}$$

$$\begin{aligned}\Delta(\underline{u}^m) = {} & \frac{C^m}{a}\zeta^m\eta\,\frac{m}{(m-1)^2}\,\underline{b}^2\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1} \\ & - \frac{C^m}{a}\zeta^m\eta\left(\frac{m}{m-1}\right)^2\underline{b}^2\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}} \\ & + \frac{C^m}{a}\zeta^m\eta\,\frac{m}{m-1}\,\underline{b}\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}} \\ & - \frac{C^m}{a}\zeta^m\eta\,\frac{m}{m-1}\,\underline{b}\,\frac{(\log r)^{\underline{b}-1}}{r^2}F^{\frac{1}{m-1}}(N-2)\end{aligned} \tag{5.6}$$

Observe that for any $(x,t) \in B_e(0) \times (0,T)$, we have:

$$\underline{v}_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}, \tag{5.7}$$

$$\Delta(\underline{v}^m) = \frac{C^m}{a^2}\zeta^m\frac{m}{(m-1)^2}\frac{\underline{b}^2\,r^2}{e^4}\eta^2 G^{\frac{1}{m-1}-1} - N\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}\frac{\underline{b}}{e^2}\eta G^{\frac{1}{m-1}}\,. \tag{5.8}$$

We also define

$$\begin{aligned} &\underline{\sigma}(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,k_2\left(\underline{b}\frac{m}{m-1} + N - 2\right), \\ &\underline{\delta}(t) := \frac{\zeta}{m-1}\frac{\eta'}{\eta} \\ &\underline{\gamma}(t) := C^{p-1}\zeta^p, \\ &\underline{\sigma}_0(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \rho_2\,N\,\frac{\underline{b}}{e^2}\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,, \\ &K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0. \end{aligned} \tag{5.9}$$

Proposition 5.1. *Let $T \in (0,\infty)$, ζ, $\eta \in C^1([0,T);[0,+\infty))$. Let $\underline{\sigma}, \underline{\delta}, \underline{\gamma}, \underline{\sigma}_0, K$ be defined in (5.9). Assume that, for all $t \in (0,T)$,*

$$\underline{\sigma}(t) > 0, \quad K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}\,, \tag{5.10}$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{5.11}$$

$$\underline{\sigma}_0(t) > 0, \quad K[\underline{\sigma}_0](t)^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}, \tag{5.12}$$

$$(m-1)\underline{\sigma}_0(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{5.13}$$

Then $\underline{w}$ defined in (5.2) is a subsolution of equation (5.1).

Proof of Proposition 5.1. In view of (5.5) and (5.6) we obtain

$$
\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&\quad = C\zeta' F^{\frac{1}{m-1}} + C\frac{\zeta}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\frac{\zeta}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\
&\qquad - \frac{1}{\rho}\left\{\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}\underline{b}^2\,\eta\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1}\right. \\
&\qquad + \frac{C^m}{a}\zeta^m\left(\frac{m}{m-1}\right)^2\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}} - \frac{C^m}{a}\zeta^m\frac{m}{m-1}\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}} \\
&\qquad \left.+\frac{C^m}{a}\zeta^m\frac{m}{m-1}\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-1}}{r^2}F^{\frac{1}{m-1}}\,(N-2)\right\} \\
&\qquad - C^p\zeta^p F^{\frac{p}{m-1}}, \quad \text{for all } (x,t)\in D_1\,.
\end{aligned}
\tag{5.14}
$$

In view of (H_2) and (2.10), we can infer that

$$
-\frac{1}{\rho}\frac{(\log r)^{\underline{b}-2}}{r^2} \le -\frac{k_1}{\log r} \le -k_1, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,, \tag{5.15}
$$

$$
\frac{1}{\rho}\frac{(\log r)^{\underline{b}-2}}{r^2} \le \frac{k_2}{\log r} \le k_2, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,, \tag{5.16}
$$

$$
\frac{1}{\rho}\frac{(\log r)^{\underline{b}-1}}{r^2} \le k_2, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,. \tag{5.17}
$$

