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# Increasing revenue in Bayesian posted price auctions through signaling ☆

Matteo Castiglioni *, Alberto Marchesi, Giulia Romano, Nicola Gatti

*Politecnico di Milano, Piazza Leonardo da Vinci 32, Milan, Italy*

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A B S T R A C T

We study single-item single-unit *Bayesian posted price auctions*, where buyers arrive sequentially and their valuations for the item being sold depend on a random, unknown state of nature. The seller has complete knowledge of the actual state and can send signals to the buyers so as to disclose information about it. For instance, the state of nature may reflect the condition and/or some particular features of the item, which are known to the seller only. The problem faced by the seller is about *how to partially disclose information about the state so as to maximize revenue*. Unlike classical signaling problems, in this setting, the seller must also correlate the signals being sent to the buyers with some price proposals for them. This introduces additional challenges compared to standard settings. As a preliminary step, we show that, w.l.o.g., the seller can deterministically propose a price to each buyer on the basis of the signal being sent to that buyer, rather than selecting prices stochastically and arbitrarily correlating them with signals sent to all the buyers. Next, we consider two cases: the one where the seller can only send signals *publicly* visible to all buyers, and the case in which the seller can *privately* send a different signal to each buyer. As a first step, we prove that, in both settings, the problem of maximizing the seller's revenue does *not* admit an additive FPTAS unless P = NP, even for basic instances with a single buyer. As a result, in the rest of the paper, we focus on designing additive PTASs. In order to do so, we first introduce a unifying framework encompassing both public and private signaling, whose core result is a decomposition lemma that allows focusing on a finite set of possible buyers' posteriors. This forms the basis on which our additive PTASs are developed. In particular, in the public signaling setting, our PTAS employs some *ad hoc* techniques based on linear programming, while our PTAS for the private setting relies on the ellipsoid method to solve an exponentially-sized LP in polynomial time. In the latter case, we need a custom approximate separation oracle, which we implement with a dynamic programming approach.



## 1. Introduction

In *posted price auctions*, the seller tries to sell an item by proposing *take-it-or-leave-it* prices to buyers arriving sequentially. Each buyer has to choose between declining the offer —without having the possibility of coming back— or accepting

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* Corresponding author.
*E-mail addresses:* matteo.castiglioni@polimi.it (M. Castiglioni), alberto.marchesi@polimi.it (A. Marchesi), giulia.romano@polimi.it (G. Romano), nicola.gatti@polimi.it (N. Gatti).

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it, thus ending the auction. Nowadays, posted pricing is the most used selling format in e-commerce [2], whose sales reach over $4 trillion in 2020 [3]. Posted price auctions are ubiquitous in settings such as, for example, online travel agencies (*e.g.*, *Expedia*), accommodation websites (*e.g.*, *Booking.com*), and retail platforms (*e.g.*, *Amazon* and *eBay*). As a result, growing attention has been devoted to their analysis, both in economics [4] and in computer science [5–9], within artificial intelligence and machine learning in particular [10–12].

We study *Bayesian* posted price auctions, where the buyers' valuations for the item depend on a random state of nature, which is known to the seller only. We apply the *Bayesian persuasion* framework by Kamenica and Gentzkow [13], by considering the case in which the seller (sender) can send signals to the buyers (receivers) so as to disclose information about the state. Thus, in a Bayesian auction, the seller does *not* only have to decide price proposals for the buyers, but also *how to partially disclose information about the state so as to maximize revenue*. This is achieved by the seller by first publicly committing to a *signaling scheme*, *i.e.*, a randomized mapping from states of nature to signals for the buyers, and, then, by drawing signals according to such a signaling scheme once the state of nature has been observed. Then, the buyers receive their signals and compute their posterior beliefs over possible states of nature, which they use in order to decide whether to buy the item or not.

Our Bayesian auction model finds application in several real-world scenarios. For instance, in an e-commerce platform, the state of nature may reflect the condition (or quality) of the item being sold and/or some of its features. These are usually known to the seller only, since the buyers cannot physically evaluate the item given that the auction is carried out on the web. In such a setting, the signaling scheme could represent some established policy for visualizing information on the web page, which the platform implements in order to inform potential buyers on the item being sold (*e.g.*, the platform could decide the order in which the features of the item are presented on the web page).

Over the last years, the Bayesian persuasion framework by Kamenica and Gentzkow [13] has received considerable attention from the computer science community, due to its applicability to many real-world scenarios, such as online advertising [14–16], voting [17–20], traffic routing [21–24], recommendation systems [25], security [26,27], product marketing [28,29], and sequential decision making [30–32]. In spite of that, only few works (such as, *e.g.*, [15,16,18]) addressed computational problems related to Bayesian persuasion in auctions (see Section 1.2 for more details). Moreover, to the best of our knowledge, our work is the first one studying the computational properties of Bayesian persuasion in the specific setting of posted price auctions.

### 1.1. Original contributions

We study the problem of computing a signaling scheme that maximizes the seller's revenue in single-item single-unit Bayesian posted price auctions. We focus on two different settings: *public signaling*, where the signals are publicly visible to all buyers, and *private signaling*, in which the seller can send a different signal to each buyer through private communication channels.

We start in Section 2 by introducing the notation and all the definitions that are needed in the rest of the article. We also provide two simple numerical examples that help to better clarify our model. The first one shows that the seller can increase their revenue by revealing information on the state of nature through signaling, with respect to the case in which they do not disclose anything. The second example shows a setting in which the seller gets an higher revenue by using private signaling rather than public signaling.

In Section 3, we provide a preliminary result that allows us to assume w.l.o.g. that the seller commits to price functions with specific structures. Indeed, in a Bayesian posted price auction, the seller may commit to a price function that selects the prices to be proposed to the buyers *stochastically* on the basis of the signals being sent to *all* the buyers. This introduces considerable additional challenges compared to standard posted price auctions. In order to overcome such difficulties, as a first result we show that the seller can commit to a price function that *deterministically* proposes a price to each buyer on the basis of the signal being sent to *that* buyer only, without incurring in any revenue loss. This holds in both the public and the private signaling settings.

In the remaining sections, we provide the core results of this article. In particular, in Section 4 we start with a negative result. We show that, in both the public and the private signaling settings, the problem of computing a revenue-maximizing signaling scheme does *not* admit either an *additive* or a *multiplicative* FPTAS unless $\mathsf{P} = \mathsf{NP}$, even for basic instances with a single buyer.

As in the mainstream literature on Bayesian persuasion [18,20,33,34], we focus on additive approximations, which are considered more suitable for this problem. We provide tight positive results by designing additive PTASs for the public (Section 7) and the private (Section 8) signaling settings. In order to do so, in Section 5 we first introduce a unifying framework encompassing both public and private signaling. Its core result is a *decomposition lemma* that allows us to focus on a finite set of buyers' posterior beliefs over states of nature—called $q$-uniform posteriors—rather than reasoning about signaling schemes with a (potentially) infinite number of signals. Compared to previous works on signaling, our framework has to deal with some additional challenges. The main one is that, in our model, the seller (sender) is *not* only required to choose how to send signals, but they also have to take some actions in the form of price proposals. This requires significant extensions to standard approaches based on decomposition lemmas [18,20,33]. The framework forms the basis on which we design our additive PTASs for the public and the private signaling settings.

In Section 7, we address the public signaling case, in which our framework establishes a connection between signaling schemes and probability distributions over $q$-uniform posteriors. This allows us to formulate the seller's revenue-maximizing problem as an LP of polynomial size, whose objective coefficients are *not* readily available.[1] However, they can be approximately computed in polynomial time by an algorithm for finding approximately-optimal prices in (non-Bayesian) posted price auctions, which we anticipate in a separate section (Section 6) as it may also be of independent interest. Solving the LP with approximate coefficients then gives the desired additive PTAS.

In Section 8, we study the private signaling case. In such a setting, our framework provides a connection between marginal signaling schemes of each buyer and probability distributions over $q$-uniform posteriors, which, to the best of our knowledge, is the first of its kind, since previous works are limited to public settings [18,34] (a notable exception is [20], which studies a specific case in between private and public signaling). Such a connection allows us to formulate an LP correlating marginal signaling schemes together and with price proposals. Although the LP has an exponential number of variables, we show that it can still be approximately solved in polynomial time by means of the ellipsoid method. This requires the implementation of a problem-specific *approximate separation oracle* that can be implemented in polynomial time by means of a dynamic programming algorithm.

Finally, Section 9 concludes the paper and describes some open problems of scientific interest. For the sake of presentation, the proofs of all the theoretical statements are deferred in the appendices.

### 1.2. Related works

The works most related to ours study algorithmic Bayesian persuasion in standard (non-posted-price) auction settings in which the seller, having information about the item being sold, acts as a sender in order to influence buyers' decisions. These works mostly focus on *second-price auctions* [15,16,18]. Two notable exceptions that study different auction models are the work by Bacchiocchi et al. [35], who address *ad auctions* using Vickrey–Clarke–Groves mechanisms, and the work by Bergemann et al. [36], who focus on ad auctions under the generalized second-price mechanism.

Emek et al. [15] study second-price auctions assuming that the seller knows the buyers' valuations. They provide an LP to compute a revenue-maximizing public signaling scheme. Moreover, they show that it is NP-hard to compute a revenue-maximizing signaling scheme in settings with Bayesian valuations.

Cheng et al. [18] provide a PTAS for the Bayesian valuations setting, complementing the hardness result in [15].

Badanidiyuru et al. [16] study algorithms whose running time does *not* depend on the number of states of nature. Moreover, they initiate the study of private signaling schemes, showing that, in second-price auctions, private signaling introduces non-trivial equilibrium selection problems.

Finally, Bacchiocchi et al. [35] provide two generalizations of the result by Cheng et al. [18] to ad auctions using Vickrey–Clarke–Groves mechanisms. In particular, they provide a PTAS for the Bayesian valuations setting with a constant number of slots, and a QPTAS for the case in which the bidder's valuations are bounded away from zero. Moreover, they provide an FPTAS working when the number of states of nature is constant.

Our work is also related to the line of research that employs distributions over $q$-uniform posteriors in Bayesian persuasion problems that do *not* involve auctions [18,20,33,34]. Cheng et al. [18] introduce $q$-uniform posteriors for the first time, showing that, under some assumptions, distributions over $q$-uniform posteriors approximate a sender-optimal public signaling scheme. They use this argument to provide a PTAS working for second-price auctions with Bayesian valuations. Xu [33] focuses on the case of receivers' binary actions and employs $q$-uniform posteriors in order to show that the problem of computing an optimal public signaling scheme admits a bi-criteria PTAS for monotone submodular sender's utility functions. Castiglioni et al. [34] consider the case of receivers' binary actions and general sender's utility functions, and they provide a tight bi-criteria QPTAS for the problem. All the works mentioned above focus on public signaling. The only exception is [20], which studies a specific case in between private and public signaling, though restricted to a voting scenario.

It is also worth citing a line of work that addresses Bayesian persuasion in posted price auctions, thought in problems (also known as *price discrimination* problems) where the sender is neither the seller nor the buyer, but an intermediary that discloses information about the item with the goal of optimizing a linear combination of buyer's surplus and seller's revenue [37–40]. Thought related to ours, these works address a fundamentally different problem, since in our setting the seller also plays the role of sender, and, thus, their goal is to jointly optimize prices and signaling schemes in order to maximize their revenue.

In conclusion, there are also some works addressing Bayesian persuasion problems in sequential decision-making settings which are related to posted price auctions, namely *prophet inequalities* [41] and *secretary problems* [42].

## 2. Preliminaries

In this section, we introduce the notation and all the definitions required for the rest of this article. Section 2.1 introduces Bayesian posted price auctions and signaling-related definitions. Section 2.2 formally states the computational problems that

[1] Notice that the objective of the LP for the public setting is *not* readily available since, as it is common in the literature, we assume that the distributions of buyers' valuations are only accessible through an oracle providing random samples drawn i.i.d. from them.

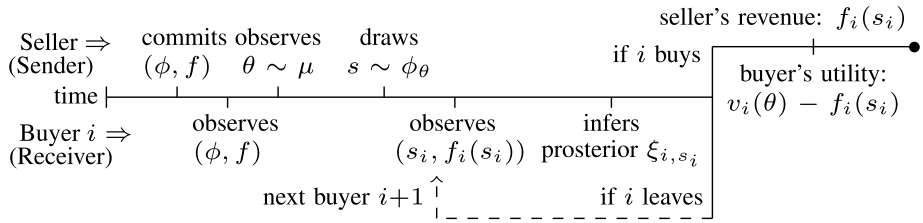


**Fig. 1.** Interaction between the seller and the buyers.

we tackle in this article. Finally, Section 2.3 provides a numerical example showing that the seller can increase their revenue by revealing information on the state of nature through signaling, while Section 2.4 provides another example in which the seller's revenue by using private signaling is better than that obtained with public signaling.

### *2.1. Bayesian posted price auctions and signaling*

In a *posted price auction*, the seller tries to sell an item to a finite set $\mathcal{N} := \{1, \dots, n\}$ of buyers arriving sequentially according to a fixed ordering. W.l.o.g., we let buyer $i \in \mathcal{N}$ be the $i$-th buyer according to such ordering. The seller chooses a price proposal $p_i \in [0, 1]$ for each buyer $i \in \mathcal{N}$. Then, each buyer in turn has to decide whether to buy the item for the proposed price or not. Buyer $i \in \mathcal{N}$ buys only if their item valuation is at least the proposed price $p_i$.[2] In that case, the auction ends and the seller gets revenue $p_i$ for selling the item, otherwise the auction continues with the next buyer.

We study *Bayesian* posted price auctions, characterized by a finite set of $d$ states of nature, namely $\Theta := \{\theta_1, \dots, \theta_d\}$. Each buyer $i \in \mathcal{N}$ has a valuation vector $v_i \in [0, 1]^d$, with $v_i(\theta)$ representing buyer $i$'s valuation when the state is $\theta \in \Theta$. Each valuation $v_i$ is independently drawn from a probability distribution $\mathcal{V}_i$ supported on $[0, 1]^d$. For the ease of presentation, we let $V \in [0, 1]^{n\times d}$ be the matrix of buyers' valuations, whose entries are $V(i, \theta) := v_i(\theta)$ for all $i \in \mathcal{N}$ and $\theta \in \Theta$. Sometimes, we also write $V_i := v_i^\top$ to denote the $i$-th row of matrix $V$, which is the valuation of buyer $i \in \mathcal{N}$. Moreover, by letting $\mathcal{V} := \{\mathcal{V}_i\}_{i\in\mathcal{N}}$ be the collection of all the distributions of buyers' valuations, we write $V \sim \mathcal{V}$ to denote that $V$ is built by drawing each $v_i$ independently from $\mathcal{V}_i$.

We model signaling with the *Bayesian persuasion* framework by Kamenica and Gentzkow [13]. We consider the case in which the seller —having knowledge of the state of nature— acts as a *sender* by issuing signals to the buyers (the *receivers*), so as to partially disclose information about the state and increase revenue. As customary in the literature, we assume that the state is drawn from a common prior distribution $\mu \in \Delta_\Theta$, explicitly known to both the seller and the buyers.[3] We denote by $\mu_\theta$ the probability of state $\theta \in \Theta$. The seller *commits to a signaling scheme* $\phi$, which is a randomized mapping from states of nature to signals for the buyers. Letting $\mathcal{S}_i$ be the finite set of signals for buyer $i \in \mathcal{N}$, a signaling scheme is a function $\phi : \Theta \to \Delta_\mathcal{S}$, where $\mathcal{S} := \times_{i\in\mathcal{N}} \mathcal{S}_i$. An element $s \in \mathcal{S}$ —called *signal profile*— is a tuple specifying a signal for each buyer. We use $s_i$ to refer to the $i$-th component of any $s \in \mathcal{S}$ (*i.e.*, the signal for buyer $i$), so that $s = (s_1, \dots, s_n)$. We let $\phi_\theta(s)$ be the probability of drawing signal profile $s \in \mathcal{S}$ when the state is $\theta \in \Theta$. Furthermore, we let $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ be the marginal signaling scheme of buyer $i \in \mathcal{N}$, with $\phi_i(\theta)$ being the marginalization of $\phi(\theta)$ with respect to buyer $i$'s signals. As for general signaling schemes, $\phi_{i,\theta}(s_i) := \sum_{s'\in\mathcal{S}:s'_i=s_i} \phi_\theta(s')$ denotes the probability of drawing signal $s_i \in \mathcal{S}_i$ when the state is $\theta \in \Theta$.

As we show in Section 3, we can assume w.l.o.g. that the seller adopts a *price function* that is *deterministic* and *factorized*, meaning that the price proposed to each buyer deterministically depends on the signal being sent to that buyer (and *not* on the signals that are sent to other buyers). Formally, the seller *commits to* a (deterministic and factorized) price function $f : \mathcal{S} \to [0, 1]^n$ with $f(s) \in [0, 1]^n$ being the price vector when the signal profile is $s \in \mathcal{S}$. In particular, the fact that $f : \mathcal{S} \to [0, 1]^n$ is factorized means that it can also be encoded by functions $f_i : \mathcal{S}_i \to [0, 1]$ defining prices for each buyer $i \in \mathcal{N}$ independently, with $f_i(s_i)$ denoting the $i$-th component of $f(s)$ for all $s \in \mathcal{S}$ and $i \in \mathcal{N}$.[4]

The interaction involving the seller and the buyers in a Bayesian posted price auction goes on as follows (see also Fig. 1):

(i) the seller commits to a signaling scheme $\phi : \Theta \to \Delta_\mathcal{S}$ and a price function $f : \mathcal{S} \to [0, 1]^n$, and the buyers observe such commitments;
(ii) the seller observes the state of nature $\theta \sim \mu$;
(iii) the seller draws a signal profile $s \sim \phi(\theta)$; and
(iv) the buyers arrive sequentially, with each buyer $i \in \mathcal{N}$ observing their signal $s_i$ and being proposed price $f_i(s_i)$.

[2] As customary in the literature, we assume that buyers always buy when they are offered a price that is equal to their valuation.
[3] In this work, given a finite set $X$, we denote with $\Delta_X$ the $(|X| - 1)$-dimensional simplex defined over the elements of $X$.
[4] Let us remark that the structure of the seller's price function ensures that a buyer does *not* get additional information about the state of nature by observing the proposed price, since the latter only depends on the signal which is revealed to them anyway.

Then, each buyer rationally updates their prior belief over states according to Bayes rule, and buys the item only if their expected valuation for the item is greater than or equal to the offered price. The interaction terminates whenever a buyer decides to buy the item or there are no more buyers arriving.

**Remark 1.** In this work, we assume that the buyers have no information about the buyers that played previously in the auction. Notice that, when such information is available, the buyers' reasoning problem may raise a number of technical issues in their belief updating procedure. Remarkably, this scenario requires some stringent assumptions on what the buyers know about the other buyers and the auction that are hardly satisfied in concrete applications. In particular, from the perspective of a given buyer, there is no direct rationality model incorporating information on the choices of the preceding buyers, unless the buyer knows: (1) the distributions of valuations of all the buyers, (2) the price functions and the signaling schemes of all the buyers, and (3) the ordering of the buyers. Moreover, even if (1), (2), and (3) are available to the buyer, incorporating preceding buyers' behavior in the belief updates would require non-trivial nested reasoning. For example, with three buyers, the third buyer should consider that the second buyer updated their beliefs with knowledge of the behavior of the first buyer. Thus, while incorporating external information sources in addition to signals may be of scientific interest, its investigation is beyond the scope of this paper.

In the following, we formally define the elements involved in step (iv).

