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RESEARCH

Acyclic Toric Sheaves

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Abstract
Let $\mathcal{E}$ be a torus-linearised reflexive sheaf over a smooth projective toric variety. Generalising a theorem of Perlman and Smith, we prove an explicit sufficient condition for $\mathcal{E}$ to be acyclic via Weil decorations.

Keywords Cohomology of sheaves · Polyhedra · Reflexive sheaves · Toric varieties · Weil decorations

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1 Introduction

For a given sheaf it is natural to ask whether it is *acyclic*, that is, it has no higher cohomology except possibly for H^0, or even *immaculate*, that is, it has no cohomology at all. In the setting of toric geometry, Perlman and Smith [9] give a neat condition for the acyclity of *toric vector bundles*, that is, locally free sheaves with a linearised torus action.

1.1 The Perlman-Smith Theorem

To state it, we let X be a d-dimensional, smooth and projective toric variety over $\mathbb{C}$ with embedded torus $j_\mathrm{T} \colon \mathrm{T} \hookrightarrow X$ and fan Σ. As usual, M denotes the character lattice of T

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and N its dual. Following Klyachko [6], a toric vector bundle $\mathcal{E}$ can be described in terms of $\mathbb{Z}$-descending filtrations of the $\mathbb{C}$-vector space E given by the torus-invariant sections in $\Gamma(\mathrm{T}, \mathcal{E})$, that is,

$$E_\rho^\bullet = \left[\ldots \supseteq E_\rho^{\ell-1} \supseteq E_\rho^{\ell} \supseteq E_\rho^{\ell+1} \supseteq \ldots \right]$$

with $E_\rho^\ell = E$ for $\ell \ll 0$ and $= 0$ for $\ell \gg 0$. They are parametrised by the one-dimensional cones or *rays* $\rho \in \Sigma(1)$ of the fan. This datum is supposed to satisfy a further compatibility condition, see for instance also [8] for a short and concise statement. The filtrations induce the quantities

$$\lambda_\rho := \max \left\{ \ell \in \mathbb{Z} \mid E_\rho^\ell \neq 0 \right\} \quad \text{and} \quad \mu_\rho := \max \left\{ \ell \in \mathbb{Z} \mid E_\rho^\ell = E \right\}.$$

Finally, we recall that a *primitive collection* $\mathcal{P}$ is a subset of rays $\{\rho_1, \ldots, \rho_k\}$ defining a minimal non-face, that is, every strict subset of $\mathcal{P}$ is a set of rays $\tau(1)$ for some cone $\tau \in \Sigma$, but $\mathcal{P}$ itself is not. We let $\sigma(\mathcal{P})$ be the uniquely determined cone in Σ for which $\sum_{\rho \in \mathcal{P}} \rho$ lies in the interior; we note in passing that we always identify a ray with its primitive generator. Consider the natural map $\pi : \mathbb{Z}^{\Sigma(1)} \to N$ that assigns the base vector e_ρ of $\mathbb{Z}^{\Sigma(1)}$ to $\rho \in N$. The *focus of* $\mathcal{P}$ is the unique element

$$f(\mathcal{P}) := \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot e_\rho \in \mathbb{Z}_{\geq 1}^{\sigma(\mathcal{P})(1)}$$

such that the so-called *primitive relation*

$$\mathcal{R}(\mathcal{P}) := \sum_{\rho \in \mathcal{P}} e_\rho - f(\mathcal{P})$$

lies in the kernel of π which is $\mathrm{Cl}(X)^\vee$, the dual of the class group of X.

Theorem (Perlman-Smith [9]) *Let $\mathcal{E}$ be a toric vector bundle on X and $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a* T*-invariant divisor such that for all primitive collections $\mathcal{P} \subseteq \Sigma(1)$, the inequality*

$$\sum_{\rho \in \mathcal{P}} (a_\rho + \mu_\rho) \geq \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot (a_\rho + \lambda_\rho) \tag{1}$$

holds for the focus $f(\mathcal{P}) = \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot \rho$ of $\mathcal{P}$. Then $\mathcal{E}(D) = \mathcal{E} \otimes \mathcal{O}_X(D)$ is acyclic, that is, $\mathrm{H}^i(X, \mathcal{E}(D)) = 0$ *for all $i \geq 1$.*

Remark 1.1 The notion of a primitive collection goes back to Batyrev [4]. As he showed and we use below, the set of primitive relations $\mathcal{R}(\mathcal{P})$ coming from the primitive collections spans the Mori cone $\mathrm{NE}(X) \subseteq \mathrm{Cl}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ which is dual to the nef cone $\mathrm{Nef}(X) \subseteq \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

1.2 Weil Decorations

So-called *Weil decorations* were introduced in the preprint [2] and provide an alternative description of toric vector bundles and in fact of toric sheaves, that is, reflexive sheaves with a T-linearised action. Roughly speaking, a Weil decoration of $\mathcal{E}$ consists of a finite collection of T-invariant divisors $\{D_1, \ldots, D_n\}$ relating to the vector space E from

Section 1.1. From [2], we immediately derive the following vanishing theorem and its corollary, see Section 2.4.