From (5.14), (5.15), (5.16) and (5.17) we have

$$
\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&\le CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\underline{b}\,\eta k_2\left(N-2+\underline{b}\frac{m}{m-1}\right)\right]\right. \\
&\qquad \left. -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}.
\end{aligned}
\tag{5.18}
$$

Thanks to (5.9), (5.18) becomes

$$
\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \le CF^{\frac{1}{m-1}-1}\varphi(F),
$$

where, for each $t\in(0,T)$,

$$
\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}.
$$

Our goal is to find suitable C, a, ζ, η such that, for each $t\in(0,T)$,

$$
\varphi(F) \le 0 \quad \text{for any } F\in(0,1)\,.
$$

To this aim, we impose that

$$\sup_{F\in(0,1)} \varphi(F) = \max_{F\in(0,1)} \varphi(F) = \varphi(F_0) \leq 0\,,$$

for some $F_0 \in (0,1)$. We have

$$\begin{aligned}\frac{d\varphi}{dF} = 0 \iff & \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0 \\ \iff & F = F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{m-1}{p-1}}\,.\end{aligned}$$

Then

$$\varphi(F_0) = K\,\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t)\,,$$

where the coefficient K depending on m and p has been defined in (5.9). By (5.10) and (5.11), for each $t \in (0,T)$,

$$\varphi(F_0) \leq 0\,, \quad F_0 \leq 1\,. \tag{5.19}$$

So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \leq 0 \quad \text{in } \; D_1. \tag{5.20}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_e(0)] \times (0,T))$, due to Lemma 3.6 (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_e(0) \cup D_1], u_1 = \underline{u}, u_2 = 0, u = \underline{u}$), it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in } \; [\mathbb{R}^N \setminus B_e(0)] \times (0,T),$$

in the sense of Definition 3.5.

Let

$$D_2 := \{(x,t) \in B_e(0) \times (0,T) \,:\, 0 < G(r,t) < 1\}\,.$$

Using (2.9), (5.1) yields, for all $(x,t) \in D_2$,

$$\begin{aligned}&\underline{v}_t - \frac{1}{\rho}\Delta(\underline{v}^m) - \underline{v}^p \\ &\quad\leq CG^{\frac{1}{m-1}-1}\Big\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + N\,\rho_2\,\frac{b}{e^2}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\right] \\ &\qquad - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\} \\ &\quad= CG^{\frac{1}{m-1}-1}\left[\underline{\sigma}_0(t)G - \underline{\delta}(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right]\,.\end{aligned} \tag{5.21}$$

Now, by the same arguments used to obtain (5.20), in view of (5.13) and (5.14) we can infer that

$$\underline{v}_t - \frac{1}{\rho}\Delta \underline{v}^m \leq \underline{v}^p \quad \text{for any } (x,t) \in D_2\,. \tag{5.22}$$

Moreover, since $\underline{v}^m \in C^1(B_e(0) \times (0,T))$, in view of Lemma 3.6 (applied with $\Omega_1 = D_2, \Omega_2 = B_e(0) \setminus D_2, u_1 = \underline{v}, u_2 = 0, u = \underline{v}$), we get that $\underline{v}$ is a subsolution to equation

$$\underline{v}_t - \frac{1}{\rho}\Delta \underline{v}^m = \underline{v}^p \quad \text{in } B_e(0) \times (0,T)\,, \tag{5.23}$$

in the sense of Definition 3.5. Now, observe that $\underline{w} \in C(\mathbb{R}^N \times [0,T))$; indeed,

$$\underline{u} = \underline{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e(0) \times (0,T)\,.$$

Moreover, $\underline{w}^m \in C^1(\mathbb{R}^N \times [0,T))$; indeed,

$$(\underline{u}^m)_r = (\underline{v}^m)_r = -C^m\zeta(t)^m \frac{m}{m-1}\,\frac{\eta(t)}{a}\,\frac{\underline{b}}{e}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e(0) \times (0,T)\,. \tag{5.24}$$

