

ON A GENERALIZATION OF SYMMETRIC EDGE POLYTOPES TO REGULAR MATROIDS

ALESSIO D'ALÌ, MARTINA JUHNKE-KUBITZKE, AND MELISSA KOCH

ABSTRACT. Starting from any finite simple graph, one can build a reflexive polytope known as a symmetric edge polytope. The first goal of this paper is to show that symmetric edge polytopes are intrinsically matroidal objects: more precisely, we prove that two symmetric edge polytopes are unimodularly equivalent precisely when they share the same graphical matroid. The second goal is to show that one can construct a generalized symmetric edge polytope starting from every *regular* matroid. Just like in the usual case, we are able to find combinatorial ways to describe the facets and an explicit regular unimodular triangulation of any such polytope. Finally, we show that the Ehrhart theory of the polar of a given generalized symmetric edge polytope is tightly linked to the structure of the lattice of flows of the dual regular matroid.

1. INTRODUCTION

Symmetric edge polytopes are a class of centrally symmetric reflexive lattice polytopes which has seen a lot of interest in the last few years due to their fascinating combinatorial properties [HJM19, OT21a, OT21b, BB22, BBK22, CDK23, KT22a, DJKKV23] and their connections to various branches of mathematics and physics [CDM18, DH20, ÇJM⁺21, DDM22].

Given a finite simple graph G on vertex set $V = [n] := \{1, 2, \dots, n\}$, the *symmetric edge polytope* associated with G is the lattice polytope

$$\mathcal{P}_G := \operatorname{conv}\{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid \{i, j\} \in E(G)\} \subseteq \mathbb{R}^{|V|},$$

where \mathbf{e}_i denotes the i -th standard basis vector.

Equivalently, after assigning an arbitrary orientation to each of the edges of G , one has that

$$\mathcal{P}_G = \operatorname{conv}[M_G \mid -M_G],$$

where $M_G \in \mathbb{Z}^{|V| \times |E|}$ is the signed incidence matrix of G with respect to the chosen orientation (i.e., the matrix whose (v, e) -entry is 1 if v is the head of e , -1 if v is the tail of e , and 0 otherwise). The matrix M_G also serves as a representation of the *graphic matroid* \mathcal{M}_G associated with G . Several objects associated with \mathcal{P}_G , for instance the facets or some triangulations, can be described via the combinatorial features of the graph G , and one can rephrase many of these characterizations in terms of the matroid \mathcal{M}_G only. This is not by accident; in fact, we will prove in Theorem 4.6 that two symmetric edge polytopes \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent precisely when the graphical matroids \mathcal{M}_G and \mathcal{M}_H are isomorphic. In particular, if G and H are both 3-connected, applying Whitney's 2-isomorphism theorem yields that \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent if and only if G and H are isomorphic. We remark that the characterization in Theorem 4.6 corrects an

2020 *Mathematics Subject Classification.* Primary: 52B40; Secondary: 52B20, 05B35, 13P10.

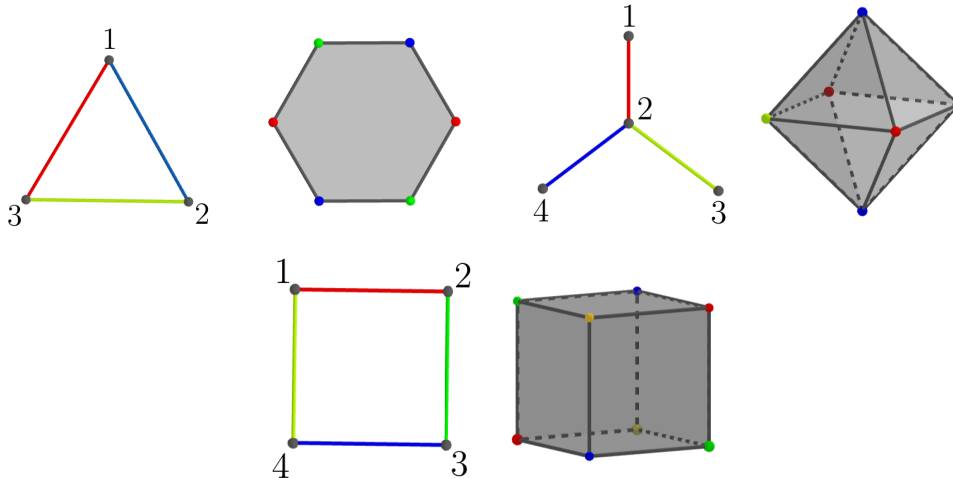


FIGURE 1. Some examples of symmetric edge polytopes.
The pictures were obtained using GeoGebra [HBA⁺18].

erroneous statement of Matsui, Higashitani, Nagazawa, Ohsugi and Hibi [MHN⁺11, Lemma 4.4, Theorem 4.5]: see Remark 4.8 for the details.

It is tempting to ask what happens if we take the polytope defined by the convex hull of the columns of $[M \mid -M]$ for a more general matrix M , and whether this object bears any relation to the matroid represented by M . The former question was investigated by Ohsugi and Hibi in [OH14], while the latter appears to be new. In general, changing the representation of a given matroid and applying the above “symmetrization” will produce wildly different polytopes, so this question might seem to be too far-fetched at first sight. However, as we show in Theorem 3.2, the construction described above does indeed yield a unique lattice polytope (up to some unimodular equivalence not involving any translation) when we consider any *regular* matroid \mathcal{M} and restrict to its (full-rank) *weakly unimodular* representations. We will call any polytope arising from such a setting a *generalized symmetric edge polytope*. Throughout the paper, if we need the concrete polytope associated with a specific representation M , we will denote it by \mathcal{P}_M ; if instead it is enough for our purposes to just deal with the equivalence class, we will write $\mathcal{P}_{\mathcal{M}}$.

Most of the properties that make the usual symmetric edge polytopes pleasant are preserved in this wider environment: for instance, generalized symmetric edge polytopes are reflexive (as already observed in [OH14]) and terminal, and it is possible to describe their facets in a purely combinatorial fashion (Theorem 5.8). We remark here that Kálmán and Tóthmérész have been working on a similar statement for extended root polytopes in the recent preprint [KT23].

The polars of generalized symmetric edge polytopes are special instances of *Lipschitz polytopes* and enjoy a rich Ehrhart theory. More precisely, in the spirit of work by Beck and Zaslavsky [BZ06], we show that the lattice points in the k -th dilation of the polar $\mathcal{P}_{\mathcal{M}}^{\Delta}$ are in bijection with the $(k+1)$ -cuts of \mathcal{M} or, equivalently, with the $(k+1)$ -flows of the dual matroid \mathcal{M}^* (Proposition 5.1).

Finally, the existence of a regular unimodular triangulation for \mathcal{P}_M had already been proved by Ohsugi and Hibi [OH14], while an explicit one had been provided in the case of graphs by Higashitani, Jochemko and Michałek [HJM19]. We show that, via a careful

analysis of signed circuits, it is possible to extend the latter result to generalized symmetric edge polytopes (Theorem 6.11).

The paper is organized as follows. Section 2 contains some preliminaries about matroids, polytopes and toric ideals, while Section 3 is devoted to define generalized symmetric edge polytopes and prove that any two full-rank weakly unimodular representations of the same regular matroid will yield unimodularly equivalent polytopes (Theorem 3.2). Section 4 studies properties of generalized symmetric edge polytopes, including a partial converse to Theorem 3.2. Section 5 focuses on the polytopes polar to generalized symmetric edge polytopes and their Ehrhart theory; the obtained results are then used to derive a facet description for generalized symmetric edge polytopes, extending the one for the graphical case from [HJM19]. Section 6 is devoted to the explicit description of a regular unimodular triangulation of any generalized symmetric edge polytope. Finally, we collect some open questions and suggestions for future work in Section 7.

2. PRELIMINARIES

2.1. Regular matroids, cuts and flows. The aim of this subsection is to briefly introduce regular matroids and their properties. We direct the reader to [Oxl11] for a more complete treatment and for general matroid terminology.

Notation 2.1. If \mathcal{M} is a matroid, we will denote by $\mathfrak{B}(\mathcal{M})$ and $\mathfrak{C}(\mathcal{M})$ the sets of its bases and circuits, respectively. If M is a matrix, we will sometimes write “basis/circuit of M ” to refer to a basis/circuit of the matroid represented by M . In this case, the ground set of such a matroid will consist of the column indices of M .

Definition 2.2. Let $M \in \mathbb{Z}^{m \times n}$ be an integer matrix. We will say that M is:

- *totally unimodular* if the determinant of every square submatrix of M lies in $\{0, \pm 1\}$;
- *weakly unimodular* if the determinant of every square submatrix of M of size $\max\{m, n\}$ lies in $\{0, \pm 1\}$.

Definition/Theorem 2.3. A matroid \mathcal{M} of rank $r > 0$ is called *regular* if it satisfies any of the following equivalent properties:

- (i) \mathcal{M} can be represented via a totally unimodular matrix;
- (ii) \mathcal{M} can be represented via a full-rank weakly unimodular matrix;
- (iii) \mathcal{M} is representable over any field.

Proof. The equivalence of (i) and (iii) is a well-known fact, see for instance [Oxl11, Theorem 6.6.3]. We will now prove for clarity’s sake that (i) and (ii) are equivalent: for a source in the literature, the reader can check [Whi87, Theorem 3.1.1].

To see that (i) implies (ii) it is enough to show that, if \mathcal{M} is represented by a totally unimodular matrix, then it is also represented by a totally unimodular (and hence weakly unimodular) matrix of the form $[I_r \mid D]$, where r is the rank of \mathcal{M} . For a proof of this claim, see [Oxl11, Lemma 2.2.21].

Let us now prove that (ii) implies (i). By assumption, there exists a full-rank matrix $M \in \mathbb{Z}^{r \times n}$ that is weakly unimodular and represents \mathcal{M} . We now proceed as in [SW10, Subsection 2.2]: after choosing a basis \mathcal{B} for \mathcal{M} , we can shuffle the columns of M so that the elements of \mathcal{B} correspond to the first r columns. This amounts to multiplying M on the right by an $(n \times n)$ -permutation matrix P , an operation preserving the weakly unimodular property. Now consider the invertible submatrix N of MP obtained by taking the first

r columns. Since MP is weakly unimodular and N is invertible, the determinant of the integer matrix N is either 1 or -1 ; in other words, $N \in \text{GL}_r(\mathbb{Z})$. By construction, one has that $N^{-1}MP = [I_r \mid D]$ represents \mathcal{M} and is weakly unimodular; however, since it contains the identity matrix I_r , it must actually be totally unimodular ([SW10, Lemma 3] or [Oxl11, Exercise 10.1.1]), as desired. \square

We illustrate the content of the previous definition with an example that will also serve as a running example throughout.

Example 2.4. Let \mathcal{M} be the rank 3 simple matroid with ground set $[5]$, bases $\mathfrak{B}(\mathcal{M}) = \{123, 124, 134, 135, 145, 234, 235, 245\}$, and circuits $\mathfrak{C}(\mathcal{M}) = \{125, 345, 1234\}$, where we are using the shorthand $i_1 i_2 \dots i_m$ for $\{i_1, i_2, \dots, i_m\}$. It is easy to check that \mathcal{M} is represented by the full-rank totally unimodular matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

and thus \mathcal{M} is regular. In fact, in this case \mathcal{M} is also graphic.

Remark 2.5. The assumption about the rank of \mathcal{M} being nonzero is not part of the usual definition of regular matroid in the literature: we include it to avoid nuisances with representability, see for instance [Oxl11, Lemma 2.2.21].

The class of regular matroids (including those of rank zero) is closed under duality and contains all graphic matroids.