*Buyers' posteriors* In step (iv), a buyer $i \in \mathcal{N}$ receiving a signal $s_i \in \mathcal{S}_i$ infers a posterior belief over states (also called *posterior*), which we denote by $\xi_{i,s_i} \in \Delta_\Theta$, with $\xi_{i,s_i}(\theta)$ being the posterior probability of state $\theta \in \Theta$. Formally,

$$\xi_{i,s_i}(\theta) := \frac{\mu_\theta \phi_{i,\theta}(s_i)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i)}. \tag{1}$$

Thus, after receiving signal $s_i \in \mathcal{S}_i$, buyer $i$'s expected valuation for the item is $\sum_{\theta \in \Theta} v_i(\theta)\, \xi_{i,s_i}(\theta)$, and the buyer buys it only if such value is at least as large as the price $f_i(s_i)$. In the following, given a signal profile $s \in \mathcal{S}$, we denote by $\xi_s$ a tuple defining all buyers' posteriors resulting from observing signals in $s$; formally, $\xi_s := (\xi_{1,s_1}, \ldots, \xi_{n,s_n})$.

We conclude the subsection with some notation and definitions that are useful in the rest of this article.

*Distributions over posteriors* In single-receiver Bayesian persuasion models, it is oftentimes useful to represent signaling schemes as convex combinations of the posteriors they can induce. In our setting, a marginal signaling scheme $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ of buyer $i \in \mathcal{N}$ *induces* a probability distribution $\gamma_i$ over posteriors in $\Delta_\Theta$, with $\gamma_i(\xi_i)$ denoting the probability of posterior $\xi_i \in \Delta_\Theta$. Formally, it holds that

$$\gamma_i(\xi_i) := \sum_{s_i \in \mathcal{S}_i : \xi_{i,s_i} = \xi_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i).$$

Intuitively, $\gamma_i(\xi_i)$ denotes the probability that buyer $i$ has posterior $\xi_i$. Indeed, it is possible to directly reason about distributions $\gamma_i$ rather than marginal signaling schemes, provided that such distributions are *consistent* with the prior. Formally, by letting $\text{supp}(\gamma_i) := \{\xi_i \in \Delta_\Theta \mid \gamma_i(\xi_i) > 0\}$ be the support of $\gamma_i$, it must be required that

$$\sum_{\xi_i \in \text{supp}(\gamma_i)} \gamma_i(\xi_i)\, \xi_i(\theta) = \mu_\theta \quad \forall \theta \in \Theta. \tag{2}$$

*Public signaling schemes* A *public* signaling scheme is such that the same signal is sent to all the buyers. Formally, a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ is public if: (i) $\mathcal{S}_i = \mathcal{S}_j$ for all $i, j \in \mathcal{N}$; and (ii) for every $\theta \in \Theta$, $\phi_\theta(s) > 0$ only for signal profiles $s \in \mathcal{S}$ such that $s_i = s_j$ for $i, j \in \mathcal{N}$. Since, given a signal profile $s \in \mathcal{S}$, under a public signaling scheme all the buyers always share the same posterior (*i.e.*, $\xi_{i,s_i} = \xi_{j,s_j}$ for all $i, j \in \mathcal{N}$), we overload notation and sometimes use $\xi_s \in \Delta_\Theta$ to denote the unique posterior appearing in $\xi_s = (\xi_{1,s_1}, \ldots, \xi_{n,s_n})$. Similarly, in the public setting, given a posterior $\xi \in \Delta_\Theta$ we sometimes write $\xi$ in place of a tuple of $n$ copies of $\xi$.

### 2.2. Computational problems

We focus on the problem of computing a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ and a price function $f : \mathcal{S} \to [0, 1]^n$ that maximize the seller's expected revenue, considering both public and private signaling settings.

We denote by $\textsc{Rev}(\mathcal{V}, p, \xi)$ the expected revenue of the seller when the distributions of buyers' valuations are given by $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, the proposed prices are defined by the vector $p \in [0, 1]^n$, and the buyers' posteriors are those specified by the tuple $\xi = (\xi_1, \ldots, \xi_n)$ containing a posterior $\xi_i \in \Delta_\Theta$ for each buyer $i \in \mathcal{N}$. Then, the seller's expected revenue is:

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{s \in \mathcal{S}} \phi_\theta(s) \textsc{Rev}(\mathcal{V}, f(s), \xi_s)\,. \tag{3}$$

| | | State $\theta_H$ ($\mu_{\theta_H} = 0.3$) | | State $\theta_L$ ($\mu_{\theta_L} = 0.7$) | |
|---|---|---|---|---|---|
| $\mathcal{A}$ | A | $p$ | $3/4 - p$ | $p$ | $1/4 - p$ |
| | R | 0 | 0 | 0 | 0 |

| | | State $\theta_H$ ($\mu_{\theta_H} = 0.3$) | | State $\theta_L$ ($\mu_{\theta_L} = 0.7$) | |
|---|---|---|---|---|---|
| $\mathcal{A}$ | A | $p$ | $1 - p$ | $p$ | $-p$ |
| | R | 0 | 0 | 0 | 0 |

**Fig. 2.** Payoffs of the game between the seller and the buyer in the example in Section 2.3. Rows represent the buyer's actions. For each possible state of nature $\theta \in \Theta = \{\theta_H, \theta_L\}$, the first column is the seller's revenue and the second one is the buyer's utility. **Left**: Payoffs of a buyer with valuation vector $v_1 = \left(\frac{3}{4}, \frac{1}{4}\right)$. **Right**: Payoffs of a buyer with valuation vector $v_2 = (1, 0)$.

In the following, we denote by $OPT$ the value of the seller's expected revenue for a revenue-maximizing $(\phi, f)$ pair.

In this work, we assume that algorithms have access to a black-box oracle to sample buyers' valuations according to the probability distributions specified by $\mathcal{V}$ (rather than actually knowing such distributions). Thus, we look for algorithms that output pairs $(\phi, f)$ such that

$$\mathbb{E}\left[\sum_{\theta \in \Theta} \mu_\theta \sum_{s \in \mathcal{S}} \phi_\theta(s) \textsc{Rev}(\mathcal{V}, f(s), \xi_s)\right] \geq OPT - \lambda,$$

where $\lambda \geq 0$ is an additive error. Notice that the expectation above is with respect to the randomness of the algorithm, which originates from using the black-box sampling oracle.

*Notation for non-Bayesian posted price auctions* When we study non-Bayesian posted price auctions, we stick to our notation, with the following differences: valuations are scalars rather than vectors, namely $v_i \in [0, 1]$; distributions $\mathcal{V}_i$ are supported on $[0, 1]$ rather than $[0, 1]^d$; the matrix $V$ is indeed a column vector whose components are buyers' valuations; and the price function $f$ is replaced by a single price vector $p \in [0, 1]^n$, with its $i$-th component $p_i$ being the price for buyer $i \in \mathcal{N}$. Moreover, we continue to use the notation $\textsc{Rev}$ to denote seller's revenues, dropping the dependence on the tuple of posteriors. Thus, in a non-Bayesian auction in which the distributions of buyers' valuations are $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, the notation $\textsc{Rev}(\mathcal{V}, p)$ simply denotes the seller's expected revenue by selecting a price vector $p \in [0, 1]^n$.

### 2.3. The benefits of signaling

In this subsection, we provide an example that shows how the seller can increase their expected revenue by using a suitably-computed $(\phi, f)$ pair, with respect to the case in which no signaling scheme is employed.

We consider a simple scenario in which a seller tries to convince a buyer that the quality of the item being sold is high. When the item actually has high quality, disclosing information about it will help the seller. On the contrary, when the item has low quality, it is *not* convenient for the seller to provide further details about the item to the buyer. In particular, we assume that there are two states of nature, namely $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H$ intuitively corresponds to *high quality* and $\theta_L$ to *low quality*. Since there is only one buyer, in the following we drop w.l.o.g. any reference to the buyer's index appearing in our notation. We suppose that:

- the state of nature is drawn from the prior $\mu = (\mu_{\theta_H}, \mu_{\theta_L}) = (0.3, 0.7)$;
- there are two possible buyer's valuations, which are defined by the valuation vectors $v_1 = \left(\frac{3}{4}, \frac{1}{4}\right)^\top$ and $v_2 = (1, 0)^\top$, respectively;
- the distribution over buyer's valuations is $\mathcal{V}$, where $\Pr_{v \sim \mathcal{V}}\{v = v_1\} = \frac{1}{2}$ is the probability that the buyer has valuation $v_1$, while $\Pr_{v \sim \mathcal{V}}\{v = v_2\} = \frac{1}{2}$ is the probability that the buyer has valuation $v_2$.

The seller chooses a price proposal $p \in [0, 1]$ and, then, the buyer decides whether to buy the item for that price or not. Hence, for ease of presentation, we denote the set of actions available to the buyer as $\mathcal{A} = \{\text{A}, \text{R}\}$, where A corresponds to *accepting* the offer and R to *refusing* it. Fig. 2 shows the seller's revenue and the buyer's utility for each possible buyer's valuation vector and realized state of nature, depending on the action of the buyer.

In the following, we compare the revenue of the seller when they cannot disclose their private information about the item with their revenue when they can employ a signaling scheme to (partially) disclose such information, showing how signaling schemes can increase the seller's expected revenue.

*Without signaling scheme* The buyer's expected valuation for the item is:

- $\mu_{\theta_H} v_1(\theta_H) + \mu_{\theta_L} v_1(\theta_L) = 0.3 \cdot \frac{3}{4} + 0.7 \cdot \frac{1}{4} = 0.4$ for a buyer with valuation vector $v_1$;
- $\mu_{\theta_H} v_2(\theta_H) + \mu_{\theta_L} v_2(\theta_L) = 0.3 \cdot 1 + 0.7 \cdot 0 = 0.3$ for a buyer with valuation vector $v_2$.

As a result, if $p \leq 0.3$, the buyer plays action A for both the valuation vectors $v_1$ and $v_2$, while, when $0.3 < p \leq 0.4$, the buyer plays action A only for valuation vector $v_1$, otherwise they play action R. Thus, if the seller chooses a price $p \leq 0.3$, the seller's expected revenue is

| | | Realized state | |
|---|---|---|---|
| | | State $\theta_H$ | State $\theta_L$ |
| $\mathcal{S}$ | $s_1$ | 1 | 3/7 |
| | $s_2$ | 0 | 4/7 |

| | | State of nature | | |
|---|---|---|---|---|
| | | State $\theta_H$ | State $\theta_L$ | $\gamma$ |
| supp($\gamma$) | $\xi_{s_1}$ | 1/2 | 1/2 | 3/5 |
| | $\xi_{s_2}$ | 0 | 1 | 2/5 |

**Fig. 3.** Signaling scheme $\phi$ in the example in Section 2.3. **Left**: Each column describes the probability with which the two signals are drawn given the realized state of nature. **Right**: Representation of $\phi$ as a probability distribution $\gamma$ over buyer's posteriors.

$$\sum_{\theta\in\Theta}\mu_\theta \sum_{i\in\{1,2\}} \Pr_{v\sim\mathcal{V}}\{v=v_i\}\,\mathbb{I}\{v_i\geq p\}\,p = 0.3\left(\frac{1}{2}\cdot p+\frac{1}{2}\cdot p\right)+0.7\left(\frac{1}{2}\cdot p+\frac{1}{2}\cdot p\right)=p.$$

If the seller chooses a price $0.3 < p \leq 0.4$, the seller's expected revenue is

$$\sum_{\theta\in\Theta}\mu_\theta \sum_{i\in\{1,2\}} \Pr_{v\sim\mathcal{V}}\{v=v_i\}\,\mathbb{I}\{v_i\geq p\}\,p = 0.3\left(\frac{1}{2}\cdot p\right)+0.7\left(\frac{1}{2}\cdot p\right)=\frac{1}{2}\cdot p.$$

Furthermore, if the seller chooses any price $p > 0.4$, the seller's expected revenue is zero. As a consequence, the revenue-maximizing price is $p = 0.3$, which corresponds to an expected revenue of 0.3 for the seller.

*With signaling scheme* Suppose that the seller adopts the signaling scheme described in Fig. 3, which is such that $\phi_{\theta_H}(s_1) = 1$, $\phi_{\theta_L}(s_1) = \frac{3}{7}$, $\phi_{\theta_H}(s_2) = 0$, and $\phi_{\theta_L}(s_2) = \frac{4}{7}$. Moreover, the price function $f$ is such that $f(s_1) = 0.5$ and $f(s_2) = 0.25$. Then, the buyer's expected valuations for the item are:

- $v_1^\top \xi_{s_1} = v_1(\theta_H)\,\xi_{s_1}(\theta_H) + v_1(\theta_L)\,\xi_{s_1}(\theta_L) = \frac{3}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{2}$ for a buyer with valuation vector $v_1$ observing signal $s_1$;
- $v_1^\top \xi_{s_2} = v_1(\theta_H)\,\xi_{s_2}(\theta_H) + v_1(\theta_L)\,\xi_{s_2}(\theta_L) = \frac{3}{4}\cdot 0+\frac{1}{4}\cdot 1=\frac{1}{4}$ for a buyer with valuation vector $v_1$ observing signal $s_2$;
- $v_2^\top \xi_{s_1} = v_2(\theta_H)\,\xi_{s_1}(\theta_H) + v_2(\theta_L)\,\xi_{s_1}(\theta_L) = 1\cdot\frac{1}{2}+0\cdot\frac{1}{2}=\frac{1}{2}$ for a buyer with valuation vector $v_2$ observing signal $s_1$;
- $v_2^\top \xi_{s_2} = v_2(\theta_H)\,\xi_{s_2}(\theta_H) + v_2(\theta_L)\,\xi_{s_2}(\theta_L) = 1\cdot 0+0\cdot 1=0$ for a buyer with valuation vector $v_2$ observing signal $s_2$.

Thus, the seller's expected revenue is

$$\begin{aligned}
&\sum_{\theta\in\Theta}\mu_\theta\sum_{s\in\mathcal{S}}\phi_\theta(s)\textsc{Rev}\left(\mathcal{V}, f(s), \xi_s\right) = \\
&\left(\mu_{\theta_H}\phi_{\theta_H}(s_1)+\mu_{\theta_L}\phi_{\theta_L}(s_1)\right)\\
&\qquad\left(\Pr_{v\sim\mathcal{V}}\{v=v_1\}\mathbb{I}\{v_1^\top\xi_{s_1}\geq f(s_1)\}f(s_1)+\Pr_{v\sim\mathcal{V}}\{v=v_2\}\mathbb{I}\{v_2^\top\xi_{s_1}\geq f(s_1)\}f(s_1)\right)\\
&+\left(\mu_{\theta_H}\phi_{\theta_H}(s_2)+\mu_{\theta_L}\phi_{\theta_L}(s_2)\right)\\
&\qquad\left(\Pr_{v\sim\mathcal{V}}\{v=v_1\}\mathbb{I}\{v_1^\top\xi_{s_2}\geq f(s_2)\}f(s_2)+\Pr_{v\sim\mathcal{V}}\{v=v_2\}\mathbb{I}\{v_2^\top\xi_{s_2}\geq f(s_2)\}f(s_2)\right)\\
&=\left(0.3\cdot 1+0.7\cdot\frac{3}{7}\right)\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\right)+\left(0.3\cdot 0+0.7\cdot\frac{4}{7}\right)\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot 0\right)\\
&=0.35.
\end{aligned}$$

As a result, the seller's expected revenue with the chosen $(\phi, f)$ pair is 0.35, which is larger than the maximum expected revenue achievable without signaling scheme, which is 0.3.

### 2.4. Private versus public signaling

In this subsection, we provider an example in which private signaling schemes provide a higher revenue to the seller than public ones.

We consider a Bayesian posted price auction in which there are two states of nature $\Theta = \{\theta_1, \theta_2\}$, and two buyers $\mathcal{N} = \{1, 2\}$. Moreover:

- The prior distribution over the states of nature is $\mu = (\mu_{\theta_1}, \mu_{\theta_2}) = (\frac{1}{2}, \frac{1}{2})$.
- The first buyer's valuation can be either $v_{1,1} = (0, 0)^\top$ or $v_{1,2} = (1, 0)^\top$, while the second buyer's one is either $v_{2,1} = (\frac{1}{2}, \frac{1}{2})^\top$ or $v_{2,2} = (0, 1)^\top$.
- The distribution over the first buyer's valuations is $\mathcal{V}_1$, where $\Pr_{v_1\sim\mathcal{V}_1}\left\{v_1 = v_{1,1}\right\} = \frac{2}{3}$ is the probability that the buyer has valuation $v_{1,1}$, while $\Pr_{v_1\sim\mathcal{V}_1}\left\{v_1 = v_{1,2}\right\} = \frac{1}{3}$ is the probability that the buyer has valuation $v_{1,2}$.
- The distribution over the second buyer's valuations is $\mathcal{V}_2$, where $\Pr_{v_2\sim\mathcal{V}_2}\left\{v_2 = v_{2,1}\right\} = \frac{1}{2}$ is the probability that the buyer has valuation $v_{2,1}$, while $\Pr_{v_2\sim\mathcal{V}_2}\left\{v_2 = v_{2,2}\right\} = \frac{1}{2}$ is the probability that the buyer has valuation $v_{2,2}$.

*Private signaling* We show that the sellers' revenue with private signaling is at least $\frac{7}{12}$. Let us consider a private signaling scheme using two signals $s_{1,1}$ and $s_{1,2}$ such that $\phi_{1,\theta_1}(s_{1,1}) = 1$ and $\phi_{1,\theta_2}(s_{1,2}) = 1$, *i.e.*, the sender fully reveals the state to the first buyer, while $\phi_{2,\theta_1}(s_{2,1}) = \phi_{2,\theta_2}(s_{2,1}) = 1$, *i.e.*, the sender reveals no information to the second buyer. Moreover, let us consider a price function such that the price proposed by the seller to the first buyer is 1, namely $f_1(s_{1,1}) = f_1(s_{1,2}) = 1$, while the price proposed to the second buyer is $\frac{1}{2}$, namely $f_2(s_{2,1}) = \frac{1}{2}$. It is immediate to check that, when the state of nature is $\theta_1$ and the first buyer's valuation is $v_{1,1}$, then the buyer buys the item, otherwise, in all the other cases, they never buy the item. Moreover, the second buyer always buys the item, since their expected valuation for it is always greater than or equal to $\frac{1}{2}$. Then, the seller's revenue is:

$$
\begin{aligned}
&\mu_{\theta_1}\phi_{1,\theta_1}(s_{1,1})\phi_{2,\theta_2}(s_{2,1})\Bigg[\Pr_{v_1\sim\mathcal{V}_1}\{v_1 = v_{1,1}\}\Bigg(\Pr_{v_2\sim\mathcal{V}_2}\{v_2 = v_{2,1}\}\mathbb{I}\{v_{2,1}^\top \xi_{2,s_{2,1}} \geq f_2(s_{2,1})\}f_2(s_{2,1}) \\
&\qquad + \Pr_{v_2\sim\mathcal{V}_2}\{v_2 = v_{2,2}\}\mathbb{I}\{v_{2,2}^\top \xi_{2,s_{2,1}} \geq f_2(s_{2,1})\}f_2(s_{2,1})\Bigg) \\
&\qquad + \Pr_{v_1\sim\mathcal{V}_1}\{v_1 = v_{1,2}\}\mathbb{I}\{v_{1,2}^\top \xi_{1,s_{1,1}} \geq f_1(s_{1,1})\}f_1(s_{1,1})\Bigg] \\
&+ \mu_{\theta_2}\phi_{1,\theta_2}(s_{1,2})\phi_{2,\theta_2}(s_{2,1}) \\
&\qquad \Bigg(\Pr_{v_2\sim\mathcal{V}_2}\{v_2 = v_{2,1}\}\mathbb{I}\{v_{2,1}^\top \xi_{s_{2,1}} \geq f_2(s_{2,1})\}f_2(s_{2,1}) \\
&\qquad + \Pr_{v_2\sim\mathcal{V}_2}\{v_2 = v_{2,2}\}\mathbb{I}\{v_{2,2}^\top \xi_{s_{2,1}} \geq f_2(s_{2,1})\}f_2(s_{2,1})\Bigg) \\
&= \frac{1}{2}\left(\frac{2}{3}\left(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\right) + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\right) \\
&= \frac{7}{12},
\end{aligned}
$$

where we omitted all the terms that are equal to zero, for ease of exposition.

