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A NOTE ON THE MOORE-GIBSON-THOMPSON EQUATION WITH MEMORY OF TYPE II

FILIPPO DELL'ORO, IRENA LASIECKA AND VITTORINO PATA

Abstract. We consider the Moore-Gibson-Thompson equation with memory of type II

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(t-s)A\partial_t u(s)\mathrm{d}s = 0$$

where A is a strictly positive selfadjoint linear operator (bounded or unbounded) and $\alpha, \beta, \gamma > 0$ satisfy the relation $\gamma \leq \alpha\beta$. First, we prove well-posedness of finite energy solutions, without requiring any restriction on the total mass ϱ of g. This extends previous results in the literature, where such a restriction was imposed. Second, we address an open question within the context of longtime behavior of solutions. We show that an "overdamping" in the memory term can destabilize the originally stable dynamics. In fact, it is always possible to find memory kernels g, complying with the usual mass restriction $\varrho < \beta$, such that the equation admits solutions with energy growing exponentially fast, even in the regime $\gamma < \alpha\beta$ where the corresponding model without memory is exponentially stable. In particular, this provides an answer to a question recently raised in the literature.

1. Introduction

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let

$$A : \mathfrak{D}(A) \subset H \to H$$

be a strictly positive selfadjoint linear operator (bounded or unbounded). We consider the Moore-Gibson-Thompson (MGT) equation with memory of type II

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(t-s)A\partial_t u(s)\mathrm{d}s = 0, \tag{1.1}$$

where α, β, γ are strictly positive fixed constants subject to the structural constraint

$$\gamma \leq \alpha\beta, \tag{1.2}$$

and the so-called memory kernel $g : [0,\infty) \to [0,\infty)$ is an absolutely continuous nonincreasing function of total mass

$$\varrho = \int_0^\infty g(s)\mathrm{d}s > 0.$$

The MGT equation (1.1) is an abstract version of a wave-type equation studied in the context of acoustic wave propagation with the so-called second sound, where the paradox of the infinite speed of propagation is eliminated by replacing the Fourier law by

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the Maxwell-Cattaneo one. As a consequence, the resulting PDE accounts for a small time relaxation parameter $\tau > 0$, appearing in the unscaled equation in the form $\tau\partial_{ttt}u$, contributing to the presence of the third-order time derivative in the model. The model itself, originally introduced by Stokes [23], has received considerable attention in recent years, due to plethora of applications in nonlinear acoustic and related inverse problems, including lithotripsy, high intensity focused ultrasounds (see the review article [9] with many references therein). On the other hand, the mathematical properties of the resulting dynamics are intricate and drastically different from the one derived via the Fourier law, the latter exhibiting parabolic effects.

The MGT equation *without* memory (i.e. with $g \equiv 0$) has been intensively studied in the recent literature (see e.g. [9, 10, 11, 16, 20]). In particular, the consequences of the hyperbolicity induced by the second sound have been shown to have profound implications on existence/stability results in both linear and nonlinear versions of the model. Much less is known about the effects of the memory on the dynamics, due to molecular relaxation [8], and intrinsically present in an acoustic environment. It is a well-known (although nonintuitive) fact that the effects of the memory may compromise the stability properties of the original dynamics. This has been eloquently demonstrated in [6] for a *second-order* hyperbolic dynamics, where the decay rates for the energy of a wave equation with frictional damping are compromised by the memory.

In the case of a *third-order* equation, the modeling of memory effects is more complex and subtle. The memory may affect the variable u, or the variable $\partial_t u$, or even a combination of both. Accordingly, the corresponding models are labeled as memory of type I, type II, and type III. This raises a fundamental question on the impact of the memory terms on the stability properties of the corresponding evolutions. Stabilizing effects of the memory are well known in the literature (see e.g. [2, 18, 22]). However, a quantification of the stability results (via decay rates) reveals that the memory may deteriorate the stability properties of an otherwise stable systems. An already mentioned classical example is the damped wave equation which, under the influence of memory, may loose exponential stability to become only polynomially stable, as shown by Fabrizio and Polidoro [6]. In the case of the third-order equation, the scenario is much richer and complex. Indeed, for the conservative MGT dynamics in the critical case corresponding to $\gamma = \alpha\beta$, the introduction of memory of type I can only produce strong stability (not exponential), a fact recently discovered in [3, Theorem 4.1 and Corollary 6.2]. However, an addition of memory of type I to a stable dynamics (i.e. where $\gamma < \alpha\beta$) retains the stability of the system, and it does not lead to a deterioration of the rates [14, Theorem 1.4]. Memory of type II is shown to sustain stability for an already stable dynamics, provided however that the size of ϱ is *sufficiently small* [14, Theorem 1.6]. This raises the following questions:

- What happens when a larger amount of type II memory term is introduced?
- Would it be sufficient to stabilize a conservative dynamics?
- Would stability of an already stable dynamics be retained?

These issues have been raised in [3] for the case of finite memory, and in [1] for the case of infinite memory.

In this paper, we show that there is threshold of type II memory, where the memory itself starts acting as an antidamper, destabilizing completely an otherwise stable dynamics, a much stronger negative effect than in the case of the second-order dynamics. However, it is not true that the memory of type II is always acting as a destabilizer. Indeed, if it is paired with the memory of type I, it may stabilize a critically unstable wave. These results are neither intuitive nor predictable. Their proofs rely on a careful quantitative analysis of the damping mechanism, with the aid of energy methods and some counterexamples.