We now introduce cuts and flows of a regular matroid, following Su and Wagner's treatment in [SW10]. Let $M \in \mathbb{R}^{r \times n}$ (where $0 < r \leq n$) be a full-rank weakly unimodular matrix. We define the *lattice of integer cuts* of M , denoted $\Gamma(M)$, and the *lattice of integer flows* of M , denoted $\Lambda(M)$, as

$$\begin{aligned} \Gamma(M) &:= \text{row}(M) \cap \mathbb{Z}^n, \\ \Lambda(M) &:= \ker(M) \cap \mathbb{Z}^n. \end{aligned}$$

The lattices of integer cuts and flows are orthogonal to each other with respect to the usual dot product. In particular, if $A = [I_r \mid D] \in \mathbb{R}^{r \times n}$ (with $0 < r < n$) is totally unimodular and $A^* := [-D^T \mid I_{n-r}]$, one has that

$$(1) \quad \Lambda(A) = \Gamma(A^*).$$

Definition 2.6. If $\mathbf{v} \in \mathbb{R}^n$, the *support* of \mathbf{v} , denoted by $\text{supp}(\mathbf{v})$, is the set of indices $i \in [n]$ such that $v_i \neq 0$.

Given a full-rank weakly unimodular matrix $M \in \mathbb{R}^{r \times n}$ with $r \leq n$, we will call a flow $\boldsymbol{\lambda} \in \Lambda(M)$ (respectively, a cut $\boldsymbol{\lambda} \in \Gamma(M)$)

- *nowhere-zero* if $\lambda_i \neq 0$ for every $i \in [n]$, i.e., if $\text{supp}(\boldsymbol{\lambda}) = [n]$;
- a *k-flow* (respectively, a *k-cut*) if $|\lambda_i| < k$ for every $i \in [n]$;
- a *signed circuit* or *simple flow* if it is a 2-flow and its support is a circuit of M . We denote the set of signed circuits of M by $\vec{\mathfrak{C}}(M)$.

For a regular matroid \mathcal{M} , we are able to talk about *the* lattice of integer cuts of \mathcal{M} *up to isometry*: in fact, if M and M' are two full-rank weakly unimodular matrices representing \mathcal{M} , then the elements of $\Gamma(M')$ correspond to elements of $\Gamma(M)$ via multiplication by a

signed permutation matrix (compare, e.g., [SW10, proof of Lemma 10]; their argument is stated for totally unimodular matrices, but goes through for full-rank weakly unimodular matrices as well). In particular, an element of $\Gamma(M')$ will be a nowhere-zero cut, a k -cut or a signed circuit if and only if the corresponding element of $\Gamma(M)$ is. Moreover, due to (1), all the above statements go through for flows as well.

Example 2.7. The matrix M from Example 2.4 has

- seventeen 2-cuts: $(0, 0, 0, 0, 0)$, $(1, 0, 0, -1, 1)$, $(-1, 0, 0, 1, -1)$, $(0, 1, 0, -1, 1)$, $(0, -1, 0, 1, -1)$, $(0, 0, 1, -1, 0)$, $(0, 0, -1, 1, 0)$, $(1, -1, 0, 0, 0)$, $(-1, 1, 0, 0, 0)$, $(1, 0, -1, 0, 1)$, $(-1, 0, 1, 0, -1)$, $(0, 1, -1, 0, 1)$, $(0, -1, 1, 0, -1)$, $(1, -1, 1, -1, 0)$, $(-1, 1, -1, 1, 0)$, $(1, -1, -1, 1, 0)$, $(-1, 1, 1, -1, 0)$.
- seven 2-flows: $(0, 0, 0, 0, 0)$, $(1, 1, 0, 0, -1)$, $(-1, -1, 0, 0, 1)$, $(0, 0, 1, 1, 1)$, $(0, 0, -1, -1, -1)$, $(1, 1, 1, 1, 0)$, $(-1, -1, -1, -1, 0)$.
- six signed circuits: all the 2-flows except for the origin.

We record here for further reference some useful facts:

Proposition 2.8. *Let M be a full-rank weakly unimodular matrix.*

- (i) *If $\lambda \in \Lambda(M)$ and $\text{supp}(\lambda)$ is a circuit of M , then every coordinate of λ has the same absolute value.*
- (ii) *If $\mathcal{C} \in \mathfrak{C}(M)$, then there are exactly two signed circuits (differing by a global sign) with support \mathcal{C} .*

Proof. Part (i) can be derived directly from [Stu96, Lemma 4.9] and constitutes a strengthening of [SW10, Lemma 7]. The proof of part (ii) is almost verbatim the same as the one of [SW10, Lemma 8], using part (i) instead of [SW10, Lemma 7]. \square

Definition 2.9. Let \mathcal{M} be a regular matroid with ground set E . If \mathcal{B} is a basis of \mathcal{M} and $e \in E \setminus \mathcal{B}$, then the *fundamental circuit of e with respect to \mathcal{B}* is the unique circuit $\mathcal{C}(e, \mathcal{B}) \in \mathfrak{C}(\mathcal{M})$ contained in $\mathcal{B} \cup \{e\}$. Note that $e \in \mathcal{C}(e, \mathcal{B})$.

If, moreover, M is a full-rank weakly unimodular representation of \mathcal{M} , then by Proposition 2.8 there is a unique signed circuit $\vec{\mathcal{C}}(e, \mathcal{B}) \in \Lambda(M)$ supported at $\mathcal{C}(e, \mathcal{B})$ whose e -th entry equals 1. We will call such a signed circuit the *fundamental signed circuit of e with respect to \mathcal{B} and M* .

Example 2.10. Let \mathcal{M} and M be as in Example 2.4. Then, for $\mathcal{B} = \{1, 2, 3\}$, one has that $\vec{\mathcal{C}}(4, \mathcal{B}) = (1, 1, 1, 1, 0)$ and $\vec{\mathcal{C}}(5, \mathcal{B}) = (-1, -1, 0, 0, 1)$.

Lemma 2.11. *Let $M \in \mathbb{R}^{r \times n}$ (where $0 < r \leq n$) be a full-rank weakly unimodular matrix and assume that the first r columns of M are linearly independent. Then, for any $a_1, \dots, a_r \in \mathbb{Z}$, there exist unique $a_{r+1}, \dots, a_n \in \mathbb{Z}$ such that $(a_1, \dots, a_n) \in \Gamma(M)$. In other words, there exists a unique cut $\gamma \in \Gamma(M)$ having a_1, \dots, a_r as its first r entries.*

Proof. Call \mathcal{M} the regular matroid represented by M .

If $r = n$, then $\Gamma(M) = \mathbb{Z}^n$ and the claim is true. Assume now that $r < n$.

To prove existence, define $\gamma \in \mathbb{Z}^n$ in the following way:

- $\gamma_i = a_i$ for every $i \in [r]$;
- for every $j \in \{r+1, \dots, n\}$, we determine γ_j by imposing that $\gamma \cdot \vec{\mathcal{C}}(j, [r]) = 0$, where $\vec{\mathcal{C}}(j, [r])$ is the fundamental signed circuit of j with respect to the basis $[r]$ of \mathcal{M} and the representation M .

To prove that the integer vector $\gamma \in \mathbb{Z}^n$ we have just defined is indeed a cut, it is enough to show that $\gamma \in \text{row}(M)$; but since $\text{row}(M)$ and $\ker(M)$ are orthogonal with respect to the standard dot product, this amounts to proving that $\gamma \cdot \mathbf{v} = 0$ for every $\mathbf{v} \in \ker(M)$. Since the fundamental signed circuits of M with respect to $[r]$ form an \mathbb{R} -basis of $\ker(M)$ (being $n - r$ many linearly independent vectors by construction), the claim follows.

To prove uniqueness, assume there is another cut γ' such that $\gamma'_i = a_i$ for every $i \in [r]$, and consider $\beta := \gamma' - \gamma \in \Gamma(M)$. By assumption, $\beta_i = 0$ for every $i \in [r]$. Since $\text{row}(M) = \ker(M)^\perp$, it follows that $\beta \cdot \lambda = 0$ for every $\lambda \in \ker(M)$. In particular, for every $j \in \{r + 1, \dots, n\}$, one has that $\beta \cdot \vec{\mathcal{C}}(j, [r]) = 0$, and thus $\beta_j = 0$. This proves that $\gamma' = \gamma$. \square

Given a matroid \mathcal{M} , denote by \mathcal{M}° the matroid obtained from \mathcal{M} by deleting all its loops. Su and Wagner proved in [SW10] that knowing the lattice of integer cuts of \mathcal{M} is enough to determine \mathcal{M}° up to isomorphism. In particular, if we know beforehand that \mathcal{M} is loopless (for instance, if \mathcal{M} is simple), we can reconstruct completely \mathcal{M} from the data of its lattice of integer cuts. This idea will serve as a blueprint for the constructions in this paper.

2.2. Polarity. Let $P \subseteq \mathbb{R}^d$ be a full-dimensional lattice polytope with $\mathbf{0} \in \mathring{P} \cap \mathbb{Z}^d$ (here \mathring{P} denotes the interior of P with respect to the Euclidean topology). We recall that the *polar* of P is the polytope

$$P^\Delta := \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{x} \leq 1 \text{ for every } \mathbf{x} \in P\},$$

where we are using the usual dot product to identify \mathbb{R}^d and its dual $(\mathbb{R}^d)^*$.

The polar P^Δ will not be a lattice polytope in general. If P^Δ happens to be a lattice polytope, then P is called *reflexive*.

For the rest of this subsection we fix a full-dimensional reflexive polytope $P \subseteq \mathbb{R}^d$ with $\mathring{P} \cap \mathbb{Z}^d = \{\mathbf{0}\}$.

Notation/Remark 2.12. If $\mathbf{u} \in P^\Delta \cap \mathbb{Z}^d$, we denote by $F_{\mathbf{u}}$ the face of P obtained as $\text{conv}\{\mathbf{v}_i \mid \mathbf{v}_i \cdot \mathbf{u} = 1\}$, where the \mathbf{v}_i 's are the vertices of P . Indeed, the polytope P lies entirely inside one of the halfspaces defined by the hyperplane $H_{\mathbf{u}} := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{x} = 1\}$.

By polarity, facets of the polytope P correspond to the vertices of the polar polytope P^Δ ; in particular, any facet of P will be of the form $F_{\mathbf{u}}$ for some $\mathbf{u} \in P^\Delta \cap \mathbb{Z}^d$, and such a \mathbf{u} will be a vertex of P^Δ .

2.3. Toric ideals. We introduce some basic notation about toric ideals. For the concepts not explained here and to get further insight, see for instance [Stu96, Chapters 4 and 8].

Definition 2.13. If $M \in \mathbb{Z}^{r \times n}$ is an integer matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, we will denote by I_M the *toric ideal* associated with M , i.e. the kernel of the map

$$\begin{aligned} \pi: K[x_1, \dots, x_n] &\rightarrow K[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \\ x_i &\mapsto \mathbf{t}^{\mathbf{v}_i} := t_1^{m_{1,i}} t_2^{m_{2,i}} \dots t_r^{m_{r,i}}, \end{aligned}$$

where K is a field. Every $\lambda \in \mathbb{Z}^n$ can be uniquely written as $\lambda^+ - \lambda^-$, where λ^+ and λ^- are in \mathbb{N}^n and have disjoint supports. Any column vector $\lambda \in \ker(M) \cap \mathbb{Z}^n$ gives rise to a binomial $\mathbf{x}^{\lambda^+} - \mathbf{x}^{\lambda^-} \in \ker(\pi)$, and the ideal I_M is generated by binomials of this form. In what follows, with a slight abuse of notation, we will use the expression ‘‘signed circuit’’ to denote both an element $\lambda \in \{0, \pm 1\}^n$ as in Definition 2.6 and the associated binomial $\mathbf{x}^{\lambda^+} - \mathbf{x}^{\lambda^-}$ in $\ker(\pi)$.