*Public signaling* We show that, when the seller is forced to use public signaling schemes, their revenue is at most $\frac{1}{2}$. In order to show this, we prove that the seller's revenue is at most $\frac{1}{2}$ in each posterior. By an averaging argument, this implies that also the revenue of an optimal public signaling scheme is upper bounded by $\frac{1}{2}$. Let $\xi \in \Delta_\Theta$ be a posterior. It is easy to see that, when the induced posterior is $\xi$, an optimal price function is such that either the price proposed to the first buyer is $\xi_{\theta_1}$, so that the first buyer buys at the largest possible price when their valuation is $v_{1,2}$, or, only if $\xi_{\theta_1} < 1$, it is 1, so that the first buyer never buys the item. Suppose that the price proposed to the first buyer is $\xi_{\theta_1}$, then the buyer buys the item with probability $\frac{1}{3}$, resulting in a seller's expected revenue of $\frac{1}{3}\xi_{\theta_1}$, while they do *not* buy the item with probability $\frac{2}{3}$. In the latter case, the revenue that can be extracted from the second buyer is as follows:

- if $\xi_{\theta_1} < \frac{1}{2}$, an optimal price to be proposed to the second buyer is $\frac{1}{2}$, so that the revenue is $\frac{1}{2}$;
- if $\xi_{\theta_1} \in [\frac{1}{2}, \frac{3}{4}]$, an optimal price is $\xi_{\theta_1}$, so that the revenue is $1 - \xi_{\theta_1}$;
- if $\xi_{\theta_1} > \frac{3}{4}$, an optimal price is $\frac{1}{2}$, and the revenue is $\frac{1}{4}$.

Now, we upper bound the seller's revenue in each case:

- in the first case, the maximum total revenue is $\frac{1}{3}\xi_{\theta_1} + \frac{2}{3}\frac{1}{2} \leq \frac{1}{2}$;
- in the second case, it is $\frac{1}{3}\xi_{\theta_1} + \frac{2}{3}(1 - \xi_{\theta_1}) \leq \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$;
- in the third case, the maximum total revenue is $\frac{1}{3}\xi_{\theta_1} + \frac{2}{3}\frac{1}{4} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Thus, in all the cases, the seller's revenue is at most $\frac{1}{2}$. Finally, in the case in which the price proposed to the first buyer 1 (and $\xi_{\theta_1} < 1$), the seller extracts revenue only from the second buyer (since the first buyer does *not* buy the item). It is easy to see that the largest revenue that can be extracted is $\frac{1}{2}$. This shows that the seller's revenue when the seller is forced to employ public signaling schemes is at most $\frac{1}{2}$.

## 3. On the generality of deterministic and factorized price functions

Before presenting the main results of the article, in this section we show that restricting the attention to *deterministic* and *factorized* price functions is without loss of generality. Formally, we prove that, by using a deterministic and factorized price function, the sender can achieve the same revenue as the one obtained with an *arbitrary* price function.

In general, the seller may commit to a $(\phi, \mathsf{f})$ pair, where $\phi$ is a signaling scheme (as defined in Section 2) and $\mathsf{f}$ is a function mapping signal profiles $s \in \mathcal{S}$ to probability distributions over price vectors, *i.e.*, over $[0,1]^n$. Given a $(\phi, \mathsf{f})$ pair, each buyer $i \in \mathcal{N}$ computes their posterior beliefs in a way that is different from the one defined in Equation (1). In particular, the posterior beliefs also depend on proposed prices. In particular, a signal $s_i \in \mathcal{S}_i$ and a price $p_i \in [0,1]$ induce a buyer $i$'s posterior $\xi_{i,s_i,p_i} \in \Delta_\Theta$, which is defined so that, for every $\theta \in \Theta$[5]:

$$\xi_{i,s_i,p_i}(\theta) := \frac{\mu_\theta \sum_{s' \in \mathcal{S}: s'_i = s_i} \phi_\theta(s') \Pr_{p' \sim \mathsf{f}(s')} \left\{ p'_i = p_i \right\}}{\sum_{\theta' \in \Theta} \mu_{\theta'} \sum_{s' \in \mathcal{S}: s'_i = s_i} \phi_{\theta'}(s') \Pr_{p' \sim \mathsf{f}(s')} \left\{ p'_i = p_i \right\}}.$$

By letting $\xi_{s,p} := (\xi_{1,s_1,p_1}, \dots, \xi_{n,s_n,p_n})$ be the tuple of induced buyers' posteriors for a signal profile $s \in \mathcal{S}$ and a price vector $p \in [0,1]^n$, the seller's expected revenue for a $(\phi, \mathsf{f})$ pair is:

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{s \in \mathcal{S}} \sum_{p \in \text{supp}(\mathsf{f}(s))} \phi_\theta(s) \Pr_{p' \sim \mathsf{f}(s)} \{p' = p\} \text{Rev}(\mathcal{V}, p, \xi_{s,p}).$$

It is easy to see that when the price function is factorized and deterministic, it is the case that $\xi_{i,s_i,p_i} = \xi_{i,s_i}$, where $\xi_{i,s_i}$ is defined as in Equation (1), and that the seller's expected revenue simplifies to Equation (3).

The following two theorems, which focus on public and private signaling problems, respectively, show that there always exists a revenue-maximizing $(\phi, f)$ pair that employs a deterministic and factorized price function $f$.

**Theorem 1.** *In Bayesian posted price auctions with public signaling, there always exists a revenue-maximizing $(\phi, f)$ pair such that $f$ is a deterministic and factorized price function.*

**Theorem 2.** *In Bayesian posted price auctions with private signaling, there always exists a revenue-maximizing $(\phi, f)$ pair such that $f$ is a deterministic and factorized price function.*

## 4. Hardness of signaling with a single buyer

We start presenting the main result of this article with a negative result: there is no additive FPTAS for the problem of computing a revenue-maximizing $(\phi, f)$ pair unless $\mathsf{P} = \mathsf{NP}$, in both public and private signaling settings. Notice that this negative result directly extends to multiplicative FPTASs. Our result holds even in the basic case with only one buyer, where public and private signaling are equivalent. Notice that, in the reduction that we use to prove our result, we assume that the support of the distribution of valuations of the (single) buyer is finite and that such distribution is perfectly known to the seller. This represents an even simpler setting than that in which the seller has only access to a black-box oracle returning samples drawn from the buyer's distribution of valuations. The result formally reads as follows:

**Theorem 3.** *In Bayesian posted price auctions, there is no additive FPTAS for the problem of computing a revenue-maximizing $(\phi, f)$ pair unless $\mathsf{P} = \mathsf{NP}$, even when there is a single buyer.*

## 5. Unifying public and private signaling

In this section, we introduce a general mathematical framework related to buyers' posteriors and distributions over them, proving some results that will be crucial in the rest of this work, both in public and private signaling scenarios.

One of the main difficulties in computing sender-optimal signaling schemes is that they might need a (potentially) infinite number of signals, resulting in infinitely-many receiver's posteriors. The trick commonly used to circumvent this issue in settings with a finite number of valuations is to use direct signals, which explicitly specify action recommendations for each receiver's valuation [43–46]. However, in our auction setting, this solution is *not* viable, since a direct signal for a buyer $i \in \mathcal{N}$ should represent a recommendation for every possible $v_i \in [0,1]^d$, and these are infinitely many. An alternative technique, which can be employed in our setting, is to restrict the number of possible posteriors.

Our core idea is to focus on a small set of posteriors, which are those encoded as particular $q$-uniform probability distributions, as formally stated in the following definition. Notice that, in all the definitions and results of this section, we denote by $\xi \in \Delta_\Theta$ a generic posterior common to all the buyers and with $\gamma$ a probability distribution over $\Delta_\Theta$ (*i.e.*, over posteriors).

**Definition 1** *(q-uniform posterior).* A posterior $\xi \in \Delta_\Theta$ is *$q$-uniform* if it can be obtained by averaging the elements of a multiset defined by $q \in \mathbb{N}_{>0}$ canonical basis vectors of $\mathbb{R}^d$.

[5] In this article, for the ease of exposition, we assume that $\mathsf{f}$ maps each signal profile $s \in \mathcal{S}$ to a *discrete* probability distribution over price vectors. Our results in Section 3 can be easily generalized to the case in which such distributions are continuous.

We denote the set of all $q$-uniform posteriors as $\Xi^q \subset \Delta_\Theta$. Notice that the set $\Xi^q$ has size $|\Xi^q| = O\left(d^q\right)$.

The existence of an approximately-optimal signaling scheme that only uses $q$-uniform posteriors is usually proved by means of so-called *decomposition lemmas* (see [18,20,33]). The goal of these lemmas is to show that, given some signaling scheme encoded as a distribution over posteriors, it is possible to obtain a new signaling scheme whose corresponding distribution is supported only on $q$-uniform posteriors, and such that the sender's utility only decreases by a small amount. At the same time, these lemmas must also ensure that the distribution over posteriors corresponding to the new signaling scheme is still consistent (according to Equation (2)).

The main result of our framework (Theorem 4) is a decomposition lemma that is suitable for our setting. Before stating the result, we need to introduce some preliminary definitions.

**Definition 2** *($(\alpha,\epsilon)$-decreasing distribution).* Let $\alpha,\epsilon > 0$. A probability distribution $\gamma$ over $\Delta_\Theta$ is *$(\alpha,\epsilon)$-decreasing around a given posterior* $\xi \in \Delta_\Theta$ if the following condition holds for every matrix $V \in [0,1]^{n\times d}$ of buyers' valuations:

$$\Pr_{\tilde{\xi}\sim\gamma}\left\{V_i\tilde{\xi} \geq V_i\xi - \epsilon\right\} \geq 1-\alpha \quad \forall i \in \mathcal{N}.$$

Intuitively, a probability distribution $\gamma$ as in Definition 2 can be interpreted as a perturbation of the given posterior $\xi$ such that, with high probability, buyers' expected valuations in $\gamma$ are at most $\epsilon$ less than those in posterior $\xi$.

Definition 2 is similar to analogous ones in the literature [20,33], where the distance is usually measured in both directions, as $|V_i\tilde{\xi} - V_i\xi| \leq \epsilon$. We look only at the direction of decreasing values, since in our setting, if a buyer's valuation increases, then the seller's revenue also increases.

The second definition we need is about functions mapping vectors in $[0,1]^n$ —defining a valuation for each buyer— to seller's revenues. For instance, one such function could be the seller's revenue given price vector $p \in [0,1]^n$. In particular, we define the stability of a function $g$ compared to another function $h$. Intuitively, $g$ is stable compared to $h$ if the value of $g$, in expectation over buyers' valuations and posteriors drawn from a probability distribution $\gamma$ that is $(\alpha,\epsilon)$-decreasing around $\xi$, is "close" to the value of $h$ given $\xi$, in expectation over buyers' valuations. Formally:

**Definition 3** *($(\delta,\alpha,\epsilon)$-stability).* Let $\alpha,\epsilon,\delta > 0$. Given a posterior $\xi \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, and two functions $g,h : [0,1]^n \to [0,1]$, $g$ is *$(\delta,\alpha,\epsilon)$-stable compared to $h$ for $(\xi,\mathcal{V})$* if, for every probability distribution $\gamma$ over $\Delta_\Theta$ that is $(\alpha,\epsilon)$-decreasing around $\xi$, it holds:

$$\mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right] \geq (1-\alpha)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\xi)\right] - \delta\epsilon.$$

The notion of compared stability has been already used [18,20]. However, previous works consider the case in which $g$ is a relaxation of $h$. Instead, our definition is conceptually different, as $g$ and $h$ represent two different functions corresponding to different price vectors of the seller.

Now, we are ready to state our main result. We show that, for any buyer's posterior $\xi \in \Delta_\Theta$, if a function $g$ is stable compare to $h$, then there exists a suitable probability distribution over $q$-uniform posteriors such that the expected value of $g$ given such distribution is "close" to that of $h$ given $\xi$.[6]

**Theorem 4.** *Let $\alpha,\epsilon,\delta > 0$, and set $q := \frac{32}{\epsilon^2}\log\frac{4}{\alpha}$. Given a posterior $\xi \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, and two functions $g,h : [0,1]^n \to [0,1]$, if $g$ is $(\delta,\alpha,\epsilon)$-stable compared to $h$ for $(\xi,\mathcal{V})$, then there exists $\gamma \in \Delta_{\Xi^q}$ such that, for every $\theta \in \Theta$, it holds $\sum_{\tilde{\xi}\in\text{supp}(\gamma)}\gamma(\tilde{\xi})\tilde{\xi}(\theta) = \xi(\theta)$ and*

$$\mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[\tilde{\xi}(\theta)g(V\tilde{\xi})\right] \geq \xi(\theta)\left((1-\alpha)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\xi)\right] - \delta\epsilon\right). \tag{4}$$

The crucial feature of Theorem 4 is that Equation (4) holds for every state. This is fundamental for proving our results in the private signaling scenario. On the other hand, with public signaling, we will make use of the following (weaker) corollary, obtained by summing Equation (4) over all $\theta \in \Theta$.

**Corollary 1.** *Let $\alpha,\epsilon,\delta > 0$, and set $q := \frac{32}{\epsilon^2}\log\frac{4}{\alpha}$. Given a posterior $\xi \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, and two functions $g,h : [0,1]^n \to [0,1]$, if $g$ is $(\delta,\alpha,\epsilon)$-stable compared to $h$ for $(\xi,\mathcal{V})$, then there exists $\gamma \in \Delta_{\Xi^q}$ such that, for every $\theta \in \Theta$, it holds $\sum_{\tilde{\xi}\in\text{supp}(\gamma)}\gamma(\tilde{\xi})\tilde{\xi}(\theta) = \xi(\theta)$ and*

$$\mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right] \geq (1-\alpha)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\xi)\right] - \delta\epsilon. \tag{5}$$

[6] Notice that, as the techniques previously developed in the Bayesian persuasion literature, our techniques cannot be applied to derive multiplicative approximations. In particular, the concentration inequalities that we employ in the proof of Theorem 4 guarantee an additive error and do not provide any guarantee on multiplicative errors. Notably, the adaptation of our techniques to get multiplicative approximations would need concentration inequalities with multiplicative guarantees, but, to the best of our knowledge, no such tools are known in the literature.

## 6. Auxiliary results on non-Bayesian posted price auctions

In this section, we focus on non-Bayesian posted price auctions, proving some results that will be useful in the rest of the paper. In particular, we study what happens to the seller's expected revenue when buyers' valuations are "slightly decreased", proving that the revenue also decreases, but only by a small amount. This result will be crucial when dealing with public signaling, and it also allows to design a poly-time algorithm for finding approximately-optimal price vectors in non-Bayesian auctions, as we show at the end of this section.

In the following, we extensively use distributions of buyers' valuations as specified in the definition below.

**Definition 4.** Given $\epsilon > 0$, we denote by $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$ and $\mathcal{V}^\epsilon = \{\mathcal{V}_i^\epsilon\}_{i \in \mathcal{N}}$ two collections of distributions of buyers' valuations such that, for every price vector $p \in [0,1]^n$, it holds:

$$\Pr_{v_i \sim \mathcal{V}_i^\epsilon} \{v_i \geq p_i - \epsilon\} \geq \Pr_{v_i \sim \mathcal{V}_i} \{v_i \geq p_i\} \quad \forall i \in \mathcal{N}.$$

Intuitively, valuations drawn from $\mathcal{V}^\epsilon$ are "slightly decreased" with respect to those drawn from $\mathcal{V}$, since the probability with which any buyer $i \in \mathcal{N}$ buys the item at the (reduced) price $[p_i - \epsilon]_+$ when their valuation is drawn from $\mathcal{V}_i^\epsilon$ is at least as large as the probability of buying at price $p_i$ when their valuation is drawn from $\mathcal{V}_i$.[7]

Our main contribution in this section is Lemma 2. The lemma shows that

$$\max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}^\epsilon, p) \geq \max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \epsilon.$$

By letting $p^*$ be any revenue-maximizing price vector under distributions $\mathcal{V}$, *i.e.*, $p^* \in \arg\max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p)$, one may naïvely think that, since under distributions $\mathcal{V}^\epsilon$ and price vector $[p^* - \epsilon]_+$ each buyer would buy the item at least with the same probability as with distributions $\mathcal{V}$ and price vector $p^*$, while paying a price that is only $\epsilon$ less, then $\textsc{Rev}(\mathcal{V}^\epsilon, [p^* - \epsilon]_+) \geq \textsc{Rev}(\mathcal{V}, p^*) - \epsilon$, proving the result. However, this line of reasoning does *not* work, since, as shown by the following example, it has a major flaw.

**Example 1.** Consider a posted price auction with two buyers. In the first case (distributions $\mathcal{V}$), buyer 1 has valuation $v_1 = \frac{1}{2}$ and buyer 2 has valuation $v_2 = 1$. In such setting, an optimal price vector $p^*$ is such that $p_1^* = \frac{1}{2} + \epsilon$ and $p_2^* = 1$, so that the revenue of the seller, namely $\textsc{Rev}(\mathcal{V}, p^*)$, is 1. In the second case (distributions $\mathcal{V}^\epsilon$), buyer 1 has valuations $v_1 = \frac{1}{2}$ and buyer 2 has valuation $v_2 = 1 - \epsilon$. Thus, the revenue of the seller for the price vector $p^{*,\epsilon}$ (with $p_1^{*,\epsilon} = \frac{1}{2}$ and $p_2^{*,\epsilon} = 1 - \epsilon$), namely $\textsc{Rev}(\mathcal{V}^\epsilon, p^{*,\epsilon})$, is $\frac{1}{2}$, since buyer 1 will buy the item.

The crucial feature of the setting described in Example 1 is that there is an optimal price vector in which one buyer (buyer 1) is offered a price that is too low, and, thus, the seller prefers not to sell the item to them, but rather to another buyer (buyer 2). This prevents a direct application of the line of reasoning outlined above. However, one could circumvent this issue by choosing an optimal price vector such that the seller is never upset if some buyer buys the item. In other words, prices must be such that each buyer is proposed a price that is at least as large as the seller's expected revenue in the posted price auction restricted to buyers following them. In Example 1, the optimal price vector $p^*$ such that $p_1^* = p_2^* = 1$ would be fine. Next, we show that there always exists a $p^*$ with such desirable property.

Letting $\textsc{Rev}_{>i}(\mathcal{V}, p)$ be the seller's revenue for price vector $p \in [0,1]^n$ and distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$ in the auction restricted to buyers $j \in \mathcal{N} : j > i$, we prove the following:

**Lemma 1.** *For any $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, there exists a revenue-maximizing price vector $p^* \in \arg\max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p)$ such that $p_i^* \geq \textsc{Rev}_{>i}(\mathcal{V}, p^*)$ for every buyer $i \in \mathcal{N}$.*

The proof of Lemma 2 builds upon the existence of a revenue-maximizing price vector $p^* \in [0,1]^n$ as in Lemma 1 and the fact that, under distributions $\mathcal{V}^\epsilon$, the probability with which each buyer buys the item given price vector $[p^* - \epsilon]_+$ is greater than that with which they would buy given $p^*$. Since the seller's expected revenue is larger when a buyer buys compared to when they do *not* buy (since $p_i^* \geq \textsc{Rev}_{>i}(\mathcal{V}, p^*)$), the seller's expected revenue decreases by at most $\epsilon$. Formally:

**Lemma 2.** *Given $\epsilon > 0$, let $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$ and $\mathcal{V}^\epsilon = \{\mathcal{V}_i^\epsilon\}_{i \in \mathcal{N}}$ satisfying the conditions of Definition 4. Then,*

$$\max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}^\epsilon, p) \geq \max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \epsilon.$$

Lemma 2 will be useful to prove Lemma 3 and to show the compared stability of a suitably-defined function that is used to design a PTAS in the public signaling scenario.

[7] In this work, given $x \in \mathbb{R}$, we let $[x]_+ := \max\{x, 0\}$. We extend the $[\cdot]_+$ operator to vectors by applying it component-wise.