In conclusion, in view of (5.24) and Lemma 3.6 (applied with $\Omega_1 = B_e(0), \Omega_2 = \mathbb{R}^N \setminus B_e(0), u_1 = \underline{v}, u_2 = \underline{u}, u = \underline{w}$), we can infer that $\underline{w}$ is a subsolution to equation (5.1), in the sense of Definition 3.5. □

Remark 5.2. Let $p > m$ and assumptions (H_2) and (2.9) be satisfied. Let define $\omega := C^{m-1}/a$. In Theorem 2.2, the precise hypotheses on parameters $C > 0$, $a > 0$, $\omega > 0$ and $T > 0$ are the following.

$$\begin{aligned}\max\left\{1 + m\,k_2\,\underline{b}\,\frac{C^{m-1}}{a}\left(N - 2 + \underline{b}\frac{m}{m-1}\right); 1 + m\rho_2\frac{C^{m-1}}{a}\,\underline{b}\,\frac{N}{e^2}\right\} \\ \leq (p+m-2)C^{p-1}\,,\end{aligned} \tag{5.25}$$

$$\begin{aligned}\frac{K}{(m-1)^{\frac{p+m-2}{p-1}}}\,\max\Bigg\{\left[1 + m\,k_2\underline{b}\,\frac{C^{m-1}}{a}\left(N - 2 + \underline{b}\frac{m}{m-1}\right)\right]^{\frac{p+m-2}{p-1}};\\ \left(1 + m\,\rho_2\,\frac{C^{m-1}}{a}\,\underline{b}\,\frac{N}{e^2}\right)^{\frac{p+m-2}{p-1}}\Bigg\} \leq \frac{p-m}{(m-1)(p-1)}C^{m-1}\,.\end{aligned} \tag{5.26}$$

Lemma 5.3. *All the conditions in Remark 5.2 can hold simultaneously.*

Proof. We can take $\omega > 0$ such that

$$\omega_0 \leq \omega \leq \omega_1$$

for suitable $0 < \omega_0 < \omega_1$ and we can choose $C > 0$ sufficiently large to guarantee (5.25) and (5.26) (so, $a > 0$ is fixed, too). □

Proof of Theorem 2.2. We now prove Theorem 2.2, by means of Proposition 5.1. In view of Lemma 5.3 we can assume that all conditions of Remark 5.2 are fulfilled. Set

$$\zeta = (T-t)^{-\beta}\,, \quad \eta = (T-t)^{\lambda}\,, \quad \text{for all} \quad t>0\,,$$
$$\beta = \frac{1}{p-1}\,, \qquad \lambda = \frac{m-p}{p-1}\,.$$

Then

$$\begin{aligned}
\underline{\sigma}(t) &:= \left[\frac{1}{m-1} + \frac{C^{m-1}}{a}\,\frac{m}{m-1}\underline{b}\,k_2\left(\underline{b}\,\frac{m}{m-1}+N-2\right)\right](T-t)^{-\frac{p}{p-1}}\,,\\
\underline{\delta}(t) &:= \frac{p-m}{(m-1)(p-1)}(T-t)^{-\frac{p}{p-1}}\,,\\
\underline{\gamma}(t) &:= C^{p-1}(T-t)^{-\frac{p}{p-1}}\,,\\
\underline{\sigma_0}(t) &:= \frac{1}{m-1}\left[1+\frac{\rho_2\,N\,m\,\underline{b}}{e^2}\,\frac{C^{m-1}}{a}\right](T-t)^{-\frac{p}{p-1}}\,.
\end{aligned} \tag{5.27}$$

Let $p>m$. Condition (5.25) implies (5.11), (5.13), while condition (5.26) implies (5.10), (5.12). Hence by Propositions 5.1 and 3.4 the thesis follows. □

6. Proof of Theorem 2.3

We assume (1.2), (2.16) and (2.17). In order to construct a suitable family of supersolutions of (4.1), we define, for all $(x,t)\in\mathbb{R}^N\times(0,+\infty)$,

$$\bar u(x,t) \equiv \bar u(r(x),t) := C\zeta(t)\left[1-\frac{(\log(r+r_0))^{\bar b}}{a}\eta(t)\right]_+^{\frac{1}{m-1}}\,, \tag{6.1}$$

where $\eta,\,\zeta\in C^1([0,+\infty);[0,+\infty))$, $C>0$, $a>0$, $r_0>e$ and $\bar b$ as in (2.18).