Remark 2.14. When M is full-rank weakly unimodular, the toric ideal I_M is remarkably well-behaved: in fact, the set $\vec{\mathcal{C}}(M)$ of signed circuits is a universal Gröbner basis for I_M

(and hence, in particular, the signed circuits of M generate I_M). Actually, an even stronger result is true, as $\vec{\mathfrak{C}}(M)$ turns out to be the *Graver basis* of I_M [Stu96, Theorem 8.11]. (In fact, since two signed circuits only differing by a global sign give rise to the same binomial up to sign, one usually picks a representative for every pair; in particular, the Graver basis of I_M will have cardinality $|\mathfrak{C}(M)| = \frac{1}{2}|\vec{\mathfrak{C}}(M)|$.)

Example 2.15. Let M be as in Example 2.4. The enumeration of signed circuits in Example 2.7 shows that the polynomials $x_1x_2 - x_5$, $x_3x_4x_5 - 1$ and $x_1x_2x_3x_4 - 1$ are the Graver basis (and a universal Gröbner basis) of the toric ideal I_M .

Throughout the paper, when we say that a certain polynomial inside a polynomial ring is homogeneous, we are using the standard grading: i.e., each variable has degree 1. We record here for further reference a useful observation.

Lemma 2.16. *Let $B \in \mathbb{Z}^{m \times n}$ and let $B' \in \mathbb{Z}^{(m+1) \times (n+1)}$ be the matrix defined via*

$$b'_{ij} := \begin{cases} b_{ij} & \text{if } i \leq m \text{ and } j \leq n \\ 0 & \text{if } i \leq m \text{ and } j = n + 1 \\ 1 & \text{if } i = m + 1 \end{cases}.$$

Let $I_B \subseteq K[x_1, \dots, x_n]$ and $I_{B'} \subseteq K[x_1, \dots, x_n, z]$ be the respective toric ideals (here x_i corresponds to the i -th column and z to the $(n+1)$ -st, when available). Then $I_{B'} = I_B^{\text{hom}}$, where the homogenization is taken with respect to the variable z .

Proof. Let us first prove that $I_{B'} \subseteq I_B^{\text{hom}}$. The toric ideal $I_{B'}$ is generated by the set of its primitive binomials, i.e. its Graver basis. Let f be a primitive binomial of $I_{B'}$. Due to primitivity, the variable z can appear at most on one side of the binomial; without loss of generality, we can hence write $f = \mathbf{x}^{\lambda^+} - \mathbf{x}^{\lambda^-} z^k$, where $\lambda = \lambda^+ - \lambda^- \in \mathbb{Z}^n$, $k \geq 0$ and $(\lambda, k) \in \ker(B')$. By construction, $\lambda \in \ker(B)$ and f is homogeneous; more precisely, f is the homogenization of a binomial in I_B with respect to the variable z . It follows that $I_{B'} \subseteq I_B^{\text{hom}}$.

Let us now prove that $I_B^{\text{hom}} \subseteq I_{B'}$. By [CLO15, Theorem 8.4.4], in order to find a generating set for I_B^{hom} it is enough to homogenize a set of polynomials forming a Gröbner basis of I_B with respect to a graded monomial order. Primitive polynomials provide such a set: in fact, the Graver basis of I_B contains the universal Gröbner basis of I_B [Stu96, Lemma 4.6]. Let $g = \mathbf{x}^{\lambda^+} - \mathbf{x}^{\lambda^-}$ be a primitive binomial in I_B . We can assume without loss of generality that $k := |\lambda^+| - |\lambda^-| \geq 0$. By construction, the homogenized polynomial $\mathbf{x}^{\lambda^+} - \mathbf{x}^{\lambda^-} z^k$ lies in $I_{B'}$. This shows that $I_B^{\text{hom}} \subseteq I_{B'}$. \square

3. UNIQUENESS UP TO UNIMODULAR EQUIVALENCE

The main aim of this section is to describe how to extend the definition of a symmetric edge polytope from the context of graphs to that of regular matroids. Let us first fix some notation:

Notation 3.1. For any integer matrix $M \in \mathbb{Z}^{r \times n}$ with $0 < r \leq n$, we denote by \mathcal{P}_M the lattice polytope of \mathbb{R}^r obtained as $\text{conv}[M \mid -M]$.

For $F \in \text{GL}_r(\mathbb{Z})$, we denote by $\psi_F : \mathbb{R}^r \rightarrow \mathbb{R}^r$ the affine map sending \mathbf{x} to $F\mathbf{x}$.

It is our goal to show that any two full-rank weakly unimodular representations of a regular matroid \mathcal{M} produce the same lattice polytope (in the sense specified in Notation 3.1) up to

unimodular equivalence, and the same holds for the polytopes obtained via polarity. More precisely, we show the following:

Theorem 3.2. *Let \mathcal{M} be a regular matroid of rank $r > 0$ on n elements and let $M_1, M_2 \in \mathbb{R}^{r \times n}$ be two full-rank weakly unimodular representations of \mathcal{M} . Then there exists $F \in \text{GL}_r(\mathbb{Z})$ such that*

$$\mathcal{P}_{M_2} = \psi_F(\mathcal{P}_{M_1}) \quad \text{and} \quad \mathcal{P}_{M_2}^\Delta = \psi_{(F^T)^{-1}}(\mathcal{P}_{M_1}^\Delta).$$

We will show how to handle the case when the matrices representing \mathcal{M} are not full-rank in Remark 3.4. Moreover, a partial converse to Theorem 3.2 will be proved later, see Theorem 4.5.

Proof. Pick two weakly unimodular full-rank $(r \times n)$ -matrices M_1 and M_2 both representing \mathcal{M} . For each $i \in \{1, 2\}$, write $\mathcal{P}_i := \mathcal{P}_{M_i}$. Multiplying M_i on the right by a (signed) permutation matrix has no effect on the polytope \mathcal{P}_i : permuting the columns just permutes the list L of points we are taking the convex hull of, and changing the sign of a column is harmless because the list L consists of the columns of both M_i and $-M_i$. After some permutation of the columns of M_1 and M_2 , we can hence assume without loss of generality the following two statements:

- the identity map $[n] \rightarrow [n]$ yields an isomorphism between the matroids represented by M_1 and M_2 ;
- the submatrices N_1 and N_2 obtained by selecting the first r columns of respectively M_1 and M_2 are both invertible.

Proceeding as in the proof of “(ii) implies (i)” in Definition/Theorem 2.3, we can now multiply each M_i on the left by $N_i^{-1} \in \text{GL}_r(\mathbb{Z})$, obtaining the totally unimodular matrix $[I_r | D_i]$. Since the identity map still yields an isomorphism between the matroids represented by $[I_r | D_1]$ and $[I_r | D_2]$, we can apply [Oxl11, Proposition 6.4.1] to get that D_1 and D_2 are congruent modulo 2, and hence so are $[I_r | D_1]$ and $[I_r | D_2]$. Since $[I_r | D_1]$ and $[I_r | D_2]$ are both totally unimodular, we are now in the position to use Camion’s signing lemma [Oxl11, Lemma 10.1.7]: i.e., we can obtain the matrix $[I_r | D_2]$ by changing the signs of some rows and columns of $[I_r | D_1]$. In other words, there exist diagonal matrices $R \in \text{GL}_r(\mathbb{Z})$ and $C \in \text{GL}_n(\mathbb{Z})$ with only 1’s and -1 ’s on the diagonal and such that $[I_r | D_2] = R \cdot [I_r | D_1] \cdot C$.

Now let $F := N_2 \cdot R \cdot N_1^{-1} \in \text{GL}_r(\mathbb{Z})$. It follows from the discussion above that $\mathcal{P}_2 = \psi_F(\mathcal{P}_1)$, as desired (note that C , being a signed permutation matrix, does not enter the picture).

The polar statement can now be derived like this:

$$\begin{aligned} \mathcal{P}_2^\Delta &= \{\mathbf{u} \in \mathbb{R}^r \mid \mathbf{u} \cdot \mathbf{x} \leq 1 \text{ for every } \mathbf{x} \in \mathcal{P}_2\} \\ &= \{\mathbf{u} \in \mathbb{R}^r \mid \mathbf{u} \cdot \mathbf{x} \leq 1 \text{ for every } \mathbf{x} \in \psi_F(\mathcal{P}_1)\} \\ &= \{\mathbf{u} \in \mathbb{R}^r \mid \mathbf{u} \cdot F\mathbf{y} \leq 1 \text{ for every } \mathbf{y} \in \mathcal{P}_1\} \\ &= \{\mathbf{u} \in \mathbb{R}^r \mid F^T \mathbf{u} \cdot \mathbf{y} \leq 1 \text{ for every } \mathbf{y} \in \mathcal{P}_1\} \\ &= \{(F^T)^{-1} \mathbf{v} \in \mathbb{R}^r \mid \mathbf{v} \cdot \mathbf{y} \leq 1 \text{ for every } \mathbf{y} \in \mathcal{P}_1\} \\ &= \psi_{(F^T)^{-1}}(\mathcal{P}_1^\Delta). \end{aligned}$$

□

Example 3.3. Let \mathcal{M} be the uniform matroid $U_{2,3}$. The two full-rank totally unimodular matrices

$$M_1 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } M_2 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

both represent \mathcal{M} . Changing the signs of both the second row and the second column of M_1 yields M_2 ; in formulas,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that the two polytopes \mathcal{P}_1 and \mathcal{P}_2 are unimodularly equivalent, as guaranteed by Theorem 3.2.

The usual symmetric edge polytope associated with a graph G is defined as \mathcal{P}_{A_G} , where A_G is any signed incidence matrix associated with G . The matrix A_G provides a totally unimodular representation of the graphic matroid \mathcal{M}_G , but is *not* full-rank. However, this is not really an issue, as we now explain.

Remark 3.4. Let \mathcal{M} be a regular matroid of rank $r > 0$ and let $M \in \mathbb{Z}^{m \times n}$ be a totally unimodular representation of \mathcal{M} with $m > r$. Possibly after permuting the columns of M , we can assume without loss of generality that the first r columns of M are linearly independent. Pivoting repeatedly we can then reach a matrix

$$M' := \begin{bmatrix} I_r & D \\ \mathbf{0}_{m-r,r} & \mathbf{0}_{m-r,n-r} \end{bmatrix},$$

which will again be totally unimodular by [Oxl11, Lemma 2.2.20].

The two polytopes \mathcal{P}_M and $\mathcal{P}_{M'}$ are unimodularly equivalent, and projecting onto the first r coordinates shows that $\mathcal{P}_{M'}$ is in turn unimodularly equivalent to $\mathcal{P}_{M''}$, where $M'' := [I_r \mid D]$ is a full-rank totally unimodular representation of \mathcal{M} .

Example 3.5. Let $G = C_3$ be the cycle graph on three vertices and pick

$$A_G := \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then \mathcal{P}_{A_G} is the symmetric edge polytope \mathcal{P}_G . Successive row operations on A_G yield that

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so \mathcal{P}_G is unimodularly equivalent to the full-dimensional polytope $\mathcal{P}_{M_2} \subseteq \mathbb{R}^2$, with M_2 as in Example 3.3.

Remark 3.6. Selecting a directed spanning tree inside a connected finite simple graph G on $r + 1$ vertices and n edges yields an explicit full-rank totally unimodular representation for \mathcal{M}_G in the following way (compare [Oxl11, Section 5.1, p. 137]):

- fix an orientation for each edge of G ;
- pick a spanning tree \mathcal{T} for G and number its edges from 1 to r ;

- assign the i -th standard basis vector $\mathbf{e}_i \in \mathbb{R}^r$ to the i -th edge in \mathcal{T} (taken with the orientation selected at the beginning);
- for any edge \bar{e} in G taken with its orientation, consider the unique directed path $\mathcal{P}_{\bar{e}}$ from the starting vertex to the ending vertex that only uses edges of \mathcal{T} ;
- assign to $\mathcal{P}_{\bar{e}}$ the vector $\mathbf{v}_{\bar{e}} = (\lambda_1, \dots, \lambda_r)$, where λ_i equals 1 if the i -th edge of \mathcal{T} appears in $\mathcal{P}_{\bar{e}}$ with its “correct” orientation, -1 if it is traversed backwards, 0 if it does not appear at all.