**Algorithm 1** Find-APX-Prices.

**Inputs:** # of samples $K \in \mathbb{N}_{>0}$; # of discretization steps $b \in \mathbb{N}_{>0}$

**for** $i \in \mathcal{N}$ **do**
  **for** $k = 1, \dots, K$ **do**
    $v_i^k \leftarrow$ Sample buyer $i$'s valuation using oracle for $\mathcal{V}_i$
  $\mathcal{V}_i^K \leftarrow$ Empirical distribution of the $K$ i.i.d. samples $v_i^K$
$\mathcal{V}^K \leftarrow \{\mathcal{V}_i^K\}_{i \in \mathcal{N}}$; $p \leftarrow \mathbf{0}_n$; $r \leftarrow 0$
**for** $i = n, \dots, 1$ (in reversed order) **do**
  $p_i \leftarrow \arg\max_{p'_i \in P^b} p'_i \Pr_{v_i \sim \mathcal{V}_i^K} \{v_i \geq p'_i\} + \left(1 - \Pr_{v_i \sim \mathcal{V}_i^K} \{v_i \geq p'_i\}\right) r$
  $r \leftarrow p_i \Pr_{v_i \sim \mathcal{V}_i^K} \{v_i \geq p_i\} + \left(1 - \Pr_{v_i \sim \mathcal{V}_i^K} \{v_i \geq p_i\}\right) r$
**return** $(p, r)$

*Finding approximately-optimal prices* Algorithm 1 computes (in polynomial time) an approximately-optimal price vector for any non-Bayesian posted price auction. It samples $K \in \mathbb{N}_{>0}$ matrices of buyers' valuations, each one drawn according to the distributions $\mathcal{V}$. Then, it finds an optimal price vector $p$ in the discretized set $\mathcal{P}^b$, assuming that buyers' valuations are drawn according to the empirical distribution resulting from the sampled matrices.[8] This last step can be done by backward induction, as it is well known in the literature (see, *e.g.*, [47]). The following Lemma 3 establishes the correctness of Algorithm 1, also providing a bound on its running time. The key ideas of its proof are: (i) the sampling procedure constructs a good estimation of the actual distributions of buyers' valuations; and (ii) even if the algorithm only considers discretized prices, the components of the computed price vector are at most $1/b$ less than those of an optimal (unconstrained) price vector. As shown in the proof, this is strictly related to reducing buyer's valuations by $\frac{1}{b}$. Thus, it follows by Lemma 2 that the seller's expected revenue is at most $1/b$ less than the optimal one.

**Lemma 3.** *For any $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$ and $\epsilon, \tau > 0$, there exist $K \in \text{poly}\left(n, \frac{1}{\epsilon}, \log\frac{1}{\tau}\right)$ and $b \in \text{poly}\left(\frac{1}{\epsilon}\right)$ such that, with probability at least $1-\tau$, Algorithm 1 returns $(p, r)$ satisfying $\textsc{Rev}(\mathcal{V}, p) \geq \max_{p' \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p') - \epsilon$ and $r \in [\textsc{Rev}(\mathcal{V}, p) - \epsilon, \textsc{Rev}(\mathcal{V}, p) + \epsilon]$ in time $\text{poly}\left(n, \frac{1}{\epsilon}, \log\frac{1}{\tau}\right)$.*

## 7. Public signaling

In the following, we design a PTAS for computing a revenue-maximizing $(\phi, f)$ pair in the public signaling setting. Notice that, by Theorem 3, there is no additive FPTAS for the problem unless $\mathsf{P} = \mathsf{NP}$.

As a first intermediate result, we prove the compared stability of suitably-defined functions, which are intimately related to the seller's revenue. In particular, for every price vector $p \in [0,1]^n$, we conveniently let $g_p : [0,1]^n \to [0,1]$ be a function that takes a vector of buyers' valuations and outputs the seller's expected revenue achieved by selecting $p$ when the buyers' valuations are those specified as input. The following Lemma 4 shows that, given some distributions of buyers' valuations $\mathcal{V}$ and a posterior $\xi \in \Delta_\Theta$, there always exists a price vector $p \in [0,1]^n$ such that $g_p$ is stable compared with $g_{p'}$ for every other $p' \in [0,1]^n$. This result crucially allows us to decompose any posterior $\xi \in \Delta_\Theta$ by means of the decomposition lemma in Corollary 1, while guaranteeing a small loss in terms of seller's expected revenue.

**Lemma 4.** *Given $\alpha, \epsilon > 0$, a posterior $\xi \in \Delta_\Theta$, and some distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, there exists $p \in [0,1]^n$ such that, for every other $p' \in [0,1]^n$, the function $g_p$ is $(1, \alpha, \epsilon)$-stable compared with $g_{p'}$ for $(\xi, \mathcal{V})$.*

Our PTAS leverages the fact that public signaling schemes can be represented as probability distributions over buyers' posteriors (recall that, in the public signaling setting, all the buyers share the same posterior, as they all observe the same signal). In particular, the algorithm returns a pair $(\gamma, f^\circ)$, where $\gamma$ is a probability distribution over $\Delta_\Theta$ satisfying consistency constraints (see Equation (2)), while $f^\circ : \Delta_\Theta \to [0,1]^n$ is a function mapping each posterior to a price vector. In single-receiver settings, it is well known (see Subsection 2.1) that using distributions over posteriors rather than signaling schemes $\phi$ is without loss of generality. The following lemma shows that the same holds in our case, *i.e.*, given a pair $(\gamma, f^\circ)$, it is always possible to obtain a pair $(\phi, f)$ providing the seller with the same expected revenue.

**Lemma 5.** *Given a pair $(\gamma, f^\circ)$, where $\gamma$ is a probability distribution over $\Delta_\Theta$ with $\sum_{\xi \in \text{supp}(\gamma)} \gamma(\xi)\xi(\theta) = \mu_\theta$ for all $\theta \in \Theta$ and $f^\circ : \Delta_\Theta \to [0,1]^n$, there is a pair $(\phi, f)$ such that:*

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{s \in \mathcal{S}} \phi_\theta(s) \textsc{Rev}(\mathcal{V}, f(s), \xi_s) = \sum_{\xi \in \text{supp}(\gamma)} \gamma(\xi) \textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi).$$

[8] In this work, for a discretization step $b \in \mathbb{N}_{>0}$, we let $P^b \subset [0,1]$ be the set of prices multiples of $1/b$, while $\mathcal{P}^b := \bigtimes_{i \in \mathcal{N}} P^b$.

Next, we show that, in order to find an approximately-optimal pair $(\gamma, f^\circ)$, we can restrict the attention to $q$-uniform posteriors (with $q$ suitably defined). First, we introduce the following LP that computes an optimal probability distribution restricted over $q$-uniform posteriors.

$$\max_{\gamma} \sum_{\xi \in \Xi^q} \gamma(\xi) \max_{p \in [0,1]^n} \textsc{Rev}(\mathcal{V}, p, \xi) \quad \text{s.t.} \tag{6a}$$

$$\sum_{\xi \in \Xi^q} \gamma(\xi)\, \xi(\theta) = \mu_\theta \quad \forall \theta \in \Theta \tag{6b}$$

$$\sum_{\xi \in \Xi^q} \gamma(\xi) = 1 \tag{6c}$$

$$\gamma(\xi) \geq 0 \quad \forall \xi \in \Xi^q. \tag{6d}$$

The following Lemma 6 shows the optimal value of LP (6) is "close" to $OPT$. Its proof is based on the following core idea. Given the signaling scheme $\phi$ in a revenue-maximizing pair $(\phi, f)$, letting $\gamma$ be the distribution over $\Delta_\Theta$ induced by $\phi$, we can decompose each posterior in the support of $\gamma$ according to Corollary 1. Then, the obtained distributions over $q$-uniform posteriors are consistent according to Equation (2), and, thus, they satisfy Constraints (6b). Moreover, since such distributions are also decreasing around the decomposed posteriors, by Lemma 4 each time a posterior is decomposed there exists a price vector resulting in a small revenue loss. These observations allow us to conclude that the seller's expected revenue provided by an optimal solution to LP (6) is within some small additive loss of $OPT$.

**Lemma 6.** *Given $\eta > 0$ and letting $q = \frac{1}{\eta^2} 128 \log \frac{6}{\eta}$, an optimal solution to LP (6) has value at least $OPT - \eta$.*

Finally, we are ready to provide our PTAS. Its main idea is to solve LP (6) (of polynomial size) for the value of $q$ in Lemma 6. As shown in Lemma 6, the solution to LP (6) has value at least $OPT - \eta$. Hence, restricting to $q$-uniform posteriors results in a small loss in revenue. Our algorithm works by solving an approximate version of LP (6), where an approximation of the expected revenue in each $q$-uniform posterior is computed upfront. Thus, the last missing part for defining the algorithm is the computation of the terms appearing in the objective of LP (6), *i.e.*, an $\epsilon$-approximation of the revenue-maximizing price vector (together with its revenue) in each $q$-uniform posterior. In order to do so, we use Algorithm 1 (see also Lemma 3), which allows us to obtain in polynomial time good approximations of such price vectors, with arbitrarily high probability $1 - |\Xi^q|\tau$. Overall, the algorithm loses a factor $\eta$ by restricting to $q$-uniform posteriors, a factor $\epsilon$ by approximating optimal price vectors and a factor $|\Xi^q|\tau$ by the fact that the computed prices are "good" approximations only with probability $1 - |\Xi^q|\tau$. Formally:

**Theorem 5.** *There exists an additive PTAS for computing a revenue-maximizing $(\phi, f)$ pair with public signaling.*

## 8. Private signaling

With private signaling, computing a $(\phi, f)$ pair amounts to specifying a pair $(\phi_i, f_i)$ for each buyer $i \in \mathcal{N}$ —composed by a marginal signaling scheme $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ and a price function $f_i : \mathcal{S}_i \to [0, 1]$ for buyer $i$— and, then, correlating the $\phi_i$ so as to obtain a (non-marginal) signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$. We leverage this fact to design our PTAS.

In Subsection 8.1, we first show that it is possible to restrict the set of marginal signaling schemes of a given buyer $i \in \mathcal{N}$ to those encoded as distributions over $q$-uniform posteriors, as we did with public signaling. Then, we provide an LP formulation for computing an approximately-optimal $(\phi, f)$ pair, dealing with the challenge of correlating marginal signaling schemes in a non-trivial way. Finally, in Subsection 8.2, we show how to compute a solution to the LP in polynomial time, which requires the application of the ellipsoid method in a non-trivial way, due to the features of the formulation.

### *8.1. LP for approximate signaling schemes*

Before providing the LP, we show that restricting marginal signaling schemes to $q$-uniform posteriors results in a buyer's behavior which is similar to the one with arbitrary posteriors. This amounts to showing that suitably-defined functions related to the probability of buying are comparatively stable.

For $i \in \mathcal{N}$ and $p_i \in [0, 1]$, let $g_{i,p_i} : [0, 1]^n \to \{0, 1\}$ be a function that takes as input a vector of buyers' valuations and outputs 1 if and only if $v_i \geq p_i$ (otherwise it outputs 0).

**Lemma 7.** *Given $\alpha, \epsilon > 0$ and some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, for every buyer $i \in \mathcal{N}$, posterior $\xi_i \in \Delta_\Theta$, and price $p_i \in [0, 1]$, the function $g_{i,[p_i - \epsilon]_+}$ is $(0, \alpha, \epsilon)$-stable compared with $g_{i,p_i}$ for $(\xi_i, \mathcal{V})$.*

The following remark will be crucial for proving Lemma 9. It shows that, if for every $i \in \mathcal{N}$ we decompose buyer $i$'s posterior $\xi_i \in \Delta_\Theta$ by means of a distribution over $q$-uniform posteriors $(\alpha, \epsilon)$-decreasing around $\xi_i$, then the probability with which buyer $i$ buys only decreases by a small amount. In this section, for the ease of presentation, we abuse notation and use $\Xi_i^q$ to denote the (all equal) sets of $q$-uniform posteriors (Definition 1), one per buyer $i \in \mathcal{N}$, while $\Xi^q := \bigtimes_{i \in \mathcal{N}} \Xi_i^q$ is the set of tuples $\xi = (\xi_1, \dots, \xi_n)$ specifying a $\xi_i \in \Xi_i^q$ for each $i \in \mathcal{N}$.

**Remark 2.** Lemma 7 and Theorem 4 imply that, given a tuple of posteriors $\xi = (\xi_1, \dots, \xi_n) \in \bigtimes_{i \in \mathcal{N}} \Delta_\theta$ and some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, for every buyer $i \in \mathcal{N}$ and price $p_i \in [0, 1]$, there exists $\gamma_i \in \Delta_{\Xi_i^q}$ with $q = \frac{32}{\epsilon^2} \log \frac{4}{\alpha}$ s.t.

$$\mathop{\mathbb{E}}_{\tilde{\xi}_i \sim \gamma_i} \left[ \tilde{\xi}_i(\theta) \Pr_{V \sim \mathcal{V}} \left\{ V_i \tilde{\xi}_i \geq [p_i - \epsilon]_+ \right\} \right] \geq \xi_i(\theta)(1 - \alpha) \Pr_{V \sim \mathcal{V}} \{ V_i \xi_i \geq p_i \}$$

and $\sum_{\tilde{\xi}_i \in \Xi_i^q} \gamma_i(\tilde{\xi}_i) \tilde{\xi}_i(\theta) = \xi_i(\theta)$ for all $\theta \in \Theta$.

Next, we show that an approximately-optimal pair $(\phi, f)$ can be found by solving LP (7) instantiated with suitably-defined $q \in \mathbb{N}_{>0}$ and $b \in \mathbb{N}_{>0}$.

LP (7) employs:

- Variables $\gamma_{i,\xi_i}$ (for $i \in \mathcal{N}$ and $\xi_i \in \Xi_i^q$), which encode the distributions over posteriors representing the marginal signaling schemes $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ of the buyers.
- Variables $t_{i,\xi_i,p_i}$ (for $i \in \mathcal{N}$, $\xi_i \in \Xi_i^q$, and $p_i \in P^b$), with $t_{i,\xi_i,p_i}$ encoding the probability that the seller offers price $p_i$ to buyer $i$ and buyer $i$'s posterior is $\xi_i$.
- Variables $y_{\theta,\xi,p}$ (for $\theta \in \Theta$, $\xi \in \Xi^q$, and $p \in \mathcal{P}^b$), with $y_{\theta,\xi,p}$ encoding the probability that the state is $\theta$, the buyers' posteriors are those specified by $\xi$, and the prices that the seller offers to the buyers are those given by vector $p$.

$$\max_{\gamma,t,y} \sum_{\theta \in \Theta} \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b} y_{\theta,\xi,p} \, \textsc{Rev}(\mathcal{V}, p, \xi) \quad \text{s.t.} \tag{7a}$$

$$\xi_i(\theta) t_{i,\xi_i,p_i} = \sum_{\substack{\xi' \in \Xi^q: \\ \xi'_i = \xi_i}} \sum_{\substack{p' \in \mathcal{P}^b: \\ p'_i = p_i}} y_{\theta,\xi',p'} \quad \forall \theta \in \Theta, \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q, \forall p_i \in P^b \tag{7b}$$

$$\sum_{p_i \in P^b} t_{i,\xi_i,p_i} = \gamma_{i,\xi_i} \quad \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q \tag{7c}$$

$$\sum_{\xi_i \in \Xi_i^q} \gamma_{i,\xi_i} \xi_i(\theta) = \mu_\theta \quad \forall i \in \mathcal{N}, \forall \theta \in \Theta \tag{7d}$$

$$y_{\theta,\xi,p} \geq 0 \quad \forall \theta \in \Theta, \forall \xi \in \Xi^q, \forall p \in \mathcal{P}^b. \tag{7e}$$

Variables $t_{i,\xi_i,p_i}$ represent marginal signaling schemes, allowing for multiple signals inducing the same posterior. This is needed since signals may correspond to different price proposals.[9] One may think of marginal signaling schemes in LP (7) as if they were using signals defined as pairs $s_i = (\xi_i, p_i)$, with the convention that $f_i(s_i) = p_i$. Variables $y_{\theta,\xi,p}$ and Constraints (7b) ensure that marginal signaling schemes are correctly correlated together, by directly working in the domain of the distributions over posteriors.

To show that an optimal solution to LP (7) provides an approximately-optimal $(\phi, f)$ pair, we need the following two lemmas. Lemma 8 proves that, given a feasible solution to LP (7), we can recover a pair $(\phi, f)$ providing the seller with an expected revenue equal to the value of the LP solution. Lemma 9 shows that the optimal value of LP (7) is "close" to $OPT$. These two lemmas imply that an approximately-optimal $(\phi, f)$ pair can be computed by solving LP (7).

**Lemma 8.** *Given a feasible solution to LP (7), it is possible to recover a pair $(\phi, f)$ that provides the seller with an expected revenue equal to the value of the solution.*

**Lemma 9.** *For every $\eta > 0$, there exist $b(\eta), q(\eta) \in \mathbb{N}_{>0}$ such that LP (7) has optimal value at least $OPT - \eta$.*

[9] Notice that, in a classical setting in which the sender does *not* have to propose a price (or, in general, select some action after sending signals), there always exists a signaling scheme with no pair of signals inducing the same posterior. Indeed, two signals that induce the same posterior can always be joined into a single signal. This is *not* the case in our setting, where we can only join signals that induce the same posterior and correspond to the same price.

**Algorithm 2** FPTAS for MAX-LINREV.

**Inputs:** Discretization error tolerance $\delta > 0$; vector of linear components $w \in [0,1]^{n \times |\Xi_i^q| \times |P^b|}$; finite-support distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$

$c \leftarrow \lceil \frac{n}{\delta} \rceil$

$A \leftarrow \left\{0, \frac{1}{c}, \frac{2}{c}, \dots, \frac{n\delta}{c}\right\}$

Initialize an empty matrix $M$ with dimension $n \times |A|$

**for** $a \in A$ **do**

  $M(n,a) \leftarrow \max_{\xi_n \in \Xi_n^q, p_n \in P^b : w_{n,\xi_n,p_n} \geq a} \left\{ \Pr_{v_n \sim \mathcal{V}_n} \left\{ v_n^\top \xi_n \geq p_n \right\} p_n \right\}$

**for** $i = n-1, \dots, 1$ (in reversed order) **do**

  **for** $a \in A$ **do**

    $M(i,a) \leftarrow \max_{\substack{\xi_i \in \Xi_i^q, p_i \in P^b, a' \in A: \\ w_{i,\xi_i,p_i} + a' \geq a}} \left\{ \Pr_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \xi_i \geq p_i \right\} p_i \right.$

    $\left. + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \xi_i \geq p_i \right\}\right) M(i+1, a') \right\}$

**return** $\max_{a \in A} \{M(1,a) + a\}$

### 8.2. PTAS

We provide an algorithm that approximately solves LP (7) in polynomial time, which completes our PTAS for computing a revenue-maximizing pair $(\phi, f)$ in the private setting. The core idea of our algorithm is to apply the ellipsoid method on the dual of LP (7).[10] In particular, our implementation of the ellipsoid algorithm uses an approximate separation oracle that needs to solve the following optimization problem.

**Definition 5** *(MAX-LINREV).* Given some distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$ such that each $\mathcal{V}_i$ has finite support and a vector $w \in [0,1]^{n \times |\Xi_i^q| \times |P^b|}$, solve

$$\max_{\xi \in \Xi^q, p \in \mathcal{P}^b} \textsc{Rev}(\mathcal{V}, p, \xi) + \sum_{i \in \mathcal{N}} w_{i,\xi_i,p_i}.$$

As a first step, we provide an FPTAS for MAX-LINREV using a dynamic programming approach. This will be the main building block of our approximate separation oracle. Notice that, since MAX-LINREV takes as input distributions with a *finite support*, we can safely assume that such distributions can be explicitly represented in memory. In our PTAS, the inputs to the dynamic programming algorithm are obtained by building empirical distributions through samples from the actual distributions of buyers' valuations, thus ensuring finiteness of the supports.