2. The Model and the Literature

The MGT equation without memory, i.e.

$$\partial_{ttt}u + \alpha\partial_{tt}u + \beta A\partial_t u + \gamma Au = 0, \tag{2.1}$$

is a model arising in acoustics and accounting for the second sound effects and the associated thermal relaxations in viscous fluids [7, 17, 23, 24]. The case $\gamma < \alpha\beta$ is referred to as *subcritical*, since in this regime the associated solution semigroup exhibits an exponential decay in the natural weak energy space

$$\mathcal{H} = \mathfrak{D}(A^{\frac{1}{2}}) \times \mathfrak{D}(A^{\frac{1}{2}}) \times H.$$

On the contrary, the case $\gamma = \alpha\beta$ is *critical*, since stability (even the nonuniform one) is lost [10, 16]. Finally, in the *supercritical* case $\gamma > \alpha\beta$, there exist trajectories whose energy blows up exponentially [4, 10, 16]. If additional molecular relaxation phenomena are taken into account, integral terms pop up in the MGT equation, leading to (1.1) with a nonnull memory kernel [8, 12, 13, 14, 15, 19]. In more generality, the convolution term appearing in (1.1) can be taken of the form $\int_0^t g(t-s)Aw(s)\,\mathrm{d}s$, where the variable w is of the following three types:

$$w(s) = \begin{cases} u(s) & \text{(type I)},\\ \partial_t u(s) & \text{(type II)},\\ ku(s) + \partial_t u(s) \quad \text{for } k > 0 & \text{(type III)}. \end{cases}$$

For the memory of type I, the picture is quite understood. As shown in [14], in the subcritical case and under proper decay assumptions on the kernel g all the solutions converge exponentially to zero. Instead, in the critical case the decay is only strong, with a counterexample to exponential stability if the operator A is unbounded [3]. An interesting question becomes what is the effect of the memory of type II. Here, in the subcritical case and with strong restrictions on the mass of memory kernel including $\varrho \ll \beta$, one shows the exponential decay of the energy [14]. In the critical case, exponential stability holds, but with a very special choice of memory of type III, namely $w = \gamma\beta^{-1}u + \partial_t u$.

It is worth noting that in all the results mentioned above a structural restriction on ϱ is required. For the case of memory of type II, such a restriction reads

$$\varrho < \beta.$$

To better understand this issue, an interesting comparison can be made with the MGT equation without memory (2.1), which is shown to be ill-posed in $\mathcal{H}$ if A is unbounded and $\beta = 0$ (but the same is true if $\beta \leq 0$), in the sense that the equation does not generate a strongly continuous semigroup (see [10, Theorem 1.1]). And indeed, equation (2.1) with $(\beta - \varrho)A\partial_t u$ in place of the term $\beta A\partial_t u$ can be considered the limiting case of (1.1) when the kernel g converges to a multiple ϱ of the Dirac mass at 0^+. This would somehow indicate that some problems might arise when $\varrho \geq \beta$. Quite unexpectedly, as it will be shown in this work, it is instead possible to have existence and uniqueness of solutions in the natural weak energy space $\mathcal{H}$, no matter how is the size of ϱ. In fact, the same picture occurs for the MGT equation with memory of type I or III.

A second intriguing problem is to fully understand the effects of the memory of type II on the longtime dynamics, *within* the restriction $\varrho < \beta$ (indeed, if $\varrho \geq \beta$, blow up at infinity is the general rule). In particular, whether this damping alone is able to stabilize the equation in the critical case. As we shall see in this paper, the answer is negative. Even more is true: a *subcritical* MGT equation can be exponentially destabilized by "large" effects of the memory of type II. Hence, a posteriori, for such an equation we may say that no critical value changing the asymptotic dynamics exists, in the sense that blow up of solutions appears, both in the subcritical and in the critical regimes. In particular, this provides an answer to a question raised in [3]. Actually, in the same paper [3] a heuristic explanation was given, by noting that the action of the memory of type II can be interpreted as an addition of a "stabilizer" and of an "antidamper" to the MGT equation. To wit, observe that

$$-\int_0^t g(t-s)A\partial_t u(s)\mathrm{d}s = -g(0)Au(t) + g(t)Au(0) - \int_0^t g'(t-s)Au(s)\mathrm{d}s.$$

The above formula indicates that the memory of type II provides two opposite effects. The static damping term $g(0)Au$ moves the original critical value $\gamma = \alpha\beta$ to the noncritical region $\gamma = \alpha\beta - g(0)$. On the other hand, the viscoelastic term $-\int_0^t g'(t-s)Au(s)\mathrm{d}s$ acts as an antidamper, due to the negative sign of g'. This renders the issue quite interesting, as it is not clear which "damping" wins the game. Of course, the value $g(0)$ and $g'(t)$ will play a crucial role.

Comparison with the previous literature. The recent paper [1] is concerned with the existence, uniqueness and stability of the MGT equation with *infinite memory*, i.e. with a more general convolution term of the form $\int_{-\infty}^t g(t-s)Aw(s)\mathrm{d}s$. In the case of memory of type I, Theorem 3.7 therein proves the exponential decay of solutions in the subcritical case. This result has been already shown for *finite memory* of type I in [14] (see also [13] for more general relaxation kernels leading to uniform but not exponential decays). In the case of memory of type II, the same [1, Theorem 3.7] establishes the exponential decay of the energy in the subcritical case, but under strong "smallness" type restrictions imposed on the mass of kernel ϱ. Here, again, this is an extension to infinite memory of the results obtained in [14] for the case of *finite memory*. In short, this "smallness" condition requires a rather fast decay of the kernel g with respect to the strictly positive value $\alpha\beta - \gamma$. For exponentially decaying kernels of the form $g(t) = \varrho\delta e^{-\delta t}$ with $\delta > 0$, this condition translates into $\varrho < \beta - \gamma\alpha^{-1}$.