Theorem 1.2 *If k_0 is an integer such that $\mathrm{H}^k\big(X, \mathcal{O}(D_i)\big) = 0$ for $i = 1, \ldots, n$ and all $k \geq k_0$, then also $\mathrm{H}^k(X, \mathcal{E}) = 0$ for all $k \geq k_0$. In particular, this implies that if*

(i) *all $\mathcal{O}_X(D_i)$ are acyclic, then so is $\mathcal{E}$.*
(ii) *all $\mathcal{O}_X(D_i)$ are immaculate, then so is $\mathcal{E}$.*

Remark 1.3 Acyclicity or immaculacy of a line bundle $\mathcal{O}_X(D)$ can be easily read off the divisor by writing D as a (non-unique) difference $D_+ - D_-$ of two torus-invariant nef Cartier divisors. These correspond to lattice polytopes ∇_+ and ∇_- sitting inside $M_{\mathbb{R}}$. By [1, 3] we find

$$\mathrm{H}^k\big(X, \mathcal{O}_X(D)\big)_m = \widetilde{\mathrm{H}}^{k-1}\big(\nabla_- \setminus (\nabla_+ - m)\big),$$

for the cohomology of $\mathcal{O}_X(D)$ in degree $m \in M$, where $\widetilde{\mathrm{H}}^{k-1}$ denotes the $(k-1)$-th reduced singular cohomology with complex coefficients. In particular, we recover *Demazure vanishing*. Namely, a nef line bundle $\mathcal{O}_X(D)$ is acyclic as follows from taking $D_- = 0$.

Corollary 1.4 *If the divisors $D_1, \ldots, D_n$ are nef, then $\mathcal{E}$ is acyclic.*

Remark 1.5 Since ample toric vector bundles are not necessarily acyclic, see for instance [5, Example 4.10] or [2, Remark 6.4], the corollary can be regarded as a substitute for Demazure vanishing of a line bundle, at least if X is smooth and projective.

Corollary 1.4 also yields a straightforward proof for the following generalisation of the Perlman-Smith theorem, see Section 3.1.

Theorem 1.6 *Let $\mathcal{E}$ be a toric sheaf and $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a* T*-invariant divisor such that Inequality* (1) *holds for all primitive collections defining an extremal ray of the Mori cone. Then $\mathcal{E}(D)$ is acyclic.*

We shall give a geometric interpretation of the assumption of Theorem 1.6 in Section 3.2.

Remark 1.7 Theorem 1.6 will follow from the first implication in

$$(1) \implies \text{nefly decorated} \implies \text{acyclicly decorated} \implies \text{acyclic}.$$

The converses are all wrong in general. The twisted tangent bundle $\mathcal{T}_{\mathbb{P}^2}(-4)$ is acyclic with non-acyclic Weil decoration [2, Section 8.5]; $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(D_1 - D_2)$ is acyclicly decorated, but not nefly [2, Example 3.9]. In Example 3.1 we will construct an example of a nefly decorated toric vector bundle which doesn't satisfy (1).

2 Weil Decorations

We first recall some technical background from [2]. In addition to the notation introduced in Section 1, we let $\mathrm{Div}_{\mathrm{T}}(X)$ be the group of *torus-invariant Weil divisors*, that is, the free abelian group generated by the orbit closures

$$D_\rho = \overline{\mathrm{orb}(\rho)}, \quad \rho \in \Sigma(1).$$

For us, a divisor will always mean a torus-invariant Weil divisor, i.e., a divisor in $\mathrm{Div}_{\mathrm{T}}(X)$. Since X is smooth, divisors D stand in 1-1 correspondence with invertible subsheaves $\mathcal{L} = \mathcal{O}_X(D)$ of $j_{\mathrm{T}*}\mathcal{O}_{\mathrm{T}} = \mathbb{C}[M]$ (we slightly abuse notation and drop here and in the sequel the sheafification of $\mathbb{C}[M]$).

2.1 Polytopes and Weil Divisors

A lattice polytope ∇ is *compatible with* Σ if its normal fan $\mathcal{N}(\nabla)$ is refined by Σ. With Minkowski sum as addition,

$$\mathrm{Pol}^+(\Sigma) := \text{the set of compatible lattice polytopes}$$

becomes a cancellative monoid whose associated Grothendieck group is

$$\mathrm{Pol}(\Sigma) := \text{the group of } \textit{virtual} \text{ polytopes.}$$

We think of the latter as formal differences $\nabla = \nabla_+ - \nabla_-$ with $\nabla_\pm \in \mathrm{Pol}^+(\Sigma)$.

Compatibility entails that for every cone $\sigma \in \Sigma$ there is a unique face of $\nabla \in \mathrm{Pol}^+(\Sigma)$ determined by

$$\langle \mathrm{face}(\nabla, \sigma), a\rangle = \min\langle \nabla, a\rangle \quad \text{for all } a \in \sigma. \tag{2}$$

If σ is full-dimensional, then $\mathrm{face}\big(\nabla, \sigma\big)$ is a vertex that we denote $\nabla(\sigma)$. The induced monomials $x^{\nabla(\sigma)} \in \mathbb{C}[M]$ define the invertible subsheaf $\mathcal{O}_X(\nabla)$ of $(j_{\mathrm{T}})_*\mathcal{O}_{\mathrm{T}}$ by

$$\mathcal{O}_X(\nabla)|_{U_\sigma} := x^{\nabla(\sigma)} \cdot \mathcal{O}_{U_\sigma} = \mathbb{C}[\nabla_\sigma \cap M] \subseteq \mathbb{C}[M]$$

where we used the *local polyhedron* $\nabla_\sigma := \nabla(\sigma) + \sigma^\vee = \nabla + \sigma^\vee$. Furthermore,

$$x^{\nabla(\sigma)} \in \mathrm{H}^0(X, \mathcal{O}_X(\nabla)) = \mathbb{C}[\nabla \cap M],$$

which implies that $\mathcal{O}_X(\nabla)$ is globally generated. Equivalently, its associated divisor

$$D_\nabla = -\sum_{\rho \in \Sigma(1)} \min\langle \nabla, \rho\rangle \cdot D_\rho$$

is basepoint free, which for toric varieties is equivalent to being nef.