Putting together all the vectors $\mathbf{v}_{\bar{e}}$ as columns of a matrix yields a full-rank totally unimodular matrix $[I_r \mid D]$ representing \mathcal{M}_G . By the results in this section, this also produces a full-dimensional polytope $\mathcal{P}_{[I_r \mid D]} \subseteq \mathbb{R}^r$ unimodularly equivalent to the symmetric edge polytope of G (compare this to Example 3.5). If the graph G is not connected, one can select a directed spanning tree for each connected component and argue analogously.

4. FIRST PROPERTIES OF GENERALIZED SYMMETRIC EDGE POLYTOPES

Due to Theorem 3.2, if we are given a regular matroid \mathcal{M} of rank $r > 0$ on n elements, we know how to define a full-dimensional polytope $\mathcal{P}_{\mathcal{M}} \subseteq \mathbb{R}^r$ which is defined up to some unimodular equivalence not involving any translation. We now wish to prove some results about $\mathcal{P}_{\mathcal{M}}$: in the proofs we will often need to fix a specific full-rank totally unimodular representation of \mathcal{M} .

We begin by noting that the polytope $\mathcal{P}_{\mathcal{M}}$ does not see potential loops or parallel elements inside the matroid \mathcal{M} , in analogy to the usual symmetric edge polytopes (see [DDM22, Remark 60]).

Remark 4.1. Let \mathcal{M} be a regular matroid of rank $r > 0$ and let M be a full-rank weakly unimodular matrix representing \mathcal{M} . Let \overline{M} be the submatrix of M obtained by keeping only the nonzero columns \mathbf{c}_i such that $\mathbf{c}_i \neq \pm \mathbf{c}_j$ for every $j < i$. Then $\mathcal{P}_{\overline{M}} = \mathcal{P}_M$, since the redundant columns in M do not affect the structure of \mathcal{P}_M and, as $\mathbf{0}$ always lies in the interior of \mathcal{P}_M , the same holds for the zero columns. Hence, the polytope $\mathcal{P}_{\mathcal{M}}$ does not see loops or parallel elements of \mathcal{M} ; as a consequence, we can replace \mathcal{M} by its *simplification*¹ $\overline{\mathcal{M}}$.

Notation 4.2. In the setting of Remark 4.1, we will say M has *irredundant columns* if $\overline{M} = M$, i.e., if the regular matroid represented by M is simple.

We now wish to collect some properties of generalized symmetric edge polytopes. We point out that parts (i) to (iii) of Theorem 4.3 below were essentially already known to Ohsugi and Hibi [OH14, Lemma 2.11].

Theorem 4.3. *Let \mathcal{M} be a regular matroid of rank $r > 0$. The following properties hold:*

- (i) $\mathcal{P}_{\mathcal{M}}$ is centrally symmetric;
- (ii) $\dim(\mathcal{P}_{\mathcal{M}}) = \text{rk}(\mathcal{M})$;
- (iii) $\mathcal{P}_{\mathcal{M}}$ is reflexive;
- (iv) $\mathcal{P}_{\mathcal{M}}$ is terminal, i.e., the only points of $\mathcal{P}_{\mathcal{M}}$ with integer coordinates are its vertices and the origin;
- (v) The vertices of $\mathcal{P}_{\mathcal{M}}$ are twice as many as the atoms of the lattice of flats of \mathcal{M} . In particular, if \mathcal{M} is simple, every antipodal pair of vertices of $\mathcal{P}_{\mathcal{M}}$ corresponds to an element in the ground set of \mathcal{M} .

¹In [AHK18, Definition 4.6] this is called the *combinatorial geometry* of \mathcal{M} .

Proof. Part (i) is immediate by definition, no matter which representation M we choose for \mathcal{M} .

Due to Theorem 3.2 we know that, if M and M' are two full-rank weakly unimodular matrices representing \mathcal{M} , we can go from \mathcal{P}_M to $\mathcal{P}_{M'}$ and from \mathcal{P}_M^Δ to $\mathcal{P}_{M'}^\Delta$ via unimodular maps that do not involve any translation. In particular, it is enough to prove statements (ii)–(v) for \mathcal{P}_M , where M is a full-rank totally unimodular $r \times n$ matrix representing \mathcal{M} .

Part (ii) is now immediate and, together with part (i), implies that the origin lies in the interior of \mathcal{P}_M ; hence, the polar polytope \mathcal{P}_M^Δ is well-defined, and an \mathcal{H} -presentation for it is given by $[M \mid -M]^T \mathbf{x} \leq \mathbf{1}$. Since M is totally unimodular, so is $[M \mid -M]^T$; the polar \mathcal{P}_M^Δ must then be a lattice polytope (see for instance [Sch86, Theorem 19.1]), and hence \mathcal{P}_M is reflexive. This proves part (iii).

As regards part (iv), pick a lattice point $\mathbf{x} = (x_1, \dots, x_r)$ of \mathcal{P}_M different from the origin. Then we can write $\mathbf{x} = \sum_i \lambda_i \mathbf{v}_i$, where $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and the vertices \mathbf{v}_i form a set of pairwise distinct nonzero columns of $[M \mid -M]$. If $r = 1$, the claim is obvious. Assume hence that $r > 1$. Since $\mathbf{x} \neq \mathbf{0}$ and \mathcal{P}_M is a centrally symmetric subset of the hypercube $[-1, 1]^r$, we can assume without loss of generality that $x_1 = 1$. Then the first coordinate of every \mathbf{v}_i must also equal 1. If $\mathbf{x} = \mathbf{v}_1$, there is nothing to prove. Assume otherwise. Then there is a coordinate (without loss of generality, the second one) in which \mathbf{x} and \mathbf{v}_1 differ. This can happen only if $x_2 = 0$ and $(\mathbf{v}_1)_2 \in \{1, -1\}$. But then there must exist some $j > 1$ such that $(\mathbf{v}_j)_2 = -(\mathbf{v}_1)_2$. As a consequence, the totally unimodular matrix $[M \mid -M]$ contains the submatrix

$$\begin{bmatrix} 1 & 1 \\ (\mathbf{v}_1)_2 & -(\mathbf{v}_1)_2 \end{bmatrix}$$

with determinant 2 or -2 . This yields a contradiction.

Finally, it is enough to prove the statement of part (v) when \mathcal{M} is simple. When this is the case, then M has irredundant columns; denote by $\mathbf{c}_1, \dots, \mathbf{c}_{2n}$ the columns of $[M \mid -M]$. Assume by contradiction that a column of $[M \mid -M]$ (without loss of generality, the first one) can be expressed as a convex combination of the other ones; i.e., $\mathbf{c}_1 = \sum_{j \in J} \lambda_j \mathbf{c}_j$ for some $J \subseteq \{2, 3, \dots, 2n\}$, $\lambda_j > 0$, $\sum_{j \in J} \lambda_j = 1$. Since M has no zero columns and \mathcal{P}_M is a centrally symmetric subset of the hypercube $[-1, 1]^r$, we can assume without loss of generality that $(\mathbf{c}_1)_1 = 1$, and this in turn implies that $(\mathbf{c}_j)_1 = 1$ for every $j \in J$. Arguing in a similar way to part (iv), one can then build a submatrix of $[M \mid -M]$ with determinant 2 or -2 , which in turn yields the desired contradiction. \square

Example 4.4. Let \mathcal{M} and M be as in Example 2.4. Then \mathcal{P}_M is the polytope shown in Figure 2. One has that $\dim \mathcal{P}_M = \text{rk}(M) = 3$; since the matroid \mathcal{M} is simple, the lattice points of \mathcal{P}_M are the origin and the columns of $[M \mid -M]$.

Theorem 4.3 gives us the tools to establish a partial converse to Theorem 3.2.

Theorem 4.5. *Let $M, N \in \mathbb{Z}^{r \times n}$ (where $0 < r \leq n$) be two full-rank weakly unimodular matrices with irredundant columns, and assume that the polytopes \mathcal{P}_M and \mathcal{P}_N are unimodularly equivalent. Then there exist $F \in \text{GL}_r(\mathbb{Z})$ and a signed permutation matrix $P \in \mathbb{Z}^{n \times n}$ such that $N = FMP$. In particular, N and M represent the same simple regular matroid \mathcal{M} .*

Proof. By assumption there exist $F \in \text{GL}_r(\mathbb{Z})$ and $\mathbf{v} \in \mathbb{Z}^r$ such that $\mathcal{P}_N = \psi_F(\mathcal{P}_M) + \mathbf{v}$. Since $\mathbf{0}$ is the only interior point of the reflexive polytopes \mathcal{P}_N and \mathcal{P}_M , it must be that $\mathbf{v} = \mathbf{0}$, so that no translation is actually involved. Moreover, one can easily check that $\mathcal{P}_N = \psi_F(\mathcal{P}_M) = \mathcal{P}_{FM}$.

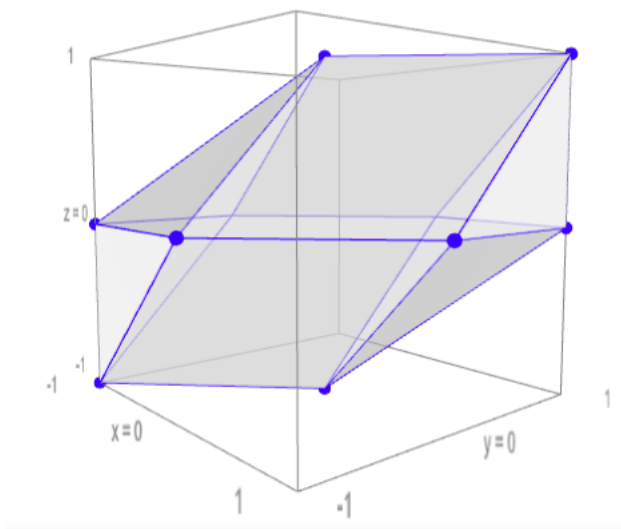


FIGURE 2. The polytope \mathcal{P}_M , where M is the matrix from Example 2.4. Plot generated by SageMath [The23].

Since the matrices FM and N are both full-rank and weakly unimodular, Theorem 4.3(v) implies that the columns of both $[FM \mid -FM]$ and $[N \mid -N]$ correspond to the vertices of $\mathcal{P}_{FM} = \mathcal{P}_N$. As a consequence, the matrices FM and N can only differ by a signed permutation of their columns; in other words, there exists a signed permutation matrix $P \in \mathbb{Z}^{n \times n}$ such that $N = FMP$, as desired. \square

As a consequence, we obtain that the matroidal setting is the “right” one to study even the usual symmetric edge polytopes.

Theorem 4.6. *Let G and H be finite simple graphs. Then the symmetric edge polytopes \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent if and only if the graphic matroids \mathcal{M}_G and \mathcal{M}_H are isomorphic.*

Proof. The “if” part follows from Remark 3.6 and Theorem 3.2. The “only if” part follows from Remark 3.6 and Theorem 4.5, noting that any signed incidence matrix of a simple graph has irredundant columns by construction. \square

Corollary 4.7. *Let G and H be finite simple 3-connected graphs. Then the symmetric edge polytopes \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent if and only if G and H are isomorphic.*

Proof. This follows directly from Theorem 4.6 and Whitney’s 2-isomorphism theorem [Oxl11, Theorem 5.3.1]. \square

Remark 4.8. It was claimed in [MHN+11, Lemma 4.4] that, if G and H are finite simple graphs and G is 2-connected, then \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent if and only if G and H are isomorphic. Unfortunately, this claim is erroneous and affects the validity of [MHN+11, Theorem 4.5] as well: indeed, there exist non-isomorphic 2-connected graphs giving rise to the same graphic matroid, and thus having unimodularly equivalent symmetric edge polytopes by Theorem 4.6. The key to build such objects is the *Whitney twist* operation, see [Oxl11, Section 5.3]. We provide here an explicit example.