Regarding the negative result in the case of memory of type II (conjectured in [3]), the paper [1] evokes the lack of dissipativity of the generator for larger values of ϱ. One should note that dissipativity is a property of the considered inner product and, alone, cannot prove the conjecture stated in [3, Remark 1.3], i.e. to disprove exponential stability. In summary, the analysis carried out in [1] is inconclusive with respect to the open question under consideration. The goal of the present paper is to provide a rigorous proof not only of the lack of exponential stability, but of the existence of solutions which exhibit *exponential blow up at infinity*, which would have been exponentially stable without the effects of the memory of type II. This gives the definite answer to the question asked.

Coming instead to the MGT equation with memory of type II in the critical case, it is still unknown whether exponential stability could be achieved with suitably calibrated relaxation kernel (fast decay and small mass). Positive results are available in the literature but only in either subcritical regime (see [1, 14]) or critical with an addition of suitably calibrated memory of type I.

Notation. We define the family of nested Hilbert spaces depending on a parameter $r \in \mathbb{R}$

$$H^r = \mathfrak{D}(A^{\frac{r}{2}}), \qquad \langle u, v\rangle_r = \langle A^{\frac{r}{2}}u, A^{\frac{r}{2}}v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}}u\|.$$

The index r will be always omitted whenever zero. Along the paper, the Hölder, Young and Poincaré inequalities will be tacitly used in several occasions. The phase space of our problem is

$$\mathcal{H} = H^1 \times H^1 \times H,$$

endowed with the (Hilbert) product norm

$$\|(u, v, w)\|^2_{\mathcal{H}} = \|u\|^2_1 + \|v\|^2_1 + \|w\|^2.$$

3. Well-Posedness

The existence and uniqueness result for (1.1) is ensured by the following theorems.

Theorem 3.1. *If the derivative g' is bounded on bounded intervals, then for every initial datum $U_0 \in \mathcal{H}$, equation* (1.1) *admits a unique weak solution*

$$U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0, T], \mathcal{H})$$

on the interval $[0, T]$, for any $T > 0$.

Theorem 3.2. *Assume the mass restriction $\varrho < \beta$. Then, for every initial datum $U_0 \in \mathcal{H}$, equation* (1.1) *admits a unique weak solution*

$$U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0, \infty), \mathcal{H})$$

whose corresponding energy

$$\mathsf{F}(t) = \|u(t)\|^2_1 + \|\partial_t u(t)\|^2_1 + \|\partial_{tt} u(t)\|^2 + \int_0^t g(t - s)\|\partial_t u(t) - \partial_t u(s)\|^2_1 \mathrm{d}s$$

satisfies the energy inequality

$$\mathsf{F}(t) \le K\mathsf{F}(0)\mathrm{e}^{\omega t}, \tag{3.1}$$

for some structural constants $K, \omega > 0$ and for all $t \ge 0$.

As already mentioned in the Introduction, the first Theorem 3.1 above, within a very mild assumption on the derivative g' of the memory kernel, provides existence and uniqueness of solutions in the space $\mathcal{H}$ *without* imposing the usual restriction $\varrho < \beta$ on the size of the mass of g. In which case, the solutions will be typically unbounded in time and exhibit a “rough” asymptotic behavior as $t \to \infty$. To the best of our knowledge, this is the first well-posedness result obtained for the MGT equation with memory without assuming "smallness" restrictions imposed on the relaxation kernel.

If instead we assume $\varrho < \beta$, then the second Theorem 3.2 provides existence and uniqueness of solutions in $\mathcal{H}$ which enjoy an exponential-type growth at infinity.

In order to prove the theorems, we first show a well-posedness result in the more regular space

$$\hat{\mathcal{H}} = H^2 \times H^1 \times H,$$

by constructing solutions to a memoryless nonhomogeneous MGT equation exploiting the so-called MacCamy trick (see e.g. [21]).

Lemma 3.3. *For every initial datum $U_0 \in \hat{\mathcal{H}}$ (and every $\varrho > 0$), equation* (1.1) *admits a unique weak solution*

$$U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0,T], \hat{\mathcal{H}})$$

on the interval $[0,T]$, *for any* $T > 0$.

Proof. Let $T > 0$ be arbitrarily fixed, and let $R_\mu(t)$ denote the resolvent operator associated with the kernel

$$\mu(s) = -\frac{1}{\beta} g(s).$$

This means that R_μ solves the equation

$$R_\mu(t) + \int_0^t \mu(t-s) R_\mu(s)\,\mathrm{d}s = \mu(t), \quad \forall t \geq 0.$$

We now rewrite (1.1) in the form

$$A^{\frac{1}{2}} \partial_t u(t) + \int_0^t \mu(t-s) A^{\frac{1}{2}} \partial_t u(s) \mathrm{d}s = Y(t),$$

where

$$Y(t) = -\frac{1}{\beta}\big[A^{-\frac{1}{2}} \partial_{ttt} u(t) + \alpha A^{-\frac{1}{2}} \partial_{tt} u(t) + \gamma A^{\frac{1}{2}} u(t)\big].$$