Conversely, starting with a divisor D, we consider the (possibly empty) *section polyhedron*

$$\mathrm{H}^0_{\mathbb{R}}\left(\sum_{\rho \in \Sigma(1)} a_\rho D_\rho\right) := \big\{u \in M_{\mathbb{R}} \mid \langle u, \rho\rangle \geq -a_\rho,\ \rho \in \Sigma(1)\big\};$$

the integral points $\mathrm{H}^0_{\mathbb{R}}(D) \cap M$ correspond to a $\mathbb{C}$-basis $\{x^m\}$ of global sections of $\mathcal{O}_X(D)$. Then, $\mathrm{H}^0_{\mathbb{R}}(D_\nabla) = \nabla$ for $\nabla \in \mathrm{Pol}^+(\Sigma)$. In particular, the restriction of the map $\mathrm{H}^0_{\mathbb{R}}$ to nef divisors induces a monoid isomorphism onto $\mathrm{Pol}^+(\Sigma)$. By projectivity, the Grothendieck group of $\mathrm{Nef}(\Sigma)$ is $\mathrm{Div}_{\mathrm{T}}(X)$ so that $\mathrm{H}^0_{\mathbb{R}}|_{\mathrm{Nef}}$ extends to a group isomorphism

$$\mathrm{P}\colon\ \mathrm{Div}_{\mathrm{T}}(X) \to \mathrm{Pol}(\Sigma).$$

2.2 The Lattice Structure

The usual poset relation

$$D \leq D' \quad \text{if and only if} \quad D' - D \text{ is effective}$$

Table 1 Relationship between divisors, invertible sheaves and virtual polyhedra

$\mathrm{Div}(\Sigma)$	Invertible sheaves $\subseteq (j_{\mathbf{T}})_*\mathcal{O}_{\mathbf{T}} = \mathbb{C}[M]$	$\mathrm{Pol}(\Sigma)$
$D + D'$	$\mathcal{O}_X(D) \cdot \mathcal{O}_X(D') = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$	$\mathrm{P}(D) + \mathrm{P}(D')$
$-D$	$\mathcal{O}_X(D)^{-1} = \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X(D), \mathcal{O}_X\big)$	$-\mathrm{P}(D)$
$D \leq D'$	$\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D')$	$\mathrm{P}(D) \subseteq \mathrm{P}(D')$
$\min(D, D')$	$\mathcal{O}_X(D) \cap \mathcal{O}_X(D')$	$\mathrm{P}(D) \Cap \mathrm{P}(D')$
$\max(D, D')$	$\big(\mathcal{O}_X(D) + \mathcal{O}_X(D')\big)^{\vee\vee}$	$\mathrm{P}(D) \Cup \mathrm{P}(D')$

turns $\mathrm{Div}_{\mathrm{T}}(X)$ into a *lattice* with *meet*

$$\min\Big(\sum a_\rho D_\rho, \sum a'_\rho D_\rho\Big) := \sum \min\left(a_\rho, a'_\rho\right) D_\rho.$$

and *join*

$$\max\Big(\sum a_\rho D_\rho, \sum a'_\rho D_\rho\Big) := \sum \max(a_\rho, a'_\rho) D_\rho.$$

By [2, Subsection 2.3.2], we have

$$D \leq D' \quad \text{if and only if} \quad \mathrm{P}(D) \subseteq \mathrm{P}(D')$$

for *virtual inclusion* $\subseteq$; it is honest inclusion for polytopes in $\mathrm{Pol}^+(\Sigma)$. Furthermore,

$$\mathcal{O}_X(D \wedge D') = \mathcal{O}_X(D) \cap \mathcal{O}_X(D') \subseteq \mathbb{C}[M],$$

so we define the *virtual intersection* in $\mathrm{Pol}(\Sigma)$ by

$$\nabla \Cap \nabla' := \mathrm{P}\big(\min(D_\nabla, D_{\nabla'})\big).$$

For the join in $\mathrm{Div}_{\mathrm{T}}(X)$, we have

$$\mathcal{O}_X(D \vee D') = \big(\mathcal{O}_X(D) + \mathcal{O}_X(D')\big)^{\vee\vee},$$

for $\mathcal{O}_X(D \vee D')$ is the smallest reflexive subsheaf of $K(X)$ containing both $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$, which by definition is the double dual of $\mathcal{O}_X(D) + \mathcal{O}_X(D')$. We therefore define the *virtual union* of two virtual polyhedra in $\mathrm{Pol}(\Sigma)$ by

$$\nabla \Cup \nabla' := \mathrm{P}\big(\max(D_\nabla, D_{\nabla'})\big).$$

In particular, we obtain the

Proposition 2.1 $\mathrm{P}\colon (\mathrm{Div}_{\mathrm{T}}(X), \leq, \min, \max) \to (\mathrm{Pol}(\Sigma), \subseteq, \Cap, \Cup)$ *defines a lattice isomorphism.*

Table 1 summarises the relationship between divisors, invertible sheaves and virtual polyhedra.

Remark 2.2 Unlike $\subseteq$, $\Cap$ and $\Cup$ do *not* restrict to the usual union and intersection $\cup$ and $\cap$ for nef polytopes.