The FPTAS works as follows (see Algorithm 2). Given an error tolerance $\delta > 0$, it first defines a step size $\frac{1}{c}$, with $c = \lceil \frac{n}{\delta} \rceil$, and builds a set $A = \left\{0, \frac{1}{c}, \frac{2}{c}, \dots, n\right\}$ of possible discretized values for the linear term appearing in the MAX-LINREV objective. Then, for every buyer $i \in \mathcal{N}$ (in reversed order) and value $a \in A$, the algorithm computes $M(i,a)$, which is an approximation of the largest seller's revenue provided by a pair $(\xi, p)$ when considering buyers $i, \dots, n$ only, and restricted to pairs $(\xi, p)$ such that the inequality $\sum_{j \in \mathcal{N}: j \geq i} w_{j,\xi_j,p_j} \geq a$ is satisfied. By letting $z_i := \Pr_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \xi_i \geq p_i \right\}$, the value $M(i,a)$ can be defined by the following recursive formula[11]:

$$M(i,a) := \max_{\xi_i \in \Xi_i^q, p_i \in P^b, a' \in A : w_{i,\xi_i,p_i} + a' \geq a} z_i p_i + (1 - z_i) M(i+1, a').$$

Finally, the algorithm returns $\max_{a \in A} \{M(1,a) + a\}$. Thus:

**Lemma 10.** *For any $\delta > 0$, there exists a dynamic programming algorithm (Algorithm 2) that provides a $\delta$-approximation (in the additive sense) to* MAX-LINREV. *Moreover, the algorithm runs in time polynomial in the size of the input and $\frac{1}{\delta}$.*

Then, we provide the following relaxation of LP (7):

$$\max_{\gamma, x, y} \sum_{\theta \in \Theta} \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b} y_{\theta,\xi,p} \textsc{Rev}(\mathcal{V}, \xi, p) \qquad \text{s.t.} \tag{8a}$$

[10] To be precise, we apply the ellipsoid method to the dual of a relaxed version of LP (7), since we need an over-constrained dual.

[11] Notice that, given a pair $(\xi, p)$ with $\xi \in \Xi^q$ and $p \in \mathcal{P}^b$, it is possible to compute in polynomial time the probability with which a buyer $i \in \mathcal{N}$ buys the item.

**Algorithm 3** PTAS for the private signaling setting.

**Input:** Error $\beta > 0$; approximation error for the approximate separation oracle $\delta > 0$; $q \in \mathbb{N}_{>0}$ defining the set of $q$-uniform posteriors; # of discretization steps $b \in \mathbb{N}_{>0}$, # of samples $K \in \mathbb{N}_{>0}$

**Initialization**: $\rho_1 \leftarrow 0$, $\rho_2 \leftarrow 1$, $H \leftarrow \varnothing$, $H^* \leftarrow \varnothing$.
Build an empirical distribution of valuations $\mathcal{V}^K$ by sampling $K$ samples using oracles for distributions $\mathcal{V}_i$ (see Algorithm 1).
**while** $\rho_2 - \rho_1 > \beta$ **do**
    $\rho_3 \leftarrow \frac{\rho_1+\rho_2}{2}$
    Run ellipsoid method on Ⓕ with objective $\rho_3$ and error $\delta$
    $H \leftarrow$ {violated constraints returned by the ellipsoid method}
    **if** ellipsoid method returned unfeasible **then**
        $\rho_1 \leftarrow \rho_3$
        $H^* \leftarrow H$
    **else**
        $\rho_2 \leftarrow \rho_3$
**return** solution to LP (8) with only variable relative to constraints in $H^*$

$$\xi_i(\theta) t_{i,\xi_i,p_i} \geq \sum_{\substack{\xi' \in \Xi: \\ \xi'_i = \xi_i}} \sum_{\substack{p' \in \mathcal{P}^b: \\ p'_i = p_i}} y_{\theta,\xi',p'} \qquad \forall \theta \in \Theta, \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q, \forall p_i \in P^b \tag{8b}$$

$$\sum_{p_i \in P^b} t_{i,\xi_i,p_i} = \gamma_{i,\xi_i} \qquad \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q \tag{8c}$$

$$\sum_{\xi_i \in \Xi^q} \gamma_{i,\xi_i} \xi_i(\theta) = \mu_\theta \qquad \forall i \in \mathcal{N}, \forall \theta \in \Theta \tag{8d}$$

$$y_{\theta,\xi,p} \geq 0 \qquad \forall \theta \in \Theta, \forall \xi \in \Xi^q, \forall p \in \mathcal{P}^b. \tag{8e}$$

The PTAS that we build in the rest of the section works with the dual of LP (8) so as to take advantage of the fact that it is more constrained than that of the original LP (7). The following lemma shows that LP (7) and LP (8) are equivalent.

**Lemma 11.** *LP (7) and LP (8) have the same optimal value. Moreover, given a feasible solution to LP (8), it is possible compute in polynomial time a feasible solution to LP (7) with a greater or equal value.*

Now, we are ready to prove the main result of this section. In particular, the following theorem shows that Algorithm 3 provides an additive PTAS for computing a revenue-maximizing $(\phi, f)$ pair with private signaling. The algorithm works by repeatedly solving the following feasibility problem, which we call Ⓕ and is related to the dual of LP (8) for different objective values $\rho_3$:

$$\sum_{i \in \mathcal{N}} \sum_{\theta \in \Theta} \mu_\theta a_{i,\theta} \leq \rho_3 \tag{9a}$$

$$\textsc{Rev}(\mathcal{V}, p, \xi) + \sum_{i \in \mathcal{N}} w_{\theta,i,\xi_i,p_i} \leq 0 \qquad \forall \theta \in \Theta, \forall \xi \in \Xi^q, \forall p \in \mathcal{P}^b \tag{9b}$$

$$\sum_{\theta \in \Theta} \xi_i(\theta) w_{\theta,i,\xi_i,p_i} + c_{i,\xi} \geq 0 \qquad \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q, \forall p_i \in P^b \tag{9c}$$

$$-c_{i,\xi_i} + \sum_{\theta \in \Theta} \xi_i(\theta) a_{i,\theta} \geq 0 \qquad \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q \tag{9d}$$

$$w_{\theta,i,\xi_i,p_i} \leq 0 \qquad \forall \theta \in \Theta, \forall i \in \mathcal{N}, \forall \xi_i \in \Xi_i^q, \forall p_i \in P^b. \tag{9e}$$

Then, the algorithm returns the solution to LP (8) with a (polynomial) subset of variables.[12] In particular, it employs the variables relative to the violated constraints returned by the ellipsoid method when run on the highest value of $\rho_3$ for which LP Ⓕ is returned unfeasible.

Algorithm 3 incurs in many approximation errors, which can all be bounded as sufficiently small values. By Lemma 9, given an $\eta > 0$, the optimal value of LP (7) is at least $OPT - \eta$. Moreover, by Lemma 11 solving LP (8) is equivalent to solving LP (7). The algorithm approximately solves LP Ⓕ. In doing so, it suffers a bisection error $\beta$ on the value of LP (7). Moreover, it suffers an additional error since it employs Algorithm 2 in the separation oracle, and such an algorithm has an approximation error $\delta$. Finally, it suffers an additional error since it approximates the valuations distribution with $K$ samples. By taking sufficiently small approximation errors, we obtain the desired result. Formally:

[12] By Lemma 11 we can recover a solution to the original LP (7) with at least the same revenue in polynomial time.

**Theorem 6.** *There exists an additive PTAS for computing a revenue-maximizing $(\phi, f)$ pair with private signaling.*

## 9. Discussion and future work

In this paper, we focus on single-item single-unit *Bayesian posted price auctions*, where buyers arrive sequentially and their valuations on the item being sold depend on a random, unknown state of nature. In particular, we study the problem faced by a seller that has complete knowledge of the state of nature. The seller aims to maximize the revenue by disclosing information about the state of nature and at the same time proposing take-it-or-leave-it prices to the buyers. We show that our problem does not admit either an additive or a multiplicative FPTAS unless $\mathsf{P} = \mathsf{NP}$, even for basic instances with a single buyer. Furthermore, we show that there are additive PTASs that approximate the optimal public and private signaling schemes.

Our work leaves several questions open. First, we assume that the buyers have limited rationality and do not take into account information other than that provided by signals. Our model can be widely extended including the possibility that a buyer has information about the other buyers acting before. This model would require every buyer to have perfect knowledge of other buyers' distributions over the valuations. In this case, a buyer could infer a more accurate posterior belief than that they can infer by only the signal. The investigation of suitable rationality models and of how they affect the optimization problem is undoubtedly of high interest. Another interesting open question is about multiplicative approximations. Indeed, while we characterized the computational complexity of computing additive approximations, it is not clear what can be achieved in the case of multiplicative approximations. Our negative result only implies that there is no multiplicative FPTAS unless $\mathsf{P} = \mathsf{NP}$. However, our techniques cannot be employed to design multiplicative PTASs since $q$-uniform posteriors provide only additive guarantees. It would be interesting to investigate the existence of multiplicative PTASs.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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## Appendix A. Proof omitted from Section 3

**Theorem 1.** *In Bayesian posted price auctions with public signaling, there always exists a revenue-maximizing $(\phi, f)$ pair such that $f$ is a deterministic and factorized price function.*

**Proof.** We show that given an arbitrary couple $(\bar{\phi}, \mathsf{f})$, we can recover a couple $(\phi, f)$ that has the same revenue and such that $f$ is a deterministic and factorized price function. This is sufficient to prove the statement.

Let $\bar{\mathcal{S}}$ be the set of signals employed by signaling scheme $\bar{\phi}$. Recall that in a public signaling scheme $\bar{\mathcal{S}}_i = \bar{\mathcal{S}}_j$ for each $i \neq j \in \mathcal{N}$ and for every $\theta \in \Theta$, $\phi_\theta(s) > 0$ only for signal profiles $s \in \bar{\mathcal{S}}$ such that $s_i = s_j$ for $i, j \in \mathcal{N}$. Hence, with abuse of notation, we use $s \in \bar{\mathcal{S}}$ to denote the signal observed by all buyers $i \in \mathcal{N}$.

Now, we can define the set of signals $\mathcal{S}$ employed by the signaling scheme $\phi$. In particular, we use a signal for each pair $s \in \bar{\mathcal{S}}$ and $p \in \text{supp}(\mathsf{f}(s))$. For the ease of notation, we introduce a function $\sigma : \bar{\mathcal{S}} \times [0, 1]^n \to \mathcal{S}$, where $\sigma(s, p) \in \mathcal{S}$ is the signal corresponding to the couple $(s, p)$. Finally, we define the signaling scheme $\phi$ as follows:

$$\phi_\theta(\sigma(s, p)) = \bar{\phi}_\theta(s) \Pr_{p' \sim \mathsf{f}(s)}\{p' = p\} \qquad \forall s \in \bar{\mathcal{S}}, \forall p \in \text{supp}(\mathsf{f}(s)), \forall \theta \in \Theta.$$

Moreover, we define $f_i(\sigma(s, p)) = p_i$ for each $s \in \bar{\mathcal{S}}$ and $p \in \text{supp}(\mathsf{f}(s))$.

As a first step, we show that given a buyer $i$, a signal $s \in \mathcal{S}$, and a price vector $p \in \text{supp}(\mathsf{f}(s))$, $\xi_{i,s,p_i} = \xi_{i,\sigma(s,p)}$, *i.e.*, the posterior induced by the signal $s$ and price $p_i$ under $(\bar{\phi}, \mathsf{f})$ is the same of the posterior induced by signal $\sigma(s, p)$ under $(\phi, f)$. Recall that for a deterministic and factorized price function the induced posterior is independent from the observed price $p_i$. Formally, for each $\theta \in \Theta$,

$$\begin{aligned}\xi_{i,\sigma(s,p)}(\theta) &= \frac{\mu_\theta \phi_\theta(\sigma(s,p))}{\sum_{\theta'\in\Theta}\mu_{\theta'}\phi_{\theta'}(\sigma(s,p))}\\ &= \frac{\mu_\theta \bar\phi_\theta(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\}}{\sum_{\theta'\in\Theta}\mu_{\theta'}\bar\phi_{\theta'}(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\}}\\ &= \frac{\mu_\theta \bar\phi_\theta(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'_i=p_i\}}{\sum_{\theta'\in\Theta}\mu_{\theta'}\bar\phi_{\theta'}(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'_i=p_i\}} = \xi_{i,s,p_i}(\theta),\end{aligned}$$

where in the third equality we add the term $\Pr_{p'\sim \mathsf{f}(s)}\{p'_i=p_i\}$ and remove the term $\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\}$ from both the numerator and the denominator.

Now we show that the two signaling schemes provide the same revenue. Formally:

$$\begin{aligned}&\sum_{\theta\in\Theta}\mu_\theta \sum_{\sigma(s,p)\in\mathcal{S}} \phi_\theta(\sigma(s,p))\textsc{Rev}(\mathcal{V}, f(\sigma(s,p)), \xi_{\sigma(s,p)})\\ &\qquad = \sum_{\theta\in\Theta}\mu_\theta \sum_{\sigma(s,p)\in\mathcal{S}} \phi_\theta(\sigma(s,p))\textsc{Rev}(\mathcal{V}, p, \xi_{s,p})\\ &\qquad = \sum_{\theta\in\Theta}\mu_\theta \sum_{\sigma(s,p)\in\mathcal{S}} \bar\phi_\theta(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\}\textsc{Rev}(\mathcal{V}, p, \xi_{s,p})\\ &\qquad = \sum_{\theta\in\Theta}\mu_\theta \sum_{s\in\bar{\mathcal{S}}}\sum_{p\in\mathrm{supp}(\mathsf{f}(s))} \bar\phi_\theta(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\}\textsc{Rev}(\mathcal{V}, p, \xi_{s,p}),\end{aligned}$$

where we recall $\xi_{\sigma(s,p)} := (\xi_{1,\sigma(s,p)},\dots,\xi_{n,\sigma(s,p)})$ and $\xi_{s,p} := (\xi_{1,s,p_1},\dots,\xi_{n,s,p_n})$. This concludes the proof. □

**Theorem 2.** *In Bayesian posted price auctions with private signaling, there always exists a revenue-maximizing $(\phi, f)$ pair such that $f$ is a deterministic and factorized price function.*

**Proof.** We prove that, given a generic couple composed by a price function $\mathsf{f}$ and a signaling scheme $\bar\phi$, it is always possible to find another couple $(\phi, f)$ composed by a signaling scheme $\phi$ and a deterministic and factorized price function $f$ with the same revenue. This is sufficient to prove the statement.

Given a bidder $i\in\mathcal{N}$ and a signal $s_i\in\bar{\mathcal{S}}$, let

$$\omega_i(s_i) := \left\{p_i \mid \exists p'\in[0,1]^n, s'\in\bar{\mathcal{S}} : p'_i = p_i \wedge p'\in\mathrm{supp}(\mathsf{f}(s'))\wedge s'_i = s_i\right\}$$

be the set of possible prices proposed to the buyer $i$ when the received signal is $s_i$. Moreover, let $\bar{\mathcal{S}}$ be the set of signal profiles employed by signaling scheme $\bar\phi$ and $\bar{\mathcal{S}}_i$ be the set of signals for a buyer $i\in\mathcal{N}$. In particular, for each buyer $i$, we employ a signal for each couple $(s_i, p_i)$ with $s_i\in\bar{\mathcal{S}}$ and $p_i\in\omega_i(s_i)$. For the ease of notation, we introduce a function $\sigma_i : \bar{\mathcal{S}}_i\times[0,1]\to\mathcal{S}$, where $\sigma_i(s_i,p_i)\in\mathcal{S}$ is the signal corresponding to the couple $(s_i,p_i)$. Moreover, given an $s\in\mathcal{S}$ and a $p\in\mathrm{supp}(\mathsf{f}(s))$, we define the signal profile $\sigma(s,p)\in\mathcal{S}$ as $\sigma(s,p) := (\sigma_i(s_i,p_i))_{i\in\mathcal{N}}$. The signaling scheme is defined as follows:

$$\phi_\theta(\sigma(s,p)) = \bar\phi_\theta(s)\Pr_{p'\sim \mathsf{f}(s)}\{p'=p\} \qquad \forall s\in\bar{\mathcal{S}}, \forall p\in\mathrm{supp}(\mathsf{f}(s)), \forall\theta\in\Theta.$$

Finally, we define $f_i(\sigma_i(s_i,p_i)) = p_i$ for each $s_i\in\bar{\mathcal{S}}$ and $p_i\in\omega_i(s_i)$.

As a first step, we show that given a buyer $i\in\mathcal{N}$, a signal $s_i\in\bar{\mathcal{S}}_i$, and a price $p_i\in\omega_i(s_i)$, $\xi_{i,s_i,p_i} = \xi_{i,\sigma_i(s_i,p_i)}$, *i.e.*, the posterior induced by signal $s_i$ and price $p_i$ under $(\bar\phi, \mathsf{f})$ is the same of the posterior induced by signal $\sigma_i(s_i,p_i)$ under $(\phi, f)$ (recall that for a deterministic and factorized price function the induced posterior is independent from the observed price $p_i$). Formally, for each $\theta\in\Theta$,

$$\begin{aligned}\xi_{i,\sigma_i(s_i,p_i)}(\theta) &= \frac{\mu_\theta\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\phi_\theta(\sigma(s',p'))}{\sum_{\theta'\in\Theta}\mu_{\theta'}\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\phi_{\theta'}(\sigma(s',p'))}\\ &= \frac{\mu_\theta\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\bar\phi_\theta(s')\Pr_{p''\sim\mathsf{f}(s')}\{p''=p'\}}{\sum_{\theta'\in\Theta}\mu_{\theta'}\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\bar\phi_{\theta'}(s')\Pr_{p''\sim\mathsf{f}(s')}\{p''=p'\}}\\ &= \frac{\mu_\theta\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\bar\phi_\theta(s')\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\Pr_{p''\sim\mathsf{f}(s')}\{p''=p'\}}{\sum_{\theta'\in\Theta}\mu_{\theta'}\sum_{s'\in\bar{\mathcal{S}}:s'_i=s_i}\bar\phi_{\theta'}(s')\sum_{p'\in\mathrm{supp}(\mathsf{f}(s')):p_i=p'_i}\Pr_{p''\sim\mathsf{f}(s')}\{p''=p'\}}\end{aligned}$$

$$= \frac{\mu_\theta \sum_{s' \in \bar{\mathcal{S}}: s'_i = s_i} \bar{\phi}_\theta(s') \Pr_{p' \sim f(s')}\{p'_i = p_i\}}{\sum_{\theta' \in \Theta} \mu_{\theta'} \sum_{s' \in \bar{\mathcal{S}}: s'_i = s_i} \bar{\phi}_{\theta'}(s') \Pr_{p' \sim f(s')}\{p'_i = p_i\}}$$
$$= \xi_{i, s_i, p_i}(\theta).$$

Now we show that the two signaling schemes provide the same revenue. Formally:

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\sigma(s,p) \in \mathcal{S}} \phi_\theta(\sigma(s,p)) \textsc{Rev}(\mathcal{V}, f(\sigma(s,p)), \xi_{\sigma(s,p)})$$
$$= \sum_{\theta \in \Theta} \mu_\theta \sum_{\sigma(s,p) \in \mathcal{S}} \phi_\theta(\sigma(s,p)) \textsc{Rev}(\mathcal{V}, p, \xi_{s,p})$$
$$= \sum_{\theta \in \Theta} \mu_\theta \sum_{\sigma(s,p) \in \mathcal{S}} \bar{\phi}_\theta(s) \Pr_{p' \sim f(s)}\{p' = p\} \textsc{Rev}(\mathcal{V}, p, \xi_{s,p}),$$

where we recall $\xi_{\sigma(s,p)} := (\xi_{1,\sigma_1(s_1,p_1)}, \ldots, \xi_{n,\sigma_n(s_n,p_n)})$ and $\xi_{s,p} := (\xi_{1,s_1,p_1}, \ldots, \xi_{n,s_n,p_n})$. This concludes the proof. □

## Appendix B. Proofs omitted from Section 4

**Theorem 3.** *In Bayesian posted price auctions, there is no additive FPTAS for the problem of computing a revenue-maximizing* $(\phi, f)$ *pair unless* $\mathsf{P} = \mathsf{NP}$*, even when there is a single buyer.*

**Proof.** We employ a reduction from an NP-hard problem originally introduced by Khot and Saket [48], which we formally state in the following. For any positive integer $k \in \mathbb{N}_{>0}$, integer $l \in \mathbb{N}$ such that $l \geq 2^k + 1$, and arbitrarily small constant $\epsilon > 0$, the problem reads as follows. Given an undirected graph $G := (U, E)$, distinguish between:

- *Case 1*. There exists a $l$-colorable induced subgraph of $G$ containing a $1 - \epsilon$ fraction of all vertices, where each color class contains a $\frac{1-\epsilon}{l}$ fraction of all vertices.[13]
- *Case 2*. Every independent set of $G$ contains less than a $\frac{1}{l^{k+1}}$ fraction of all vertices.[14]

We reduce from such problem for $k = 2, l = 5$, and $\epsilon = \frac{1}{2}$. Our reduction works as follows:

- *Completeness*. If *Case 1* holds, then there exists a signaling scheme, price function pair $(\phi, f)$ that provides the seller with an expected revenue at least as large as some threshold $\eta$ (see Equation (B.1) below for its definition).
- *Soundness*. If *Case 2* holds, then the seller's expected revenue for any signaling scheme, price function pair $(\phi, f)$ is smaller than $\eta - \delta$ with $\delta := \frac{1}{m^2}$, where $m$ denotes the number or vertices of the graph $G$.