If we knew in advance that $Y \in \mathcal{C}([0,T], H)$, then the function $X(t) = A^{\frac{1}{2}} \partial_t u(t)$, being the solution to the Volterra equation on $[0,T]$

$$X(t) + \int_0^t \mu(t-s) X(s)\,\mathrm{d}s = Y(t),$$

has the explicit representation

$$X(t) = Y(t) - \int_0^t R_\mu(t-s) Y(s)\,\mathrm{d}s.$$

Applying $\beta A^{\frac{1}{2}}$ to both sides, we conclude that the function $U = (u, \partial_t u, \partial_{tt} u)$ satisfies the equation

$$\partial_{ttt} u + \alpha \partial_{tt} u + \beta A \partial_t u + \gamma A u = Q_U, \tag{3.2}$$

having set

$$Q_U(t) = \int_0^t R_\mu(t-s)[\partial_{ttt} u(s) + \alpha \partial_{tt} u(s) + \gamma A u(s)]\,\mathrm{d}s.$$

Integrating by parts the first term above, we obtain

$$Q_U(t) = \int_0^t R'_\mu(t-s)\partial_{tt}u(s)\,\mathrm{d}s - R_\mu(t)\partial_{tt}u(0) + R_\mu(0)\partial_{tt}u(t) + \int_0^t R_\mu(t-s)[\alpha\partial_{tt}u(s) + \gamma Au(s)]\,\mathrm{d}s.$$

At this point, we observe that $R_\mu(0) = \mu(0) < 0$. Thus, calling $\hat{\alpha} = \alpha - R_\mu(0) > 0$ and

$$\hat{Q}_U(t) = \int_0^t R'_\mu(t-s)\partial_{tt}u(s)\,\mathrm{d}s - R_\mu(t)\partial_{tt}u(0) + \int_0^t R_\mu(t-s)[\alpha\partial_{tt}u(s) + \gamma Au(s)]\,\mathrm{d}s, \tag{3.3}$$

equation (3.2) reads

$$\partial_{ttt}u + \hat{\alpha}\partial_{tt}u + \beta A\partial_t u + \gamma Au = \hat{Q}_U. \tag{3.4}$$

Introducing the three-component vector

$$\mathcal{Q}_U = (0, 0, \hat{Q}_U),$$

we can write (3.4) in the abstract form

$$\frac{\mathrm{d}}{\mathrm{d}t}U = \mathbb{A}U + \mathcal{Q}_U, \tag{3.5}$$

where

$$\mathbb{A}U = (\partial_t u, \partial_{tt}u, -\hat{\alpha}\partial_{tt}u - \beta A\partial_t u - \gamma Au).$$

It is known from [10, Theorem 1.4] that the MGT equation without memory generates a strongly continuous semigroup $S(t) = \mathrm{e}^{\mathbb{A}t}$ on $\hat{\mathcal{H}}$. Hence, its nonhomogeneous version (3.5) driven by a generic forcing term $\mathcal{Q} \in L^1(0,T;\hat{\mathcal{H}})$ admits a unique solution

$$U = (u, \partial_t u, \partial_{tt}u) \in \mathcal{C}([0,T], \hat{\mathcal{H}}),$$

and the map $\mathcal{Q} \mapsto U$ is continuous from $L^1(0,T;\hat{\mathcal{H}})$ into $\mathcal{C}([0,T];\hat{\mathcal{H}})$. We shall use this fact in order to construct a fixed point solution to the equation (3.5). To this end, note that for $U \in \mathcal{C}([0,T],\hat{\mathcal{H}})$ one has $\mathcal{Q}_U \in L^1(0,T;\hat{\mathcal{H}})$, as this is the same as saying that $\hat{Q}_U \in L^1(0,T;H)$, and the latter relation follows from (3.3). Indeed, if $U \in \mathcal{C}([0,T],\hat{\mathcal{H}})$ since R_μ is bounded and R'_μ is summable on $[0,T]$, it is even true that $\hat{Q}_U \in L^\infty(0,T;H)$ (note that if $U \in \mathcal{C}([0,T],\hat{\mathcal{H}})$ then $u \in \mathcal{C}([0,T],H^2)$ and $\partial_{tt}u \in \mathcal{C}([0,T],H)$ by the very definition of $\hat{\mathcal{H}}$). Accordingly, from the variation-of-constant formula we end up with

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\mathcal{Q}_U(s)\mathrm{d}s,$$

to which we apply Banach contraction principle first on the space

$$\mathcal{X} = \big\{U \in \mathcal{C}([0,T_0],\hat{\mathcal{H}}) \,:\, U(0) = U_0\big\},$$

with T_0 sufficiently small, and then reiterated (due to the linearity) to the intervals $[nT_0,(n+1)T_0]$ until T is reached. $\square$

The next step is extending the result of Lemma 3.3 to the whole space $\mathcal{H}$. This will be easily accomplished by standard density arguments, once a priori estimates involving initial data belonging to $\mathcal{H}$ are established. Here, different arguments are needed depending whether we are in the framework of Theorem 3.1 or of Theorem 3.2.