For instance, take the two ample polytopes ∇ and ∇' for the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in two points fixed by the torus, see Fig. 1 below. They intersect in the shaded square determined by the origin and the point $[2, 2]$. The D_{ρ_4}-coefficient of its corresponding divisor is 4 while the D_{ρ_4}-coefficient of the divisors D_∇ and $D_{\nabla'}$ is 5. Hence

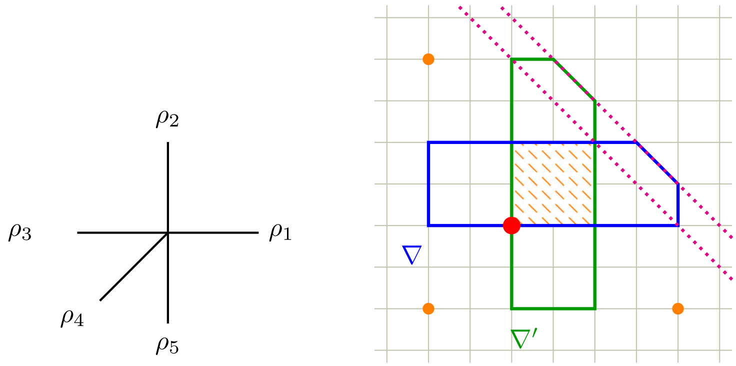

Fig. 1 Left hand side: The fan of the del Pezzo surface. Right-hand side: The big red dot indicates the origin in $M_{\mathbb{R}}$ and fixes the position of the polytopes

$D_{\nabla\cap\nabla'} \neq \min(D_\nabla, D_{\nabla'})$. Note, however, that making ∇ and ∇' sufficiently ample, $\nabla \Cap \nabla'$ actually becomes the honest intersection $\nabla \cap \nabla'$.

Similarly, we see that the actual union of ∇ and ∇' misses out the corners marked in orange, so again $\nabla \Cup \nabla' \neq \nabla \cup \nabla'$. Even worse, this inequality persists under making ∇ and ∇' arbitrarily ample as the union is not even a polytope.

2.3 Weil Decorations

We recall Weil decorations and some of their properties from [2].

Definition 2.3 A *Weil decoration* on a $\mathbb{C}$-vector space E is a map

$$\mathcal{D}\colon \boxed{E^\times := E \setminus \{0\}} \to \mathrm{Div}(\Sigma)$$

which factorises over the projectivisation $\mathbb{P}(E)$ of E and such that for all $e, e' \in E^\times$ with $e + e' \neq 0$, the inequality

$$\mathcal{D}(e+e') \geq \min\big(\mathcal{D}(e), \mathcal{D}(e')\big) \tag{3}$$

holds true.

One can show that (3) implies that the image of $\mathcal{D}$ is finite (see [2, Proposition 3.2] for a proof). Therefore, it has a lower bound which we denote $\mathcal{D}(\eta)$. The notation hints at the fact that $\mathcal{D}(e) = \mathcal{D}(\eta)$ for generic $e \in E^\times$. Furthermore, the divisor

$$\widehat{D} := \bigvee \{\mathcal{D}(e) \mid e \in E^\times\}.$$

is well-defined.

Remark 2.4 In [2] we extended the Weil decoration to all of E by setting formally $\mathcal{D}(0) = \sum \infty \cdot D_\rho$ and working with the usual arithmetic in $\mathbb{Z} \cup \{\infty\}$. While this is convenient for the course of many proofs, it is often more natural to work over $E^\times$. At any rate, we can replace (∞) by any upper bound in $\mathrm{Div}_T(X)$, for instance by $\widehat{D}$ in view of the Perlman-Smith theorem.

Given a toric sheaf $\mathcal{E}$ on X we can build its *associated Weil decoration* as follows. By torsionfreeness of reflexive sheaves, the natural morphism $\mathcal{E} \to (j_{\mathbf{T}})_* j_{\mathbf{T}}^* \mathcal{E} = (j_{\mathbf{T}})_* \mathcal{E}|_{\mathbf{T}}$ is actually injective. Let

$$E := \text{ the torus invariant subspace of } \mathcal{E}(\mathbf{T}) = \Gamma(\mathbf{T}, \mathcal{E});$$

in particular, $\mathcal{E}|_{\mathbf{T}}$ is the sheafification of $E \otimes \mathbb{C}[M]$. As for rank 1 sheaves we therefore consider

$$\mathcal{E} \subseteq (j_{\mathbf{T}})_* \mathcal{E}|_{\mathbf{T}} = E \otimes (j_{\mathbf{T}})_* \mathcal{O}_{\mathbf{T}} = E \otimes \mathbb{C}[M]$$

as a subsheaf of $E \otimes \mathbb{C}[M]$. For $e \in E^\times$ we get the saturated, hence reflexive subsheaf

$$\mathcal{E}(e) := \mathcal{E} \cap \big(\mathbb{C} \cdot e \otimes (j_{\mathbf{T}})_* \mathcal{O}_{\mathbf{T}}\big) \tag{4}$$

which means that $\mathcal{E}(e)(U_\sigma) = \mathcal{E}(U_\sigma) \cap (\mathbb{C} \cdot e \otimes \mathbb{C}[M])$. The isomorphic subsheaf $\mathcal{L}(e)$ of $(j_{\mathbf{T}})_* \mathcal{O}_{\mathbf{T}} = \mathbb{C}[M]$ resulting via

$$\begin{array}{ccc} \mathcal{E}(e) & \hookrightarrow & e \cdot \mathbb{C}[M] \\ {\scriptstyle \cdot 1/e}\downarrow \cong & & \cong \downarrow {\scriptstyle \cdot 1/e} \\ \mathcal{L}(e) & \hookrightarrow & \mathbb{C}[M] \end{array}$$

induces a well-defined Weil divisor $D(e)$ with $\mathcal{O}_X(D(e)) = \mathcal{L}(e)$. Furthermore, the assignment

$$\mathcal{D}_{\mathcal{E}} \colon E^\times \to \mathrm{Div}(X), \quad \mathcal{D}_{\mathcal{E}}(e) := D(e)$$

defines a Weil decoration [2, Subsection 3.2].

