SOME UNEXPLORED QUESTIONS ARISING IN LINEAR VISCOELASTICITY

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Abstract. We consider the abstract integrodifferential equation

$$\ddot{u}(t) + A\Big[u(t) + \int_0^\infty \mu(s)[u(t) - u(t-s)]ds\Big] = 0$$

modeling the dynamics of linearly viscoelastic solids. The equation is known to generate a semigroup $S(t)$ on a certain phase space, whose asymptotic properties have been the object of extensive studies in the last decades. Nevertheless, some relevant questions still remain open, with particular reference to the decay rate of the semigroup compared to the decay of the memory kernel μ, and to the structure of the spectrum of the infinitesimal generator of $S(t)$. This paper intends to provide some answers.

Contents

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1. Introduction

Let H be a real Hilbert space, and let $A : H \to H$ be a strictly positive selfadjoint linear operator with $\mathrm{dom}(A) \subset H$. For $t > 0$, we consider the abstract integrodifferential equation in the unknown $u = u(t) : \mathbb{R} \to H$

$$\ddot{u}(t) + A\Big[u(t) + \int_0^\infty \mu(s)[u(t) - u(t-s)]ds\Big] = 0, \tag{1.1}$$

which serves as a model in the description of the dynamics of linearly viscoelastic solids. Here, the *dot* stands for the derivative with respect to time, and the convolution -or memory- kernel

$$\mu : (0, \infty) \to [0, \infty)$$

is a summable nonincreasing function whose properties will be specified below. The function u for negative times is regarded as an initial datum. As detailed later in this paper, equation (1.1), properly translated in a suitable framework, gives rise to a strongly continuous semigroup

$$S(t) = e^{t\mathbb{A}}$$

acting on a certain Hilbert space $\mathcal{H}$, that embodies the information on the past history of the variable u. Defining the function

$$g(s) = 1 + \int_s^\infty \mu(r)dr,$$

equation (1.1) is sometimes written in the equivalent form

$$\ddot{u}(t) + A\Big[g(0)u(t) + \int_0^\infty g'(s)u(t-s)]ds\Big] = 0. \tag{1.2}$$

Remark 1.1. A concrete realization of (1.2) is the boundary value problem

$$\begin{cases} \partial_{tt}u(t) - g(0)\Delta u(t) - \displaystyle\int_0^\infty g'(s)\Delta u(t-s)ds = 0, \\ u(t)_{|\partial\Omega} = 0, \end{cases}$$

ruling the evolution of the relative displacement in a linearly viscoelastic solid occupying a volume $\Omega \subset \mathbb{R}^N$ at rest [37]. In this case $H = L^2(\Omega)$ and

$$A = -\Delta \qquad \text{with} \qquad \mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Another relevant model, used for instance in the description of the vibrations of thin viscoelastic rods, is obtained by adding an additional term to the equation, namely,

$$\begin{cases} \partial_{tt}u(t) - \nu\Delta\partial_{tt}u(t) - g(0)\Delta u(t) - \displaystyle\int_0^\infty g'(s)\Delta u(t-s)ds = 0, \\ u(t)_{|\partial\Omega} = 0, \end{cases}$$

with $\nu > 0$ small (see [28]). The latter equation can be given the form (1.1) by setting

$$A = -(1 - \nu\Delta)^{-1}\Delta.$$

Observe that now A is (or, more precisely, extends to) a bounded linear operator on $L^2(\Omega)$.

The analysis of the semigroup $S(t)$ and its asymptotic features has been carried out by several authors since the Seventies of the last century. We mention in this direction the works [2, 3, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 21, 24, 25, 26, 27, 28, 31, 32, 33, 34, 37] but the list is far from being exhaustive. Concerning the decay properties of the semigroup, necessary and sufficient conditions on the kernel μ, among a reasonably large class of admissible kernels, are nowadays well known in order for $S(t)$ to be stable or exponentially stable (see [2, 34]). Nonetheless, in spite of the vast literature on the subject, there are still some relevant issues that so far appear to be completely unexplored:

- How the decay rate of $S(t)$ depends on the decay rate of the memory kernel μ?
- In particular, does a fast decay of μ translate into a fast decay of the semigroup norm?
- If μ has a superexponential decay (e.g., μ is compactly supported), is it generally true that the semigroup norm decays faster than any exponential? In other words, are superexponential decays possible in the context of linear viscoelasticity? Incidentally, this would give some insight on the persistence of the elastic force versus viscoelastic effects.
- Are there nontrivial trajectories of $S(t)$ vanishing in finite time? When this is not the case, some authors talk of impossibility of localization of solutions (see e.g., [36]). Again, from the physical viewpoint this is related to the persistence of the elastic force, and in mathematical terms to the backward uniqueness property of the semigroup.

One of the reasons that might explain why the questions above have not been addressed so far is that we have a very poor knowledge of the spectrum $\sigma(\mathbb{A})$ of the infinitesimal generator $\mathbb{A}$, whose structure has a deep influence on the decay of the trajectories. Indeed, as we will see, the presence of the memory, that confers the equation a nonlocal character, introduces a certain degree of complexity. Such a complexity completely disappears in the limit situations where $g(s) - 1$ equals the Dirac mass at zero. In which case, (1.2) formally becomes

$$\ddot{u}(t) + A\big[u(t) + \dot{u}(t)\big] = 0,$$

known also as the Kelvin-Voigt model for viscoelasticity, whose semigroup possesses an infinitesimal generator with a very simple spectrum (see, e.g., [9]).

The aim of the present paper is to address these points, providing in particular the full characterization of $\sigma(\mathbb{A})$, at least for the case of the exponential kernel.

Remark 1.2. One might argue that, for more generality, and since we are mostly interested in the decay properties of the solutions depending on the parameters in play, we should have considered in place of (1.1) the equation

$$\ddot{u}(t) + A\Big[\alpha u(t) + \int_0^\infty \mu(s)[u(t) - u(t-s)]ds\Big] = 0,$$

with $\alpha > 0$, which is exactly the value of $g(\infty)$. But redefining the operator A to be αA and $\mu(s)$ to be $\frac{1}{\alpha}\mu(s)$, we boil down to (1.1). In particular, a multiplication of A or μ by a positive constant does not affect in any way the decay properties of the semigroup.

2. Preliminaries and Notation

2.1. **Functional setting.** We denote the inner product and the norm in the space H by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Besides, we introduce the further Hilbert space

$$H^1 = \mathrm{dom}(A^{\frac{1}{2}}), \qquad \langle u, v\rangle_1 = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle, \qquad \|u\|_1 = \|A^{\frac{1}{2}}u\|.$$

Definition 2.1. A function $\mu : (0,\infty) \to [0,\infty)$ is said to be an *admissible kernel* if the following hold:

- μ is nonincreasing and maps Lebesgue nullsets into nullsets;
- the set of the (possibly infinitely many) jump points of μ has empty interior;
- μ is summable with total mass

$$\varkappa = \int_0^\infty \mu(s)ds > 0.$$

We agree to choose the right-continuous representative of μ.

In what follows, μ is understood to be an admissible kernel. Notice that μ is a nonnull and piecewise absolutely continuous function vanishing at infinity. In particular, μ is differentiable almost everywhere with $\mu' \leq 0$, but it can be unbounded about zero. We define the number

$$s_\star = \sup\Big\{s > 0 :\ \mu(s) > 0\Big\} \in (0,\infty], \tag{2.1}$$

namely, the supremum of the support of μ. The two situations $s_\star < \infty$ (i.e., μ compactly supported) and $s_\star = \infty$ (i.e., μ strictly positive) are referred to as *finite* and *infinite memory*, respectively.

Definition 2.2. An admissible kernel μ is said to fulfill the δ*-condition*, for some $\delta > 0$, if there exists $C \geq 1$ such that

$$\mu(t+s) \leq Ce^{-\delta t}\mu(s), \tag{2.2}$$

for every $t \geq 0$ and $s > 0$.

Remark 2.3. It is readily seen that if $s_\star < \infty$ then μ fulfills the δ-condition for every $\delta > 0$. Observe also that (2.2) implies that μ exhibits an exponential decay of rate δ (at least). Indeed, for $s \geq 1$ we have

$$\mu(s) = \mu(1+s-1) \leq Qe^{-\delta s},$$

with $Q = Ce^{\delta}\mu(1)$.

For any given μ, we consider the L^2-weighted space on $(0, s_\star)$

$$\mathcal{M} = L^2_\mu(0, s_\star; H^1),$$

referred to as *memory space*, endowed with the inner product and norm

$$\langle\eta,\zeta\rangle_{\mathcal{M}} = \int_0^{s_\star}\mu(s)\langle\eta(s),\zeta(s)\rangle_1 ds, \qquad \|\eta\|_{\mathcal{M}} = \Big(\int_0^{s_\star}\mu(s)\|\eta(s)\|_1^2 ds\Big)^{\frac{1}{2}}.$$

Finally, we define the *extended memory space*

$$\mathcal{H} = H^1 \times H \times \mathcal{M},$$

endowed with the product norm

$$\|(u, v, \eta)\|_{\mathcal{H}}^2 = \|u\|_1^2 + \|v\|^2 + \|\eta\|_{\mathcal{M}}^2.$$

We will also denote the Banach spaces of bounded linear operators on $\mathcal{M}$ and $\mathcal{H}$ by $L(\mathcal{M})$ and $L(\mathcal{H})$, respectively.

2.2. **A convolution lemma.** A key tool for our analysis is the following technical result.

Lemma 2.4. *Let $s_\star < \infty$, and let $q \in L^1(0, s_\star - \varepsilon)$ for every $\varepsilon > 0$ small. If the equality*

$$\int_t^{s_\star} \mu(s) q(s-t) ds = 0$$

holds for every $t \in [0, s_\star]$, then q identically vanishes on $[0, s_\star]$.

Proof. Without loss of generality, we assume $s_\star = 1$. For $\varepsilon > 0$ small, define

$$q_\varepsilon(s) = \begin{cases} q(s) & \text{if } s \le 1 - \varepsilon, \\ 0 & \text{if } s > 1 - \varepsilon. \end{cases}$$

Since $\mu(s) = 0$ for $s > 1$, setting

$$\mu_\varepsilon(s) = \mu(s + \varepsilon),$$

the equality above implies that

$$\int_t^1 \mu_\varepsilon(s) q_\varepsilon(s-t) ds = 0, \quad \forall t \in [0, 1]. \tag{2.3}$$

Indeed, since $\mu_\varepsilon(s) = 0$ for $s > 1 - \varepsilon$, the latter integral vanishes if $t > 1 - \varepsilon$, whereas if $t \le 1 - \varepsilon$ by a change of variable we get

$$\int_t^1 \mu_\varepsilon(s) q_\varepsilon(s-t) ds = \int_{t+\varepsilon}^1 \mu(s) q(s - (t+\varepsilon)) ds = 0,$$

thanks to our hypotheses. Here we used the fact that, when s runs in the integration interval, the argument $s - (t + \varepsilon)$ of q remains less than or equal to $1 - \varepsilon$, meaning that q and q_ε coincide. Extending now μ_ε and q_ε on $\mathbb{R}$ to be zero outside $[0, 1]$, and defining then the reflection

$$\tilde{q}_\varepsilon(s) = q_\varepsilon(-s),$$

we can rephrase (2.3) in terms of convolution on the real line as

$$\operatorname{supp}(\mu_\varepsilon * \tilde{q}_\varepsilon) \subset [-1, 0].$$

Since $\mu_\varepsilon, \tilde{q}_\varepsilon \in L^1(\mathbb{R})$, the Titchmarsh convolution theorem [42] provides the equality

$$\max \operatorname{supp}(\mu_\varepsilon * \tilde{q}_\varepsilon) = \max \operatorname{supp}(\mu_\varepsilon) + \max \operatorname{supp}(\tilde{q}_\varepsilon).$$

Since $\max \operatorname{supp}(\mu_\varepsilon) = 1 - \varepsilon$, we learn that

$$\max \operatorname{supp}(\tilde{q}_\varepsilon) \le -1 + \varepsilon.$$

On the other hand, by construction,

$$\operatorname{supp}(\tilde{q}_\varepsilon) \subset [-1 + \varepsilon, 0].$$

This tells that $\tilde{q}_\varepsilon$ is identically zero, and so is q_ε. Meaning that the original function q vanishes on $[0, 1 - \varepsilon]$. Since $\varepsilon > 0$ is arbitrary, we are done. $\square$

3. The Right-Translation Semigroup on the Memory Space

We now introduce the right-translation semigroup $R(t)$ acting on the memory space $\mathcal{M}$, which will play a crucial role in the definition of the dynamical system generated by the viscoelastic equation (1.1). This is the one-parameter map

$$\eta \in \mathcal{M} \mapsto R(t)\eta \in \mathcal{M}$$

defined as follows:

$$[R(t)\eta](s) = \begin{cases} 0 & 0 < s \leq t, \\ \eta(s-t) & s > t. \end{cases}$$

Indeed, this map is well defined as μ is nonincreasing. We recall some known facts, which hold true for any given admissible kernel μ (see [2, 19, 33]).