Proof of Theorem 3.1. Let $T > 0$ be arbitrarily fixed. We start from equation (3.4) of the previous proof, that is,

$$\partial_{ttt}u + \hat{\alpha}\partial_{tt}u + \beta A\partial_t u + \gamma Au = \hat{Q}_U,$$

whose solution U (which is in fact the solution to the original equation) exists in $\hat{\mathcal{H}}$ for initial data $U_0 \in \hat{\mathcal{H}}$. Then, we multiply by $2\partial_{tt}u$ in H, and we add to both sides the term $2m\langle u, \partial_t u\rangle_1$ for $m > 0$ to be fixed later. We obtain the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{V}_m + 2\hat{\alpha}\|\partial_{tt}u\|^2 = 2\gamma\|\partial_t u\|_1^2 + 2m\langle u, \partial_t u\rangle_1 + 2\langle \hat{Q}_U, \partial_{tt}u\rangle,$$

where we set

$$\mathsf{V}_m(t) = \|\partial_{tt}u(t)\|^2 + \beta\|\partial_t u(t)\|_1^2 + 2\gamma\langle u(t), \partial_t u(t)\rangle_1 + m\|u(t)\|_1^2.$$

It is readily seen that, up to choosing $m > 0$ sufficiently large, there exist $\kappa_2 > \kappa_1 > 0$ such that

$$\kappa_1\|U(t)\|_{\mathcal{H}}^2 \leq \mathsf{V}_m(t) \leq \kappa_2\|U(t)\|_{\mathcal{H}}^2. \tag{3.6}$$

Accordingly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{V}_m \leq C\mathsf{V}_m + 2\langle \hat{Q}_U, \partial_{tt}u\rangle, \tag{3.7}$$

for some $C > 0$. We now claim that the inequality

$$\int_0^t \langle \hat{Q}_U(s), \partial_{tt}u(s)\rangle \mathrm{d}s \leq \frac{1}{4}\mathsf{V}_m(t) + C_T\mathsf{V}_m(0) + C_T\int_0^t \mathsf{V}_m(s)\mathrm{d}s \tag{3.8}$$

holds for every $t \in [0, T]$. Here and till the end of the proof, $C_T > 0$ denotes a *generic* constant, independent of the initial data, but depending on T. Then, integrating (3.7) on $[0, t]$, we end up with

$$\mathsf{V}_m(t) \leq C_T\mathsf{V}_m(0) + C_T\int_0^t \mathsf{V}_m(s)\mathrm{d}s, \quad \forall t \in [0, T],$$

and the standard Gronwall lemma together with (3.6) entail

$$\|U(t)\|_{\mathcal{H}} \leq C_T\|U(0)\|_{\mathcal{H}}, \quad \forall t \in [0, T].$$

We are left to prove (3.8). Recalling (3.3), we limit ourselves to show the more difficult estimate of the higher-order term, namely,

$$\mathfrak{I} := \int_0^t \int_0^s R_\mu(s - y)\langle u(y), \partial_{tt}u(s)\rangle_1 \mathrm{d}y\mathrm{d}s.$$

We write

$$\int_0^s R_\mu(s-y)\langle u(y),\partial_{tt}u(s)\rangle_1 \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}s}\bigg[\int_0^s R_\mu(s-y)\langle u(y),\partial_t u(s)\rangle_1 \mathrm{d}y\bigg] - R_\mu(0)\langle u(s),\partial_t u(s)\rangle_1 - \int_0^s R'_\mu(s-y)\langle u(y),\partial_t u(s)\rangle_1 \mathrm{d}y.$$

Note that, within the boundedness assumption on g', we have that R'_μ (as well as R_μ) is bounded on $[0,T]$. Then, integrating on $[0,t]$ the identity above, we are led to

$$\begin{aligned}\mathfrak{I} = &\int_0^t R_\mu(t-s)\langle u(s),\partial_t u(t)\rangle_1 \mathrm{d}s - R_\mu(0)\int_0^t \langle u(s),\partial_t u(s)\rangle_1 \mathrm{d}s \\ &- \int_0^t\int_0^s R'_\mu(s-y)\langle u(y),\partial_t u(s)\rangle_1 \mathrm{d}y\mathrm{d}s.\end{aligned}$$

We finally estimate the three terms in the right-hand side as follows:

$$\begin{aligned}&\int_0^t R_\mu(t-s)\langle u(s),\partial_t u(t)\rangle_1 \mathrm{d}s - R_\mu(0)\int_0^t \langle u(s),\partial_t u(s)\rangle_1 \mathrm{d}s \\ &\leq \varepsilon\|U(t)\|^2_{\mathcal{H}} + \frac{C_T}{\varepsilon}\int_0^t \|U(s)\|^2_{\mathcal{H}}\mathrm{d}s,\end{aligned}$$

for any $\varepsilon > 0$ small, and

$$\begin{aligned}-\int_0^t\int_0^s R'_\mu(s-y)\langle u(y),\partial_t u(s)\rangle_1 \mathrm{d}y\mathrm{d}s &\leq C_T\int_0^t \|U(s)\|_{\mathcal{H}}\int_0^s \|U(y)\|_{\mathcal{H}}\mathrm{d}y\mathrm{d}s \\ &\leq C_T\bigg(\int_0^t \|U(s)\|_{\mathcal{H}}\mathrm{d}s\bigg)^2 \\ &\leq C_T\int_0^t \|U(s)\|^2_{\mathcal{H}}\mathrm{d}s.\end{aligned}$$

Therefore,

$$\mathfrak{I} \leq \varepsilon\|U(t)\|^2_{\mathcal{H}} + \frac{C_T}{\varepsilon}\int_0^t \|U(s)\|^2_{\mathcal{H}}\mathrm{d}s.$$