This shows that it is NP-hard to approximate the optimal seller's expected revenue up to within an additive error $\delta$. Thus, since $\delta$ depends polynomially on the size of the problem instance, this also shows that there is no additive FPTAS for the problem of computing a revenue-maximizing $(\phi, f)$ pair, unless $\mathsf{P} = \mathsf{NP}$.

*Construction* Given an undirected graph $G := (U, E)$, with vertices $U := \{u_1, \ldots, u_m\}$, we build a single-buyer Bayesian posted price auction as follows.[15] There is one state of nature $\theta_u \in \Theta$ for each vertex $u \in U$, and the prior belief over states $\mu \in \Delta_\Theta$ is such that $\mu_{\theta_u} = \frac{1}{m}$ for all $u \in U$. There is a finite set of possible buyer's valuations. For every vertex $u \in U$, there is a valuation vector $v_u \in [0, 1]^m$ such that:

- $v_u(\theta_u) = 1$;
- $v_u(\theta_{u'}) = \frac{1}{2}$ for all $u' \in U : (u, u') \notin E$; and
- $v_u(\theta_{u'}) = 0$ for all $u' \in U : (u, u') \in E$.

Each valuation $v_u$ has probability $\frac{1}{m^2}$ of occurring according to the distribution $\mathcal{V}$. Moreover, there is an additional valuation vector $v_o \in [0, 1]^m$ such that $v_o(\theta) = \frac{1}{2} + \frac{l}{(1-\epsilon)2m}$ for all $\theta \in \Theta$, having probability $1 - \frac{1}{m}$.

[13] A *l-colorable induced subgraph* is identified by a subset of vertices such that it is possible to assign one among $l$ different colors to each vertex, in such a way that there are *no* two adjacent vertices having the same color. Given some color, its associated color class is the subset of all vertices in the subgraph having that color.

[14] An *independent set* of $G$ is a subset of vertices such that there are *no* two adjacent vertices.

[15] In a single-buyer setting, we always omit the subscript $i$ from symbols, as it is clear that they refer to the unique buyer. Moreover, with an overload of notation, we use buyer's signals as if they were signal profiles.

*Completeness* Assume that a $l$-colored induced subgraph of $G$ is given, and that it contains a fraction $1-\epsilon$ of vertices, while each color class is made up of a fraction $\frac{1-\epsilon}{l}$ of all vertices. We let $L := \{1, \dots, l\}$ be the set of possible colors, with $j \in L$ denoting a generic color. In the following, we show how to build a signaling scheme, price function pair $(\phi, f)$ that provides the seller with an expected revenue greater than or equal to a suitably-defined threshold $\eta$ (Equation (B.1)). The seller has $l+1$ signals available, namely $\mathcal{S} := \{s_j\}_{j \in L} \cup \{s_o\}$. For every vertex $u \in U$, if $u$ has been assigned some color $j \in L$ (that is, $u$ belongs to the induced subgraph), then we set $\phi_{\theta_u}(s_j) = 1$ and $\phi_{\theta_u}(s) = 0$ for all $s \in \mathcal{S} : s \neq s_j$; otherwise, if $u$ has no color (that is, $u$ does not belong to the given subgraph), then we set $\phi_{\theta_u}(s_o) = 1$ and $\phi_{\theta_u}(s) = 0$ for all $s \in \mathcal{S} : s \neq s_o$. Moreover, the price function is such that

$$f(s) = p_o := \frac{1}{2} + \frac{l}{(1-\epsilon)2m} \text{ for every signal } s \in \mathcal{S}.$$

Next, we prove that, after receiving a signal $s_j \in \mathcal{S}$ associated with some color $j \in L$, if the buyer has valuation $v_u$ for a node $u \in U$ colored of color $j$, then they will buy the item. In particular, the buyer's posterior $\xi_{s_j} \in \Delta_\Theta$ induced by signal $s_j$ is such that only state $\theta_u$ and states $\theta_{u'}$ for $u' \in U : (u, u') \notin E$ have positive probability (since, when the seller sends signal $s_j$, it must be the case that the vertex corresponding to the actual state of nature is colored of color $j$). Moreover, such probabilities are equal to $\xi_{s_j}(\theta_u) = \xi_{s_j}(\theta_{u'}) = \frac{l}{(1-\epsilon)m}$ (by applying Equation (1) and using the fact that each color class has a fraction $\frac{1-\epsilon}{l}$ of vertices). Thus, since $v_u(\theta_u) = 1$ and $v_u(\theta_{u'}) = \frac{1}{2}$ for all $u' \in U : (u, u') \notin E$, the expected valuation of the buyer given the posterior $\xi_{s_j}$ is

$$\sum_{\theta \in \Theta} v_u(\theta)\, \xi_{s_j}(\theta) = \frac{1}{2}\left[1 - \frac{l}{(1-\epsilon)m}\right] + \frac{l}{(1-\epsilon)m} = \frac{1}{2} + \frac{l}{(1-\epsilon)2m} = p_o,$$

and the buyer will buy the item. Furthermore, when the seller sends signal $s_o$, their expected revenue is at least $(1 - \frac{1}{m})p_o$, as it is always the case that the buyer buys the item when they have valuation $v_o$. Since the total probability of sending signals $s_j \in \mathcal{S}$ for $j \in L$ is $1-\epsilon$ (given that the subgraph contains a fraction $1-\epsilon$ of vertices) and the probability of sending signal $s_o$ is $\epsilon$, we have that the seller's expected revenue is at least

$$\eta := (1-\epsilon)\left[\frac{1-\epsilon}{ml} + 1 - \frac{1}{m}\right] p_o + \epsilon\left(1 - \frac{1}{m}\right) p_o = \left[\frac{(1-\epsilon)^2}{ml} + \left(1 - \frac{1}{m}\right)\right] p_o. \tag{B.1}$$

where the factor $\frac{1-\epsilon}{ml} + 1 - \frac{1}{m}$ represents the probability that the buyer buys when sending a signal $s_j$ (this happens when either the buyer has valuation $v_u$ for a vertex $u \in U$ colored of color $j$ or the buyer has valuation $v_o$).

*Soundness* By contradiction, we show that, if there exists a signaling scheme, price function pair $(\phi, f)$ with seller's expected revenue exceeding $\eta - \delta$, then the graph $G$ admits an independent set of size $\frac{1}{2l}(1-\epsilon)^2 m > \frac{m}{l^{k+1}}$ (recall the choice of values for $k$, $l$, and $\epsilon$). If the seller's revenue is greater than $\eta - \delta$, by an averaging argument there must be at least one signal $s^* \in \mathcal{S}$ whose contribution to the revenue $\sum_{\theta \in \Theta} \mu_\theta \phi_\theta(s^*) \textsc{Rev}(\mathcal{V}, f(s^*), \xi_{s^*})$ is more than $\eta - \delta$, where $\xi_{s^*} \in \Delta_\Theta$ is the buyer's posterior induced by signal $s^*$. Since the expected revenue cannot exceed the expected payment, the price $p^* := f(s^*)$ that the seller proposes to the buyer when signal $s^*$ is sent must be greater than

$$\begin{aligned}
\eta - \delta &= \left[\frac{(1-\epsilon)^2}{ml} + \left(1 - \frac{1}{m}\right)\right] p_o - \delta \\
&= \left[\frac{(1-\epsilon)^2}{ml} + \left(1 - \frac{1}{m}\right)\right]\left[\frac{1}{2} + \frac{l}{(1-\epsilon)2m}\right] - \delta \\
&\geq \left(1 - \frac{1}{m}\right)\left[\frac{1}{2} + \frac{l}{(1-\epsilon)2m}\right] - \delta \\
&\geq \frac{1}{2} + \frac{l}{(1-\epsilon)2m} - \frac{1}{m} - \frac{l}{(1-\epsilon)2m^2} - \delta > \frac{1}{2},
\end{aligned}$$

where the last inequality holds for $m \geq 2$ since we set $\epsilon = \frac{1}{2}$, $l = 5$, and $\delta = \frac{1}{m^2}$. Additionally, the price $p^*$ must be smaller than $p_o$ (see the completeness proof), otherwise, when the buyer has valuation $v_o$, they would never buy the item, resulting in a contribution to the seller's revenue at most of $\frac{1}{m}$ (recall that $v_o$ happens with probability $1 - \frac{1}{m}$ and all other buyer's valuations do not exceed 1). As a result, it must be the case that $p^* \in \left(\frac{1}{2}, p_o\right]$. Next, we prove that, after receiving signal $s^*$, the buyer will buy the item in all the cases in which their valuation belongs to a subset of valuations $v_u$ containing at least a fraction $\frac{1}{2l}(1-\epsilon)^2$ of all the valuations $v_u$. Indeed, if this is not the case, then the contribution to the seller's expected revenue due to signal $s^*$ would be less than

$$p^* \left[1 - \frac{1}{m} + \frac{(1-\epsilon)^2}{2ml}\right] \le p_o \left[1 - \frac{1}{m} + \frac{(1-\epsilon)^2}{2ml}\right] = \eta - p_o \frac{(1-\epsilon)^2}{2ml} \le \eta - \delta,$$

where the last inequality holds since $\delta = \frac{1}{m^2} \le p_o \frac{(1-\epsilon)^2}{2ml}$ for $m$ large enough. Let $U^* \subseteq U$ be the subset of vertices $u \in U$ such that the buyer will buy the item for their corresponding valuations $v_u$. We have that $|U^*| \ge \frac{m}{2l}(1-\epsilon)^2$. Next, we show that $U^*$ constitutes an independent set of $G$. First, since $p^* > \frac{1}{2}$, when the buyer's valuation is $v_u$ such that $u \in U^*$, then the buyer must value the item more than $\frac{1}{2}$, otherwise they would not buy. By contradiction, suppose that there is a couple of vertices $u, u' \in U^*$ such that $(u, u') \in E$. W.l.o.g., let us assume that $\xi_{s^*}(\theta_u) \le \xi_{s^*}(\theta_{u'})$. Then, the buyer's expected valuation induced by posterior $\xi_{s^*}$ is

$$\xi_{s^*}(\theta_u) + \frac{1}{2}\left(1 - \xi_{s^*}(\theta_u) - \xi_{s^*}(\theta_{u'})\right) = \frac{1 + \xi_{s^*}(\theta_u) - \xi_{s^*}(\theta_{u'})}{2} \le \frac{1}{2},$$

which is a contradiction. Given that, by our initial assumption, the size of every independent set must be smaller than $\frac{m}{l^{k+1}} < \frac{1}{2l}(1-\epsilon)^2 m$, we reach the final contradiction proving the result. □

## Appendix C. Proofs omitted from Section 5

**Theorem 4.** *Let $\alpha, \epsilon, \delta > 0$, and set $q := \frac{32}{\epsilon^2} \log \frac{4}{\alpha}$. Given a posterior $\xi \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{N}}$, and two functions $g, h : [0,1]^n \to [0,1]$, if $g$ is $(\delta, \alpha, \epsilon)$-stable compared to $h$ for $(\xi, \mathcal{V})$, then there exists $\gamma \in \Delta_{\Xi^q}$ such that, for every $\theta \in \Theta$, it holds $\sum_{\tilde{\xi} \in \text{supp}(\gamma)} \gamma(\tilde{\xi})\tilde{\xi}(\theta) = \xi(\theta)$ and*

$$\mathbb{E}_{\tilde{\xi} \sim \gamma, V \sim \mathcal{V}}\left[\tilde{\xi}(\theta) g(V\tilde{\xi})\right] \ge \xi(\theta)\left((1-\alpha)\mathbb{E}_{V \sim \mathcal{V}}\left[h(V\xi)\right] - \delta\epsilon\right). \tag{4}$$

**Proof.** The probability distribution $\gamma \in \Delta_{\Xi^q}$ over $q$-uniform posteriors in the statement is defined as follows. Let $\xi^q \in \Xi^q$ be a buyer's posterior defined as the empirical mean of $q$ vectors built form $q$ i.i.d. samples drawn from the given posterior $\xi$. In particular, each sample is obtained by randomly drawing a state of nature, with each state $\theta \in \Theta$ having probability $\xi(\theta)$ of being selected, and, then, a $d$-dimensional vector is built by letting all its components equal to 0, except for that one corresponding to $\theta$, which is set to 1. Notice that $\xi^q$ is a random vector supported on $q$-uniform posteriors, whose expected value is posterior $\xi$. Then, $\gamma$ is such that, for every $\tilde{\xi} \in \Xi^q$, it holds $\gamma(\tilde{\xi}) = \Pr\left\{\xi^q = \tilde{\xi}\right\}$.

It is easy to check that $\xi(\theta) = \sum_{\tilde{\xi} \in \Xi^q} \gamma(\tilde{\xi})\tilde{\xi}(\theta) = \mathbb{E}_{\tilde{\xi} \sim \gamma}\left[\tilde{\xi}(\theta)\right]$ for all $\theta \in \Theta$, proving the first condition needed.

Next, we prove that $\gamma$ satisfies Equation (4). To do so, we first introduce some useful definitions. For every $q$-uniform posterior $\tilde{\xi} \in \Xi^q$, with an overload of notation we let $\gamma(\tilde{\xi}, \theta, j)$ be the conditional probability of having drawn $\tilde{\xi}$ from $\gamma$ given that the drawn posterior assigns probability $\frac{j}{q}$ to state $\theta \in \Theta$ with $j \in \{0, \dots, q\}$. Formally, for every $\tilde{\xi} \in \Xi^q$:

$$\gamma(\tilde{\xi}, \theta, j) := \begin{cases} \dfrac{\gamma(\tilde{\xi})}{\sum_{\xi' \in \Xi^q : \xi'(\theta) = j/q} \gamma(\xi')} & \text{if } \tilde{\xi}(\theta) = \frac{j}{q} \\ 0 & \text{otherwise} \end{cases}.$$

Then, for every $\theta \in \Theta$ and $j \in \{0, \dots, q\}$, we let $\gamma^{\theta, j}$ be a probability distribution over $\Delta_\Theta$ supported on $q$-uniform posteriors such that $\gamma^{\theta, j}(\tilde{\xi}) := \gamma(\tilde{\xi}, \theta, j)$ for all $\tilde{\xi} \in \Xi^q$. Moreover, for every buyer $i \in \mathcal{N}$ and matrix $V \in [0,1]^{n \times d}$ of buyers' valuations, we let $\Xi^{i,V} \subseteq \Xi^q$ be the set of $q$-uniform posteriors that do *not* change buyer $i$'s expected valuation by more than an additive factor $\epsilon$ with respect to their valuation in posterior $\xi$. Formally,

$$\Xi^{i,V} := \left\{\tilde{\xi} \in \Xi^q \mid \left|V_i \xi - V_i \tilde{\xi}\right| \le \epsilon\right\}.$$

In order to complete the proof, we introduce the following three lemmas (with Lemmas 12 and 13 being adapted from [49]). The first lemma shows that, for every state of nature $\theta \in \Theta$, it is possible to bound the cumulative probability mass that the distribution $\gamma$ assigns to $q$-uniform posteriors $\tilde{\xi} \in \Xi^q$ such that $\tilde{\xi}(\theta)$ differs from $\xi(\theta)$ by at least $\frac{\epsilon}{4}$ (in absolute terms). Formally:

**Lemma 12** *(Essentially Lemma 5 by Castiglioni and Gatti [49]). Given $\xi \in \Delta_\Theta$, for every $\theta \in \Theta$ it holds:*

$$\sum_{j : \left|\frac{j}{q} - \xi(\theta)\right| \ge \frac{\epsilon}{4}} \; \sum_{\tilde{\xi} \in \Xi^q : \hat{\xi}(\theta) = \frac{j}{q}} \gamma(\tilde{\xi}) \le \frac{\alpha}{2}\xi(\theta),$$

*where $\gamma \in \Delta_{\Xi^q}$ is the probability distribution over $q$-uniform posteriors introduced at the beginning of the proof.*

The second lemma, which is useful to prove Lemma 14, shows that, for $q$-uniform posteriors $\tilde{\xi} \in \Xi^q$ such that $\tilde{\xi}(\theta)$ is sufficiently close to $\xi(\theta)$ for a state of nature $\theta \in \Theta$, the expected utility of each buyer is close to their utility in the given posterior $\xi$ with high probability. Formally:

**Lemma 13** *(Essentially Lemma 6 by Castiglioni and Gatti [49]). Given $\xi \in \Delta_\Theta$, matrix $V \in [0,1]^{n\times d}$ of buyers' valuations, state of nature $\theta \in \Theta$, and $j : \left|\frac{j}{q} - \xi(\theta)\right| \le \frac{\epsilon}{4}$, the following holds for every buyer $i \in \mathcal{N}$:*

$$\sum_{\tilde{\xi} \in \Xi^{i,V} : \tilde{\xi}(\theta) = \frac{j}{q}} \gamma(\tilde{\xi}) \ge \left(1 - \frac{\alpha}{2}\right) \sum_{\tilde{\xi} \in \Xi^q : \tilde{\xi}(\theta) = \frac{j}{q}} \gamma(\tilde{\xi}),$$

*where $\gamma \in \Delta_{\Xi^q}$ is the probability distribution over q-uniform posteriors introduced at the beginning of the proof.*

Finally, the third lemma that we need reads as follows:

**Lemma 14.** *Given $\xi \in \Delta_\Theta$, for every state of nature $\theta \in \Theta$ and $j : \left|\frac{j}{q} - \xi(\theta)\right| \le \frac{\epsilon}{4}$, the probability distribution $\gamma^{\theta,j}$ defined at the beginning of the proof is $\left(\frac{\alpha}{2}, \epsilon\right)$-decreasing around posterior $\xi$.*