Now, we compute

$$\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p.$$

To this aim, set

$$F(r,t) := 1-\frac{(\log(r+r_0))^{\bar b}}{a}\,\eta(t)\,,$$

and

$$D_1 := \left\{(x,t)\in[\mathbb{R}^N\setminus\{0\}]\times(0,+\infty)\mid 0<F(r,t)<1\right\}.$$

For any $(x,t)\in D_1$, we have:

$$\begin{aligned}
\bar u_t &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{(\log(r+r_0))^{\bar b}}{a}\eta'\right)\\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}.
\end{aligned} \tag{6.2}$$

$$\begin{aligned}\Delta(\bar u^m) = {} & \frac{(N-1)}{r}\left(-\bar b\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)}\eta\right)\\ & -\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\left[\bar b\frac{m}{m-1}-1\right]\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\ & +\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\ & +\bar b^2\frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}-1}.\end{aligned}\tag{6.3}$$

We also define

$$\begin{aligned}&\bar\sigma(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \bar b\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_1\left(\bar b\frac{m}{m-1}+N-3\right),\\ &\bar\delta(t) := \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \bar b^2\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{(m-1)^2}\eta k_2\,,\\ &\bar\gamma(t) := C^{p-1}\zeta^p\,.\end{aligned}\tag{6.4}$$

Proposition 6.1. *Let $\zeta,\eta\in C^1([0,+\infty);[0,+\infty))$. Let $\bar\sigma$, $\bar\delta$, $\bar\gamma$ be as defined in (6.4). Assume (H_2), (2.16), (2.17), (2.18) and that, for all $t\in(0,+\infty)$,*

$$-\frac{\eta'}{\eta^2} \geq \bar b^2\,\frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}k_2,\tag{6.5}$$

and

$$\begin{aligned}\zeta' + \bar b\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[\left(\bar b\frac{m}{m-1}+N-3\right)k_1 - \frac{\bar b}{(m-1)}k_2\right]&\\ - C^{p-1}\zeta^p \geq 0\,.&\end{aligned}\tag{6.6}$$

Then $\bar u$ defined in (6.1) is a supersolution of equation (4.1).

Proof of Proposition 6.1. In view of (6.2) and (6.3), for any $(x,t)\in D_1$,

$$\begin{aligned}&\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p\\ &\geq C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}\\ &\quad + \frac{1}{\rho}(N-2)\,\bar b\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}\eta\\ &\quad + \frac{1}{\rho}\,\bar b\,\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\left[\bar b\frac{m}{m-1}-1\right]\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\ &\quad - \frac{1}{\rho}\,\bar b^2\frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}-1} - C^p\zeta^pF^{\frac{p}{m-1}},\end{aligned}\tag{6.7}$$

where we have used the inequality

$$\frac{1}{r(r+r_0)} \geq \frac{1}{(r+r_0)^2}\,.$$

Thanks to (2.16) and (2.18), we have

$$\frac{1}{\rho}\,\frac{(\log(r+r_0))^{\bar{b}-2}}{(r+r_0)^2} \geq k_1 \quad \text{for all } \; x\in\mathbb{R}^N\,, \tag{6.8}$$

$$-\,\frac{1}{\rho}\,\frac{(\log(r+r_0))^{\bar{b}-2}}{(r+r_0)^2} \geq -k_2 \quad \text{for all } \; x\in\mathbb{R}^N\,, \tag{6.9}$$

$$\frac{1}{\rho}\,\frac{(\log(r+r_0))^{\bar{b}-1}}{(r+r_0)^2} \geq k_1\log(r+r_0) \geq k_1 \quad \text{for all } \; x\in\mathbb{R}^N\,. \tag{6.10}$$