The remaining terms of $\int_0^t \langle \hat{Q}_U(s),\partial_{tt}u(s)\rangle \mathrm{d}s$, as we said, are controlled in a similar (in fact easier) way, and at the end one has to use (3.6). Only at that point, one fixes ε in order to get the desired coefficient $1/4$ (or smaller) in front of $\mathsf{V}_m(t)$. This finishes the proof. □

Proof of Thorem 3.2. We only need to show the energy inequality (3.1). To this aim, similarly to the proof of Theorem 3.1, we take the product in H of (1.1) and $2\partial_{tt}u$, and we add to both sides the term $2m\langle u, \partial_t u\rangle_1$ for $m > 0$ to be fixed later. This yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\partial_{tt}u(t)\|^2 + \beta\|\partial_t u(t)\|_1^2 + m\|u(t)\|_1^2 + 2\gamma\langle u(t), \partial_t u(t)\rangle_1\big] \\
&\quad - 2\int_0^t g(t-s)\langle \partial_t u(s), \partial_{tt}u(t)\rangle_1 \,\mathrm{d}s \\
&= 2\gamma\|\partial_t u(t)\|_1^2 - 2\alpha\|\partial_{tt}u(t)\|^2 + 2m\langle u(t), \partial_t u(t)\rangle_1 \\
&\leq 2(\gamma + m)\mathsf{F}(t).
\end{aligned}$$

Next, calling $G(t) = \int_0^t g(s)\,\mathrm{d}s$, we compute the integral in the left-hand side as

$$\begin{aligned}
&- 2\int_0^t g(t-s)\langle \partial_t u(s), \partial_{tt}u(t)\rangle_1 \,\mathrm{d}s \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_0^t g(t-s)\|\partial_t u(t) - \partial_t u(s)\|_1^2 \,\mathrm{d}s - G(t)\|\partial_t u(t)\|_1^2\Big] \\
&\quad + g(t)\|\partial_t u(t)\|_1^2 - \int_0^t g'(t-s)\|\partial_t u(t) - \partial_t u(s)\|_1^2 \,\mathrm{d}s.
\end{aligned}$$

Setting

$$\begin{aligned}
\mathsf{E}_m(t) &= \|\partial_{tt}u(t)\|^2 + (\beta - G(t))\|\partial_t u(t)\|_1^2 + m\|u(t)\|_1^2 + 2\gamma\langle u(t), \partial_t u(t)\rangle_1 \\
&\quad + \int_0^t g(t-s)\|\partial_t u(t) - \partial_t u(s)\|_1^2 \,\mathrm{d}s,
\end{aligned}$$

since g is nonnegative and nonincreasing we arrive at the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_m \leq 2(\gamma + m)\mathsf{F}.$$

Recalling that $G(t) \leq \varrho < \beta$, is then clear that, up to choosing m large enough,

$$\kappa_1 \mathsf{F}(t) \leq \mathsf{E}_m(t) \leq \kappa_2 \mathsf{F}(t),$$

for some $\kappa_2 > \kappa_1 > 0$. The desired conclusion follows by an application of the Gronwall lemma. □

Remark 3.4. One might wonder why the fixed point argument cannot be directly applied to find a solution in the space $\mathcal{C}([0,T],\mathcal{H})$. The reason is that, in order to apply the Banach contraction principle to (3.5), we need a forcing term $\mathcal{Q}_U \in L^1(0,T;\mathcal{H})$. But in general this is not the case, since if we take initial data in $\mathcal{H}$ we only know that the component u belongs to H^1. Therefore the term $\mathcal{Q}_U$ is not a priori in the energy space $\mathcal{H}$, as well as the multipliers employed in the proof of Theorem 3.1. Hence the calculations would be only formal. Accordingly, the strategy is first finding a solution in the more regular space $\mathcal{C}([0,T],\hat{\mathcal{H}})$, and then drawing the desired conclusion by means of energy estimates.

Remark 3.5. If A is a bounded operator, the conclusions of Theorem 3.2 are easily attained removing the restriction $\varrho < \beta$.

Remark 3.6. As a final comment, it is interesting to observe that the trick of multiplying both sides of the equation by $\langle u, \partial_t u\rangle_1$, employed in the proofs above, allows to provide a two-line proof of the well-posedness of the strongly damped wave equation with the "wrong" sign of Au, namely,

$$\partial_{tt}u + A\partial_t u - Au = 0,$$

which highlights the essential parabolicity of the original equation. This is not the case if one has a lower-order dissipation. Indeed, the equation

$$\partial_{tt}u + A^{\vartheta}\partial_t u - Au = 0$$

is ill-posed for $\vartheta < 1$, as the real part of the spectrum of the associated linear operator is not bounded above.

4. The Case of the Exponential Kernel

We now dwell on the particular case of the exponential kernel

$$g(s) = \varrho\delta \mathrm{e}^{-\delta s},$$

with

$$\varrho \in (0, \beta) \qquad \text{and} \qquad \delta > 0.$$

For this choice, equation (1.1) reads

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \varrho\delta \int_0^t \mathrm{e}^{-\delta(t-s)} A\partial_t u(s)\mathrm{d}s = 0. \tag{4.1}$$

In the same spirit of [3], taking the sum ∂_t(4.1)+δ(4.1) we obtain the fourth-order equation

$$\partial_{tttt}u + (\alpha+\delta)\partial_{ttt}u + \alpha\delta\partial_{tt}u + \beta A\partial_{tt}u + (\gamma + \delta\beta - \varrho\delta)A\partial_t u + \gamma\delta Au = 0. \tag{4.2}$$