**Proof.** According to Definition 2 and by reversing the inequalities, we need to prove that, for every matrix $V \in [0,1]^{n\times d}$ of buyers' valuations and $i \in \mathcal{N}$, it holds $\Pr_{\tilde{\xi}\sim\gamma^{\theta,j}}\left\{V_i\tilde{\xi} \ge V_i\xi - \epsilon\right\} \ge 1 - \frac{\alpha}{2}$. By using the definition of the set $\Xi^{i,V}$, Lemma 13, and the definition of $\gamma^{\theta,j}$, we can write the following:

$$\Pr_{\tilde{\xi}\sim\gamma^{\theta,j}}\left\{V_i\tilde{\xi} \ge V_i\xi - \epsilon\right\} = \sum_{\tilde{\xi}\in\Xi^{i,V}} \gamma^{\theta,j}(\tilde{\xi}) = \sum_{\tilde{\xi}\in\Xi^{i,V}:\tilde{\xi}(\theta)=\frac{j}{q}} \frac{\gamma(\tilde{\xi})}{\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}}\gamma(\xi')}$$
$$\ge \left(1 - \frac{\alpha}{2}\right) \sum_{\tilde{\xi}\in\Xi^q:\tilde{\xi}(\theta)=\frac{j}{q}} \frac{\gamma(\tilde{\xi})}{\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}}\gamma(\xi')} = 1 - \frac{\alpha}{2},$$

which proves the lemma. □

Now, we are ready to prove the theorem, by means of the following inequalities:

$$\mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[\tilde{\xi}(\theta)\, g(V\tilde{\xi})\right] = \sum_{\tilde{\xi}\in\Xi^q} \gamma(\tilde{\xi})\,\tilde{\xi}(\theta)\mathbb{E}_{V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right]$$
$$\ge \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\le\frac{\epsilon}{4}} \frac{j}{q} \sum_{\tilde{\xi}\in\Xi^q:\tilde{\xi}(\theta)=\frac{j}{q}} \gamma(\tilde{\xi})\,\mathbb{E}_{V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right] \quad \text{(By dropping terms from the sum)}$$
$$= \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\le\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}} \gamma(\xi')\right) \sum_{\tilde{\xi}\in\Xi^q:\tilde{\xi}(\theta)=\frac{j}{q}} \frac{\gamma(\tilde{\xi})}{\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}}\gamma(\xi')}\mathbb{E}_{V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right]$$
$$= \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\le\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}} \gamma(\xi')\right) \mathbb{E}_{\tilde{\xi}\sim\gamma^{\theta,j}, V\sim\mathcal{V}}\left[g(V\tilde{\xi})\right]$$
$$= \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\le\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}} \gamma(\xi')\right) \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\tilde{\xi})\right] - \delta\epsilon\right] \quad \text{(By Definition 3 – Lemma 14)}$$
$$= \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\tilde{\xi})\right] - \delta\epsilon\right] \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\le\frac{\epsilon}{4}} \frac{j}{q} \sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}} \gamma(\xi')$$
$$= \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\tilde{\xi})\right] - \delta\epsilon\right] \left(\xi(\theta) - \sum_{j:\left|\frac{j}{q}-\xi(\theta)\right|\ge\frac{\epsilon}{4}} \sum_{\xi'\in\Xi^q:\xi'(\theta)=\frac{j}{q}} \gamma(\xi')\right)$$

$$\geq \xi(\theta)\left[(1-\alpha)\,\mathbb{E}_{V\sim\mathcal{V}}\left[h(V\tilde{\xi})\right]-\delta\epsilon\right]. \quad \text{(By Lemma 12), } (1-\alpha/2)^2 \geq 1-\alpha, \text{ and } \alpha < 1)$$

This concludes the proof. □

## Appendix D. Proofs omitted from Section 6

**Lemma 1.** *For any $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, there exists a revenue-maximizing price vector $p^* \in \arg\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p)$ such that $p^*_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p^*)$ for every buyer $i \in \mathcal{N}$.*

**Proof.** In order to prove the lemma, we show an even stronger result: for every price vector $p \in [0,1]^n$, it is always possible to recover another price vector $p' \in [0,1]^n$ that provides the seller with an expected revenue at least as large as that provided by $p$, and such that $p'_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p')$ for every $i \in \mathcal{N}$. Let us assume that $p$ does *not* satisfy the required condition for some buyer $i \in \mathcal{N}$. Then, let $p'$ be such that $p'_i = \textsc{Rev}_{>i}(\mathcal{V}, p) > p_i$ and $p'_j = p_j$ for all $j \in \mathcal{N} : j \neq i$. Since by construction $\textsc{Rev}_{>i}(\mathcal{V}, p') = \textsc{Rev}_{>i}(\mathcal{V}, p)$, the condition $p'_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p')$ holds. Moreover, the seller's expected revenue for $p'$ in the auction restricted to all buyers $j \in \mathcal{N} : j \geq i$, namely $\textsc{Rev}_{\geq i}(\mathcal{V}, p')$, is such that:

$$\begin{aligned}
\textsc{Rev}_{\geq i}(\mathcal{V}, p') &= \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p'_i\right\} p'_i + \left(1-\Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p'_i\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p') \\
&= \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p'_i\right\} p'_i + \left(1-\Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p'_i\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p) \\
&= \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p_i\right\} p'_i + \left(1-\Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p_i\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p) \\
&\geq \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p_i\right\} p_i + \left(1-\Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p_i\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p) \\
&= \textsc{Rev}_{\geq i}(\mathcal{V}, p),
\end{aligned}$$

where the first equality and the last one holds by definition of $\textsc{Rev}_{\geq i}(\mathcal{V}, p')$, the second one follows from $\textsc{Rev}_{>i}(\mathcal{V}, p') = \textsc{Rev}_{>i}(\mathcal{V}, p)$, the third one holds since $p'_i = \textsc{Rev}_{>i}(\mathcal{V}, p)$, while the inequality follows from $p'_i \geq p_i$. As a result, we can conclude that $\textsc{Rev}(\mathcal{V}, p') \geq \textsc{Rev}(\mathcal{V}, p)$. The lemma is readily proved by iteratively applying the procedure described above until we get a price vector $p' \in [0,1]^n$ such that $p'_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p')$ for every buyer $i \in \mathcal{N}$, starting from an optimal price vector $p^* \in \arg\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p)$. □

**Lemma 2.** *Given $\epsilon > 0$, let $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$ and $\mathcal{V}^\epsilon = \{\mathcal{V}^\epsilon_i\}_{i\in\mathcal{N}}$ satisfying the conditions of Definition 4. Then,*

$$\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}^\epsilon, p) \geq \max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \epsilon.$$

**Proof.** Let $p^* \in [0,1]^n$ be a price vector such that $p^* \in \arg\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p)$ and $p^*_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p^*)$ for every $i \in \mathcal{N}$. Such price vector is guaranteed to exist by Lemma 1. We show by induction that $\textsc{Rev}_{\geq i}(\mathcal{V}^\epsilon, p^{*,\epsilon}) \geq \textsc{Rev}_{\geq i}(\mathcal{V}, p) - \epsilon$. As a base case, it is easy to check that

$$\begin{aligned}
\textsc{Rev}_{\geq n}(\mathcal{V}^\epsilon, p^{*,\epsilon}) &= [p^*_n - \epsilon]_+ \Pr_{v_n\sim\mathcal{V}^\epsilon_n}\left\{v_n \geq [p^*_n - \epsilon]_+\right\} \\
&\geq (p^*_n - \epsilon)\Pr_{v_n\sim\mathcal{V}_n}\left\{v_n \geq p^*_n\right\} \\
&\geq p^*_n \Pr_{v_n\sim\mathcal{V}_n}\left\{v_n \geq p^*_n\right\} - \epsilon \\
&= \textsc{Rev}_{\geq n}(\mathcal{V}, p^*) - \epsilon.
\end{aligned}$$

By induction, assume that the condition holds for $i+1$ (notice that $\textsc{Rev}_{>i}(\cdot,\cdot) = \textsc{Rev}_{\geq i+1}(\cdot,\cdot)$), then

$$\begin{aligned}
&\textsc{Rev}_{\geq i}(\mathcal{V}^\epsilon, p^{*,\epsilon}) \\
&= [p^*_i - \epsilon]_+ \Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\} + \left(1-\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}^\epsilon, p^{*,\epsilon}) \\
&\geq (p^*_i - \epsilon)\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\} + \left(1-\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\}\right)\left(\textsc{Rev}_{>i}(\mathcal{V}, p^*) - \epsilon\right) \\
&= p^*_i \Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\} + \left(1-\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p^*) - \epsilon \\
&\geq p^*_i \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p^*_i\right\} + \left(1-\Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p^*_i\right\}\right)\textsc{Rev}_{>i}(\mathcal{V}, p^*) - \epsilon \\
&= \textsc{Rev}_{\geq i}(\mathcal{V}, p^*) - \epsilon,
\end{aligned}$$

where the last inequality follows from $p^*_i \geq \textsc{Rev}_{>i}(\mathcal{V}, p^*)$ and $\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq [p^*_i - \epsilon]_+\right\} \geq \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p^*_i\right\}$. □

**Lemma 3.** *For any $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$ and $\epsilon, \tau > 0$, there exist $K \in \text{poly}\left(n, \frac{1}{\epsilon}, \log\frac{1}{\tau}\right)$ and $b \in \text{poly}\left(\frac{1}{\epsilon}\right)$ such that, with probability at least $1-\tau$, Algorithm 1 returns $(p, r)$ satisfying $\textsc{Rev}(\mathcal{V}, p) \geq \max_{p'\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p') - \epsilon$ and $r \in [\textsc{Rev}(\mathcal{V}, p) - \epsilon, \textsc{Rev}(\mathcal{V}, p) + \epsilon]$ in time $\text{poly}\left(n, \frac{1}{\epsilon}, \log\frac{1}{\tau}\right)$.*

**Proof.** Letting $b := \lceil \frac{2}{\epsilon} \rceil$ and $K := \frac{8}{\epsilon^2} \log \frac{2b^n}{\tau} \in \text{poly}\left(n, \frac{1}{\epsilon}, \log\frac{1}{\tau}\right)$, the proof unfolds in two steps.

The first step is to show that restricting price vectors to those in the discretized set $\mathcal{P}^b$ results in a small reduction of the seller's expected revenue. Formally, we prove that:

$$\max_{p\in\mathcal{P}^b} \textsc{Rev}(\mathcal{V}, p) \geq \max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \frac{\epsilon}{2}.$$

To do so, we define some modified distributions of buyers' valuations, namely $\mathcal{V}^b = \{\mathcal{V}_i^b\}_{i\in\mathcal{N}}$, which are supported on the discretized set $\mathcal{P}^b$ and are obtained by mapping each valuation $v_i \in [0, 1]$ in the support of $\mathcal{V}_i$ (for any $i \in \mathcal{N}$) to a discretized valuation $\frac{x}{b}$, where $x$ is the greatest integer such that $\frac{x}{b} \leq v_i$. It is easy to see that, since an optimal price vector for distributions $\mathcal{V}^b$ must specify prices that are multiples of $\frac{1}{b}$, then

$$\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}^b, p) = \max_{p\in\mathcal{P}^b} \textsc{Rev}(\mathcal{V}^b, p).$$

Moreover, by definition of $b$, distributions $\mathcal{V}^b$ are such that it holds $\Pr_{v_i\sim\mathcal{V}_i^b}\left\{v_i \geq p_i - \frac{\epsilon}{2}\right\} \geq \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p_i\}$ for every $i \in \mathcal{N}$ and possible price $p_i \in [0, 1]$. Thus, by Lemma 2, $\max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}^b, p) \geq \max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \frac{\epsilon}{2}$, which implies that $\max_{p\in\mathcal{P}^b} \textsc{Rev}(\mathcal{V}, p) \geq \max_{p\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p) - \frac{\epsilon}{2}$. This proves that we can restrict the attention to price vectors in $\mathcal{P}^b \subset [0, 1]^n$, loosing only an additive factor $\frac{\epsilon}{2}$ of the seller's optimal expected revenue.

The second step of the proof is to show that replacing distributions $\mathcal{V}$ with the empirical distributions $\mathcal{V}^K$ built by Algorithm 1 only reduces the seller's optimal expected revenue by a small amount, with high probability. For any price vector $p \in [0, 1]^n$, by using an Hoeffding's bound we obtain that

$$\Pr\left\{\left|\textsc{Rev}(\mathcal{V}, p) - \textsc{Rev}(\mathcal{V}^K, p)\right| \geq \frac{\epsilon}{4}\right\} \leq 2e^{-K\epsilon^2/8},$$

where the probability is with respect to the stochasticity of the algorithm (as a result of the sampling steps). Since the number of elements in the discretized set $\mathcal{P}^b$ is $b^n$, by a union bound we get

$$\Pr\left\{\left|\textsc{Rev}(\mathcal{V}, p) - \textsc{Rev}(\mathcal{V}^K, p)\right| < \frac{\epsilon}{4} \quad \forall p \in \mathcal{P}^b\right\} \geq 1 - 2b^n e^{-K\epsilon^2/8} = 1 - \tau.$$

Letting $p \in \mathcal{P}^b$ be the price vector returned by Algorithm 1, it is the case that $p \in \arg\max_{p'\in\mathcal{P}^b} \textsc{Rev}(\mathcal{V}^K, p')$, given the correctness and optimality of the backward induction procedure with which the vector $p$ is built [47]. Moreover, letting $p^* \in \arg\max_{p'\in\mathcal{P}^b} \textsc{Rev}(\mathcal{V}, p')$ be an optimal price vector over the discretized set $\mathcal{P}^b$ for the actual distributions of buyers' valuations $\mathcal{V}$, with probability at least $1-\tau$ it holds that

$$\textsc{Rev}(\mathcal{V}, p) \geq \textsc{Rev}(\mathcal{V}^K, p) - \frac{\epsilon}{4} \geq \textsc{Rev}(\mathcal{V}^K, p^*) - \frac{\epsilon}{4} \geq \textsc{Rev}(\mathcal{V}, p^*) - \frac{\epsilon}{2}.$$

Hence, with probability at least $1-\tau$, it holds

$$\textsc{Rev}(\mathcal{V}, p) \geq \textsc{Rev}(\mathcal{V}, p^*) - \frac{\epsilon}{2} \geq \max_{p'\in[0,1]^n} \textsc{Rev}(\mathcal{V}, p') - \epsilon,$$

where the last step has been proved in the first part of the proof.

In order to conclude the proof, it is sufficient to notice that, with probability at least $1-\tau$, it also holds that $r = \textsc{Rev}(\mathcal{V}^K, p) \in \left[\textsc{Rev}(\mathcal{V}, p) - \frac{\epsilon}{4}, \textsc{Rev}(\mathcal{V}, p) + \frac{\epsilon}{4}\right]$. □

## Appendix E. Proofs omitted from Section 7

**Lemma 4.** *Given $\alpha, \epsilon > 0$, a posterior $\xi \in \Delta_\Theta$, and some distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, there exists $p \in [0, 1]^n$ such that, for every other $p' \in [0, 1]^n$, the function $g_p$ is $(1, \alpha, \epsilon)$-stable compared with $g_{p'}$ for $(\xi, \mathcal{V})$.*

**Proof.** As a first step, we prove the following: given any matrix $V \in [0, 1]^{n\times d}$ of buyers' valuations and any price vector $p' \in [0, 1]^n$, for every distribution $\gamma$ over $\Delta_\Theta$ that is $(\alpha, \epsilon)$-decreasing around $\xi$ (see Definition 2) it holds that

$$\mathbb{E}_{\tilde{\xi}\sim\gamma}\left[g_{p'}(V\tilde{\xi})\right] \geq \mathbb{E}_{\tilde{\xi}\sim\gamma}\left[g_{p'}\left(\max\left\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\right\}\right)\right] - \alpha\, g_{p'+\epsilon\mathbb{1}}(V\xi). \tag{E.1}$$

W.l.o.g., let $i \in \mathcal{N}$ be the buyer that buys the item when buyers' valuations are specified by the vector $V\xi - \epsilon\mathbb{1}$ and the proposed prices are those specified by $p'$, that is, it must be the case that $p'_i \leq V_i\xi - \epsilon$ and $p'_j > V_j\xi - \epsilon$ for all $j \in \mathcal{N} : j < i$.

Since $\gamma$ is $(\alpha, \epsilon)$-decreasing around $\xi$, by sampling a posterior $\tilde{\xi} \in \Delta_\Theta$ according to $\gamma$, with probability at least $1-\alpha$ it holds that $V_i\tilde{\xi} \geq V_i\xi - \epsilon$ (see Definition 2). Moreover, let $\tilde{\Xi} := \{\tilde{\xi} \in \Delta_\Theta \mid V_i\tilde{\xi} \geq V_i\xi - \epsilon\}$ be the set of posteriors which result in a buyer $i$'s valuation that is at most $\epsilon$ less than that for $\xi$ (notice that $\sum_{\tilde{\xi}\in\tilde{\Xi}} \gamma(\tilde{\xi}) \geq 1-\alpha$). Then, we split the posteriors in $\Delta_\Theta$ into three groups, as follows:

- $\Xi^1 \subseteq \tilde{\Xi}$ is composed of all the posteriors $\tilde{\xi} \in \tilde{\Xi}$ such that, for every $j \in \mathcal{N} : j < i$, it holds $V_j\tilde{\xi} < p'_j$;
- $\Xi^2 \subseteq \tilde{\Xi}$ is composed of all the posteriors $\tilde{\xi} \notin \tilde{\Xi}$ such that, for every $j \in \mathcal{N} : j < i$, it holds $V_j\tilde{\xi} < p'_j$;
- $\Xi^3 \subseteq \Delta_\Theta$ is composed of all the posteriors $\tilde{\xi} \in \Delta_\Theta$ for which there exists a buyer $j(\tilde{\xi}) \in \mathcal{N} : j < i$ (notice the dependence on $\tilde{\xi}$) such that $j(\tilde{\xi}) = \min\{j \in \mathcal{N} \mid V_j\tilde{\xi} \geq p'_j\}$.

Next, we show that, for every posterior $\tilde{\xi} \in \Xi^1 \cup \Xi^3$, it holds $g_p(V\tilde{\xi}) = g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\})$, while, for every $\tilde{\xi} \in \Xi^2$, it holds $g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}) \leq g_{p'+\epsilon\mathbb{1}}(V\xi)$. First, let us consider a posterior $\tilde{\xi} \in \Xi^1$. For each $j \in \mathcal{N} : j < i$, it holds $V_j\tilde{\xi} \leq \max\{V_j\tilde{\xi}, V_j\xi - \epsilon\} < p'_j$ (by definition of $\Xi^1$, and since buyer $j$ does not buy the item for price $p'_j$). Moreover, since $V_i\tilde{\xi} \geq V_i\xi - \epsilon$, it holds $V_i\tilde{\xi} = \max\{V_i\tilde{\xi}, V_i\xi - \epsilon\} \geq p'_i$. Hence, both when buyers' valuations are specified by the vector $V\tilde{\xi}$ and when they are given by $\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}$ (with max applied component-wise), it is the case that buyer $i$ buys the item at price $p'_i$, resulting in

$$g_{p'}(V\tilde{\xi}) = g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}).$$

Now, let us consider a posterior $\tilde{\xi} \in \Xi^2$. In this case, $\max\{V_i\tilde{\xi}, V_i\xi - \epsilon\} = V_i\xi - \epsilon \geq p'_i$, while $\max\{V_j\tilde{\xi}, V_j\xi - \epsilon\} < p'_j$ for every $j \in \mathcal{N} : j < i$. Thus, both when buyers' valuations are specified by $\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}$ and when they are given by $V\xi - \epsilon\mathbb{1}$, it is the case that buyer $i$ buys the item at price $p'_i$, resulting in

$$g_{p'}(\max\{V\tilde{\xi}, V\xi - \mathbb{1}\}) = g_{p'}(V\xi - \epsilon\mathbb{1}) \leq g_{p'+\epsilon\mathbb{1}}(V\xi),$$

where the inequality holds since buyer $i$ buys the item at price $p'_i$ for valuations $V\xi - \epsilon\mathbb{1}$ and price vector $p'$, while the buyer would still buy the item, though at price $p'_i + \epsilon \geq p'_i$, for valuations $V\xi$ and price vector $p' + \epsilon\mathbb{1}$. Finally, let us consider a posterior $\tilde{\xi} \in \Xi^3$. We have that, for every $j \in \mathcal{N} : j < j(\tilde{\xi})$, it holds $V_j\tilde{\xi} \leq \max\{V_j\tilde{\xi}, V_j\xi - \epsilon\} < p'_j$, while $\max\{V_{j(\tilde{\xi})}\tilde{\xi}, V_{j(\tilde{\xi})}\xi - \epsilon\} \geq V_{j(\tilde{\xi})}\tilde{\xi} \geq p'_{j(\tilde{\xi})}$. As a result, both when buyers' valuations are specified by $V\tilde{\xi}$ and when they are given by $\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}$, it is the case that buyer $j(\tilde{\xi})$ buys the item at price $p'_{j(\tilde{\xi})}$, resulting in

$$g_{p'}(V\tilde{\xi}) = g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}).$$

This allows us to prove Equation (E.1), as follows:

$$\mathbb{E}_{\tilde{\xi}\sim\gamma}\left[g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\})\right] - \mathbb{E}_{\tilde{\xi}\sim\gamma}\left[g_{p'}(V\tilde{\xi})\right] \leq \sum_{\tilde{\xi}\in\Xi^2} \gamma(\tilde{\xi}) g_{p'+\epsilon\mathbb{1}}(V\xi) \leq \alpha g_{p'+\epsilon\mathbb{1}}(V\xi),$$

where the first inequality comes from the fact that, as previously proved, $g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}) = g_{p'}(V\tilde{\xi})$ for every posterior $\tilde{\xi} \in \Xi^1 \cup \Xi^3$ and $g_{p'}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}) \leq g_{p'+\epsilon\mathbb{1}}(V\xi)$ for every posterior $\tilde{\xi} \in \Xi^2$, while the second inequality is readily obtained by noticing that $\sum_{\tilde{\xi}\in\Xi^2} \gamma(\tilde{\xi}) \leq \sum_{\tilde{\xi}\notin\tilde{\Xi}} \gamma(\tilde{\xi}) \leq \alpha$.