From (6.8), (6.9) and (6.10) we get

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\
&\geq CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \bar{b}\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\, k_1\left(\bar{b}\frac{m}{m-1}+N-3\right)\right]\right.\\
&\quad\left. -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - \bar{b}^2\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{(m-1)^2}\eta\, k_2 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}
\end{aligned} \tag{6.11}$$

From (6.11) and (6.4), we have

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq CF^{\frac{1}{m-1}-1}\left[\bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}\right]\,. \tag{6.12}$$

For each $t>0$, set

$$\varphi(F) := \bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}\,, \quad F\in(0,1)\,.$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t>0$,

$$\varphi(F)\geq 0 \quad \text{for any } \; F\in(0,1)\,.$$

We observe that $\varphi(F)$ is concave in the variable F. Hence it is sufficient to have that $\varphi(F)$ is positive at the extrema of the interval $(0,1)$. This reduces, for any $t>0$, to the conditions

$$\begin{aligned}\varphi(0)&\geq 0\,,\\ \varphi(1)&\geq 0\,.\end{aligned}$$

These are equivalent to

$$-\bar{\delta}(t)\geq 0\,, \quad \bar{\sigma}(t)-\bar{\delta}(t)-\bar{\gamma}(t)\geq 0\,,$$

that is

$$-\frac{\eta'}{\eta^2} \geq \bar{b}^2 \frac{C^{m-1}}{a} \zeta^{m-1} \frac{m}{m-1} k_2 \,,$$
$$\zeta' + \bar{b}\,\frac{C^{m-1}}{a} \zeta^m \frac{m}{m-1} \eta \left[\left(\bar{b}\frac{m}{m-1} + N - 3 \right) k_1 - \frac{\bar{b}}{(m-1)} k_2 \right] - C^{p-1}\zeta^p \geq 0 \,.$$

which are guaranteed by (2.17), (6.5) and (6.6). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in } D_1 \,.$$

Now observe that

$\bar{u} \in C(\mathbb{R}^N \times [0, +\infty))$,

$\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0, +\infty))$, and by the definition of $\bar{u}$,

$\bar{u} \equiv 0$ in $[\mathbb{R}^N \setminus D_1] \times [0, +\infty))$.

Hence, by Lemma 3.6 (applied with $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N \setminus D_1$, $u_1 = \bar{u}$, $u_2 = 0$, $u = \bar{u}$), $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty)$$

in the sense of Definition 3.5. Thanks to a Kato-type inequality, since $\bar{u}^m_r(0, t) \leq 0$, we can easily infer that $\bar{u}$ is a supersolution of equation (4.1) in the sense of Definition 3.5. □

Remark 6.2. Let $p > m$ and assumption (2.17) be satisfied. Let $\omega := C^{m-1}/a$. In Theorem 2.3 the precise hypotheses on parameters $C > 0$, $\omega > 0$, $T > 0$ are the following:

$$\frac{p-m}{p-1} \geq \bar{b}^2\,\omega \frac{m}{m-1} k_2, \tag{6.13}$$

$$\bar{b}\,\omega \frac{m}{m-1} \left[k_1 \left(\bar{b}\frac{m}{m-1} + N - 3 \right) - \frac{k_2}{(m-1)} \bar{b} \right] \geq C^{p-1} + \frac{1}{p-1} \,. \tag{6.14}$$

Lemma 6.3. *All the conditions in Remark 6.2 can be satisfied simultaneously.*

Proof. Since $p > m$ the left-hand-side of (6.13) is positive. By (2.17), we can select $\omega > 0$ so that (6.13) holds and

$$\bar{b}\,\omega \frac{m}{m-1} \left[k_1 \left(\bar{b}\frac{m}{m-1} + N - 3 \right) - \frac{k_2}{(m-1)} \bar{b} \right] \geq \frac{1}{p-1} \,.$$

Then we take $C > 0$ so small that (6.14) holds (and so $a > 0$ is accordingly fixed). □