Note that

$$\gamma + \delta\beta - \varrho\delta > 0,$$

as $\varrho < \beta$. Introducing the 4-component space

$$\mathcal{V} = H^1 \times H^1 \times H^1 \times H,$$

it is known from [5, Theorem 3.1] that (4.2) admits a unique (weak) solution

$$\hat{U} = (u, \partial_t u, \partial_{tt}u, \partial_{ttt}u) \in \mathcal{C}([0,\infty), \mathcal{V}),$$

for every initial datum $\hat{U}_0 \in \mathcal{V}$. Besides, the analysis in [5] provides necessary and sufficient conditions in order for (4.2) to be (exponentially) stable, depending on two *stability numbers* $\varkappa$ and ϖ, which in turn depend only on the (positive) structural constants of the equation. For this particular case, the two stability numbers read

$$\varkappa = \frac{\alpha\beta - \gamma + \varrho\delta}{\alpha+\delta} > 0 \qquad \text{and} \qquad \varpi = \frac{\alpha\beta\delta^2 - \gamma\delta^2 - \alpha\varrho\delta^2}{\gamma + \delta\beta - \varrho\delta}.$$

In particular, if $\varrho \in (\beta - \frac{\gamma}{\alpha}, \beta)$ and δ is large enough, then

$$\varpi < -\lambda_1\varkappa$$

where $\lambda_1 > 0$ is the smallest element of the spectrum $\sigma(A)$ of the operator A. In this regime, the results of [5] predict the existence of solutions growing exponentially fast, which gives a clear indication that our energy F might blow up exponentially for some initial data. At the same time, the equivalence between (4.1) and (4.2) is, at this stage, only formal. The next proposition establishes such an equivalence in a rigorous way.

Proposition 4.1. *Let $U_0 = (u_0, v_0, w_0) \in H^1 \times H^1 \times H^1$ be an arbitrarily fixed vector satisfying the further regularity assumption*

$$\beta v_0 + \gamma u_0 \in H^2.$$

Then the projection $U = (u, \partial_t u, \partial_{tt} u)$ onto the first three components of the solution $\hat{U} = (u, \partial_t u, \partial_{tt} u, \partial_{ttt} u)$ to (4.2) with initial datum

$$(u_0, v_0, w_0, -\alpha w_0 - A(\beta v_0 + \gamma u_0)) \in \mathcal{V}$$

is the unique solution to (4.1) with initial datum U_0.

Proof. We introduce the auxiliary variable

$$\phi(t) = \partial_{ttt} u(t) + \alpha \partial_{tt} u(t) + \beta A \partial_t u(t) + \gamma A u(t).$$

Since u solves (4.2), the function ϕ fulfils the identity

$$\partial_t \phi + \delta \phi - \varrho \delta A \partial_t u = 0.$$

A multiplication by $\mathrm{e}^{\delta t}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{\delta t}\phi(t)] - \varrho\delta \mathrm{e}^{\delta t} A \partial_t u(t) = 0.$$

Noting that $\phi(0) = 0$, an integration on $[0, t]$ leads at once to (4.1). □

Still, this is not enough to conclude that F can grow exponentially fast, since one has to verify that this occurs for a particular trajectory of (4.2), with initial conditions complying with the assumptions above.

5. Exponentially Growing Solutions

In this section, we state and prove the second main result of the paper. Namely, we produce an example of memory kernel g for which equation (1.1) admits solutions with energy growing exponentially fast. To this end, we consider the exponential kernel $g(s) = \varrho\delta \mathrm{e}^{-\delta s}$ of the previous section. For simplicity, we also assume that the spectrum of the operator A contains at least one eigenvalue $\lambda > 0$, which is always the case in concrete situations.

Theorem 5.1. *Let $\varrho \in (\beta - \frac{\gamma}{\alpha}, \beta)$ be arbitrarily fixed. Then, for every $\delta > 0$ sufficiently large, there exist $\varepsilon > 0$, an initial datum $U_0 \in \mathcal{H}$ and a sequence $t_n \to \infty$ such that the energy $\mathsf{F}(t)$ associated to the solution to (4.1) originating from U_0 satisfies the estimate*

$$\mathsf{F}(t_n) \geq \lambda \mathrm{e}^{\varepsilon t_n}, \quad \forall n \in \mathbb{N}.$$

Remark 5.2. According to [14], for this particular kernel exponential stability occurs in the subcritical case, within the following assumption: there exist $k \in (\frac{\gamma}{\beta}, \alpha)$ and $\theta > \frac{k}{\delta}$ such that

$$\varrho \leq \Big(\beta - \frac{\gamma}{k}\Big) \min\Big\{1, \frac{2}{k(2+\theta)}\Big\}.$$

The reader will have no difficulties to check that the condition above implies that $\varrho < \beta - \frac{\gamma}{\alpha}$, which contradicts $\varrho \in (\beta - \frac{\gamma}{\alpha}, \beta)$ assumed in Theorem 5.1.

In order to prove the theorem, we introduce the fourth-order polynomial in the complex variable ξ

$$\mathsf{P}(\xi) = \xi^4 + (\alpha+\delta)\xi^3 + (\alpha\delta+\beta\lambda)\xi^2 + \lambda(\gamma+\delta\beta-\varrho\delta)\xi + \lambda\gamma\delta. \tag{5.1}$$

Moreover, for all $x \geq 0$, we set

$$q(x) = \sqrt{\frac{4x^3 + 3(\alpha+\delta)x^2 + 2(\alpha\delta+\beta\lambda)x + \lambda(\gamma+\delta\beta-\varrho\delta)}{\alpha+\delta+4x}} > 0. \tag{5.2}$$

The next algebraic result will be crucial for our purposes.