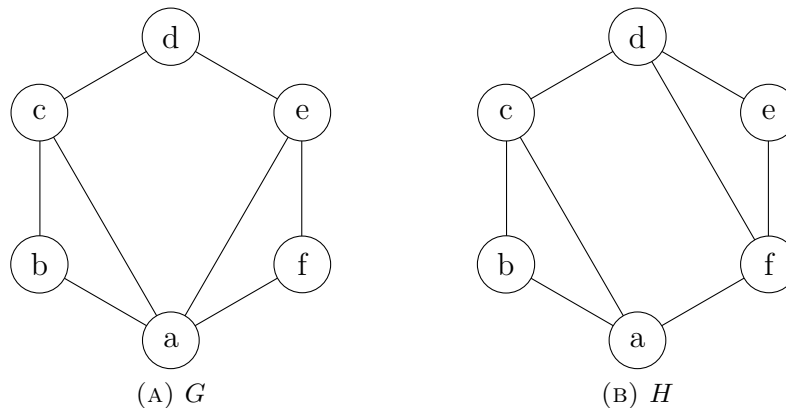


FIGURE 3. Two non-isomorphic 2-connected graphs G and H giving rise to unimodularly equivalent symmetric edge polytopes.

Let G and H be the 6-vertex graphs depicted in Figure 3. Both G and H are 2-connected; moreover, since the vertex a has degree 4 in G and all vertices have degree at most 3 in H , the graphs G and H are not isomorphic. After matching the i -th letter of the English alphabet with the i -th coordinate of \mathbb{R}^6 , consider the 5-dimensional symmetric edge polytopes \mathcal{P}_G and \mathcal{P}_H in \mathbb{R}^6 . Letting

$$F = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \in \text{GL}_6(\mathbb{Z}),$$

one checks that the unimodular map $\psi_F : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ sending \mathbf{x} to $F\mathbf{x}$ transforms \mathcal{P}_G into \mathcal{P}_H , and thus \mathcal{P}_G and \mathcal{P}_H are unimodularly equivalent.

5. FACETS OF $\mathcal{P}_{\mathcal{M}}$ AND THE EHRHART THEORY OF THE POLAR POLYTOPE

After defining generalized symmetric edge polytopes and investigating their first structural properties, it is our next goal to find a combinatorial characterization of their facets. In order to achieve this, it is fruitful to focus on the Ehrhart theory of the polar polytope. Unless specified differently, in this section we will only consider *simple* regular matroids of positive rank, so that by Theorem 4.3(v) the vertices of $\mathcal{P}_{\mathcal{M}}$ will correspond to the columns of $[M \mid -M]$ for any full-rank weakly unimodular matrix M representing \mathcal{M} .

Inspired by work of Beck and Zaslavsky [BZ06], we begin by providing a description of the lattice points in the k -th dilation of $\mathcal{P}_{\mathcal{M}}^{\Delta}$.

Proposition 5.1. *Let k be a positive integer, \mathcal{M} be a simple regular matroid of rank $r > 0$ and M be a full-rank weakly unimodular matrix representing \mathcal{M} . Then the map*

$$\begin{aligned} (k \cdot \mathcal{P}_{\mathcal{M}}^{\Delta}) \cap \mathbb{Z}^r &\rightarrow \{(k+1)\text{-cuts of } M\} \\ \mathbf{u} &\mapsto M^T \mathbf{u} \end{aligned}$$

is a bijection.

Proof. Let us first describe in more detail the polar polytope \mathcal{P}_M^Δ . A facet description of \mathcal{P}_M^Δ is given by $[M \mid -M]^T \mathbf{u} \leq \mathbf{1}$, which in turn implies that

$$(2) \quad k \cdot \mathcal{P}_M^\Delta = \{\mathbf{u} \in \mathbb{R}^r : -k \cdot \mathbf{1} \leq M^T \mathbf{u} \leq k \cdot \mathbf{1}\},$$

where the inequalities are meant to be taken componentwise. This implies that, if $\mathbf{u} \in (k \cdot \mathcal{P}_M^\Delta) \cap \mathbb{Z}^r$, then $(M^T \mathbf{u})$ is an element of $\text{row}(M) \cap \mathbb{Z}^n$ such that $|(M^T \mathbf{u})_i| \leq k$ for every $i \in [n]$. This means precisely that $M^T \mathbf{u}$ is a $(k+1)$ -cut of M .

Vice versa, let γ be a $(k+1)$ -cut of M . Since $\gamma \in \text{row}(M)$, there exists $\mathbf{u} \in \mathbb{Z}^r$ such that $M^T \mathbf{u} = \gamma$. Since M is full-rank, the linear map $\mathbb{Z}^r \rightarrow \mathbb{Z}^n$ defined by M^T is injective, and thus \mathbf{u} is uniquely determined. Since \mathbf{u} satisfies the inequalities in (2), we have that \mathbf{u} is a lattice point of $k \cdot \mathcal{P}_M^\Delta$, and this finishes the proof. \square

Example 5.2. Let M be as in Example 2.4. Then the polar polytope \mathcal{P}_M^Δ is shown in Figure 4. The lattice points of \mathcal{P}_M^Δ are obtained from the 2-cuts in Example 2.7 by throwing away the last two coordinates.

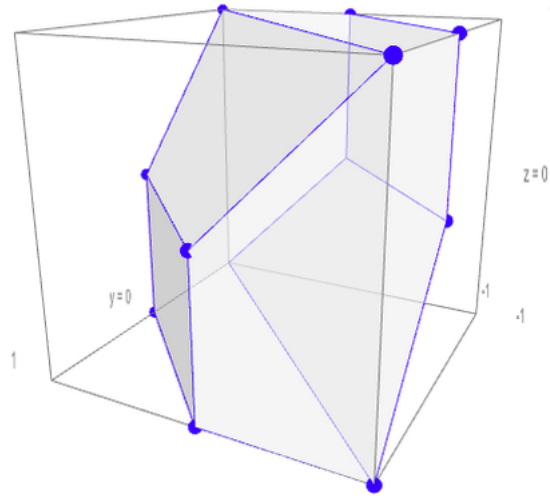


FIGURE 4. The polar polytope \mathcal{P}_M^Δ , where M is the matrix from Example 2.4. Plot generated by SageMath [The23].

Remark 5.3. A consequence of Proposition 5.1 is that the lattice of cuts $\Gamma(M)$ can be thought of as the union of the lattice points of $k \cdot \mathcal{P}_M^\Delta$ as k varies in \mathbb{N} (with the convention that $0 \cdot \mathcal{P}_M^\Delta = \{\mathbf{0}\}$). This gives us an interpretation of the lattice of cuts as a “limit object”.

It follows from the argument in Notation/Remark 2.12 that, if \mathbf{u} is a lattice point of \mathcal{P}_M^Δ , then

$$F_{\mathbf{u}} = \text{conv}(\{M\mathbf{e}_i \mid M\mathbf{e}_i \cdot \mathbf{u} = 1\} \cup \{-M\mathbf{e}_i \mid M\mathbf{e}_i \cdot \mathbf{u} = -1\})$$

is a face of \mathcal{P}_M with supporting hyperplane

$$H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^r \mid \mathbf{x} \cdot \mathbf{u} = 1\}$$

(where we are using the fact that the columns of $[M \mid -M]$ correspond to the vertices of \mathcal{P}_M). Since by Proposition 5.1 $\gamma := M^T \mathbf{u}$ is a 2-cut of M , we can rewrite $F_{\mathbf{u}}$ in the following way:

$$F_{\mathbf{u}} = \text{conv}(\{M\mathbf{e}_i \mid \gamma_i = 1\} \cup \{-M\mathbf{e}_i \mid \gamma_i = -1\}).$$

In other words, the 2-cut $\boldsymbol{\gamma} = M^T \mathbf{u}$ acts as an indicator vector for $F_{\mathbf{u}}$, in the following sense: the i -th entry of $\boldsymbol{\gamma}$ equals $+1$ (respectively, -1) if and only if the vertex $M\mathbf{e}_i$ (respectively, $-M\mathbf{e}_i$) belongs to $F_{\mathbf{u}}$.

Next, we are going to define a partial order on $\{0, \pm 1\}$ -tuples that will enable us to give a first characterization of the facets of $\mathcal{P}_{\mathcal{M}}$.

Definition 5.4. Let $\mathbf{u}, \mathbf{v} \in \{0, \pm 1\}^m$. We will write that $\mathbf{u} \leq \mathbf{v}$ if for every $i \in [m]$ it holds that $u_i = 0$ or $u_i = v_i$. Equivalently, \leq is the partial order induced componentwise by the relations “ $0 < +1$ ”, “ $0 < -1$ ” and “ $+1$ and -1 are incomparable”.

Remark 5.5. Note that $\{0, \pm 1\}^m$ equipped with the partial order from Definition 5.4 is isomorphic to the face lattice of the m -dimensional cross-polytope: see for example [AR20, Example 2.9]. More specifically, the isomorphism maps $\boldsymbol{\gamma} \in \{0, \pm 1\}^m$ to the face obtained as $\text{conv}(\{\mathbf{e}_i \mid \gamma_i = 1\} \cup \{-\mathbf{e}_i \mid \gamma_i = -1\})$. This foreshadows the upcoming characterizations of the facets of $\mathcal{P}_{\mathcal{M}}$.

A direct consequence of the definition of \leq is that $F_{\mathbf{u}} \subseteq F_{\mathbf{u}'}$ if and only if $M^T \mathbf{u} \leq M^T \mathbf{u}'$. This immediately yields a first characterization of the facets of $\mathcal{P}_{\mathcal{M}}$ when \mathcal{M} is simple.

Corollary 5.6. *Let \mathcal{M} be a simple regular matroid of positive rank and let M be a full-rank weakly unimodular representation of \mathcal{M} . Then the facets of \mathcal{P}_M are the faces $F_{\mathbf{u}}$ of \mathcal{P}_M for which $M^T \mathbf{u}$ is a \leq -maximal 2-cut of M .*

The facet description in Corollary 5.6 is not completely satisfactory. Our next goal is to develop an alternate characterization that will be the “right” generalization of the description obtained by Higashitani, Jochemko and Michałek [HJM19, Theorem 3.1] for classical symmetric edge polytopes: see Remark 5.10 below for a more detailed discussion.

Definition 5.7. Let \mathcal{M} be a regular matroid of positive rank and let M be a full-rank weakly unimodular representation of \mathcal{M} . We will say that the cut $\boldsymbol{\gamma} \in \Gamma(M)$ is *spanning* if the support of $\boldsymbol{\gamma}$ contains a basis of \mathcal{M} .

Theorem 5.8. *Let \mathcal{M} be a simple regular matroid of rank $r > 0$ and let M be a full-rank weakly unimodular representation of \mathcal{M} . Then the facets of \mathcal{P}_M are the faces $F_{\mathbf{u}}$ of \mathcal{P}_M for which $M^T \mathbf{u}$ is a spanning 2-cut of M .*

Proof. Let us recall once more that, since \mathcal{M} is simple, the vertices of \mathcal{P}_M are in bijection with the columns of $[M \mid -M]$ by Theorem 4.3(v).

Let us show that, if $\boldsymbol{\gamma} = M^T \mathbf{u}$ is a spanning 2-cut of M , then $F_{\mathbf{u}}$ is a facet of \mathcal{P}_M . Since $\boldsymbol{\gamma}$ is spanning, by the discussion after Remark 5.3 we know that the face $F_{\mathbf{u}}$ contains r linearly independent vertices. Since $\mathbf{0} \notin F_{\mathbf{u}}$, such vertices are also affinely independent; but then, since $\dim(\mathcal{P}_M) = r$ by Theorem 4.3(ii), it follows that $F_{\mathbf{u}}$ must be a facet.