Proof of Theorem 2.3. In view of Lemma 6.3, we can assume that all the conditions in Remark 6.2 are fulfilled. Set

$$\zeta(t) = (T+t)^{-\frac{1}{p-1}}, \quad \text{for all} \quad t \geq 0\,,$$

and

$$\eta(t) = (T+t)^{-\frac{p-m}{p-1}}, \quad \text{for all} \quad t \geq 0\,.$$

Let $p > m$. Consider conditions (6.5) and (6.6) with this choice of ζ and η. They read

$$\frac{p-m}{p-1} \geq \bar{b}^2 \frac{C^{m-1}}{a} \frac{m}{m-1} k_2,$$

$$-\frac{1}{p-1} + \bar{b}\,\frac{C^{m-1}}{a} \frac{m}{m-1} \left[\left(\bar{b} \frac{m}{m-1} + N - 3 \right) k_1 - \frac{\bar{b}}{(m-1)} k_2 \right] - C^{p-1} \geq 0\,.$$

Therefore, (6.5) and (6.6) follow from assumptions (6.13) and (6.14). Hence, by Propositions 6.1 and 3.2 the thesis follows. □

References

[1] Andreucci D., Tedeev A.F.: Universal Bounds at the Blow-Up Time for Nonlinear Parabolic Equations, Adv. Differ. Equations **10**, 89–120 (2005)

[2] de Pablo A., Reyes G., Sanchez A.: The Cauchy problem for a nonhomogeneous heat equation with reaction, Discr. Cont. Dyn. Sist. A **33**, 643–662 (2013)

[3] Eidus D.: The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium, J. Differential Equations **84**, 309–318 (2013)

[4] Eidus D., Kamin S.: The filtration equation in a class of functions decreasing at infinity, Proc. Amer. Math. Soc. **120**, 825–830 (1994)

[5] Fujita H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13**, 109–124 (1966)

[6] Galaktionov V.A., Vázquez J.L.: Continuation of blowup solutions of nonlinear heat equations in several dimensions, Comm. Pure Appl. Math. **50**, 1–67 (1997)

[7] Grillo G., Meglioli G., Punzo F.: Smoothing effects and infinite time blow-up for reaction-diffusion equations: approach via Sobolev and Poincaré inequalities, submitted (2020)

[8] Grillo G., Muratori M., Porzio M.M.: Porous media equations with two weights: existence, uniqueness, smoothing and decay properties of energy solutions via Poincaré inequalities, Discrete Contin. Dyn. Syst. A **33**, 3599–3640 (2013)

[9] Grillo G., Muratori M., Punzo F.: On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density, Discr. Cont. Dyn. Syst. A **35**, 5927–5962 (2015)

[10] Grillo G., Muratori M., Punzo F.: Fractional porous media equations: existence and uniqueness of weak solutions with measure data, Calc. Var. Part. Diff. Eq. **54**, 3303–3335 (2015)

[11] Grillo G., Muratori M., Punzo F.: Blow-up and global existence for the porous medium equation with reaction on a class of Cartan–Hadamard manifolds, J. Diff. Eq. **266**, 4305–4336 (2019)

[12] Hayakawa K.: On nonexistence of global solutions of some semilinear parabolic differential equations, Proc. Japan Acad. **49**, 503–505 (1973)

[13] Iagar R.G., Sanchez A.: Blow up profiles for a quasilinear reaction-diffusion equation with weighted reaction with linear growth, J. Dynam. Differential Equations **31**, 2061–2094 (2019)

[14] Kamin S., Kersner R., Tesei A.: On the Cauchy problem for a class of parabolic equations with variable density, Atti Accad. Naz. Lincei, Rend. Mat. Appl. **9**, 279–298 (1998)

[15] Kamin S., Pozio M.A., Tesei A.: Admissible conditions for parabolic equations degenerating at infinity, St. Petersburg Math. J. **19**, 239–251 (2008)

[16] Kamin S., Punzo F.: Prescribed conditions at infinity for parabolic equations, Comm. Cont. Math. **17**, 1–19 (2015)