Lemma 5.3. *Let $\varrho \in (\beta - \frac{\gamma}{\alpha}, \beta)$ be arbitrarily fixed. Then, for every $\delta > 0$ sufficiently large, there exists $p > 0$ such that the complex number*

$$\hat{\xi} = p + \mathrm{i}q(p)$$

solves the equation $\mathsf{P}(\hat{\xi}) = 0$.

Proof. For all $x \geq 0$, by direct calculations we find the equalities

$$\mathfrak{Im}\,[\mathsf{P}(x+\mathrm{i}q(x))] = 0 \qquad \text{and} \qquad \mathfrak{Re}\,[\mathsf{P}(x+\mathrm{i}q(x))] = f(x),$$

where

$$f(x) = x^4 + (\alpha+\delta)x^3 + (\alpha\delta+\beta\lambda)x^2 + (\gamma+\delta\beta-\varrho\delta)\lambda x + \gamma\delta\lambda + q(x)^4$$
$$- (6x^2 + 3(\alpha+\delta)x + \alpha\delta + \beta\lambda)q(x)^2.$$

By means of direct computations, with the aid of (5.2) and the assumption $\varrho > \beta - \frac{\gamma}{\alpha}$, it is readily seen that

$$\lim_{x\to+\infty} f(x) = -\infty$$

and

$$f(0) = \gamma\delta\lambda + q(0)^4 - (\alpha\delta+\beta\lambda)q(0)^2 \sim \delta\lambda(\gamma - \alpha\beta + \alpha\varrho) > 0, \quad \text{as } \delta \to +\infty.$$

As a consequence, being f continuous on $[0,\infty)$, once $\delta > 0$ has been fixed sufficiently large there exists $p > 0$ such that

$$0 = f(p) = \mathfrak{Re}\,[\mathsf{P}(p+\mathrm{i}q(p))].$$

The proof is finished. □

Proof of Theorem 5.1. Denoting by $w \in H$ the normalized eigenvector of A corresponding to λ, we consider the function

$$u(t) = \mathrm{e}^{pt}\,[r\sin(qt) + \cos(qt)]\,w.$$

Here, $p > 0$ is given by Lemma 5.3, $q = q(p) > 0$ is given by (5.2) and

$$r = r(p) = \frac{p^3 - 3pq^2 + \alpha(p^2 - q^2) + \beta\lambda p + \gamma\lambda}{q^3 - 3p^2q - 2\alpha pq - \beta\lambda q}.$$

Note that r is well defined, since (1.2) and (5.2) ensure that

$$q^3 - 3p^2q - 2\alpha pq - \beta\lambda q = \frac{-q(8p^3 + 8\alpha p^2 + 2\alpha^2 p + 2\beta\lambda p + \lambda\varrho\delta + \lambda(\alpha\beta-\gamma))}{\alpha+\delta+4p} < 0.$$

The function u defined above solves the fourth-order equation (4.2). Indeed, calling for simplicity

$$\psi(t) = \mathrm{e}^{pt}\,[r\sin(qt) + \cos(qt)]$$

and recalling that due to Lemma 5.3 the complex number $p+\mathrm{i}q$ is a root of the polynomial P defined in (5.1), we have

$$\begin{aligned}
&\partial_{tttt}u + (\alpha+\delta)\partial_{ttt}u + \alpha\delta\partial_{tt}u + \beta A\partial_{tt}u + (\gamma+\delta\beta-\varrho\delta)A\partial_t u + \gamma\delta A u \\
&= \Big[\frac{\mathrm{d}^4\psi}{\mathrm{d}t^4} + (\alpha+\delta)\frac{\mathrm{d}^3\psi}{\mathrm{d}t^3} + (\alpha\delta+\beta\lambda)\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \lambda(\gamma+\delta\beta-\varrho\delta)\frac{\mathrm{d}\psi}{\mathrm{d}t} + \lambda\gamma\delta\psi\Big]w \\
&= 0.
\end{aligned}$$

Moreover, being

$$\begin{aligned}
u(0) &= w,\\
\partial_t u(0) &= (p+rq)w,\\
\partial_{tt}u(0) &= (p^2+2rpq-q^2)w,\\
\partial_{ttt}u(0) &= (p^3+3rp^2q-3pq^2-rq^3)w,
\end{aligned}$$

thanks to the choice of r it is true that

$$\begin{aligned}
\partial_{ttt}u(0) &= -\alpha(p^2+2rpq-q^2)w - \beta\lambda(p+rq)w - \gamma\lambda w\\
&= -\alpha\partial_{tt}u(0) - \beta A\partial_t u(0) - \gamma A u(0).
\end{aligned}$$

Invoking Lemma 4.1, the function u turns out to be the unique solution to (4.1) corresponding to the initial datum

$$z_0 = (w, (p+rq)w, (p^2+2rpq-q^2)w).$$

Finally, setting

$$t_n = \frac{2n\pi}{q} \to +\infty \qquad \text{and} \qquad \varepsilon = 2p > 0,$$

we conclude that

$$\mathsf{F}(t_n) \geq \|u(t_n)\|_1^2 = \lambda\mathrm{e}^{\varepsilon t_n}.$$

The proof of Theorem 5.1 is finished. ☐

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