- **(i)** $R(t)$ is a contraction semigroup, that is, $\|R(t)\|_{L(\mathcal{M})} \leq 1$ for all $t \geq 0$. Besides, $R(t)$ is nilpotent if (and only if) $s_\star < \infty$.
- **(ii)** The infinitesimal generator of $R(t)$ is the linear operator T on $\mathcal{M}$ defined as
$$[T\eta](s) = -\eta'(s),$$
with domain
$$\mathrm{dom}(T) = \big\{\eta \in \mathcal{M} \,:\, \eta' \in \mathcal{M},\ \eta(0) = 0\big\},$$
where the *prime* stands for the distributional derivative with respect to s, and $\eta(0)$ is the limit in H^1 of $\eta(s)$ as $s \to 0$.
- **(iii)** $R(t)$ is stable, that is, $\|R(t)\eta\|_{\mathcal{M}} \to 0$ as $t \to \infty$ for any $\eta \in \mathcal{M}$.
- **(iv)** For $C \geq 1$ and $\delta > 0$, the exponential decay estimate
$$\|R(t)\|^2_{L(\mathcal{M})} \leq Ce^{-\delta t}$$
holds if and only if (2.2) holds.

In the present work we want to understand more deeply in which terms the decay of μ affects the decay of the norm of $R(t)$.

Proposition 3.1. *For every $\varepsilon > 0$ (and less than $s_\star$),*

$$\|R(t)\|^2_{L(\mathcal{M})} \geq \frac{\mu(t+\varepsilon)}{\mu(\varepsilon)}.$$

Proof. Given a unit vector $w \in H^1$, define the sequence of unit vectors $\eta_n \in \mathcal{M}$ as follows:

$$\eta_n(s) = C_n \chi_{[\varepsilon, \varepsilon + \frac{1}{n}]}(s) w \qquad \text{with} \qquad C_n = \Big(\int_\varepsilon^{\varepsilon+\frac{1}{n}} \mu(s)ds\Big)^{-\frac{1}{2}}.$$

Then, from the right-continuity of μ,

$$\|R(t)\eta_n\|^2_{\mathcal{M}} = \frac{\displaystyle\int_\varepsilon^{\varepsilon+\frac{1}{n}} \mu(t+s)ds}{\displaystyle\int_\varepsilon^{\varepsilon+\frac{1}{n}} \mu(s)ds} \geq \frac{\mu(t+\varepsilon+\frac{1}{n})}{n\displaystyle\int_\varepsilon^{\varepsilon+\frac{1}{n}} \mu(s)ds} \to \frac{\mu(t+\varepsilon)}{\mu(\varepsilon)},$$

as $n \to \infty$. Since for every n

$$\|R(t)\|_{L(\mathcal{M})} \geq \|R(t)\eta_n\|_{\mathcal{M}},$$

the claim follows.

In particular, Proposition 3.1 tells that if $\mu(0) < \infty$ then there is a constant $c > 0$, which is exactly $1/\sqrt{\mu(0)}$, such that

$$\|R(t)\|_{L(\mathcal{M})} \geq c\,\sqrt{\mu(t)}\,. \tag{3.1}$$

The lower bound (3.1) is generally false if $\mu(0) = \infty$, as the following example shows.

Example 3.2. Consider the (admissible) kernel

$$\mu(s) = \frac{e^{-s^2}}{\sqrt{s}}.$$

Given any $\eta \in \mathcal{M}$ of unit norm, we have

$$\|R(t)\eta\|^2_{\mathcal{M}} = \frac{e^{-t^2}}{t}\int_0^\infty \mu(s)\|\eta(s)\|_1^2\Phi(s,t)ds,$$

where

$$\Phi(s,t) = \frac{\sqrt{t}}{\sqrt{s+t}}\,e^{-2st}\sqrt{st} \leq e^{-2st}\sqrt{st} \leq K,$$

having set $K = \max_{x>0} e^{-2x}\sqrt{x}$. Accordingly,

$$\|R(t)\eta\|^2_{\mathcal{M}} \leq \frac{Ke^{-t^2}}{t} = \frac{K}{\sqrt{t}}\,\mu(t),$$

which tells that the decay rate of the norm of $R(t)$ is faster than $\sqrt{\mu(t)}$.

However, even if $\mu(0) < \infty$, hence (3.1) holds, one can generally find nontrivial trajectories whose energy decays faster than $\mu(t)$.

Example 3.3. Consider the kernel $\mu(s) = e^{-s^2}$, and choose $\eta(s) = w$ with $\|w\|_1 = 1$. Then

$$\|R(t)\eta\|^2_{\mathcal{M}} = \int_0^\infty e^{-(t+s)^2}ds = \frac{\sqrt{\pi}}{2} - \int_0^t e^{-s^2}ds \sim \frac{1}{2t}\,\mu(t),$$

as $t \to \infty$.

We will compare the decay rate of the norm of $R(t)$ with the one of the semigroup generated by the viscoelastic equation (1.1).

4. The Solution Semigroup: A Brief Survey

4.1. **The equation in the past history framework.** The standard way to recast equation (1.1) in order to obtain a solution semigroup is working in the so-called *history space framework*, devised by Dafermos in [5]. This amounts to introducing an auxiliary variable $\eta = \eta^t(s)$, formally defined as,

$$\eta^t(s) = u(t) - u(t-s), \quad t \geq 0,\ s \in (0, s_\star),$$

with $s_\star$ given by (2.1), which keeps track of the past history of u, and is assigned as an initial datum at $t=0$. Then (1.1) translates into the system

$$
\begin{cases} \ddot{u} + A\Big[u + \displaystyle\int_0^{s_\star} \mu(s)\eta(s)ds\Big] = 0, \\ \dot{\eta} = T\eta + \dot{u}. \end{cases} \tag{4.1}
$$

Introducing the linear operator $\mathbb{A}$ on $\mathcal{H}$ acting as

$$
\mathbb{A}(u,v,\eta) = \Big(v, -A\Big[u + \int_0^{s_\star} \mu(s)\eta(s)ds\Big], T\eta + v\Big),
$$

with domain

$$
\mathrm{dom}(\mathbb{A}) = \left\{ (u,v,\eta) \in \mathcal{H} \;\middle|\; \begin{array}{c} v \in H^1 \\ u + \int_0^{s_\star} \mu(s)\eta(s)ds \in \mathrm{dom}(A) \\ \eta \in \mathrm{dom}(T) \end{array} \right\},
$$

system (4.1) can be equivalently written as the ODE in $\mathcal{H}$

$$
\frac{d}{dt}\boldsymbol{u}(t) = \mathbb{A}\boldsymbol{u}(t).
$$

For every initial datum $\boldsymbol{u}_0 = (u_0, v_0, \eta_0) \in \mathcal{H}$, such an ODE is known to possess a unique weak solution

$$
t \mapsto \boldsymbol{u}(t) = (u(t), u_t(t), \eta^t) \in \mathcal{C}([0,\infty), \mathcal{H}).
$$

Besides, the third component η^t fulfills the explicit representation form

$$
\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ \eta_0(s-t) + u(t) - u_0 & s > t. \end{cases} \tag{4.2}
$$

Accordingly, (4.1) generates a strongly continuous semigroup $S(t) : \mathcal{H} \to \mathcal{H}$ acting by the rule [11, 35]

$$
S(t)\boldsymbol{u}_0 = \boldsymbol{u}(t),
$$

and whose infinitesimal generator is the linear operator $\mathbb{A}$. The energy at time t corresponding to the initial datum $\boldsymbol{u}_0$ reads

$$
\mathsf{E}(t) = \frac{1}{2}\|S(t)\boldsymbol{u}_0\|^2_{\mathcal{H}} = \frac{1}{2}\Big[\|u(t)\|_1^2 + \|u_t(t)\|^2 + \int_0^{s_\star} \mu(s)\|\eta^t(s)\|_1^2 ds\Big].
$$

Indeed, $S(t)$ is actually a contraction semigroup, meaning that $\mathsf{E}(t)$ is a nonincreasing function for all initial data. We address the reader to the works [19, 32, 33] for more details.

4.2. **Exponential stability.** Since $S(t)$ is contraction semigroup, either its operator norm is always equal to 1, or $S(t)$ is exponentially stable, that is,

$$
\|S(t)\|_{L(\mathcal{H})} \le M e^{-\omega t}, \tag{4.3}
$$

for some $M \ge 1$ and $\omega > 0$. For every admissible kernel μ, a necessary and sufficient condition for the exponential stability of $S(t)$ has been established. This is the content of the following theorem proved in [34].

Theorem 4.1. *Assume that μ is not completely flat, that is, the set*

$$\{s > 0 : \mu'(s) < 0\}$$

has positive measure. Then $S(t)$ is always stable, and it is exponentially stable if and only if (2.2) *holds.*

If μ is completely flat, we have a different picture, depending whether the operator A is unbounded or not. In the first case, it is generally true that $S(t)$ is not exponentially stable. Instead, when A is bounded, the semigroup turns out to be exponentially stable whenever it is stable. Lack of stability occurs for a certain class of flat kernels, called resonant, where trajectories with conserved energy arise. We will not go into deeper details here, rather addressing the reader to [2, 34] for more information.

Definition 4.2. We define the *(exponential) decay rate* of the semigroup to be the non-negative number

$$\omega_\star = \sup\big\{\omega \geq 0 : \ \|S(t)\|_{L(\mathcal{H})} \leq Me^{-\omega t}\big\},$$

for some $M = M(\omega)$. The number $-\omega_\star$ is usually called the *growth bound* of $S(t)$.

Thus, $S(t)$ is exponentially stable if and only if $\omega_\star > 0$, whereas if $\omega_\star = 0$ then it has unitary operator norm for all t.

Remark 4.3. The decay rate of $S(t)$ can also be computed via the formula (see [11, 35])

$$\omega_\star = -\lim_{t\to\infty} \frac{\log \|S(t)\|_{L(\mathcal{H})}}{t}.$$

In principle, it might happen that $\omega_\star = \infty$, meaning that the decay of the semigroup is faster than any exponential. We will discuss this situation in the final Section 10.

The next result is basically contained in [33], although not explicitly written.

Proposition 4.4. *For every admissible kernel μ, we have the inequality*

$$\|S(t)\|_{L(\mathcal{H})} \geq c\|R(t)\|_{L(\mathcal{M})},$$

for some $c > 0$.