Let us now prove that all facets of \mathcal{P}_M arise in this fashion. Let G be a facet of \mathcal{P}_M . Since $\dim(\mathcal{P}_M) = r$, the facet G must contain r linearly independent vertices $\mathbf{v}_1, \dots, \mathbf{v}_r$; these will correspond to certain columns of M or $-M$. If the i -th column of M appears among the \mathbf{v}_j 's, set $\gamma_i = 1$; if the i -th column of $-M$ does, set $\gamma_i = -1$. Possibly after some relabeling, we can assume without loss of generality that $\gamma_i \neq 0$ for every $i \in [r]$. By Lemma 2.11, there exists a unique cut $\boldsymbol{\gamma}$ compatible with the above assignments; moreover, such a cut is spanning by construction. It only remains to show that $\boldsymbol{\gamma}$ is a 2-cut. By polarity, the facet G corresponds to a vertex \mathbf{u}' of the polar polytope \mathcal{P}_M^Δ ; it then follows from Proposition 5.1 that

$$G = F_{\mathbf{u}'} = \text{conv}(\{M\mathbf{e}_i \mid \gamma'_i = 1\} \cup \{-M\mathbf{e}_i \mid \gamma'_i = -1\}),$$

where $\gamma' = M^T \mathbf{u}'$ is a 2-cut of M . Since γ and γ' coincide on a basis of \mathcal{M} , it follows from the uniqueness of the cut in Lemma 2.11 that $\gamma = \gamma'$ and hence γ is a 2-cut, as desired. \square

Example 5.9. Let M be as in Example 2.4. Twelve of the seventeen 2-cuts enumerated in Example 2.7 are spanning: these are $(1, 0, 0, -1, 1)$, $(-1, 0, 0, 1, -1)$, $(0, 1, 0, -1, 1)$, $(0, -1, 0, 1, -1)$, $(1, 0, -1, 0, 1)$, $(-1, 0, 1, 0, -1)$, $(0, 1, -1, 0, 1)$, $(0, -1, 1, 0, -1)$, $(1, -1, 1, -1, 0)$, $(-1, 1, -1, 1, 0)$, $(1, -1, -1, 1, 0)$, $(-1, 1, 1, -1, 0)$.

Hence, \mathcal{P}_M has twelve facets, and each of the spanning 2-cuts serves as an indicator vector for one of them: for instance, the 2-cut $(1, 0, 0, -1, 1)$ corresponds to the facet obtained as the convex hull of \mathbf{e}_1 , $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_1 + \mathbf{e}_2$ (respectively, the first, minus the fourth, and the fifth column of M).

Remark 5.10. Some words are needed in order to explain in which sense Theorem 5.8 generalizes the characterization of facets obtained by Higashitani, Jochemko and Michałek for classical symmetric edge polytopes [HJM19, Theorem 3.1]. If G is a connected graph, facets of the symmetric edge polytope \mathcal{P}_G were shown to be in bijection with integer vertex labelings such that

- (i) if i and j are adjacent in G , then their labels differ at most by one;
- (ii) the subgraph of G consisting of the edges $\{i, j\}$ whose vertex labels differ exactly by one contains a spanning tree of G .

(For the statement to be precise, one further needs to identify any two vertex labelings that differ by a fixed constant value on each vertex.) The first author, Delucchi and Michałek observed in [DDM22, proof of Proposition 61] that, after fixing an orientation of G , such a characterization is equivalent to asking for integer *edge* labelings such that

- (a) each label is either 1, 0 or -1 ;
- (b) the sum of the labels on each oriented cycle of G is zero;
- (c) the set of edges with nonzero labels contains a spanning tree of G .

This last characterization corresponds to Theorem 5.8 in the special case when \mathcal{M} is the graphic matroid associated with G and M is the signed incidence matrix associated with the chosen orientation of G (although, to be fully precise, such a matrix is not full-rank). Indeed, labeling the (oriented) edges of G can be thought of as labeling the columns of the matrix M . More in detail, condition (b) can be expressed more succinctly by saying that the desired edge labelings are cuts of M , while conditions (a) and (c) further specify that they must be spanning 2-cuts.

Comparing the facet characterization from Corollary 5.6 with the one found in Theorem 5.8 immediately yields the following corollary:

Corollary 5.11. *Let \mathcal{M} be a simple regular matroid of positive rank and M a full-rank weakly unimodular representation of \mathcal{M} . Then a 2-cut of M is spanning if and only if it is \leq -maximal.*

The next result generalizes [DDM22, Proposition 61]. We recall that a matroid is said to be *bipartite* if all of its circuits have even cardinality.

Proposition 5.12. *Let \mathcal{M} be a simple regular bipartite matroid of rank $r > 0$ and let M be a full-rank weakly unimodular representation of \mathcal{M} . Then the facets of \mathcal{P}_M are in bijection with the nowhere-zero 2-cuts of M .*

Proof. By Theorem 5.8, it is enough to prove that the spanning 2-cuts of M are exactly the nowhere-zero 2-cuts of M . Clearly, every nowhere-zero 2-cut must be spanning.

For the reverse containment, let n be the number of elements in the ground set of \mathcal{M} . If $r = n$, then \mathcal{M} is the uniform matroid $\mathcal{U}_{n,n}$, the polytope \mathcal{P}_M is unimodularly equivalent to the n -dimensional cross-polytope, and its 2^n facets correspond to the nowhere-zero elements of $\text{row}(M) \cap \{0, \pm 1\}^n = \{0, \pm 1\}^n$ (see also Remark 5.5).

Assume now that $r < n$. Let γ be a spanning 2-cut of M and assume without loss of generality that $\gamma_i \neq 0$ for every $i \in [r]$. Now pick any $j \in \{r+1, \dots, n\}$ and consider the fundamental signed circuit $\vec{\mathcal{C}}(j, [r])$, whose support has even cardinality because of the bipartite assumption. Since $\gamma \cdot \vec{\mathcal{C}}(j, [r]) = 0$, one has that 0 is the sum of γ_j and an odd number of elements in $\{+1, -1\}$. For parity reasons, it follows that $\gamma_j \neq 0$, which proves the claim. \square

6. A REGULAR UNIMODULAR TRIANGULATION FOR \mathcal{P}_M

It follows from a result of Ohsugi and Hibi [OH14, Theorem 2.7] that the polytope \mathcal{P}_M always admits a regular unimodular triangulation. The aim of this section is to find an explicit description generalizing what Higashitani, Jochemko and Michałek found in the context of symmetric edge polytopes [HJM19, Proposition 3.8]. Since the desired characterization involves signed circuits (see Definition 2.6), our results will be expressed in terms of a fixed full-rank weakly unimodular representation of the given (simple) regular matroid \mathcal{M} .

If M is a full-rank weakly unimodular matrix, then $[M \mid -M]$ is as well. It will be useful to describe the signed circuits of the latter in terms of the former. To achieve this goal, we need to introduce some more notation.

Notation 6.1. Let $J \subseteq [n]$. We will denote by η_J the injective map $\mathbb{Z}^n \rightarrow \mathbb{Z}^{2n}$ sending $(\lambda_1, \dots, \lambda_n)$ to $(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{2n})$, where for every $i \in [n]$

$$\tilde{\lambda}_i := \begin{cases} 0 & i \in J \\ \lambda_i & i \notin J \end{cases} \quad \text{and} \quad \tilde{\lambda}_{n+i} := \begin{cases} -\lambda_i & i \in J \\ 0 & i \notin J. \end{cases}$$

Basically, given an integer vector, the map η_J changes the sign of the entries indexed by an element of J , and then moves them to the second half of an integer vector twice as long. We will sometimes refer to this operation as a *promotion*. Note that η_J restricts to a map $\{0, \pm 1\}^n \rightarrow \{0, \pm 1\}^{2n}$.

Remark 6.2. Note that $\tilde{\lambda} \in \mathbb{Z}^{2n}$ is in the image of η_J precisely when $\tilde{\lambda}_i = 0$ for every $i \in J$ and $\tilde{\lambda}_{n+i} = 0$ for every $i \notin J$. In particular, if the support of $\tilde{\mu} \in \mathbb{Z}^{2n}$ is contained in the support of $\tilde{\lambda} \in \text{im}(\eta_J)$, then $\tilde{\mu} \in \text{im}(\eta_J)$.

Lemma 6.3. *Let $J \subseteq [n]$ and let $M \in \mathbb{Z}^{r \times n}$ (where $0 < r \leq n$) be a full-rank weakly unimodular integer matrix. Then $\lambda \in \vec{\mathcal{C}}(M)$ if and only if $\eta_J(\lambda) \in \vec{\mathcal{C}}([M \mid -M])$.*

Proof. Let $\lambda \in \mathbb{Z}^n$. By construction, one has that

$$M\lambda = \sum_{i \in [n]} \lambda_i (M\mathbf{e}_i) = \sum_{i \in [n]} (\tilde{\lambda}_i - \tilde{\lambda}_{n+i})(M\mathbf{e}_i) = \sum_{i \in [n]} \tilde{\lambda}_i (M\mathbf{e}_i) + \sum_{i \in [n]} \tilde{\lambda}_{n+i} (-M\mathbf{e}_i) = [M \mid -M] \eta_J(\lambda).$$

In particular, $\lambda \in \ker(M)$ if and only if $\eta_J(\lambda) \in \ker([M \mid -M])$, and λ is a 2-flow if and only if $\eta_J(\lambda)$ is. To prove the claim, we still need to show that $\text{supp}(\lambda)$ is a circuit of M if and only if $\text{supp}(\eta_J(\lambda))$ is a circuit of $[M \mid -M]$.

The definition of η_J implies immediately that, if $\text{supp}(\boldsymbol{\lambda})$ is not minimally dependent, then $\text{supp}(\eta_J(\boldsymbol{\lambda}))$ is not minimally dependent either. Conversely, assume $\text{supp}(\eta_J(\boldsymbol{\lambda}))$ is not minimally dependent. Then there exists $\tilde{\boldsymbol{\mu}} \in \mathbb{Z}^{2n}$ such that $[M \mid -M] \tilde{\boldsymbol{\mu}} = \mathbf{0}$ and $\text{supp}(\tilde{\boldsymbol{\mu}}) \subseteq \text{supp}(\eta_J(\boldsymbol{\lambda}))$. By Remark 6.2, $\tilde{\boldsymbol{\mu}}$ belongs to the image of η_J and hence $\text{supp}(\boldsymbol{\lambda})$ is not minimally dependent, since $\text{supp}(\eta_J^{-1}(\tilde{\boldsymbol{\mu}})) \subseteq \text{supp}(\boldsymbol{\lambda})$. This proves the claim. \square

Provided that M does not contain any zero column, the signed circuits of $[M \mid -M]$ come in two flavors: on the one hand, we have the ones of the form $\pm(\mathbf{e}_i + \mathbf{e}_{n+i})$ (reflecting the relation $M\mathbf{e}_i + (-M)\mathbf{e}_i = \mathbf{0}$), while on the other hand we have those obtained by promoting a signed circuit of M . This is the content of the technical lemma below.

Lemma 6.4. *Assume $M \in \mathbb{Z}^{r \times n}$ (where $0 < r \leq n$) is a full-rank weakly unimodular matrix not containing any zero column. Then*

$$(3) \quad \vec{\mathcal{C}}([M \mid -M]) = \{\pm(\mathbf{e}_i + \mathbf{e}_{n+i}) : i \in [n]\} \cup \bigcup_{J \subseteq [n]} \eta_J(\vec{\mathcal{C}}(M)).$$

Proof. Let us first prove that the right hand side of (3) consists of signed circuits of $[M \mid -M]$. This is clear for $\pm(\mathbf{e}_i + \mathbf{e}_{n+i})$, since

$$[M \mid -M] \mathbf{e}_i + [M \mid -M] \mathbf{e}_{n+i} = M\mathbf{e}_i + (-M)\mathbf{e}_i = \mathbf{0}$$

and M does not contain any zero column by hypothesis. Moreover, by Lemma 6.3, $\eta_J(\boldsymbol{\lambda})$ is a signed circuit of $[M \mid -M]$ for every choice of $J \subseteq [n]$ and $\boldsymbol{\lambda} \in \vec{\mathcal{C}}(M)$.