Remark 2.5 The Weil decoration $\mathcal{D}_{\mathcal{E}}$ actually determines $\mathcal{E}$. For instance, we recover the Klyachko filtrations via

$$E_\rho^\ell = \{e \in E^\times \mid \big(\mathcal{D}_{\mathcal{E}}(e)\big)_\rho := \text{the coefficient of } D_\rho \text{ in } \mathcal{D}_{\mathcal{E}}(e) \text{ is } \geq \ell\} \cup \{0\}.$$

Remark 2.6 In Section 1 we defined the quantities

$$\lambda_\rho = \max\left\{\ell \in \mathbb{Z} \mid E_\rho^\ell \neq 0\right\} \geq \mu_\rho := \max\left\{\ell \in \mathbb{Z} \mid E_\rho^\ell = E\right\}$$

which are determined by the Klyachko filtrations. Then

$$\mathcal{D}(\eta) = \min\{\mathcal{D}(e) \mid e \in E^\times\} = \sum_{\rho \in \Sigma(1)} \min\{\mathcal{D}(e)_\rho \mid e \in E^\times\} \cdot D_\rho = \sum_{\rho \in \Sigma(1)} \mu_\rho \cdot D_\rho$$

and

$$\widehat{D} = \bigvee \left\{\mathcal{D}(e) \mid e \in E^\times\right\} = \sum_{\rho \in \Sigma(1)} \lambda_\rho \cdot D_\rho.$$

Since we are free to identify $\mathrm{Div}_{\mathrm{T}}(X)$ with $\mathrm{Pol}(\Sigma)$ we will tacitly think of Weil decorations as being $\mathrm{Pol}(\Sigma)$-valued.

Definition 2.7 A toric sheaf $\mathcal{E}$ is *nefly decorated* if the image of $\mathcal{D}_{\mathcal{E}}$ lies in $\mathrm{Pol}^+(\Sigma)$.

To keep notation tight we will usually drop the index $\mathcal{E}$ and simply write $\mathcal{D}$ instead of $\mathcal{D}_{\mathcal{E}}$. Pooling together the vectors $e \in E^\times$ into $\mathbb{S}(D) = \{e \in E^\times \mid \mathcal{D}(e) = D\}$ and setting $\mathcal{S} := \{\mathbb{S}(D) \mid D \in \mathcal{D}(E^\times)\}$ yields a stratification $\mathcal{S} \cup \{0\}$ of E by linear subspaces $\overline{\mathbb{S}}$ for $\mathbb{S} \in \mathcal{S}$. Further, $\mathcal{S}$ carries the natural partial order

$$\mathbb{S} \leq \mathbb{S}' \quad \text{if and only if} \quad \mathbb{S} \subseteq \overline{\mathbb{S}'}. \tag{5}$$

The generic stratum whose divisor is the lower bound of $\mathcal{D}(E)$ is denoted by η. The minimal stratum whose closure contains $\mathbb{S}(D)$ and $\mathbb{S}(D')$ defines a join $\mathbb{S}(D) \vee \mathbb{S}(D')$ on $(\mathcal{S}, \leq)$ and one easily verifies that

$$\mathbb{S}(D) \vee \mathbb{S}(D') = \mathbb{S}(\min(D, D')).$$

Consequently, the Weil decoration $\mathcal{D}$ descends to an anti-semilattice isomorphism onto the image of

$$\mathcal{D}\colon (\mathcal{S}, \leq, \vee) \to (\mathrm{Div}(\Sigma), \leq, \min), \quad \mathbb{S}(D) \mapsto D,$$

that is,

$$\mathcal{D}(\mathbb{S} \vee \mathbb{S}') = \min\big(\mathcal{D}(\mathbb{S}), \mathcal{D}(\mathbb{S}')\big)$$

for all strata $\mathbb{S}, \mathbb{S}' \in \mathcal{S}$.

2.4 The Canonical Resolution

By [2], there is a particular resolution of toric sheaves by invertible ones which turns out to be useful here. To define our complex we consider for $\mathbb{S}, \mathbb{T} \in \mathcal{S}$ the totally split sheaf

$$\mathcal{E}_{\mathbb{S},\mathbb{T}} := \overline{\mathbb{S}} \otimes_{\mathbb{C}} \mathcal{O}_X(\mathbb{T}),$$

where $\overline{\mathbb{S}}$ denotes closure of $\mathbb{S}$ in E and $\mathcal{O}_X(\mathbb{T})$ the line bundle given by the divisor $\mathcal{D}(\mathbb{T})$. For $\mathcal{E}_{\mathbb{S},\mathbb{S}}$ we simply write $\mathcal{E}_{\mathbb{S}}$. Further, we let

$$\mathrm{ch}_\ell(\mathbb{S}, \mathbb{T}) := \{\mathbb{S} = \mathbb{S}_0 < \mathbb{S}_1 < \ldots < \mathbb{S}_{\ell-1} < \mathbb{S}_\ell = \mathbb{T} \mid \mathbb{S}_1, \ldots, \mathbb{S}_{\ell-1} \in \mathcal{S}\}$$