Proof. If $s_\star < \infty$ and $t \geq s_\star$ then $R(t) = 0$. If $t < s_\star$, choose $\boldsymbol{u}_0 = (0, 0, \eta_0) \in \mathcal{H}$ of unit norm. Exploiting (4.2),

$$\begin{aligned}
\|S(t)\boldsymbol{u}_0\|^2_{\mathcal{H}} &\geq \int_t^{s_\star} \mu(s)\|\eta_0(s-t) + u(t)\|_1^2 ds \\
&\geq \frac{1}{2}\int_t^{s_\star} \mu(s)\|\eta_0(s-t)\|_1^2 ds - \varkappa\|u(t)\|_1^2 \\
&\geq \frac{1}{2}\|R(t)\eta_0\|^2_{\mathcal{M}} - \varkappa\|S(t)\boldsymbol{u}_0\|^2_{\mathcal{H}}.
\end{aligned}$$

Hence

$$\|S(t)\|^2_{L(\mathcal{H})} \geq \|S(t)\boldsymbol{u}_0\|^2_{\mathcal{H}} \geq \frac{1}{2(1+\varkappa)}\|R(t)\eta_0\|^2_{\mathcal{M}}.$$

Taking the supremum over the $\eta_0 \in \mathcal{M}$ of unit norm, we are done.

In view of point (iv) of the previous Section 3, Proposition 4.4 has an immediate consequence.

Corollary 4.5. *If* (4.3) *holds for some* $\omega > 0$*, then* (2.2) *necessarily holds for some* $\delta \geq 2\omega$*.*

5. The Spectrum of the Infinitesimal Generator

In this section we discuss the spectrum $\sigma(\mathbb{A})$ of (the complexification of) the infinitesimal generator $\mathbb{A}$ of the semigroup $S(t)$. The knowledge of $\sigma(\mathbb{A})$ plays an important role in concern with the decay rate of $S(t)$. Indeed, defining the *spectral bound*

$$\sigma_\star = \sup\big\{\mathfrak{Re}(z) : \ z \in \sigma(\mathbb{A})\big\},$$

it is well known that the following inequality holds:

$$\omega_\star \leq -\sigma_\star,$$

providing an upper bound to the decay rate $\omega_\star$ of $S(t)$. Observe that, being $S(t)$ a contraction semigroup, $\sigma_\star \leq 0$.

With standard notation (see e.g. [40]), the spectrum $\sigma(\mathbb{A})$ decomposes into the disjoint union

$$\sigma(\mathbb{A}) = \sigma_{\rm p}(\mathbb{A}) \cup \sigma_{\rm c}(\mathbb{A}) \cup \sigma_{\rm r}(\mathbb{A}),$$

where

- $\sigma_{\rm p}(\mathbb{A})$ is the point spectrum: $z \in \sigma_{\rm p}(\mathbb{A})$ if and only if z is an eigenvalue of $\mathbb{A}$;
- $\sigma_{\rm c}(\mathbb{A})$ is the continuous spectrum: $z \in \sigma_{\rm c}(\mathbb{A})$ if and only if the operator $z - \mathbb{A}$ is injective but not surjective, and the closure of its range is $\mathcal{H}$;
- $\sigma_{\rm r}(\mathbb{A})$ is the residual spectrum: $z \in \sigma_{\rm r}(\mathbb{A})$ if and only if the operator $z - \mathbb{A}$ is injective and the closure of its range is strictly contained in $\mathcal{H}$.

Another interesting subset of $\sigma(\mathbb{A})$ is the *approximate point spectrum*

$$\sigma_{\rm ap}(\mathbb{A}) = \{z \in \mathbb{C} : \ z - \mathbb{A} \text{ is not bounded below}\},$$

whose elements are called *approximate eigenvalues*. We recall that both $\sigma_{\rm p}(\mathbb{A})$ and $\sigma_{\rm c}(\mathbb{A})$ are contained in $\sigma_{\rm ap}(\mathbb{A})$, while $\sigma_{\rm r}(\mathbb{A})$ and $\sigma_{\rm ap}(\mathbb{A})$ are generally askew. Moreover, it is a general fact that the topological boundary of $\sigma(\mathbb{A})$ is contained in $\sigma_{\rm ap}(\mathbb{A})$ (see e.g. [11]).

Let now μ be a fixed admissible kernel of total mass $\varkappa$. For $\delta > 0$, we define the open half-plane

$$\Pi_\delta = \big\{z \in \mathbb{C} : \ \mathfrak{Re}(z) > -\tfrac{\delta}{2}\big\},$$

along with the function $\mathcal{L}_\mu : \Pi_\delta \to \mathbb{C}$ given by

$$\mathcal{L}_\mu(z) = 1 + \varkappa - \int_0^{s_\star} e^{-zs}\mu(s)ds.$$

This function is certainly well defined if (2.2) holds true. Then we consider three subsets of Π_δ, depending on the spectrum $\sigma(A) = \sigma_{\rm p}(A) \cup \sigma_{\rm c}(A)$ of the operator A (recall that

$\sigma_{\mathrm{r}}(A)=\emptyset$ as A is selfadjoint); namely,

$$W_{\mathrm{p}}=\bigcup_{\lambda\in\sigma_{\mathrm{p}}(A)}\big\{z\in\Pi_\delta:\ \lambda\mathcal{L}_\mu(z)+z^2=0\big\},$$

$$W_{\mathrm{c}}=\bigcup_{\lambda\in\sigma_{\mathrm{c}}(A)}\big\{z\in\Pi_\delta:\ \lambda\mathcal{L}_\mu(z)+z^2=0\big\},$$

$$Z=\begin{cases}\big\{z\in\Pi_\delta:\ \mathcal{L}_\mu(z)=0\big\} & \text{if } A \text{ is unbounded,}\\ \emptyset & \text{if } A \text{ is bounded.}\end{cases}$$

Note that, as $\mathcal{L}_\mu(0)=1$, the value $z=0$ does not belong to either set.

If the memory kernel μ complies with (2.2), then the spectrum of $\mathbb{A}$ in the half-plane Π_δ is known explicitly. This result has been originally proved in [25], within a more restrictive condition than (2.2), and then in [7] in the version reported here below.[1]

Theorem 5.1. *Assume that* (2.2) *holds. Then* $\sigma(\mathbb{A})\cap\Pi_\delta$ *decomposes into the disjoint union*

$$\sigma(\mathbb{A})\cap\Pi_\delta=W_{\mathrm{p}}\cup W_{\mathrm{c}}\cup Z.$$

In particular, 0 *belongs to the resolvent set of* $\mathbb{A}$.

Nonetheless, the result does not say anything on the structure of the spectrum. This is done in the next theorem.

Theorem 5.2. *Assume that* (2.2) *holds. Then*

$$\sigma_{\mathrm{p}}(\mathbb{A})\cap\Pi_\delta=W_{\mathrm{p}},\qquad \sigma_{\mathrm{c}}(\mathbb{A})\cap\Pi_\delta=W_{\mathrm{c}}\cup Z,\qquad \sigma_{\mathrm{r}}(\mathbb{A})\cap\Pi_\delta=\emptyset.$$

Before entering the details of the proof, let us fix $z\in\mathbb{C}$, with $z\neq 0$. Then, for any given vector $\hat{\boldsymbol{u}}=(\hat u,\hat v,\hat\eta)\in\mathcal{H}$, we consider the resolvent equation

$$(z-\mathbb{A})\boldsymbol{u}=\hat{\boldsymbol{u}}. \tag{5.1}$$

in the unknown $\boldsymbol{u}=(u,v,\eta)\in\mathrm{dom}(\mathbb{A})$. Componentwise,

$$zu-v=\hat u, \tag{5.2}$$

$$zv+A\Big[u+\int_0^{s_\star}\mu(s)\eta(s)ds\Big]=\hat v, \tag{5.3}$$

$$z\eta-T\eta-v=\hat\eta. \tag{5.4}$$

Integrating the last equation with $\eta(0)=0$, a constraint coming from the fact that we want $\eta\in\mathrm{dom}(T)$, we find

$$\eta(s)=\frac{1}{z}v-\frac{e^{-zs}}{z}\Big[v-z\int_0^s e^{zr}\hat\eta(r)dr\Big]. \tag{5.5}$$

Substituting (5.2) and (5.5) into (5.3), we arrive at

$$z^2u+\mathcal{L}_\mu(z)Au=A\Theta(\hat u,\hat v,\hat\eta), \tag{5.6}$$

[1] This is actually proved in [7] by restricting Π_δ on the closed left complex half-plane and for A unbounded, but it is readily seen from the proof therein that the result holds as written here.

where

$$\Theta(\hat{u},\hat{v},\hat{\eta}) = A^{-1}[\hat{v}+z\hat{u}] + \Big[\frac{\mathcal{L}_\mu(z)-1}{z}\Big]\hat{u} - \int_0^{s_\star}\mu(s)\int_0^s e^{-z(s-r)}\hat{\eta}(r)dr\,ds.$$

Up to now, the calculations are only formal. However, if μ complies with (2.2) and $z \in \Pi_\delta \setminus \{0\}$, the paper [7] proves the following facts:

- $\Theta(\hat{u},\hat{v},\hat{\eta}) \in H^1$;
- if $u \in H^1$ solves (5.6), then $\boldsymbol{u} = (u, v, \eta)$ with v given by (5.2) and η given by (5.5) is the unique solution to the resolvent equation (5.1).

We can now proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. We divide the argument in four steps. In what follows, we always take $z \neq 0$, as $z = 0$ does not belong to $\sigma(\mathbb{A})$.

Step 1. We first show that

$$\sigma_{\rm p}(\mathbb{A}) \cap \Pi_\delta = W_{\rm p}.$$

Let $z \in W_{\rm p}$ be fixed, namely,

$$z^2 = -\lambda_0 \mathcal{L}_\mu(z),$$

for some $\lambda_0 \in \sigma_{\rm p}(A)$. It is readily seen from (5.2)-(5.4) that z is an eigenvalue of $\mathbb{A}$ with eigenvector $\boldsymbol{u} = (u, zu, \eta)$, where u is an eigenvector of A corresponding to λ_0 and $\eta(s) = \big[1 - e^{-zs}\big]u$. This proves that $W_{\rm p} \subset \sigma_{\rm p}(\mathbb{A}) \cap \Pi_\delta$. For the reverse inclusion, let $z \in \Pi_\delta$ be an eigenvalue of $\mathbb{A}$ with eigenvector $\boldsymbol{u} = (u, v, \eta)$. Then $\boldsymbol{u}$ solves (5.1) with $\hat{\boldsymbol{u}} = \mathbf{0}$, so that u solves (5.6) with null right-hand side. Note that $u \neq 0$ (otherwise $\boldsymbol{u} = \mathbf{0}$), forcing $\mathcal{L}_\mu(z) \neq 0$. Hence u is an eigenvector of A with eigenvalue

$$\lambda_0 = -\frac{z^2}{\mathcal{L}_\mu(z)},$$

that is, $z \in W_{\rm p}$.

Step 2. Next, we show that

$$Z \subset \sigma_{\rm c}(\mathbb{A}) \cap \Pi_\delta.$$

Assume that A is unbounded, otherwise there is nothing to prove, and take any $z \in Z$. Since $Z \cap W_{\rm p} = \emptyset$, we know from Step 1 that $z \notin \sigma_{\rm p}(\mathbb{A})$. We claim that the resolvent equation (5.1) can be solved whenever $\hat{\boldsymbol{u}} = (\hat{u},\hat{v},\hat{\eta})$ belongs to the dense subset of $\mathcal{H}$

$$\mathrm{dom}(A^{\frac{3}{2}}) \times \mathrm{dom}(A) \times L^2_\mu(0, s_\star; \mathrm{dom}(A^{\frac{3}{2}})).$$

Indeed, being $\mathcal{L}_\mu(z) = 0$, equation (5.6) becomes

$$u = \frac{1}{z^2}A\Theta(\hat{u},\hat{v},\hat{\eta}) = \frac{1}{z^2}\Theta(A\hat{u}, A\hat{v}, A\hat{\eta}) \in H^1,$$

and equation (5.1) is solved. Thus, $\overline{\mathrm{ran}(z - \mathbb{A})} = \mathcal{H}$, as desired.