Let us now prove that *every* signed circuit of $[M \mid -M]$ arises as in the right-hand side of (3). Let $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{2n}) \in \{0, \pm 1\}^{2n}$ be a signed circuit of $[M \mid -M]$. If there exists $i \in [n]$ such that $\tilde{\lambda}_i \tilde{\lambda}_{n+i} \neq 0$ then, by support minimality, $\tilde{\boldsymbol{\lambda}}$ must be equal to $\mathbf{e}_i + \mathbf{e}_{n+i}$ up to sign. Assume then that $\tilde{\lambda}_i \tilde{\lambda}_{n+i} = 0$ for every $i \in [n]$, and let $J := \{i \in [n] : \tilde{\lambda}_{n+i} \neq 0\}$. By Remark 6.2, one has that $\tilde{\boldsymbol{\lambda}} = \eta_J(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \{0, \pm 1\}^n$; moreover, by Lemma 6.3, such $\boldsymbol{\lambda}$ is a signed circuit of M . This finishes the proof. \square

We remark that the unimodularity assumption is not really crucial for Lemmas 6.3 and 6.4: one could prove similar statements by substituting “signed circuits” with “circuits” (in the toric ideal meaning). However, since in this paper we are reserving the word “circuit” for its matroidal meaning, we did not want to confuse the reader unnecessarily.

Before moving on, we need to introduce some notation about toric ideals naturally arising in this context.

Notation 6.5. Let \mathcal{M} be a *simple* regular matroid of rank $r > 0$ on n elements and let M be a full-rank weakly unimodular representation of \mathcal{M} . Then, by Theorem 4.3(iv)–(v), the lattice points of \mathcal{P}_M are the columns of $[M \mid -M]$ and the origin $\mathbf{0}$. We will denote by $I_{\mathcal{P}_M}$ the toric ideal associated with the polytope \mathcal{P}_M , i.e., the one obtained as the kernel of the map

$$\begin{aligned} K[x_1, \dots, x_n, x_{-1}, \dots, x_{-n}, z] &\rightarrow K[t_1^{\pm 1}, \dots, t_r^{\pm 1}, s] \\ x_i &\mapsto \mathbf{t}^{M\mathbf{e}_i} s \\ x_{-i} &\mapsto \mathbf{t}^{-M\mathbf{e}_i} s \\ z &\mapsto s \end{aligned}$$

and by $I_{[M \mid -M]}$ the toric ideal obtained as the kernel of the map

$$\begin{aligned} K[x_1, \dots, x_n, x_{-1}, \dots, x_{-n}] &\rightarrow K[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \\ x_i &\mapsto \mathbf{t}^{M\mathbf{e}_i} \\ x_{-i} &\mapsto \mathbf{t}^{-M\mathbf{e}_i}, \end{aligned}$$

where K is a field.

We immediately obtain the following corollary of Lemma 2.16:

Corollary 6.6. *Let \mathcal{M} be a simple regular matroid of rank $r > 0$ and let $M \in \mathbb{Z}^{r \times n}$ be a full-rank weakly unimodular matrix representing \mathcal{M} . Then the ideal $I_{\mathcal{P}_M}$ is the homogenization of $I_{[M \mid -M]}$ with respect to the variable z . In particular, the (irreducible) projective variety $V(I_{\mathcal{P}_M})$ is the projective closure of the (irreducible) affine variety $V(I_{[M \mid -M]})$.*

Lemma 6.4 and Remark 2.14 imply the following description for the universal Gröbner basis of the toric ideal $I_{[M \mid -M]}$ when M is a full-rank weakly unimodular matrix not containing any zero column:

Corollary 6.7. *Let $M \in \mathbb{Z}^{r \times n}$ (where $0 < r \leq n$) be a full-rank weakly unimodular matrix without any zero column. Then the set of signed circuits, the universal Gröbner basis and the Graver basis of $I_{[M \mid -M]}$ all coincide and consist of the following binomials:*

- $x_i x_{-i} - 1$ for every $i \in [n]$;
- $\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-}$ for every $J \subseteq [n]$, $\boldsymbol{\lambda} \in \vec{\mathcal{C}}(M)$ such that $|\eta_J(\boldsymbol{\lambda})^+| \geq |\eta_J(\boldsymbol{\lambda})^-|$.

(With a slight abuse of notation, we identify those binomials that differ only up to a global sign.)

Example 6.8. Let M be as in Example 2.4 and Example 2.7. The Graver basis of $[M \mid -M]$ contains 37 binomials: 5 of the form $x_i x_{-i} - 1$, 16 arising from the promotions of $x_1 x_2 x_3 x_4 - 1$, 8 from the promotions of $x_1 x_2 - x_5$ and another 8 from the promotions of $x_3 x_4 x_5 - 1$. For instance, the promotions of $\boldsymbol{\lambda} = (1, 1, 0, 0, -1) \in \vec{\mathcal{C}}(M)$ (corresponding to the binomial $x_1 x_2 - x_5$) give rise to the following eight signed circuits of $[M \mid -M]$: $x_1 x_2 - x_5$, $x_2 - x_{-1} x_5$, $x_1 - x_{-2} x_5$, $x_1 x_2 x_{-5} - 1$, $1 - x_{-1} x_{-2} x_5$, $x_2 x_{-5} - x_{-1}$, $x_1 x_{-5} - x_{-2}$, $x_{-5} - x_{-1} x_{-2}$. Technically, the statement of Corollary 6.7 only asks for the promotions such that $|\eta_J(\boldsymbol{\lambda})^+| \geq |\eta_J(\boldsymbol{\lambda})^-|$; however, when this is not the case, we just take $-\boldsymbol{\lambda}$ instead of $\boldsymbol{\lambda}$, and the global count is not affected.

The next proposition, reminiscent of the results in [OH14, Section 2], proves the existence of a regular unimodular triangulation for $I_{\mathcal{P}_M}$ and serves as a first step towards an explicit description. For the correspondence between regular unimodular triangulations and squarefree initial ideals, we refer the reader to [Stu96, Chapter 8] and [HPPS21, Section 2.4].

Proposition 6.9. *Let \mathcal{M} be a simple regular matroid of rank $r > 0$ and let $M \in \mathbb{Z}^{r \times n}$ be a full-rank weakly unimodular representation of \mathcal{M} . Denote by S the polynomial ring $K[x_1, \dots, x_n, x_{-1}, \dots, x_{-n}]$. Let $<$ be a graded monomial order of S and let $<_h$ be any monomial order of $S[z]$ with the property that $\text{in}_{<_h} f^h = \text{in}_< f$ for every $f \in S$. Then the toric ideal $I_{\mathcal{P}_M}$ has a squarefree initial ideal with respect to $<_h$.*

Remark 6.10. A concrete choice for $<_h$ as in Proposition 6.9 (and later Theorem 6.11) is any degrevlex order of $S[z]$ such that $z <_h v$ for every variable v in S .

Proof of Proposition 6.9. By Corollary 6.6, the toric ideal $I_{\mathcal{P}_M}$ is the homogenization of $I_{[M | -M]}$ with respect to the variable z . In order to find a Gröbner basis for $I_{\mathcal{P}_M}$, by [CLO15, Theorem 8.4.4] it is then enough to homogenize a set of polynomials forming a Gröbner basis for $I_{[M | -M]}$ with respect to a graded monomial order. The universal Gröbner basis of $I_{[M | -M]}$ is described in Corollary 6.7; since by definition signed circuits have coefficients in $\{0, \pm 1\}$, the claim follows. \square

We are finally able to generalize the Gröbner basis description obtained by Higashitani, Jochemko and Michałek for the usual symmetric edge polytopes.

Theorem 6.11. *Let \mathcal{M} be a simple regular matroid and let $M \in \mathbb{Z}^{r \times n}$ be a full-rank weakly unimodular representation of \mathcal{M} . Let $S[z]$, $<$ and $<_h$ be as in Proposition 6.9. Then the polynomials*

- (i) $x_i x_{-i} - z^2$ for every $i \in [n]$;
- (ii) $\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z$ for every $J \subseteq [n]$, $\boldsymbol{\lambda} \in \vec{\mathcal{C}}(M)$ such that $|\eta_J(\boldsymbol{\lambda})^+| = |\eta_J(\boldsymbol{\lambda})^-| + 1$;
- (iii) $\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-}$ for every $J \subseteq [n]$, $\boldsymbol{\lambda} \in \vec{\mathcal{C}}(M)$ such that $|\eta_J(\boldsymbol{\lambda})^+| = |\eta_J(\boldsymbol{\lambda})^-|$

form a Gröbner basis for $I_{\mathcal{P}_M}$ with respect to $<_h$.

Remark 6.12. The binomials of types (ii) and (iii) in Theorem 6.11 come from considering those signed circuits of $[M | -M]$ where 1's and -1's are “as balanced as possible”; this is exactly what happens for classical symmetric edge polytopes, recalling that both orientations for every edge of the original undirected graph are available in that setting.

Proof of Theorem 6.11. By the proof of Proposition 6.9, we know that homogenizing the polynomials of Corollary 6.7 with respect to the variable z yields a Gröbner basis for $I_{\mathcal{P}_M}$ with respect to $<_h$. For every $i \in [n]$, the homogenization of $x_i x_{-i} - 1$ gives us one of the polynomials of type (i). Now let $J \subseteq [n]$, $\boldsymbol{\lambda} \in \vec{\mathcal{C}}(M)$ and set $k := |\eta_J(\boldsymbol{\lambda})^+| - |\eta_J(\boldsymbol{\lambda})^-|$. After possibly swapping $\boldsymbol{\lambda}$ with $-\boldsymbol{\lambda}$, we can assume without loss of generality that $k \geq 0$.

If $k = 0$ or $k = 1$, the homogenization of $\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-}$ yields one of the binomials of type (iii) or (ii) in the list. It is then enough to show that the homogenization of $\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-}$ is redundant when $k \geq 2$. Consider such a polynomial. There exists $j \in [n]$ such that either $\eta_J(\boldsymbol{\lambda})_j = 1$ or $\eta_J(\boldsymbol{\lambda})_{n+j} = 1$. If $\eta_J(\boldsymbol{\lambda})_j = 1$, one has that

$$x_j \cdot \mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^+} = \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} \quad \text{and} \quad x_{-j} \cdot \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} = \mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^-}$$

and we can write

$$\begin{aligned} \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^k &= \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^k + x_j x_{-j} \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^{k-2} - x_j x_{-j} \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^{k-2} \\ &= x_j \cdot \mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^k + x_j x_{-j} \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^{k-2} - x_j \cdot \mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^-} z^{k-2} \\ &= x_j \cdot (\mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_{J \cup \{j\}}(\boldsymbol{\lambda})^-} z^{k-2}) + \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^{k-2} (x_j x_{-j} - z^2). \end{aligned}$$

If instead $\eta_J(\boldsymbol{\lambda})_{n+j} = 1$, one has that

$$x_{-j} \cdot \mathbf{x}^{\eta_{J \setminus \{j\}}(\boldsymbol{\lambda})^+} = \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} \quad \text{and} \quad x_j \cdot \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} = \mathbf{x}^{\eta_{J \setminus \{j\}}(\boldsymbol{\lambda})^-}$$

and an analogous computation leads to

$$\mathbf{x}^{\eta_J(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^k = x_{-j} \cdot (\mathbf{x}^{\eta_{J \setminus \{j\}}(\boldsymbol{\lambda})^+} - \mathbf{x}^{\eta_{J \setminus \{j\}}(\boldsymbol{\lambda})^-} z^{k-2}) + \mathbf{x}^{\eta_J(\boldsymbol{\lambda})^-} z^{k-2} (x_j x_{-j} - z^2).$$

Iterating this procedure as many times as possible yields the claim. \square

Example 6.13. Let M be as in Example 2.4 and Example 2.7, and pick as a term order the degree reverse lexicographic order with $x_1 > x_{-1} > x_2 > x_{-2} > x_3 > x_{-3} > x_4 > x_{-4} > x_5 > x_{-5} > z$. Then, by Theorem 6.11, there is a Gröbner basis for $I_{\mathcal{P}_M}$ consisting of the following binomials (where we underline the leading term):

- $\underline{x_1x_{-1}} - z^2, \underline{x_2x_{-2}} - z^2, \underline{x_3x_{-3}} - z^2, \underline{x_4x_{-4}} - z^2, \underline{x_5x_{-5}} - z^2$
- $\underline{x_1x_2} - x_5z, \underline{x_{-1}x_{-2}} - x_{-5}z, \underline{x_{-1}x_5} - x_2z, \underline{x_1x_{-5}} - x_{-2}z, \underline{x_{-2}x_5} - x_1z, \underline{x_2x_{-5}} - x_{-1}z$
- $\underline{x_3x_4} - x_{-5}z, \underline{x_{-3}x_{-4}} - x_5z, \underline{x_3x_5} - x_{-4}z, \underline{x_{-3}x_{-5}} - x_4z, \underline{x_4x_5} - x_{-3}z, \underline{x_{-4}x_{-5}} - x_3z$
- $\underline{x_1x_2} - x_{-3}x_{-4}, \underline{x_{-1}x_{-2}} - x_3x_4, \underline{x_1x_3} - x_{-2}x_{-4}, \underline{x_{-1}x_{-3}} - x_2x_4, -x_{-2}x_{-3} + x_1x_4, -x_2x_3 + x_{-1}x_{-4}$.