be the set of strict chains of length ℓ in $\mathcal{S}$ starting at $\mathbb{S}_0 = \mathbb{S}$ and terminating at $\mathbb{S}_\ell = \mathbb{T}$. Note that $\mathrm{ch}_\ell(\mathbb{S}, \mathbb{T}) \neq \varnothing$ implies $\mathbb{S} \leq \mathbb{T}$. For instance,

$$\mathrm{ch}_0(\mathbb{S}, \mathbb{T}) = \begin{cases} \{\mathbb{S}\} & \text{if } \mathbb{S} = \mathbb{T} \\ \varnothing & \text{if } \mathbb{S} < \mathbb{T} \end{cases} \quad \text{and} \quad \mathrm{ch}_1(\mathbb{S}, \mathbb{T}) = \begin{cases} \varnothing & \text{if } \mathbb{S} = \mathbb{T} \\ \{\mathbb{S} < \mathbb{T}\} & \text{if } \mathbb{S} < \mathbb{T} \end{cases}.$$

Proposition 2.8 (Corollary 6.10 in [2]) *For any toric sheaf $\mathcal{E}$ of rank r, the complex*

$$\boxed{0 \to \bigoplus_{\mathbb{S} \leq \mathbb{T}} \bigoplus_{\mathrm{ch}_r(\mathbb{S},\mathbb{T})} \mathcal{E}_{\mathbb{S},\mathbb{T}} \to \cdots \to \bigoplus_{\mathbb{S} \leq \mathbb{T}} \bigoplus_{\mathrm{ch}_0(\mathbb{S},\mathbb{T})} \mathcal{E}_{\mathbb{S},\mathbb{T}}} \to \mathcal{E} \to 0 \tag{6}$$

with the usual differential is exact. In particular, the line bundles occurring in this complex are nef if $\mathcal{E}$ is nefly decorated.

For the summand $\mathcal{E}_{\mathbb{S}_0,\mathbb{S}_\ell} \subseteq \bigoplus_{\mathbb{S} \leq \mathbb{T}} \bigoplus_{\mathrm{ch}_\ell(\mathbb{S},\mathbb{T})} \mathcal{E}_{\mathbb{S},\mathbb{T}}$ corresponding to the chain $\mathbb{S}_0 < \ldots < \mathbb{S}_\ell$, we could have written $\mathcal{E}_{\mathbb{S}_0,\ldots,\mathbb{S}_\ell}$ for $\mathcal{E}_{\mathbb{S}_0,\mathbb{S}_\ell}$ even though the summands do not depend on the inner terms.

The boxed complex in Eq. 6 is thus quasi-isomorphic to $\mathcal{E}$. Taking the standard spectral sequence of hypercohomology, namely,

$$E_1^{-\ell,q} = \mathrm{H}^q\left(X, \bigoplus_{\mathbb{S} \leq \mathbb{T}} \bigoplus_{\mathrm{ch}_\ell(\mathbb{S},\mathbb{T})} \mathcal{E}_{\mathbb{S},\mathbb{T}}\right) \Rightarrow \mathrm{H}^{q-\ell}(X, \mathcal{E})$$

for $0 \leq \ell \leq r = \mathrm{rk}(\mathcal{E})$ and $0 \leq q \leq d = \dim(X)$, yields the

Theorem 2.9 *If k_0 is an integer such that $\mathrm{H}^k\big(X, \mathcal{O}(D(e))\big) = 0$ for $e \in E^\times$ and all $k \geq k_0$, then also $\mathrm{H}^k(X, \mathcal{E}) = 0$ for all $k \geq k_0$. In particular, this implies that if for all $e \in E^\times$ the line bundle*

(i) $\mathcal{O}_X(D(e))$ *is acyclic, then so is* $\mathcal{E}$.
(ii) $\mathcal{O}_X(D(e))$ *is immaculate, then so is* $\mathcal{E}$.

Remark 2.10 As the set of the divisors $D(e)$ is finite, Theorem 2.9 is really Theorem 1.2 from the introduction.

Remark 1.3 or Demazure vanishing immediately implies the

Corollary 2.11 *A nefly decorated toric sheaf* $\mathcal{E}$ *is acyclic.*

Remark 2.12 As a quotient of a direct sum of nef line bundles, a nefly decorated $\mathcal{E}$ is actually also nef, cf. [7, Theorem 6.2.12].

3 Acyclicity of Toric Sheaves

3.1 Proof of Theorem 1.6

First, we may assume that without loss of generality $D = 0$ for the twisting sheaf $\mathcal{O}_X(D)$. If $D = \sum a_\rho D_\rho \neq 0$, we pass to the toric sheaf $\mathcal{E}' := \mathcal{E}(D)$. This merely translates the Weil decoration by D, namely $\mathcal{D}'(e) = \mathcal{D}(e) + D$ for all $e \in E^\times$. In particular, $\mathcal{D}'(\eta) = \mathcal{D}(\eta) + D$ and $\widehat{D}' = \widehat{D} + D$ and thus $\mu'_\rho = a_\rho + \mu_\rho$ and $\lambda'_\rho = a_\rho + \lambda_\rho$, cf. Section 2.3. The assumption of Theorem 1.6 therefore becomes

$$\sum_{\rho \in \mathcal{P}} \mu_\rho \geq \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot \lambda_\rho \tag{7}$$

for all primitive relations $\mathcal{P}$ whose primitive inequality $\mathcal{R}(\mathcal{P})$ defines an extremal ray of $\mathrm{NE}(X) \subseteq \mathrm{Cl}(X)^\vee$.