Step 3. Finally, we prove that

$$W_{\rm c} \subset \sigma_{\rm c}(\mathbb{A}) \cap \Pi_\delta.$$

To this end, let $z \in W_{\rm c}$ be fixed, namely,

$$z^2 = -\lambda_0 \mathcal{L}_\mu(z), \tag{5.7}$$

for some $\lambda_0 \in \sigma_{\rm c}(A)$. Since $W_{\rm c} \cap W_{\rm p} = \emptyset$, Step 1 tells that $z \notin \sigma_{\rm p}(\mathbb{A})$. We now consider the family of sets depending on $\varepsilon > 0$

$$\Delta_\varepsilon = (-\infty, \lambda_0 - \varepsilon] \cup [\lambda_0 + \varepsilon, \infty).$$

For every $\hat{\boldsymbol{u}} = (\hat{u}, \hat{v}, \hat{\eta}) \in \mathcal{H}$, we call

$$\hat{u}_\varepsilon = E_A(\Delta_\varepsilon)\hat{u}, \qquad \hat{v}_\varepsilon = E_A(\Delta_\varepsilon)\hat{v}, \qquad \hat{\eta}_\varepsilon = E_A(\Delta_\varepsilon)\hat{\eta}, \tag{5.8}$$

where E_A is the spectral measure of A (see e.g. [38]). Since $E_A(\{\lambda_0\}) = 0$, otherwise λ_0 would be an eigenvalue of A, we have that

$$\hat{u}_\varepsilon \to \hat{u} \quad \text{in } H^1 \qquad \text{and} \qquad \hat{v}_\varepsilon \to \hat{v} \quad \text{in } H,$$

as $\varepsilon \to 0$. By the same token, $\hat{\eta}_\varepsilon(s) \to \hat{\eta}(s)$, and since $\|\hat{\eta}_\varepsilon(s)\|_1 \leq \|\hat{\eta}(s)\|_1$, the Dominated Convergence theorem yields

$$\hat{\eta}_\varepsilon \to \hat{\eta} \quad \text{in } \mathcal{M}.$$

In summary, the set of vectors of the form (5.8), for $\varepsilon > 0$ and $\hat{\boldsymbol{u}} \in \mathcal{H}$, is dense in $\mathcal{H}$. Accordingly, in view of our scopes, we can limit ourselves to consider (5.1) with

$$\hat{\boldsymbol{u}} = (\hat{u}, \hat{v}, \hat{\eta}) \in E_A(\Delta_\varepsilon)H^1 \times E_A(\Delta_\varepsilon)H \times E_A(\Delta_\varepsilon)\mathcal{M},$$

for some $\varepsilon > 0$. Recalling (5.7), equation (5.6) takes the form

$$u - \frac{1}{\lambda_0}Au = \frac{1}{z^2}A\Theta(\hat{u}, \hat{v}, \hat{\eta}).$$

Observing that

$$\Theta(\hat{u}, \hat{v}, \hat{\eta}) \in E_A(\Delta_\varepsilon)H^1,$$

exploiting the functional calculus of A we get

$$u = \frac{\lambda_0}{z^2}\Big[\int_{\sigma(A)\cap\Delta_\varepsilon} \frac{\lambda}{\lambda_0 - \lambda} dE_A(\lambda)\Big]\Theta(\hat{u}, \hat{v}, \hat{\eta}),$$

which belongs to H^1, for

$$\sup_{\lambda \in \sigma(A)\cap\Delta_\varepsilon} \frac{\lambda}{|\lambda_0 - \lambda|} < \infty.$$

Then the resolvent equation (5.1) is solved.

Step 4. Collecting the previous steps,

$$\sigma_{\rm p}(\mathbb{A}) \cap \Pi_\delta = W_{\rm p} \qquad \text{and} \qquad W_{\rm c} \cup Z \subset \sigma_{\rm c}(\mathbb{A}) \cap \Pi_\delta.$$

Invoking Theorem 5.1, we conclude that $\sigma_{\rm c}(\mathbb{A}) \cap \Pi_\delta = W_{\rm c} \cup Z$ and $\sigma_{\rm r}(\mathbb{A}) \cap \Pi_\delta = \emptyset$. $\square$

One might ask what happens instead in the complement of Π_δ. This issue has been partially addressed in [20] for the particular case of the exponential kernel (or of a finite sum of exponentials). But in general, the question is not completely well posed, as one should first find the largest δ for which (2.2) is verified. Accordingly, the best possible result is seemingly as follows.

Theorem 5.3. *Suppose that for some $\nu > 0$*

$$\int_0^\infty e^{\nu s}\mu(s)ds = \infty.$$

If $\mathfrak{Re}(z) \le -\frac{\nu}{2}$ then $z \in \sigma_{\rm c}(\mathbb{A}) \cup \sigma_{\rm r}(\mathbb{A})$.

Proof. Let $z \in \mathbb{C}$ with $\mathfrak{Re}(z) \le -\frac{\nu}{2}$ be given. Choosing any vector of the form $\hat{\boldsymbol{u}} = (0, \hat{v}, 0)$, we look for a solution $\boldsymbol{u} = (u, v, \eta) \in \mathrm{dom}(\mathbb{A})$ to the resolvent equation (5.1). In view of (5.5),

$$\eta(s) = \frac{1}{z}\big[1 - e^{-zs}\big]v.$$

Since for large s

$$|1 - e^{-zs}|^2 \sim e^{-2\mathfrak{Re}(z)s} \ge e^{\nu s},$$

the only possibility for η to belong to $\mathcal{M}$ is that $v = 0$, hence $\eta = 0$. In which case, we learn from (5.2) that $u = 0$ as well, and (5.3) becomes $0 = \hat{v}$. Therefore, we have a solution if and only if $\hat{v} = 0$, and the solution is the trivial one. This means that z belongs to the spectrum and it is not an eigenvalue.

Example 5.4. Let $\mu(s)$ be a subexponential kernel, in the following sense

$$\int_0^\infty e^{\delta s}\mu(s)ds = \infty, \quad \forall \delta > 0.$$

For instance,

$$\mu(s) = \frac{1}{(1+s)^2}.$$

Then, according to Theorem 5.3 and using the fact that the spectrum is a closed set, we have

$$\sigma(\mathbb{A}) = \big\{z \in \mathbb{C} : \mathfrak{Re}(z) < 0\big\} \cup i\mathbb{R},$$

where the first set is contained in $\sigma_{\rm c}(\mathbb{A}) \cup \sigma_{\rm r}(\mathbb{A})$. Besides, since $S(t)$ is a contraction semigroup, by a well-known result from [1] we infer that $i\mathbb{R} \cap \sigma_{\rm r}(\mathbb{A}) = \emptyset$. If in addition $S(t)$ is stable (cf. Theorem 4.1), as in the case of the particular kernel $\mu(s)$ considered above, then $i\mathbb{R} \cap \sigma_{\rm p}(\mathbb{A}) = \emptyset$ as well.

6. Solutions of Pure Memory Type

We now dwell on some particular solutions to the viscoelastic equation.

Definition 6.1. A solution $(u(t), \dot{u}(t), \eta^t)$ to (4.1) is said to be of *pure memory type* if u is identically zero.

Clearly, for such solutions, we also have $\dot{u} \equiv 0$. Accordingly, the system exhibits an empty dynamics, since only the evolution of the memory term survives. Indeed, in light of (4.1) and (4.2), a solution of pure memory type is of the form $(0, 0, \eta^t)$, where

$$\eta^t(s) = [R(t)\eta_0](s) = \begin{cases} 0 & 0 < s \le t, \\ \eta_0(s-t) & s > t, \end{cases} \tag{6.1}$$

for $s < s_\star$ with $s_\star$ as is (2.1), and η_0 satisfies

$$\int_0^{s_\star} \mu(t+s)\eta_0(s)ds = 0, \quad \forall t \geq 0. \tag{6.2}$$

A first issue is whether or not, besides the trivial one, solutions of this kind exist.

Example 6.2. Consider the exponential kernel $\mu(s) = e^{-s}$. Let $u_0 \in H^1$, and let $\phi(s)$ be any function such that

$$\int_0^\infty e^{-s}\phi(s)ds = 0.$$

Then $(0, 0, \eta^t)$ with η^t given by (6.1) with initial datum $\eta_0(s) = \phi(s)u_0$ is of pure memory type, as it fulfills (6.2). By the same token, one can construct solutions of pure memory type for a kernel made by a linear combination of negative exponentials.

The question is now if this situation can occur with kernels of different type. If the kernel is compactly supported the answer is negative. In fact, a stronger result holds.

Theorem 6.3. *Let $s_\star < \infty$, and let $(u(t), \dot{u}(t), \eta^t)$ be a solution to (4.1) such that $u(t) = 0$ for every $t \geq t_\star$, for some $t_\star \geq 0$. Then such a solution is the trivial one.*

Proof. We preliminary show that such a solution is of pure memory type. To this end, without loss of generality, we assume $t_\star \geq s_\star$, and we prove that $u(t) = 0$ for every $t \geq t_\star - s_\star$. The thesis then follows by finite recursion. Indeed, when $u(t) = 0$ for every $t \geq t_\star$, we read from (4.1) together with (4.2) that

$$\int_0^{s_\star} \mu(s)u(t-s)ds = 0, \quad \forall t \geq t_\star.$$

Choosing any vector $w \in H$, and setting $p(s) = \langle u(s), w\rangle$, we get

$$\int_0^{s_\star} \mu(s)p(t-s)ds = 0, \quad \forall t \geq t_\star.$$

Since $p(t) = 0$ for $t \geq t_\star$, we deduce that

$$\int_{t-t_\star}^{s_\star} \mu(s)p(t-s)ds = 0, \quad \forall t \in [t_\star, t_\star + s_\star].$$

Denoting

$$q(s) = p(t_\star - s),$$

we rewrite the latter relation as

$$\int_t^{s_\star} \mu(s)q(s-t)ds = 0, \quad \forall t \in [0, s_\star].$$

Lemma 2.4 then yields $q(s) = 0$ for all $s \in [0, s_\star]$, that is,

$$p(s) = 0, \quad \forall s \in [t_\star - s_\star, t_\star].$$

Since the vector $w \in H$ is arbitrarily chosen, then

$$u(s) = 0, \quad \forall s \in [t_\star - s_\star, t_\star].$$

At this point, knowing that $u \equiv 0$, we set

$$q(s) = \langle \eta_0(s), w\rangle, \quad w \in H,$$

and we infer from (6.2) that

$$\int_t^{s_\star} \mu(s)q(s-t)ds = 0, \quad \forall t \in [0, s_\star].$$

Again, we are in the hypotheses of Lemma 2.4. Therefore, $q(s) = 0$ for all $s \in [0, s_\star]$. From the arbitrariness of $w \in H$, we conclude that $\eta_0(s) = 0$ for all $s \in [0, s_\star]$. In turn, this implies that η^t is zero for every $t \geq 0$.

The theorem has an immediate corollary.

Corollary 6.4. *If $s_\star < \infty$ then system* (4.1) *does not possess nontrivial solutions of pure memory type.*

Let us see what happens instead for a relevant nonvanishing kernel different from the exponential one.

Example 6.5. Let $\mu(s) = e^{-s^2}$. We show that in this case system (4.1) does not possess nontrivial solutions of pure memory type. Indeed, let $(0, 0, \eta^t)$ be such a solution. Then, defining for an arbitrarily given $w \in H$

$$q(s) = e^{-s^2} \langle \eta_0(s), w \rangle,$$

equation (6.2) yields

$$\int_0^\infty e^{-2ts} q(s)ds = 0, \quad \forall t \geq 0,$$

meaning that the Laplace transform of q, and hence q, is identically zero. Hence, we conclude that $\eta_0 \equiv 0$.

Still, we can produce examples of kernels that are not linear combinations of exponentials, but for which nontrivial solutions of pure memory type do exist.

Example 6.6. Consider the kernel

$$\mu(s) = e^{-s}(1 + as),$$

with $a > 0$ small enough to guarantee that μ is nonincreasing. Define

$$\phi(s) = 2 - 4s + s^2,$$

noting that

$$\int_0^\infty e^{-s}\phi(s)ds = \int_0^\infty se^{-s}\phi(s)ds = 0.$$

It is then apparent that, for any $u_0 \in H^1$, the function $\eta_0(s) = \phi(s)u_0$ fulfills (6.2). Accordingly, $(0, 0, \eta^t)$ with η^t given by (6.1) is of pure memory type.