[17] Kamin S., Punzo F.: Dirichlet conditions at infinity for parabolic and elliptic equations, Nonlin. Anal. **138**, 156–175 (2016)

[18] Kamin S., Reyes G., Vázquez J.L.: Long time behavior for the inhomogeneous PME in a medium with rapidly decaying density, Discrete Contin. Dyn. Syst. A **26**, 521–549 (2010)

[19] Kamin S., Rosenau P.: Propagation of thermal waves in an inhomogeneous medium, Comm. Pure Appl. Math. **34**, 831–852 (1981)

[20] Kamin S., Rosenau P.: Nonlinear diffusion in a finite mass medium, Comm. Pure Appl. Math. **35**, 113–127 (1982)

[21] Kamin S., Rosenau P.: Nonlinear thermal evolution in an inhomogeneous medium, J. Math. Physics **23**, 1385-1390 (1982)

[22] Lie X., Hiang Z.: Existence and nonexistence of local/global solutions for a nonhomogeneous heat equation, Commun. Pure Appl. Anal. **13** 1465–1480 (2014)

[23] Martynenko A.V., Tedeev A. F.: On the behavior of solutions of the Cauchy problem for a degenerate parabolic equation with nonhomogeneous density and a source, (Russian) Zh. Vychisl. Mat. Mat. Fiz. **48**, no. 7, 1214-1229 (2008); transl. in Comput. Math. Math. Phys. **48**, no. 7, 1145-1160 (2008)

[24] Martynenko A.V., Tedeev A.F., Shramenko V.N.: The Cauchy problem for a degenerate parabolic equation with inhomogenous density and a source in the class of slowly vanishing initial functions, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **76**, no. 3, 139-156 (2012); transl. in Izv. Math. **76**, no. 3, 563-580 (2012)

[25] Martynenko A.V., Tedeev A.F., Shramenko V.N.: On the behavior of solutions of the Cauchy problem for a degenerate parabolic equation with source in the case where the initial function slowly vanishes, Ukr. Math. J. **64**, 1698–1715 (2013)

[26] Mastrolia P., Monticelli D.D., Punzo F.: Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds, Math. Ann. **367**, 929–963 (2017)

[27] Meglioli G., Punzo F.: Blow-up and global existence for solutions to the porous medium equation with reaction and slowly decaying density, J. Diff. Eq. **269**, 8918–8958 (2020)

[28] Meglioli G., Punzo F.: Blow-up and global existence for solutions to the porous medium equation with reaction and fast decaying density, submitted (2020)

[29] Pozio M.A., Tesei A.: On the uniqueness of bounded solutions to singular parabolic problems, Discr. Cont. Din. Syst. A **13**, 117–137 (2005)

[30] Pozio M.A., Punzo F., Tesei A.: Criteria for well-posedness of degenerate elliptic and parabolic problems, J. Math. Pures Appl. **90**, 353–386 (2008)

[31] Pozio M.A., Punzo F., Tesei A.: Uniqueness and nonuniqueness of solutions to parabolic problems with singular coefficients, Discr. Cont. Din. Syst. A **30**, 891–916 (2011)

[32] Pohozaev S.I., Tesei A.: Blow-up of nonnegative solutions to quasilinear parabolic inequalities, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl. **11**, 99–109 (2000)

[33] Punzo F.: On the Cauchy problem for nonlinear parabolic equations with variable density, J. Evol. Equ. **9**, 429–447 (2009)

[34] Reyes G., Vázquez J.L.: The Cauchy problem for the inhomogeneous porous medium equation, Netw. Heterog. Media **1**, 337–351 (2006)

[35] Samarskii A.A., Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P.: Blow-up in Quasilinear Parabolic Equations, De Gruyter Expositions in Mathematics **19**, Walter de Gruyter & Co., Berlin (1995)

[36] Wang C., Zheng S.: Critical Fujita exponents of degenerate and singular parabolic equations, Proc. Roy. Soc. Edinburgh Sect. A **136**, 415–430 (2006)

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