We claim that this forces $\mathcal{E}$ to be nefly decorated; actually, even $\widehat{D}$ becomes nef. Indeed, let $\mathcal{D}(e) = \sum b_\rho D_\rho$ be a divisor of the Weil decoration. Then $\mathcal{D}(e) \geq \mathcal{D}(\eta)$ implies $\lambda_\rho \geq b_\rho \geq \mu_\rho$ and thus

$$\langle \mathcal{D}(e), \mathcal{R}(\mathcal{P}) \rangle = \sum_{\rho \in \mathcal{P}} b_\rho - \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot b_\rho \geq \sum_{\rho \in \mathcal{P}} \mu_\rho - \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot \lambda_\rho \geq 0$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing in $\mathbb{Z}^{\Sigma(1)}$. Since this holds for all extremal primitive collections and also for $\widehat{D}$, we obtain that $\mathcal{D}(e)$ and $\widehat{D}$ are nef. □

3.2 Geometric Interpretation of the Assumption (1)

Though the Perlman-Smith assumption is too strong for acyclity, it is nevertheless worthwile giving a polyhedral interpretation. Towards that end, we continue to assume $D = 0$ and write the resulting (7) as

$$\boxed{\langle \mathcal{D}(\eta), \mathcal{R}(\mathcal{P}) \rangle = \textstyle\sum_{\rho \in \mathcal{P}} \mu_\rho - \sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot \mu_\rho} \geq \boxed{\textstyle\sum_{\rho \in \sigma(\mathcal{P})(1)} f_\rho \cdot (\lambda_\rho - \mu_\rho)}. \tag{8}$$

Since the assumption of Theorem 1.6 implies that $\mathcal{E}$ is nefly decorated, we may straight away assume that $\nabla(e) \in \mathrm{Pol}^+(\Sigma)$ for all $e \in E^\times$.

3.2.1 The Left Hand Side

Since $\mathcal{R}(\mathcal{P}) \in \mathrm{Cl}(X)^\vee$, the pairing only depends on the class $[\mathcal{D}(\eta)] \in \mathrm{Cl}(X)$ and not on the concrete representative $\mathcal{D}(\eta)$.

First, for every primitive collection giving rise to an extremal ray of the Mori cone, the primitive relation $\mathcal{R}(\mathcal{P}) \in \mathrm{NE}(X) \subseteq \mathrm{Cl}(X)^\vee$ may be regarded as the class of a curve $[\overline{\mathrm{orb}}(\tau)]$ given by a wall $\tau \in \Sigma(d-1)$ of the fan. Hence

$$\langle \mathcal{D}(e), \mathcal{R}(\mathcal{P})\rangle = \langle [\mathcal{D}(e)],\ \overline{\mathrm{orb}}(\tau)\rangle$$

where the right hand side denotes the intersection product.

Second, each wall τ corresponds to an edge $k_\tau(\Delta)$ of any ample polytope Δ and more generally of a nef polytope if we allow edges to degenerate to a vertex. In particular, we obtain for every $e \in E^\times$ a (possibly degenerate) edge $k_\tau(e) \leq \nabla(e)$. Then $\langle [\mathcal{D}(e)],\ \overline{\mathrm{orb}}(\tau)\rangle$ is precisely the lattice length $\ell(k_\tau(e))$. In particular, the pairing of the left hand side is given by

$$\langle \mathcal{D}(\eta), \mathcal{R}(\mathcal{P})\rangle = \ell(k_\tau(\eta))$$

for $\mathcal{P} = \mathcal{P}(\tau)$.

3.2.2 The Right Hand Side

By definition, the focus $f(\mathcal{P}) = \sum_{\rho\in\sigma(\mathcal{P})(1)} f_\rho \cdot e_\rho$ introduced in Section 1 has the same image under $\pi : \mathbb{Z}^{\Sigma(1)} \to N$ as the incidence vector

$$e_{\mathcal{P}} := \sum_{\rho\in\mathcal{P}} e_\rho$$

of the primitive collection $\mathcal{P} \subseteq \Sigma(1)$. Next, we may regard $\widehat{D} - \mathcal{D}(\eta) = \widehat{\nabla} - \nabla(\eta)$ as a formal difference of polytopes $\widehat{\nabla} \supseteq \nabla(\eta)$ in $\mathrm{Pol}(\Sigma)$. From Eq. 2 we obtain for $\sigma = \sigma(\mathcal{P})$ a difference vector $v(\sigma)$ in M determined up to $\sigma^\perp$ pointing from $\mathrm{face}(\nabla(\eta), \sigma)$ to $\mathrm{face}(\widehat{\nabla}, \sigma)$, which we somehow abusively write

$$v(\sigma) := \mathrm{face}(\widehat{\nabla}, \sigma) - \mathrm{face}(\nabla(\eta), \sigma) \in M/\sigma^\perp.$$

Via the standard embedding of M into $\mathbb{Z}^{\Sigma(1)}$, pairing $v(\sigma)$ with $f(\mathcal{P})$ yields

$$\begin{aligned}\sum_{\rho\in\sigma(\mathcal{P})(1)} \big(\widehat{D} - \mathcal{D}(\eta)\big)_\rho \cdot f_\rho \;&=\; -\langle v(\sigma),\ e_{\mathcal{P}}\rangle \\ &=\; \min\langle \nabla(\eta),\ e_{\mathcal{P}}\rangle - \min\langle \widehat{\nabla},\ e_{\mathcal{P}}\rangle\end{aligned}$$

and so magically eliminates the distinguished cone σ of the primitive collection $\mathcal{P}$.