The example can be easily extended to any kernel of the form

$$\mu(s) = \sum_{n=1}^{N} p_n(s)e^{-b_n s}, \tag{6.3}$$

where $b_n > 0$ and p_n are polynomials. The situation is quite different if we consider an infinite sum of exponential kernels.

Example 6.7. Consider the kernel

$$\mu(s) = \sum_{n=1}^{\infty} a_n e^{-b_n s},$$

where $a_n > 0$ is the general term of a convergent series, and $b_n > 0$ is a strictly increasing sequence satisfying

$$\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty.$$

Such a b_n is called a *Münz sequence*. Let $(0, 0, \eta^t)$ be a solution of pure memory type. As before, choosing any vector $w \in H$, we set $q(s) = \langle \eta_0(s), w\rangle$. Then we have the equality

$$\sum_{n=1}^{\infty} c_n e^{-b_n t} = 0,$$

where

$$c_n = a_n \int_0^{\infty} e^{-b_n s} q(s) ds.$$

It is readily seen that the latter equality, valid for every $t \geq 0$, is possible if and only if $c_n = 0$ for every n. In other words, the Laplace transform of q is zero at every point b_n. By a famous theorem due to Münz (see [43]), this implies that $q \equiv 0$, so that $\eta_0 \equiv 0$.

We now discuss the analogue of Theorem 6.3 for $s_\star = \infty$. Here, we know that pure memory solutions can exist, at least for certain kernels. Nonetheless, one might ask if $u(t)$ eventually zero forces the solution to be of pure memory type. In general, this seems a hard question to cope with. We have a (positive) answer for the exponential kernel.

Theorem 6.8. *Let $\mu(s) = e^{-s}$, and let $(u(t), \dot{u}(t), \eta^t)$ be a solution to* (4.1) *such that $u(t) = 0$ for every $t \geq t_\star$, for some $t_\star \geq 0$. Then such a solution is of pure memory type.*

Proof. For an arbitrarily given $w \in H^1$, let us denote

$$\xi(t) = \langle u(t), w\rangle \qquad \text{and} \qquad \xi_1(t) = \langle u(t), w\rangle_1.$$

Borrowing an idea from [8], we multiply (4.1) by $e^t w$ in H. Exploiting the representation formula (4.2), we obtain the equality

$$e^t \ddot{\xi}(t) + 2e^t \xi_1(t) - \int_0^t e^s \xi_1(s) ds = G,$$

where

$$G = \int_0^{\infty} e^{-s} \big[\xi_1(0) - \langle \eta_0(s), w\rangle_1\big] ds$$

is independent of t. Taking the derivative with respect to time, and then multiplying by e^{-t}, we are led to

$$\dddot{\xi} + \ddot{\xi} + 2\dot{\xi}_1 + \xi_1 = 0.$$

This is nothing but the weak formulation of the equation

$$\dddot{u} + \ddot{u} + 2A\dot{u} + Au = 0. \tag{6.4}$$

The latter is a particular instance of the Moore-Gibson-Thompson (MGT) equation

$$\dddot{u} + \alpha\ddot{u} + \beta A\dot{u} + \gamma A u = 0,$$

which is known to generate a strongly continuous semigroup on the space $H^1 \times H^1 \times H$, for all parameters $\alpha, \beta, \gamma > 0$ (see, e.g., [8, 23, 29, 30, 39, 41]). In fact, the recent paper [4] shows that for the generation of the semigroup it is enough to require $\beta > 0$, whereas α, γ can be arbitrary real numbers. Hence, as $\dddot{u}$ and $\beta A\dot{u}$ keep the same sign under temporal inversion, the MGT equation actually generates a strongly continuous group of solutions. Accordingly, if $u(t) = 0$ for every $t \geq t_\star$ in (6.4), then the whole energy vanishes at $t_\star$, and consequently $u \equiv 0$. Meaning that the solution to (4.1) is of pure memory type. □

Incidentally, the fact that the MGT equation generates a group allows us to draw a conclusion on the viscoelastic system (4.1) which at first glance might seem rather surprising. Namely, it is possible to have a nonempty dynamics where the position $u(t)$ and the velocity $\dot{u}(t)$ simultaneously vanish at a given positive time. This is detailed in the next example.

Example 6.9. Let $\mu(s) = e^{-s}$, and select a smooth element w_1, say, $w_1 \in \mathrm{dom}(A)$. Then, there exists a solution to the MGT equation (6.4) such that

$$(u(1), \dot{u}(1), \ddot{u}(1)) = (0, 0, w_1).$$

Then, setting $\eta_0(s) = \ddot{u}(0)$, which is as regular as needed, and defining η^t via (4.2), by recasting the argument of the proof of Theorem 6.8 the other way around, we see that $(u(t), \dot{u}(t), \eta^t)$ solves (4.1). This solution cannot be of pure memory type, otherwise $u(t)$ would be the null solution to (6.4), but nonetheless it fulfills $u(1) = \dot{u}(1) = 0$.

7. The Backward Uniqueness Property

An important issue on the viscoelastic equation, which seems to have never been discussed in full detail, concerns with the backward uniqueness property. Let us recall the definition.

Definition 7.1. The semigroup $S(t)$ generated by system (4.1) satisfies the *backward uniqueness property* if the initial condition $\mathsf{E}(0) > 0$ implies that $\mathsf{E}(t) > 0$ for all $t > 0$.

In particular, if the semigroup complies with the definition above, then it cannot be nilpotent. The following theorem holds.

Theorem 7.2. *For any admissible kernel μ, the semigroup $S(t)$ satisfies the backward uniqueness property.*

For the case $s_\star = \infty$, we need a preliminary result which, although preparatory to the proof of Theorem 7.2, has a clear interest by itself.

Theorem 7.3. *Let $s_\star = \infty$, and assume that $u \not\equiv 0$. Then, there exists $\tau > 0$ and a constant $c > 0$, depending on the particular solution, such that*

$$\mathsf{E}(t) \geq c\mu(t), \quad \forall t \geq \tau.$$

Proof. Since $u \not\equiv 0$, there exists $\tau > 0$ such that

$$\nu = \int_0^\tau \|u(s)\|_1^2 ds > 0.$$

Since μ is decreasing, for $t \geq \tau$ we have

$$\begin{aligned}
2\mathsf{E}(t) &\geq \int_0^t \mu(s)\|u(t)-u(t-s)\|_1^2 ds \\
&\geq \frac{1}{2}\int_0^t \mu(s)\|u(t-s)\|_1^2 ds - \varkappa\|u(t)\|_1^2 \\
&\geq \frac{1}{2}\mu(t)\int_0^t \|u(s)\|_1^2 ds - 2\varkappa\mathsf{E}(t) \\
&\geq \frac{\nu}{2}\mu(t) - 2\varkappa\mathsf{E}(t).
\end{aligned}$$

The thesis follows by setting $c = \frac{\nu}{4(1+\varkappa)}$.

Proof of Theorem 7.2. We consider two cases separately.

Case $s_\star = \infty$. If $u \not\equiv 0$, the claim follows directly from Theorem 7.3 together with the fact that the energy is nonincreasing. If $u \equiv 0$ and $(u(t), \dot{u}(t), \eta^t)$ is not the trivial solution, then it is of pure memory type. Therefore, on account of (6.1), we infer that

$$\mathsf{E}(t) = \frac{1}{2}\int_0^\infty \mu(t+s)\|\eta_0(s)\|_1^2 ds > 0,$$

as $\eta_0 \not\equiv 0$.

Case $s_\star < \infty$. If $\mathsf{E}(t_\star) = 0$ for some $t_\star > 0$, then $\mathsf{E}(t) = 0$ for every $t \geq t_\star$. In particular, $u(t) = 0$ for every $t \geq t_\star$. Hence, $\mathsf{E} \equiv 0$ by Theorem 6.3.

Remark 7.4. Theorem 7.2 is particularly relevant for the case $s_\star < \infty$, as it tells that the energy survives even though the kernel is eventually vanishing. It is also clear from the proof that a stronger form of the backwards uniqueness property holds when $s_\star < \infty$. Namely, in order to conclude that $\mathsf{E}(0) = 0$ it is enough to know that $u(t)$ eventually vanishes.

We finally state an immediate consequence of Theorem 7.3.

Theorem 7.5. *If $s_\star = \infty$ then*

$$\|S(t)\|_{L(\mathcal{H})} \geq c\sqrt{\mu(t)},$$

for some $c > 0$ and every $t > 0$ large.

Clearly, after Theorem 7.2, the same is true, and in a much stronger form, if $s_\star < \infty$. Observe that the situation here is different from what we saw in concern with the right-translation semigroup $R(t)$ (cf. Example 3.2).

8. The Exponential Kernel: Analysis of the Decay Rate

The simplest possible situation, but at the same time the most significant from the physical viewpoint, is the one of the exponential kernel

$$\mu(s) = \varkappa\varrho e^{-\varrho s},$$

where $\varkappa, \varrho > 0$. Observe that $\varkappa$ is exactly the mass of μ. In connection with (1.2), this is the same as taking

$$g(s) = 1 + \varkappa e^{-\varrho s}.$$

Such a kernel clearly satisfies (2.2) with $C = 1$ and $\delta = \varrho$, but (2.2) does not hold for any $\delta > \varrho$. Accordingly, the related semigroup $S(t)$ is exponentially stable, and by Corollary 4.5 its decay rate fulfills the bound

$$\omega_\star \leq \frac{\varrho}{2},$$

which can also be deduced from Theorem 7.5. In fact, from our previous analysis, we know that for any initial datum $\boldsymbol{u}_0 \neq 0$ the decay of $\|S(t)\boldsymbol{u}_0\|_{\mathcal{H}}$ cannot be faster than $e^{-\frac{\varrho}{2}t}$. And such a decay is actually attained whenever $S(t)\boldsymbol{u}_0$ is a solution of pure memory type, since in that case $\boldsymbol{u}_0 = (0, 0, \eta_0)$ and

$$\|S(t)\boldsymbol{u}_0\|_{\mathcal{H}} = \|R(t)\eta_0\|_{\mathcal{M}}.$$

Then, two natural questions arise:

- *Is $\omega_\star$ exactly equal to $\frac{\varrho}{2}$?*
- *And, if not, how $\omega_\star$ depend on ϱ and $\varkappa$?*

The answer to the first question is negative. This was already noted in [25], but only for A unbounded. Here, we provide a much simpler proof, covering also the case A bounded.

Theorem 8.1. *The decay rate of the semigroup complies with the bound*

$$\omega_\star < \frac{\varrho}{3}.$$

Proof. With reference to Section 5, it is enough showing that $\sigma(\mathbb{A})$ contains an element z_0 with $\mathfrak{Re}(z_0) > -\frac{\varrho}{3}$. For this particular kernel,

$$\mathcal{L}_\mu(z) = 1 + \varkappa - \frac{\varkappa\varrho}{z + \varrho},$$

so that we infer from Theorem 5.1 that $\sigma(\mathbb{A})$ contains the set

$$\bigcup_{\lambda \in \sigma(A)} \big\{ z \in \Pi_\varrho : \, p(z) = 0 \big\},$$

where

$$p(z) = z^3 + \varrho z^2 + \lambda(1 + \varkappa)z + \lambda\varrho. \tag{8.1}$$

We will reach our conclusion by proving that, for an arbitrarily fixed $\lambda \in \sigma(A)$, the polynomial p has a root with real part greater than $-\frac{\varrho}{3}$. Incidentally, since $S(t)$ is a contraction semigroup, the real part of this root cannot be positive. Let then $\lambda \in \sigma(A)$ be fixed. Performing the change of variable

$$w = z + \frac{\varrho}{3},$$

one can check that $p(z) = 0$ turns into

$$w^3 + w\Big[\lambda(1 + \varkappa) - \frac{\varrho^2}{3}\Big] + \frac{2}{27}\varrho^3 - \frac{1}{3}\varrho\lambda(\varkappa - 2) = 0.$$

The latter is a third-order equation lacking the second-order term, and this readily implies the existence of a root w_0 with positive real part. The corresponding z_0 is then the sought root of p.