Note that this Gröbner basis is not reduced, as the monomials x_1x_2 and $x_{-1}x_{-2}$ are both featured twice as the leading term of a binomial. The associated triangulation has sixteen facets and is shown in Figure 5.

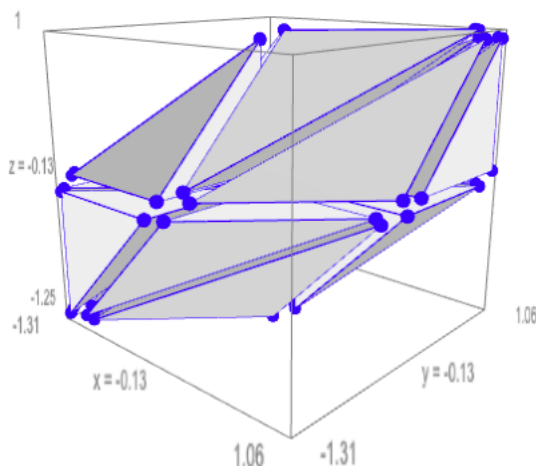


FIGURE 5. The triangulation described in Example 6.13. Plot generated by SageMath [The23].

Finally, the γ -polynomial of a symmetric edge polytope has been the object of much recent work after Ohsugi and Tsuchiya conjectured the nonnegativity of its coefficients in [OT21a]. We wish to conclude the present article by extending to the matroidal setting a characterization of γ_1 which appeared independently in [DJKKV23] and [KT22a].

Corollary 6.14. *Let \mathcal{M} be a simple regular matroid of positive rank. Then $\gamma_1(\mathcal{P}_M) = 2 \cdot \text{rk}(\mathcal{M}^*)$. In particular, $\gamma_1(\mathcal{P}_M)$ is nonnegative.*

Proof. In what follows, let E be the ground set of the matroid \mathcal{M} . By Proposition 6.9, the polytope \mathcal{P}_M admits a (regular) unimodular triangulation $\Delta_{<}$, and hence the h^* -polynomial of \mathcal{P}_M and the h -polynomial of $\Delta_{<}$ coincide. Then

$$\begin{aligned}
 \gamma_1(\mathcal{P}_M) &= h_1^*(\mathcal{P}_M) - \text{rk}(\mathcal{M}) && \text{by Theorem 4.3(ii)} \\
 &= h_1(\Delta_{<}) - \text{rk}(\mathcal{M}) && \text{by Proposition 6.9} \\
 &= (f_0(\Delta_{<}) - \text{rk}(\mathcal{M})) - \text{rk}(\mathcal{M}) && \text{by definition of } h_1 \\
 &= 2 \cdot (|E| - \text{rk}(\mathcal{M})) && \text{since } \Delta_{<} \text{ has } 2 \cdot |E| \text{ vertices} \\
 &= 2 \cdot \text{rk}(\mathcal{M}^*).
 \end{aligned}$$

□

7. FUTURE DIRECTIONS

We conclude the present paper with some questions.

Question 7.1. Are generalized symmetric edge polytopes γ -positive? A positive answer would settle the conjecture by Ohsugi and Tsuchiya on symmetric edge polytopes [OT21a]. More modestly, one could try to prove or disprove that γ_2 is always nonnegative, analogously to the classical symmetric edge polytope case treated in [DJKKV23].

Question 7.2. How do properties of the generalized symmetric edge polytope (e.g., its h^* -vector) change under operations on the associated matroid? Is there any way to use Seymour's characterization of regular matroids via 1-, 2- and 3-sums [Sey80]?

Question 7.3. Can one determine a formula for the h^* -vector of generalized symmetric edge polytopes analogous to the one found by Kálmán and Tóthmérész in [KT22b]?

Question 7.4. Are there “nice” classes of regular matroids for which the h^* -polynomial of the associated generalized symmetric edge polytope is real-rooted?

Question 7.5. To which extent can the formulas from [DDM22, Propositions 64 and 65] be generalized to the matroidal setting?

Acknowledgements. We wish to thank Emanuele Delucchi, Akihiro Higashitani, Hidefumi Ohsugi and Lorenzo Venturullo for useful comments and discussions at various stages of this project. We are grateful to Matthias Walter for helping us with using the Combinatorial Matrix Recognition Library (currently available at <http://discopt.github.io/cmr/>); in particular, in our computations we made use of the total unimodularity test described in [WT13]. We also acknowledge the use of the Macaulay2 [GS] package `Matroids` [Che19] by Justin Chen. Finally, we thank Marco Caselli and Lorenzo Venturullo for their help with coding in SageMath [The23].

REFERENCES

- [AHK18] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Ann. of Math. (2)*, 188(2):381–452, 2018.
- [AR20] Ashleigh Adams and Victor Reiner. A colorful Hochster formula and universal parameters for face rings. *arXiv preprint arXiv:2007.13021*, 2020. To appear in *J. Commut. Algebra*.
- [BB22] Benjamin Braun and Kaitlin Bruegge. Facets of symmetric edge polytopes for graphs with few edges. *arXiv preprint arXiv:2201.13303*, 2022.
- [BBK22] Benjamin Braun, Kaitlin Bruegge, and Matthew Kahle. Facets of random symmetric edge polytopes, degree sequences, and clustering. *arXiv preprint arXiv:2204.07239*, 2022.
- [BZ06] Matthias Beck and Thomas Zaslavsky. The number of nowhere-zero flows on graphs and signed graphs. *J. Combin. Theory Ser. B*, 96(6):901–918, 2006.
- [CDK23] Tianran Chen, Robert Davis, and Evgeniia Korchevskaia. Facets and facet subgraphs of symmetric edge polytopes. *Discrete Appl. Math.*, 328:139–153, 2023.
- [CDM18] Tianran Chen, Robert Davis, and Dhagash Mehta. Counting equilibria of the Kuramoto model using birationally invariant intersection index. *SIAM J. Appl. Algebra Geom.*, 2(4):489–507, 2018.
- [Che19] Justin Chen. Matroids: a Macaulay2 package. *Journal of Software for Algebra and Geometry*, 9(1):19–27, 2019.
- [ÇJM⁺21] Türkü Özlüm Çelik, Asgar Jamneshan, Guido Montúfar, Bernd Sturmfels, and Lorenzo Venturullo. Wasserstein distance to independence models. *J. Symbolic Comput.*, 104:855–873, 2021.
- [CLO15] David A. Cox, John Little, and Donal O’Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

- [DDM22] Alessio D’Alì, Emanuele Delucchi, and Mateusz Michałek. Many Faces of Symmetric Edge Polytopes. *Electron. J. Combin.*, 29(3):Paper No. 3.24–, 2022.
- [DH20] Emanuele Delucchi and Linard Hoessly. Fundamental polytopes of metric trees via parallel connections of matroids. *European J. Combin.*, 87:103098, 18, 2020.
- [DJKKV23] Alessio D’Alì, Martina Juhnke-Kubitzke, Daniel Köhne, and Lorenzo Venturello. On the gamma-vector of symmetric edge polytopes. *SIAM J. Discrete Math.*, 37(2):487–515, 2023.
- [GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [HBA⁺18] M. Hohenwarter, M. Borchers, G. Ancsin, B. Bencze, M. Blossier, J. Éliás, K. Frank, L. Gál, A. Hofstätter, F. Jordan, Z. Konečný, Z. Kovács, E. Lettner, S. Lizelfelner, B. Parrisé, C. Solyom-Gecse, C. Stadlbauer, and M. Tomaschko. GeoGebra 5.0.507.0, October 2018. <http://www.geogebra.org>.
- [HJM19] Akihiro Higashitani, Katharina Jochemko, and Mateusz Michałek. Arithmetic aspects of symmetric edge polytopes. *Mathematika*, 65(3):763–784, 2019.
- [HPPS21] Christian Haase, Andreas Paffenholz, Lindsay C. Piechnik, and Francisco Santos. Existence of unimodular triangulations—positive results. *Mem. Amer. Math. Soc.*, 270(1321):v+83, 2021.
- [KT22a] Tamás Kálmán and Lilla Tóthmérés. Ehrhart theory of symmetric edge polytopes via ribbon structures. *arXiv preprint arXiv:2201.10501*, 2022.
- [KT22b] Tamás Kálmán and Lilla Tóthmérés. h^* -vectors of graph polytopes using activities of dissecting spanning trees. *arXiv preprint arXiv:2203.17127*, 2022.
- [KT23] Tamás Kálmán and Lilla Tóthmérés. Degrees of interior polynomials and parking function enumerators. *arXiv preprint arXiv:2304.03221*, 2023.
- [MHN⁺11] Tetsushi Matsui, Akihiro Higashitani, Yuuki Nagazawa, Hidefumi Ohsugi, and Takayuki Hibi. Roots of Ehrhart polynomials arising from graphs. *J. Algebraic Combin.*, 34(4):721–749, 2011.
- [OH14] Hidefumi Ohsugi and Takayuki Hibi. Centrally symmetric configurations of integer matrices. *Nagoya Math. J.*, 216:153–170, 2014.
- [OT21a] Hidefumi Ohsugi and Akiyoshi Tsuchiya. The h^* -polynomials of locally anti-blocking lattice polytopes and their γ -positivity. *Discrete Comput. Geom.*, 66(2):701–722, 2021.
- [OT21b] Hidefumi Ohsugi and Akiyoshi Tsuchiya. Symmetric edge polytopes and matching generating polynomials. *Comb. Theory*, 1:Paper No. 9, 19, 2021.
- [Oxl11] James Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
- [Sch86] Alexander Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication.
- [Sey80] Paul D. Seymour. Decomposition of regular matroids. *J. Combin. Theory Ser. B*, 28(3):305–359, 1980.
- [Stu96] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
- [SW10] Yi Su and David G. Wagner. The lattice of integer flows of a regular matroid. *J. Combin. Theory Ser. B*, 100(6):691–703, 2010.
- [The23] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.8)*, 2023. <https://www.sagemath.org>.
- [Whi87] Neil White, editor. *Combinatorial geometries*, volume 29 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.
- [WT13] Matthias Walter and Klaus Truemper. Implementation of a unimodularity test. *Mathematical Programming Computation*, 5(1):57–73, 2013.

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, ITALY (CURRENT) AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSNABRÜCK, GERMANY

Email address: `alessio.dali@polimi.it`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSNABRÜCK, GERMANY

Email address: `juhnke-kubitzke@uni-osnabrueck.de`

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF OSNABRÜCK, GERMANY

Email address: `melikoch@uni-osnabrueck.de`