Putting everything together, Assumption (1) reads for $D = 0$ and a nefly decorated $\mathcal{E}$ in polyhedral terms as follows: For any extremal primitive collection $\mathcal{P} = \mathcal{P}(\tau) \subseteq \Sigma(1)$ associated with the wall $\tau \in \Sigma(d-1)$, the inequality

$$\ell\big(k_\tau(\eta)\big) \;\geq\; \min\langle \nabla(\eta),\ e_{\mathcal{P}}\rangle - \min\langle \widehat{\nabla},\ e_{\mathcal{P}}\rangle \tag{9}$$

holds true. Finally, the difference on the right hand side can be expressed as

$$\sum_{\rho\in\mathcal{P}} \big(\min\langle \nabla(\eta),\ \rho\rangle - \min\langle \widehat{\nabla},\ \rho\rangle\big)$$

where each summand encodes the distance between the ρ-facets of $\nabla(\eta)$ and $\widehat{\nabla}$.

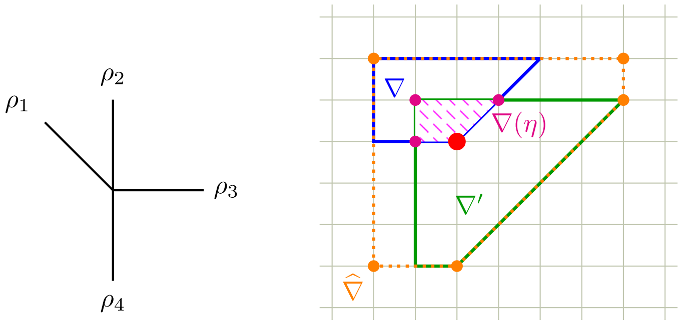

Fig. 2 Left hand side: The fan of the first Hirzebruch surface. Right-hand side: The Weil decoration of $\mathcal{O}_{\mathcal{H}}(\nabla) \oplus \mathcal{O}_{\mathcal{H}}(\nabla)'$

Example 3.1 Equation 9 always holds on a projective plane and more generally on any projective space $\mathbb{P}^n$, for there is only one primitive collection $\mathcal{P} = \{\rho_0, \ldots, \rho_n\}$ with $e_{\mathcal{P}} = 0$. But already on the first Hirzebruch surface $\mathcal{H}$, assumption (8) is strictly stronger than nefly decorated.

Indeed, for the fan given in Fig. 2 below, the Mori cone has two extremal rays R_1 and R_2 corresponding to the primitive collections $\mathcal{P}_1 = \{\rho_2, \rho_4\}$ and $\mathcal{P}_2 = \{\rho_1, \rho_3\}$, respectively. Now again $e_{\mathcal{P}_1} = 0$, but $e_{\mathcal{P}_2} = e_{\rho_2}$. Since R_2 is induced by the ray ρ_2, we obtain the possibly nontrivial inequality

$$\ell(k_{\rho_2}(\eta)) \;\geq\; \min\langle \nabla(\eta), \rho_2\rangle - \min\langle \widehat{\nabla}, \rho_2\rangle.$$

Next, let $E = \mathbb{C}v \oplus \mathbb{C}v'$ with stratification 0, $\mathbb{S} = \mathbb{C}v \setminus \{0\}$, $\mathbb{S}' = \mathbb{C}v' \setminus \{0\}$ and generic stratum η, which we decorate by the polytopes in Fig. 2, that is,

$$0 \mapsto \widehat{\nabla}, \quad \mathbb{S} \mapsto \nabla, \quad \mathbb{S}' \mapsto \nabla', \quad \eta \mapsto \nabla \cap \nabla'.$$

This defines the Weil decoration of $\mathcal{O}_{\mathcal{H}}(\nabla) \oplus \mathcal{O}_{\mathcal{H}}(\nabla)'$ on $\mathcal{H}$. Clearly, it is nefly (even amply) decorated. However,

$$\min\langle \nabla(\eta), \rho_2\rangle - \min\langle \widehat{\nabla}, \rho_2\rangle = 0 - (-3) = 3$$

while $\ell(k_{\rho_2}(\eta)) = 2$, which violates Inequality (8).

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Declarations

Ethical Approval Not applicable.

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References

1. Altmann, K., Buczyński, J., Kastner, L., Winz, A.-L.: Immaculate line bundles on toric varieties. Pure Appl. Math Quarter. **16**(4), 1147–1217 (2020)
2. Altmann, K., Hochenegger, A., Witt, F.: Toric sheaves and polyhedra. Preprint arXiv:2412.03476 (2024)
3. Altmann, K., Ploog, D.: Displaying the cohomology of toric line bundles. Izv. Math. **84**(4), 683–693 (2020)
4. Batyrev, V.: On the classification of smooth projective toric varieties. Tohoku Math. J. (2) **43**(4), 569–585 (1991)
5. Hering, M., Mustaţă, M., Payne, S.: Positivity properties of toric vector bundles. Ann. Inst. Fourier **60**(2), 607–640 (2010)
6. Klyachko, A.: Equivariant bundles on toral varieties. Math. USSR, Izv. **35**(2), 337–375 (1990)
7. Lazarsfeld, R.: Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals., volume 49 of Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer (2004)
8. Payne, S.: Moduli of toric vector bundles. Compos. Math. **144**(5), 1199–1213 (2008)
9. Smith, G.: Tbd: Toric bundles duh! In: Erman D., Hering M., Ilten N., Süß H. (eds.) Toric geometry, Oberwolfach Report, vol. 22(2), pp. 934–935 (2025)