Nonetheless, in order to understand the behavior of $\omega_\star$ in dependance of ϱ and $\varkappa$, we need to enter more deeply into the structure of the spectrum. Clearly, we already know that $\omega_\star \to 0$ as $\varrho \to 0$. It is also easy to see that the same occurs when $\varkappa \to \infty$, as in that case p exhibits a real root

$$z \sim -\frac{\varrho}{1+\varkappa}.$$

The full picture is described by the following theorem.

Theorem 8.2. *The decay rate $\omega_\star$ deteriorates to zero when*

(i) *$\varrho \to 0$ or $\varrho \to \infty$, for any fixed $\varkappa > 0$; and*
(ii) *$\varkappa \to 0$ or $\varkappa \to \infty$, for any fixed $\varrho > 0$.*

Proof. We only need to consider the limits $\varkappa \to 0$ and $\varrho \to \infty$. Again, let $\lambda \in \sigma(A)$ be given. Writing $z = x + iy$, the equation $p(z) = 0$, with p given by (8.1), is equivalent to

$$\begin{cases} x^3 - 3xy^2 + \varrho(x^2 - y^2) + \lambda(1+\varkappa)x + \lambda\varrho = 0, \\ y(3x^2 - y^2 + 2\varrho x + \lambda + \lambda\varkappa) = 0. \end{cases}$$

Looking for a solution with $y \neq 0$, from the second equation we get

$$y^2 = \mathsf{g}(x) = 3x^2 + 2\varrho x + \lambda + \lambda\varkappa,$$

which, substituted into the first one, yields

$$\mathsf{f}(x) = 8x^3 + 8\varrho x^2 + 2x(\lambda + \lambda\varkappa + \varrho^2) + \lambda\varrho\varkappa = 0.$$

Observe that $\mathsf{f}(0) = \lambda\varrho\varkappa > 0$, and the parabola $\mathsf{g}(x)$ takes its minimum at $x = -\frac{\varrho}{3}$. At this point, we split the argument in two cases.

First, we assume that $\varkappa \to 0$, and we choose

$$\xi = -\frac{\varrho}{2}\Big(\frac{\varkappa}{1+\varkappa}\Big) \to 0.$$

Since

$$\mathsf{f}(\xi) = -\frac{\varkappa\varrho^3}{(1+\varkappa)^3} < 0,$$

there exists $x_0 \in (\xi, 0)$ such that $\mathsf{f}(x_0) = 0$ and $\mathsf{g}(x_0) \sim \lambda > 0$. Accordingly, the complex conjugate numbers

$$z_0^\pm = x_0 \pm i\sqrt{g(x_0)}$$

are roots of p.

Instead, if $\varrho \to \infty$, we choose

$$\xi = -\frac{\lambda(1+\varkappa)}{2\varrho} \to 0.$$

By direct computations,

$$\mathsf{f}(\xi) = -\frac{\lambda}{\varrho^3}\big[\varrho^4 - \varrho^2(1+\varkappa)^2\lambda + (1+\varkappa)^3\lambda^2\big] \sim -\lambda\varrho < 0,$$

hence there is $x_0 \in (\xi, 0)$ such that $\mathsf{f}(x_0) = 0$. Besides, as ξ eventually satisfies $\xi > -\frac{\varrho}{3}$, we have that

$$\mathsf{g}(x_0) > \mathsf{g}(\xi) = 3\xi^2 > 0,$$

and we get $z_0^{\pm}$ as before.

In all cases, it is apparent that

$$0 > \mathfrak{Re}(z_0) = x_0 > \xi \to 0,$$

either when $\varkappa \to 0$ or when $\varrho \to \infty$.

9. The Exponential Kernel: The Spectrum

Theorem 5.2 tailored for the exponential kernel $\mu(s) = \varkappa\varrho e^{-\varrho s}$ reads

$$\sigma_{\rm p}(\mathbb{A}) \cap \Pi_\varrho = W_{\rm p}, \qquad \sigma_{\rm c}(\mathbb{A}) \cap \Pi_\varrho = W_{\rm c} \cup Z, \qquad \sigma_{\rm r}(\mathbb{A}) \cap \Pi_\varrho = \emptyset,$$

where

$$W_{\rm p} = \bigcup_{\lambda \in \sigma_{\rm p}(A)} \big\{z \in \Pi_\varrho : \, p(z) = 0\big\},$$

$$W_{\rm c} = \bigcup_{\lambda \in \sigma_{\rm c}(A)} \big\{z \in \Pi_\varrho : \, p(z) = 0\big\},$$

with $p(z)$ given by (8.1), and

$$Z = \begin{cases} \big\{-\frac{\varrho}{1+\varkappa}\big\} & \text{if } A \text{ is unbounded and } \varkappa > 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Besides, we know from Theorem 5.3 that the complement of Π_ϱ belongs to the spectrum, and more precisely to $\sigma_{\rm c}(\mathbb{A}) \cup \sigma_{\rm r}(\mathbb{A})$. Our aim here is to provide the full characterization of $\sigma(\mathbb{A})$. To this end, we introduce the special number

$$\lambda_\star = \frac{\varrho^2}{4(\varkappa - 1)},$$

clearly defined only when $\varkappa \neq 1$. For convenience, we agree to set $\lambda_\star = 0$ if $\varkappa = 1$.

Theorem 9.1. *The following hold:*

- (I) *If $z = x + iy$ with $x < -\frac{\varrho}{2}$ then $z \in \sigma_{\rm r}(\mathbb{A})$ but $z \notin \sigma_{\rm ap}(\mathbb{A})$.*
- (II) *If $z = -\frac{\varrho}{2} + iy$ with $y \neq 0$ then $z \in \sigma_{\rm c}(\mathbb{A})$.*
- (III) *If $z = -\frac{\varrho}{2}$ and $\lambda_\star \notin \sigma_{\rm p}(A)$ then $z \in \sigma_{\rm c}(\mathbb{A})$.*
- (IV) *If $z = -\frac{\varrho}{2}$ and $\lambda_\star \in \sigma_{\rm p}(A)$ then $z \in \sigma_{\rm r}(\mathbb{A})$ and $z \in \sigma_{\rm ap}(\mathbb{A})$.*

Remark 9.2. Incidentally, the result indicates that the semigroup has somehow bad spectral properties, as we know that no trajectory but the trivial one can have a decay faster than $e^{-\frac{\varrho}{2}t}$.

Defining the sets

$$X_{\rm c} = \big\{z = -\tfrac{\varrho}{2} + iy : \, y \neq 0\big\} \qquad \text{and} \qquad X_{\rm r} = \big\{z = x + iy : \, x < -\tfrac{\varrho}{2}\big\},$$

we have now the complete picture:

- If $\lambda_\star$ is not an eigenvalue of A then

$$\sigma_{\rm p}(\mathbb{A}) = W_{\rm p}, \qquad \sigma_{\rm c}(\mathbb{A}) = W_{\rm c} \cup Z \cup X_{\rm c} \cup \big\{-\tfrac{\varrho}{2}\big\}, \qquad \sigma_{\rm r}(\mathbb{A}) = X_{\rm r}.$$

- If $\lambda_\star$ is an eigenvalue of A then

$$\sigma_{\rm p}(\mathbb{A}) = W_{\rm p}, \qquad \sigma_{\rm c}(\mathbb{A}) = W_{\rm c} \cup Z \cup X_{\rm c}, \qquad \sigma_{\rm r}(\mathbb{A}) = X_{\rm r} \cup \big\{-\tfrac{\varrho}{2}\big\}.$$

- In all cases, $X_{\rm r}$ is the only part of the spectrum that does not belong to $\sigma_{\rm ap}(\mathbb{A})$.

The main tool needed in the proof of Theorem 9.1 is the linear operator $\Lambda : \mathcal{M} \to H^1$, depending on $z \in \mathbb{C}$ with $\mathfrak{Re}(z) \leq -\frac{\varrho}{2}$, defined as

$$\Lambda\eta = -z(1 + \varkappa + z^2 A^{-1})\,\Phi\eta + \varkappa\varrho z\,\Psi\eta,$$

where

$$\Phi\eta = \int_0^\infty e^{zs}\eta(s)ds,$$
$$\Psi\eta = \int_0^\infty e^{-(\varrho+z)s}\int_s^\infty e^{zr}\eta(r)dr\,ds,$$

with domain

$$\mathrm{dom}(\Lambda) = \big\{\eta \in \mathcal{M} :\ \Phi\eta \in H^1 \text{ and } \Psi\eta \in H^1\big\}.$$

It is apparent that $\mathrm{dom}(\Lambda)$ contains the set

$$\mathcal{M}_0 = \big\{\eta \in \mathcal{M} :\ \eta(s) \text{ eventually vanishes}\big\}.$$

Hence Λ is always densely defined.

A word of warning. It is understood that $\Phi\eta \in H^1$ means that the integral exists in H^1 in the sense of Bochner, namely,

$$\int_0^\infty e^{\mathfrak{Re}(z)s}\|\eta(s)\|_1 ds < \infty.$$

And the same for $\Psi\eta \in H^1$.

Notation. If $\lambda_\star$ is an eigenvalue of A, which can occur only when $\varkappa > 1$, we will denote by V the eigenspace relative to $\lambda_\star$, and by P and $P^\perp$ the projections onto V and its orthogonal complement $V^\perp$, respectively.

We need some preparatory lemmas.

Lemma 9.3. *If $\mathfrak{Re}(z) < -\frac{\varrho}{2}$ then Λ is a bounded operator.*

Proof. Let $z = x + iy \in \mathbb{C}$ with $x < -\frac{\varrho}{2}$ be given, and let $\eta \in \mathcal{M}$ be an arbitrary unit vector. Then, from the Hölder inequality, we have

$$\|\Phi\eta\|_1 \leq \int_0^\infty e^{(x+\frac{\varrho}{2})s}e^{-\frac{\varrho}{2}s}\|\eta(s)\|_1 ds \leq \bigg(\int_0^\infty e^{(2x+\varrho)s}ds\bigg)^{\frac12}\frac{1}{\sqrt{\varkappa\varrho}}\,\|\eta\|_{\mathcal{M}} = \frac{1}{\sqrt{(-2x-\varrho)\varkappa\varrho}}.$$

This proves the continuity of Φ. Concerning Ψ, we write

$$\begin{aligned}\|\Psi\eta\|_1 &\le \int_0^\infty e^{-(\varrho+x)s}\int_s^\infty e^{xr}\|\eta(r)\|_1 dr\,ds\\ &= \int_0^\infty e^{xr}\|\eta(r)\|_1\int_0^r e^{-(\varrho+x)s}ds\,dr.\end{aligned}$$

Observe that

$$\int_0^r e^{-(\varrho+x)s}ds \le \begin{cases} r & \text{if } \varrho+x\ge 0,\\ \frac{1}{-\varrho-x}\,e^{-(\varrho+x)r} & \text{if } \varrho+x<0.\end{cases}$$

Therefore, exploiting again the Hölder inequality, if $\varrho+x\ge 0$ we get

$$\|\Psi\eta\|_1 \le \int_0^\infty re^{(x+\frac{\varrho}{2})r}e^{-\frac{\varrho}{2}r}\|\eta(r)\|_1 dr \le \Big(\int_0^\infty r^2 e^{(2x+\varrho)r}dr\Big)^{\frac12}\frac{1}{\sqrt{\varkappa\varrho}} = \frac{\sqrt{2}}{(-2x-\varrho)^{\frac32}\sqrt{\varkappa\varrho}},$$

whereas if $\varrho+x<0$ we are led to

$$\|\Psi\eta\|_1 \le -\frac{1}{\varrho+x}\int_0^\infty e^{-\frac{\varrho}{2}r}e^{-\frac{\varrho}{2}r}\|\eta(r)\|_1 dr \le -\frac{1}{\varrho(\varrho+x)\sqrt{\varkappa}}.$$

In both cases the continuity of Ψ is established.

Lemma 9.4. *If either $z=-\frac{\varrho}{2}+iy$ with $y\neq 0$, or $z=-\frac{\varrho}{2}$ and $\lambda_\star\notin\sigma_{\rm p}(A)$, then for any $u\in H^1$ there is a sequence $\eta_n\in\mathcal{M}_0$ such that*

$$\eta_n\to 0 \;\text{ in } \mathcal{M} \qquad \text{and} \qquad \Lambda\eta_n\to u \;\text{ in } H^1.$$

In particular, Λ is unbounded.

Proof. We first observe that, if we can prove the lemma for u in a dense subset of H^1, by a simple argument the claim follows for every $u\in H^1$. Let then $z=-\frac{\varrho}{2}+iy$ be given, and let u belong to some dense subset of H^1. Aiming to find η_n, we consider the sequence of nonnegative functions

$$\phi_n(s)=\frac{1}{s}\chi_{[n,ne]}(s),$$

which satisfy

$$\int_s^\infty \phi_n(s)ds = \begin{cases} 1 & \text{if } s\le n,\\ \log\frac{ne}{s} & \text{if } n<s\le ne,\\ 0 & \text{if } ne<s.\end{cases}$$

For $w\in H^1$ to be chosen later, we define

$$\eta_n(s)=we^{-zs}\phi_n(s).$$

Letting $n\to\infty$, we have

$$\|\eta_n\|^2_{\mathcal{M}}=\varkappa\varrho\|w\|_1^2\int_n^{ne}\frac{1}{s^2}ds\le\frac{\varkappa\varrho}{n}\|w\|_1^2\to 0.$$

Moreover

$$\Psi\eta_n = w\int_0^n e^{-(\varrho+z)s}ds + w\int_n^{ne}e^{-(\varrho+z)s}\log\frac{ne}{s}ds \to \frac{1}{\varrho+z}w,$$

while

$$\Phi\eta_n = w\int_n^{ne} \frac{1}{s}ds = w.$$

Therefore

$$\Lambda\eta_n \to -\frac{z}{\varrho+z}\big[\varrho + z + \varkappa z + (\varrho z^2 + z^3)A^{-1}\big]w.$$

Denoting $\hat{u} = -\frac{\varrho+z}{z}u$ for simplicity, the proof is finished if we can find a solution $w \in H^1$ to the equation

$$\big[\varrho + z + \varkappa z + (\varrho z^2 + z^3)A^{-1}\big]w = \hat{u}. \tag{9.1}$$

Exploiting the functional calculus of A, as we did in Section 5, we obtain

$$w = \int_{\sigma(A)} \frac{\lambda}{p_z(\lambda)} dE_A(\lambda)\,\hat{u},$$

where E_A is the spectral measure of A, and

$$p_z(\lambda) = z^3 + \varrho z^2 + \lambda(1+\varkappa)z + \lambda\varrho$$

is exactly the third-order polynomial in z encountered in (8.1), but now considered as a function of λ. To have $w \in H^1$ it suffices that

$$\sup_{\lambda\in\sigma(A)} \frac{\lambda}{|p_z(\lambda)|} < \infty.$$

Since $\sigma(A)$ is closed, this occurs if and only if

$$z \neq -\frac{\varrho}{1+\varkappa}, \quad \text{if } A \text{ is unbounded},$$

and

$$p_z(\lambda) \neq 0, \quad \forall\lambda \in \sigma(A).$$

The first condition is always satisfied unless $z = -\frac{\varrho}{2}$ and $\varkappa = 1$. The second condition is certainly satisfied if $y \neq 0$. Indeed, recalling the proof of Theorem 8.2,

$$\mathsf{f}(-\tfrac{\varrho}{2}) = -\lambda\varrho \neq 0 \quad \Rightarrow \quad p_z(\lambda) \neq 0.$$

Hence the lemma is proved when $y \neq 0$. Let us focus on $y = 0$, that is, $z = -\frac{\varrho}{2}$. We first tackle the case $\varkappa = 1$, or equivalently $\lambda_\star = 0$, which occurs only when A is unbounded. Then (9.1) is solved by

$$w = \frac{8}{\varrho^3}A\hat{u},$$

that belongs to H^1 whenever $\hat{u} \in \mathrm{dom}(A^{\frac{3}{2}})$, a dense subspace of H^1. For the remaining cases, we see by direct computations that

$$p_z(\lambda) = 0 \quad \Leftrightarrow \quad \lambda = \lambda_\star.$$

Hence if $\lambda_\star \notin \sigma(A)$ we find $w \in H^1$. We are left to consider the case $\lambda_\star \in \sigma_{\mathrm{c}}(A)$. Introducing the family of sets depending on $\varepsilon > 0$

$$\Delta_\varepsilon = (-\infty, \lambda_\star - \varepsilon] \cup [\lambda_\star + \varepsilon, \infty),$$

we know that the space $\bigcup_{\varepsilon>0} E_A(\Delta_\varepsilon)H^1$ is dense in H^1. Given any $\varepsilon > 0$ and any $\hat{u} \in E_A(\Delta_\varepsilon)H^1$, we get

$$w = \int_{\sigma(A)\cap\Delta_\varepsilon} \frac{\lambda}{p_z(\lambda)} dE_A(\lambda)\, \hat{u},$$

which belongs to H^1 since

$$\sup_{\lambda\in\sigma(A)\cap\Delta_\varepsilon} \frac{\lambda}{|p_z(\lambda)|} < \infty.$$

This finishes the proof.

Lemma 9.5. *If $z = -\frac{\varrho}{2}$ and $\lambda_\star \in \sigma_{\rm p}(A)$, then the restriction of the operator Λ on $P\mathcal{M}$ is a bounded operator with values in PH^1.*

Proof. Integrating by parts in $\Psi\eta$, which is allowed in the domain of Λ, we get

$$\Psi\eta = \frac{2}{\varrho}\,\Phi\eta - \frac{2}{\varrho}\int_0^\infty e^{-\varrho s}\eta(s)ds,$$

so that

$$\Lambda\eta = -\frac{\varrho^3}{8\lambda_\star}(1-\lambda_\star A^{-1})\,\Phi\eta + \varkappa\varrho\int_0^\infty e^{-\varrho s}\eta(s)ds.$$

If $\eta \in P\mathcal{M}$, then $\Phi\eta$ is an eigenvector of A corresponding to the eigenvalue $\lambda_\star$. Then we reduce to

$$\Lambda\eta = \varkappa\varrho\int_0^\infty e^{-\varrho s}\eta(s)ds.$$

Using the Hölder inequality, as we did in the proof of Lemma 9.3, we conclude that Λ restricted on $P\mathcal{M}$ is a bounded operator, whose range is obviously contained in PH^1. $\square$

We can finally proceed to the

Proof of Theorem 9.1. Let $z = x + iy \in \mathbb{C}$ with $x \le -\frac{\varrho}{2}$ be given. Recall that, after Theorem 5.3, we already know that $z \in \sigma_{\rm c}(\mathbb{A}) \cup \sigma_{\rm r}(\mathbb{A})$. Take any

$$\hat{\boldsymbol{u}} = (\hat{u}, \hat{v}, \hat{\eta}) \in H^1 \times H \times \mathrm{dom}(\Lambda).$$

Such vectors are clearly dense in $\mathcal{H}$, and by Lemma 9.3 they cover the whole $\mathcal{H}$ whenever $x < -\frac{\varrho}{2}$. If $\boldsymbol{u} = (u, v, \eta) \in \mathrm{dom}(\mathbb{A})$ is the solution to the equation

$$(z - \mathbb{A})\boldsymbol{u} = \hat{\boldsymbol{u}},$$

that is, to system (5.2)-(5.4), then η must fulfills (5.5). On the other hand,

$$\lim_{s\to\infty}\int_0^s e^{zr}\hat{\eta}(r)dr = \Phi\hat{\eta},$$

where the convergence occurs in H^1. Accordingly, if $v \ne z\Phi\hat{\eta}$, then the asymptotic relation

$$\eta(s) \sim \frac{1}{z}v - \frac{e^{-zs}}{z}\big[v - z\Phi\hat{\eta}\big],$$

holds as $s \to \infty$. Since $x \le -\frac{\varrho}{2}$, this implies that $\eta \notin \mathcal{M}$. Thus, in order to have $\eta \in \mathcal{M}$, it is necessary that

$$v = z\Phi\hat{\eta}.$$

Accordingly, (5.5) becomes

$$\eta(s) = \Phi\hat{\eta} - e^{-zs}\int_s^\infty e^{zr}\hat{\eta}(r)dr. \tag{9.2}$$

Observe that $\hat{\eta} \in \mathrm{dom}(\Lambda)$ does not automatically imply $\eta \in \mathcal{M}$. This certainly true however if $\hat{\eta} \in \mathcal{M}_0$. Substituting v and η into (5.2) and (5.3) we obtain

$$u = \frac{1}{z}\hat{u} + \Phi\hat{\eta},$$

but also

$$u = -(\varkappa + z^2A^{-1})\,\Phi\hat{\eta} + \varkappa\varrho\,\Psi\hat{\eta} + A^{-1}\hat{v}.$$

Hence the solution $\boldsymbol{u}$ exists if and only if its third component η given by (9.2) belongs to $\mathcal{M}$, and the compatibility condition

$$\hat{u} = \Lambda\hat{\eta} + zA^{-1}\hat{v} \tag{9.3}$$

is satisfied. At this point, we consider cases (I)-(IV) separately.

Case (I). On account on Lemma 9.3 and the fact that $A^{-1}: H \to H^1$ is continuous, the linear functional $\Gamma : H \times \mathcal{M} \to H^1$ defined as

$$\Gamma(\hat{v}, \hat{\eta}) = \Lambda\hat{\eta} + zA^{-1}\hat{v}$$

is bounded. Hence its graph

$$\mathcal{G} = \big\{(\hat{u}, \hat{v}, \hat{\eta}) \in \mathcal{H} :\ \hat{u} = \Gamma(\hat{v}, \hat{\eta})\big\}$$

is closed and strictly contained in $\mathcal{H}$. Therefore, by (9.3) we get

$$\mathrm{ran}(z - \mathbb{A}) \subset \mathcal{G} \neq \mathcal{H},$$

implying that $z \in \sigma_{\mathrm{r}}(\mathbb{A})$. Actually, we have the equality

$$\mathrm{ran}(z - \mathbb{A}) = \mathcal{G},$$

which guarantees (by the Inverse Mapping theorem) that the operator $z - \mathbb{A}$ is bounded below, i.e., $z \notin \sigma_{\mathrm{ap}}(\mathbb{A})$. To see that, it is enough proving that, for any $\hat{\eta} \in \mathcal{M}$, the vector η given by (9.2) lies in $\mathcal{M}$ as well. Since $\Phi\hat{\eta} \in \mathcal{M}$, the conclusion follows if the same is true for

$$\zeta(s) = e^{-zs}\int_s^\infty e^{zr}\hat{\eta}(r)dr.$$

Indeed, calling $a = -x - \frac{\varrho}{2} > 0$, we have

$$\|\zeta(s)\|_1 \leq e^{(\frac{\varrho}{2}+a)s}\int_s^\infty e^{-(\frac{\varrho}{2}+a)r}\|\hat{\eta}(r)\|_1 dr.$$

Hence,

$$\|\zeta\|^2_{\mathcal{M}} \leq \varkappa\varrho\int_0^\infty |F(s)|^2 ds,$$

having set

$$F(s) = \int_s^\infty e^{a(s-r)}e^{-\frac{\varrho}{2}r}\|\hat{\eta}(r)\|_1 dr.$$

Extending $F(s)$ to be zero for $s<0$, all we need is showing that $F\in L^2(\mathbb{R})$. To this end, defining the functions

$$f(s)=\begin{cases} e^{-\frac{\varrho}{2}s}\|\hat\eta(s)\|_1 & \text{if } s\geq 0,\\ 0 & \text{if } s<0,\end{cases}$$

and

$$g(s)=\begin{cases} 0 & \text{if } s\geq 0,\\ e^{as} & \text{if } s<0,\end{cases}$$

we rewrite F as the convolution on the real line

$$F=f*g,$$

noting that $f\in L^2(\mathbb{R})$ and $g\in L^1(\mathbb{R})$. From a classical result of the convolution theory (see, e.g., [22]), we deduce that $F\in L^2(\mathbb{R})$. $\square$

Cases (II)-(III). In order to conclude that $z\in\sigma_{\rm c}(\mathbb{A})$ we show that

$$\overline{\mathrm{ran}(z-\mathbb{A})}\supset H^1\times H\times\mathcal{M}_0,$$

which readily yields

$$\overline{\mathrm{ran}(z-\mathbb{A})}=\mathcal{H}.$$

Let us then take

$$\hat{\boldsymbol{u}}=(\hat u,\hat v,\hat\eta)\in H^1\times H\times\mathcal{M}_0,$$

telling in particular that the vector η given by (9.2) belongs to $\mathcal{M}$. Exploiting Lemma 9.4, there is a sequence $\hat\eta_n\in\mathcal{M}_0$ such that

$$\hat\eta_n\to 0\ \text{ in } \mathcal{M}\qquad\text{and}\qquad \Lambda\hat\eta_n\to \hat u-\Lambda\hat\eta-zA^{-1}\hat v\ \text{ in } H^1.$$

Next, define

$$\hat{\boldsymbol{u}}_n=(\Lambda(\hat\eta+\hat\eta_n)+zA^{-1}\hat v,\hat v,\hat\eta+\hat\eta_n).$$

By construction,

$$\hat{\boldsymbol{u}}_n\to\hat{\boldsymbol{u}}\ \text{ in } \mathcal{H},$$

and $\hat{\boldsymbol{u}}_n\in\mathrm{ran}(z-\mathbb{A})$.

Case (IV). We write $\mathcal{H}$ as the orthogonal sum $\mathcal{H}=\mathcal{V}\oplus\mathcal{V}^\perp$, where

$$\mathcal{V}=PH^1\times PH\times P\mathcal{M}\qquad\text{and}\qquad \mathcal{V}^\perp=P^\perp H^1\times P^\perp H\times P^\perp\mathcal{M}.$$

Since both $\mathcal{V}$ and $\mathcal{V}^\perp$ are invariant under the action of $\mathbb{A}$, we have the decomposition

$$\mathbb{A}=\mathbb{A}_{|\mathcal{V}}\oplus\mathbb{A}_{|\mathcal{V}^\perp},$$

so that

$$\mathrm{ran}(-\tfrac{\varrho}{2}-\mathbb{A})=\mathrm{ran}(-\tfrac{\varrho}{2}-\mathbb{A}_{|\mathcal{V}})\oplus\mathrm{ran}(-\tfrac{\varrho}{2}-\mathbb{A}_{|\mathcal{V}^\perp}).$$

On the other hand, Lemma 9.5 ensures that $\Lambda_{|P\mathcal{M}}:P\mathcal{M}\to PH^1$ is bounded. Observe also that $P\mathcal{M}$ is nothing but $L^2_\mu(0,\infty;PH^1)$. Therefore, as far as $\mathbb{A}_{|\mathcal{V}}$ is concerned, we fall exactly into the situation of Case I. Accordingly,

$$\mathrm{ran}(-\tfrac{\varrho}{2}-\mathbb{A}_{|\mathcal{V}})=\mathcal{G}_\star,$$

where $\mathcal{G}_\star$ is the graph of a bounded linear operator, hence closed and strictly contained in $\mathcal{V}$. In summary,

$$\overline{\mathrm{ran}(-\tfrac{\varrho}{2}-\mathbb{A})} \subset \mathcal{G}_\star \oplus \mathcal{V}^\perp \neq \mathcal{H},$$

meaning that $-\frac{\varrho}{2} \in \sigma_{\rm r}(\mathbb{A})$. We are left to show that $-\frac{\varrho}{2} \in \sigma_{\rm ap}(\mathbb{A})$. For this we exploit once more the proof of Theorem 8.2, from which we learn that given any $z' = x' + iy'$ with $x' > -\frac{\varrho}{2}$ and $y' \neq 0$, if $\mathsf{f}(x') \neq 0$ then $z' \notin \sigma(\mathbb{A})$ (recall that $Z \subset \mathbb{R}$). Since $\mathsf{f}(-\frac{\varrho}{2}) = -\lambda\varrho$ and f is continuous, the resolvent set of $\mathbb{A}$ contains the set

$$\big\{z' = x' + iy' : -\tfrac{\varrho}{2} < x < -\tfrac{\varrho}{2} + \varepsilon,\ y \neq 0\big\},$$

for some $\varepsilon > 0$. Thus the point $-\frac{\varrho}{2}$ belongs to the topological boundary of the spectrum, besides to the spectrum, and so it is an approximate eigenvalue.

10. Superexponential Kernels

We finally want to explore the possibility for the semigroup $S(t)$ generated by system (4.1) to exhibit a decay faster than any exponential.

Definition 10.1. We say that $S(t)$ is *superstable* if it has a decay rate $\omega_\star = \infty$.

After Corollary 4.5, we know that a necessary condition for the superstability of the semigroup is that the underlying kernel μ complies with the following definition.

Definition 10.2. An admissible kernel μ is *superexponential* if the δ-condition (2.2) holds for every $\delta > 0$.

Paradigmatic examples of superexponential kernels are $\mu(s) = e^{-s^2}$ and any compactly supported admissible kernel, that is, with $s_\star < \infty$. However, it might be not so immediate to check in general if a given kernel is superexponential. Introducing the (nonincreasing) function[2]

$$\alpha(t) = \limsup_{s\to\infty} \frac{\mu(t+s)}{\mu(s)},$$

the next result provides a characterization which seems to be more handy.

Theorem 10.3. *The kernel μ is superexponential if and only if the limit*

$$\lim_{t\to\infty} e^{\delta t}\alpha(t) = 0 \tag{10.1}$$

holds for every $\delta > 0$.

Proof. One direction is pretty much obvious. Indeed, if μ satisfies the δ-condition, then for some $C = C(\delta)$ we have that

$$\frac{\mu(t+s)}{\mu(s)} \leq Ce^{-\delta t}, \quad \forall s > 0.$$

To prove the converse, define the strictly positive function

$$\beta(t) = 2\alpha(t) + e^{-t^2},$$

[2] If $s_\star < \infty$ the limit is interpreted equal to zero for every $t > 0$.

which, on account of (10.1), is easily seen to fulfill

$$\lim_{t\to\infty} \frac{\log \beta(t)}{t} = -\infty.$$

Let then $\delta > 0$ be arbitrarily fixed, and let $t_0 > 0$ large enough that

$$\frac{\log \beta(t_0)}{t_0} \leq -\delta.$$

By the very definition of α, there is $s_0 > 0$ such that

$$\mu(t_0 + s) \leq \beta(t_0)\mu(s) \leq e^{-\delta t_0}\mu(s), \quad \forall s \geq s_0.$$

Given any $t \geq 0$, we write

$$t = nt_0 + \tau,$$

for some integer n and $\tau \in [0, t_0)$. Since μ is nonincreasing, for every $s \geq s_0$ we get

$$\mu(t + s) = \mu(nt_0 + \tau + s) \leq \mu(nt_0 + s).$$

On the other hand,

$$\mu(nt_0 + s) = \mu(t_0 + (n-1)t_0 + s) \leq e^{-\delta t_0}\mu((n-1)t_0 + s) \leq \cdots \leq e^{-\delta n t_0}\mu(s).$$

Therefore

$$\mu(t + s) \leq e^{-\delta n t_0}\mu(s) \leq e^{\delta t_0}e^{-\delta t}\mu(s).$$

This is exactly the desired conclusion for $s \geq s_0$. If $s < s_0$ and $t \geq s_0$, invoking again the monotonicity of μ, we learn from the latter inequality that

$$\mu(t + s) = \mu(t - s_0 + s + s_0) \leq e^{\delta t_0}e^{-\delta(t-s_0)}\mu(s + s_0) \leq e^{\delta(t_0+s_0)}e^{-\delta t}\mu(s).$$

Finally, if $s < s_0$ and $t < s_0$,

$$\mu(t + s) \leq \mu(s) \leq e^{\delta s_0}e^{-\delta t}\mu(s).$$

Summarizing,

$$\mu(t + s) \leq Ce^{-\delta t}\mu(s),$$

for every $t \geq 0$ and $s > 0$, with $C = e^{\delta(t_0+s_0)}$.

Remark 10.4. It is easy to construct (admissible) kernels decaying faster than any exponential, but not superexponential according to our definition. Indeed, it is enough to take a kernel $\mu(s) \leq e^{-t^2}$ constant on infinitely many intervals I_n of length n. In which case, one gets $\alpha(t) = 1$ for every $t > 0$.

Remark 10.5. In concrete cases, a superexponential kernel actually satisfies a stronger condition than (10.1), namely,

$$\alpha(t) = 0, \quad \forall t > t_0, \tag{10.2}$$

for some $t_0 \geq 0$. In fact, let aside pathological situations, (10.2) usually holds for $t_0 = 0$. This is for instance the case of the superexponential kernel $\mu(s) = e^{-s^2}$, or trivially of any compactly supported kernel. It is also worth noting that (10.2) with $t_0 = 0$ is equivalent to the condition[3]

$$\lim_{t\to\infty} \frac{1}{\mu(t)} \int_t^\infty \mu(s)ds = 0.$$

[3] If $s_\star < \infty$ the limit is taken for $t \to s_\star$.

We leave the proof of this fact to the interested reader.

Nonetheless, even if μ is superexponential, the superstability phenomenon cannot occur in the context of viscoelasticity. This is perhaps one of the most interesting results of this paper.

Theorem 10.6. *For any admissible kernel μ, the decay rate $\omega_\star$ of $S(t)$ is always finite. In other words, $S(t)$ is never superstable.*

Proof. We assume μ superexponential, which we know to be a necessary condition for superstability. Recalling that $\omega_\star \leq -\sigma_\star$, it is enough showing that $\sigma(\mathbb{A})$ is nonempty. Here the superexponential kernel comes into the picture. Indeed, having (2.2) for every $\delta > 0$, the results of Section 5 hold for *every* Π_δ. In particular, we deduce from Theorem 5.1, that

$$\sigma(\mathbb{A}) \supset \big\{x \in \mathbb{R} : \psi(x) = 0\big\},$$

where

$$(x) = \lambda + \lambda\varkappa - \lambda \int_0^{s_\star} e^{-xs}\mu(s)ds + x^2,$$

for an arbitrarily given $\lambda \in \sigma(A)$, hence $\lambda > 0$. Since

$$(0) = \lambda > 0,$$

and ψ is continuous, the conclusion certainly follows if we show that $\psi(x)$ becomes negative for some $x < 0$. To this end, fixing $\varepsilon > 0$ (and $\varepsilon < s_\star$) and $x = -n$ with $n \in \mathbb{N}$, we note that

$$\int_0^{s_\star} e^{ns}\mu(s)ds \geq \int_0^{\varepsilon} e^{ns}\mu(s)ds \geq \mu(\varepsilon)\,\frac{e^{n\varepsilon}-1}{n}.$$

Hence,

$$(-n) \leq \lambda + \lambda\varkappa - \lambda\mu(\varepsilon)\,\frac{e^{n\varepsilon}-1}{n} + n^2 \to -\infty,$$

as $n \to \infty$.

Remark 10.7. As already mentioned in the proof above, if μ is superexponential then the results of Section 5, and in particular Theorem 5.2, hold in every Π_δ, hence in the whole complex plane. Accordingly, the spectrum of $\mathbb{A}$ is the disjoint union

$$\sigma(\mathbb{A}) = W_{\rm p} \cup W_{\rm c} \cup Z,$$

hence there is no residual spectrum. In particular, if H is finite dimensional, so that both A and A^{-1} are compact, then $\sigma(\mathbb{A})$ reduces to $W_{\rm p}$.

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