Given any posterior $\tilde{\xi} \in \Delta_\Theta$, the expression $\max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[\text{Rev}(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\}, p')\right]$ can be interpreted as the optimal seller's expected revenue when buyers' valuations are determined by distributions $\mathcal{V}^\epsilon = \{\mathcal{V}^\epsilon_i\}$ such that, for every buyer $i \in \mathcal{N}$, their valuation is sampled by first drawing a valuation $v_i \in [0,1]^d$ according to $\mathcal{V}_i$ and, then, taking $\max\{v_i^\top\tilde{\xi}, v_i^\top\xi - \epsilon\}$. Moreover, $\max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\text{Rev}(V\xi, p')$ can be interpreted as the optimal seller's expected revenue when buyers' valuations are determined by distributions $\mathcal{V} = \{\mathcal{V}_i\}$ such that valuations are determined by first sampling a $v_i \in [0,1]^d$ from $\mathcal{V}_i$ and, then, taking $v_i^\top\xi$. It is easy to see that $\Pr_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq p'_i - \epsilon\right\} \geq \Pr_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p'_i\right\}$ for every price $p'_i$, so that distributions $\mathcal{V}^\epsilon$ and $\mathcal{V}$ satisfy Definition 4. Then, by applying Lemma 2, we can conclude that there exists a price vector $p \in [0,1]^n$ such that $\text{Rev}(\mathcal{V}^\epsilon, p) \geq \max_{p'\in[0,1]^n} \text{Rev}(\mathcal{V}, p') - \epsilon$. Thus, for every distribution $\gamma$ over $\Delta_\Theta$ that is $(\alpha, \epsilon)$-decreasing around $\xi$, we get

$$\begin{aligned}
\mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[g_p(V\tilde{\xi})\right] &\geq \mathbb{E}_{\tilde{\xi}\sim\gamma, V\sim\mathcal{V}}\left[g_p(\max\{V\tilde{\xi}, V\xi - \epsilon\mathbb{1}\})\right] - \mathbb{E}_{V\sim\mathcal{V}}\left[\alpha\, g_{p+\epsilon\mathbb{1}}(V\xi)\right] \\
&\geq \max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[\text{Rev}(V\xi, p')\right] - \epsilon - \mathbb{E}_{V\sim\mathcal{V}}\left[\alpha\, g_{p+\epsilon\mathbb{1}}(V\xi)\right] \\
&\geq \max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[\text{Rev}(V\xi, p')\right] - \epsilon - \max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[\alpha\, \text{Rev}(V\xi, p')\right] \\
&\geq (1-\alpha) \max_{p'\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[\text{Rev}(V\xi, p')\right] - \epsilon,
\end{aligned}$$

where the first inequality holds by Equation (E.1), while the second one by Lemma 2. □

**Lemma 5.** *Given a pair $(\gamma, f^\circ)$, where $\gamma$ is a probability distribution over $\Delta_\Theta$ with $\sum_{\xi\in\text{supp}(\gamma)}\gamma(\xi)\xi(\theta)=\mu_\theta$ for all $\theta\in\Theta$ and $f^\circ:\Delta_\Theta\to[0,1]^n$, there is a pair $(\phi, f)$ such that:*

$$\sum_{\theta\in\Theta}\mu_\theta\sum_{s\in\mathcal{S}}\phi_\theta(s)\textsc{Rev}(\mathcal{V}, f(s), \xi_s)=\sum_{\xi\in\text{supp}(\gamma)}\gamma(\xi)\textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi).$$

**Proof.** The idea of the proof is to build a signaling scheme $\phi$ such that there is one-to-one correspondence between the buyers' posteriors induced by signal profiles $s\in\mathcal{S}$ under $\phi$ and the posteriors in the support of the distribution $\gamma$. Thus, in the following we can safely use the notation $\xi_s$ to the denote the posterior corresponding to signal profile $s\in\mathcal{S}$. We define the signaling scheme $\phi:\Theta\to\Delta_{\mathcal{S}}$ so that, for every state $\theta\in\Theta$, it holds $\phi_\theta(s)=\frac{\gamma(\xi_s)\xi_s(\theta)}{\mu_\theta}$ for all $s\in\mathcal{S}$. Moreover, we define $f:\mathcal{S}\to[0,1]^n$ so that $f(s)=f^\circ(\xi_s)$ for all $s\in\mathcal{S}$. First, notice that the signaling scheme $\phi$ is consistent, since, for every $\theta\in\Theta$, it holds $\sum_{s\in\mathcal{S}}\phi_\theta(s)=\sum_{s\in\mathcal{S}}\frac{\gamma(\xi_s)\xi_s(\theta)}{\mu_\theta}=\sum_{\xi\in\text{supp}(\gamma)}\frac{\gamma(\xi)\xi(\theta)}{\mu_\theta}=1$, where the last two equalities follow from the correspondence between signal profiles and posteriors in $\text{supp}(\gamma)$ and the fact that $\sum_{\xi\in\text{supp}(\gamma)}\gamma(\xi)\xi(\theta)=\mu_\theta$. It is also easy to check that each signal profile $s\in\mathcal{S}$ indeed induces its corresponding posterior $\xi_s$ under the signaling scheme $\phi$. Finally, we have

$$\begin{aligned}\sum_{\theta\in\Theta}\mu_\theta\sum_{s\in\mathcal{S}}\phi_\theta(s)\textsc{Rev}(\mathcal{V}, f(s), \xi_s)&=\sum_{\theta\in\Theta}\mu_\theta\sum_{\xi\in\text{supp}(\gamma)}\frac{\gamma(\xi)\xi(\theta)}{\mu_\theta}\textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi)\\&=\sum_{\xi\in\text{supp}(\gamma)}\gamma(\xi)\textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi),\end{aligned}$$

which concludes the proof. □

**Lemma 6.** *Given $\eta>0$ and letting $q=\frac{1}{\eta^2}128\log\frac{6}{\eta}$, an optimal solution to LP (6) has value at least $OPT-\eta$.*

**Proof.** Given a Bayesian posted price auction with prior $\mu\in\Delta_\Theta$ and distributions of buyers' valuations $\mathcal{V}$, let $(\phi^*, f^*)$ be a revenue-maximizing signaling scheme, price function pair. Then, we define $\gamma^*$ as the probability distribution over posteriors $\Delta_\Theta$ induced by $\phi^*$. Moreover, we define $f^{\circ,*}:\text{supp}(\gamma^*)\to[0,1]^n$ in such a way that, for every posterior $\xi\in\text{supp}(\gamma^*)$, it holds $f^{\circ,*}(\xi)=f^*(s)$, where $s\in\mathcal{S}$ is the signal inducing $\xi$, namely $\xi=\xi_s$.[16]

Let $\alpha=\epsilon=\frac{\eta}{2}$ and $q=\frac{32\log\frac{4}{\alpha}}{\epsilon^2}$. Then, we build a probability distribution $\gamma$ over posteriors in $\Delta_\Theta$ by decomposing each posterior $\xi\in\text{supp}(\gamma^*)$ according to Corollary 1. Additionally, each time we decompose a posterior, for every newly-introduced posterior $\xi\in\Delta_\Theta$ we define the function $f^\circ:\Delta_\Theta\to[0,1]^n$ so that $f^\circ(\xi)\in\arg\max_{p\in[0,1]^n}\textsc{Rev}(\mathcal{V}, p, \xi)$. Letting $\gamma^\xi\in\Delta_{\Xi^q}$ be the probability distribution over $q$-uniform posteriors which is obtained by decomposing posterior $\xi\in\text{supp}(\gamma^*)$ according to Corollary 1, we define $\gamma$ so that $\gamma(\xi)=\sum_{\xi'\in\text{supp}(\gamma^*)}\gamma^*(\xi')\gamma^{\xi'}(\xi)$ for every $\xi\in\Xi^q$.

First, let us notice that, for every $\theta\in\Theta$, it holds

$$\sum_{\xi\in\Xi^q}\gamma(\xi)\xi(\theta)=\sum_{\xi'\in\text{supp}(\gamma^*)}\gamma^*(\xi')\sum_{\xi\in\Xi^q}\gamma^{\xi'}(\xi)\xi(\theta)=\sum_{\xi'\in\text{supp}(\gamma^*)}\gamma^*(\xi')\xi'(\theta)=\mu_\theta,$$

where the second equality follows from the property of the decomposition in Theorem 4, while the last one from the fact that $\gamma^*$ is induced by a signaling scheme. Moreover, given any posterior $\xi\in\text{supp}(\gamma^*)$, let $p^\xi\in[0,1]^n$ be a price vector such that, for every $p\in[0,1]^n$, the function $g_{p^\xi}$ is $(1,\alpha,\epsilon)$-stable compared with the function $g_p$ in $(\mathcal{V},\xi)$. Such price vectors are guaranteed to exist by Lemma 4. Then, the pair $(\gamma, f^\circ)$ provides the seller with an expected revenue of

$$\begin{aligned}\sum_{\xi\in\Xi^q}\gamma(\xi)\textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi)&=\sum_{\xi\in\text{supp}(\gamma^*)}\gamma^*(\xi)\sum_{\xi'\in\Xi^q}\gamma^\xi(\xi')\textsc{Rev}(\mathcal{V}, f^\circ(\xi), \xi)\\&=\sum_{\xi\in\text{supp}(\gamma^*)}\gamma^*(\xi)\sum_{\xi'\in\Xi^q}\gamma^\xi(\xi')\max_{p\in[0,1]^n}\textsc{Rev}(\mathcal{V}, p, \xi)\\&\geq\sum_{\xi\in\text{supp}(\gamma^*)}\gamma^*(\xi)\sum_{\xi'\in\Xi^q}\gamma^\xi(\xi')\textsc{Rev}(\mathcal{V}, p^\xi, \xi)\end{aligned}$$

[16] W.l.o.g., we can safely assume that there is a unique signal inducing $\xi$. Indeed, if two signals $s\in\mathcal{S}$ and $s'\in\mathcal{S}$ induce the same posterior, then it is possible to build another signaling scheme, price function pair $(\phi^*, f^*)$ that joins the two signals in a new single signal $s^*\in\mathcal{S}$, by setting $\phi^*_\theta(s^*)\leftarrow\phi^*_\theta(s)+\phi^*_\theta(s')$ and $f^*(s^*)=f(s)$ if $\textsc{Rev}(\mathcal{V}, f^*(s), \xi)\geq\textsc{Rev}(\mathcal{V}, f^*(s'), \xi)$, while $f^*(s)=f^*(s')$ otherwise. It is easy to check that the new signaling scheme cannot decrease the seller's expected revenue.

$$
\begin{aligned}
&\geq \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*(\xi) \left[(1-\alpha)\text{Rev}(\mathcal{V}, f^{\circ,*}(\xi), \xi) - \epsilon\right] \\
&= (1-\alpha) \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*(\xi)\text{Rev}(\mathcal{V}, f^{\circ,*}(\xi), \xi) - \epsilon \\
&= \left(1 - \frac{\eta}{2}\right) \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*(\xi)\text{Rev}(\mathcal{V}, f^{\circ,*}(\xi), \xi) - \frac{\eta}{2} \\
&\geq \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*(\xi)\text{Rev}(\mathcal{V}, f^{\circ,*}(\xi), \xi) - \eta \\
&= \sum_{\theta\in\Theta} \mu_\theta \sum_{s\in\mathcal{S}} \phi^*_\theta(s)\text{Rev}(\mathcal{V}, f^*(s), \xi_s) - \eta,
\end{aligned}
$$

which allows us to conclude that there exists a pair $(\phi, f)$ that only uses $q$-uniform posteriors and provides the seller with an expected revenue arbitrary close to that of an optimal pair. □

**Theorem 5.** *There exists an additive PTAS for computing a revenue-maximizing $(\phi, f)$ pair with public signaling.*

**Proof.** By Lemma 6, given any constant $\eta > 0$ and letting $q = \frac{128 \log \frac{6}{\eta}}{\eta^2}$, LP (6) has optimal value at least $OPT - \eta$. The polynomial-time algorithm that proves the theorem solves an approximated version of LP (6), which is obtained by replacing the terms $\max_{p\in[0,1]^n} \text{Rev}(\mathcal{V}, p, \xi)$ with suitable values $U(\xi)$. The latter are obtained by running Algorithm 1 (the values of $\epsilon$ and $\tau$ are defined in the following) for the (non-Bayesian) auctions in which the buyers' valuations are those resulting by multiplying samples drawn from distributions $\mathcal{V}_i$ by the posterior $\xi$. We let $(p^\xi, U(\xi))$ be the pair returned by Algorithm 1. By Lemma 3, for every $q$-uniform posterior $\xi \in \Xi^q$, Algorithm 1 runs in polynomial time and the price vector $p^\xi$ is such that, with probability at least $1 - \tau$, it holds

$$
\mathbb{E}_{V\sim\mathcal{V}}\left[g_{p^\xi}(V\xi)\right] \geq \max_{p\in[0,1]^n} \mathbb{E}_{V\sim\mathcal{V}}\left[g_p(V\xi)\right] - \epsilon \text{ and} \tag{E.2}
$$

$$
U(\xi) \in \left[\mathbb{E}_{V\sim\mathcal{V}}\left[g_{p^\xi}(V\xi)\right] - \epsilon, \mathbb{E}_{V\sim\mathcal{V}}\left[g_{p^\xi}(V\xi)\right] + \epsilon\right]. \tag{E.3}
$$

As a result, with probability at least $1 - \tau|\Xi^q|$, the previous conditions hold for every $q$-uniform posterior.

Next, we show that, with probability at least $1 - \tau|\Xi^q|$, an optimal solution to LP (6) is close to an optimal solution of the following LP obtained by replacing the max terms in the objective of LP (6) with the values $U(\xi)$:

$$
\max_{\gamma\in\Delta_{\Xi^q}} \sum_{\xi\in\Xi^q} \gamma(\xi)U(\xi) \quad \text{s.t.} \tag{E.4a}
$$

$$
\sum_{\xi\in\Xi^q} \gamma(\xi)\,\xi(\theta) = \mu_\theta \qquad \forall\theta \in \Theta. \tag{E.4b}
$$

Notice that, for a constant $q \in \mathbb{N}_{>0}$, the number of $q$-uniform posteriors is at most $d^q$, so that LP (E.4) can be solved in polynomial time, as it involves $O(|\Xi^q|)$ variables and constraints.

Let $(\gamma, f^\circ)$ be such that $\gamma \in \Delta_{\Xi^q}$ is an optimal solution to LP (E.4) and $f^\circ : \Delta_\Theta \to [0,1]^n$ is such that, for every $\xi \in \Xi^q$, it holds $f^\circ(\xi) = p^\xi$, which is the price vector obtained by running Algorithm 1. Moreover, let $(\gamma^*, f^{\circ,*})$ be an optimal solution to LP (6). Then, with probability at least $1 - \tau|\Xi^q|$,

$$
\begin{aligned}
\sum_{\xi\in\Xi^q} \gamma(\xi)\mathbb{E}_{V\sim\mathcal{V}}\left[g_{f^\circ(\xi)}(V\xi)\right] &\geq \sum_{\xi\in\Xi^q} \gamma(\xi)U(\xi) - \epsilon \\
&\geq \sum_{\xi\in\Xi^q} \gamma^*(\xi)U(\xi) - \epsilon \\
&\geq \sum_{\xi\in\Xi^q} \gamma^*(\xi)\mathbb{E}_{V\sim\mathcal{V}}\left[g_{f^{\circ,*}(\xi)}(V\xi, f^{\circ,*}(\xi))\right] - 2\epsilon.
\end{aligned}
$$

In conclusion, since $\sum_{\xi\in\Xi^q} \gamma^*(\xi)\mathbb{E}_{V\sim\mathcal{V}}\left[g_{f^{\circ,*}(\xi)}(V\xi, f^{\circ,*}(\xi))\right] \geq OPT - \eta$ by Lemma 6, we have:

$$
\sum_{\xi\in\Xi^q} \gamma(\xi)\mathbb{E}_{V\sim\mathcal{V}}\left[g_p(V\xi)\right] \geq OPT - 2\epsilon - \eta
$$

with probability at least $1 - \tau|\Xi^q|$. Hence,

$$\mathbb{E}\left[\sum_{\xi\in\Xi^q}\gamma(\xi)\textsc{Rev}(\mathcal{V},f^\circ(\xi),\xi)\right]\geq\left(1-\tau d^q\right)OPT-2\epsilon-\eta,$$

where the expectation is over the randomness of the algorithm. Finally, Lemma 5 allows us to recover from $(\gamma, f^\circ)$ a signaling scheme with the same seller's expected revenue. For any additive approximation factor $\lambda > 0$, setting $\epsilon = \frac{\lambda}{6}$, $\eta = \frac{\lambda}{3}$, and $\tau = \frac{\lambda}{3d^q}$, we obtain the desired approximation bound. Moreover, the algorithm runs in polynomial time since $\eta$ is constant and the running time of the algorithm is polynomial in $\epsilon, \tau$ and the size of the problem instance. □

## Appendix F. Proofs omitted from Section 8

**Lemma 7.** *Given $\alpha, \epsilon > 0$ and some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{N}}$, for every buyer $i \in \mathcal{N}$, posterior $\xi_i \in \Delta_\Theta$, and price $p_i \in [0, 1]$, the function $g_{i,[p_i-\epsilon]_+}$ is $(0, \alpha, \epsilon)$-stable compared with $g_{i,p_i}$ for $(\xi_i, \mathcal{V})$.*

**Proof.** Let us recall that $g_{i,p_i} : [0, 1]^n \to \{0, 1\}$ is such that $g_{i,p_i}(x) = \mathbb{I}\{x_i \geq p_i\}$ for any value $x \in [0, 1]^n$. As a first step, we show that, for every valuation vector $v_i \in [0, 1]^d$ and probability distribution $\gamma$ over $\Delta_\Theta$ that is $(\alpha, \epsilon)$-decreasing around $\xi$, it holds $\mathbb{E}_{\tilde{\xi}_i\sim\gamma}\mathbb{I}\{v_i^\top\tilde{\xi}_i \geq [p_i - \epsilon]_+\} \geq (1 - \alpha)\mathbb{I}\{v_i^\top\xi_i \geq p_i\}$. Two cases are possible. If $\mathbb{I}\{v_i^\top\xi_i \geq p_i\} = 0$, then the inequality trivially holds. If $\mathbb{I}\{v_i^\top\xi_i \geq p_i\} = 1$, by Definition 1 we have that, with probability at least $1 - \alpha$, a posterior $\tilde{\xi}_i \in \Delta_\Theta$ randomly drawn according to $\gamma$ satisfies $v_i^\top\tilde{\xi}_i \geq [v_i^\top\xi_i - \epsilon]_+ \geq [p_i - \epsilon]_+$, which implies that $\mathbb{I}\{v_i^\top\tilde{\xi}_i \geq [p_i - \epsilon]_+\} = 1$. Hence, $\mathbb{E}_{\tilde{\xi}_i\sim\gamma}\mathbb{I}\{v_i^\top\tilde{\xi}_i \geq [p_i - \epsilon]_+\} \geq (1 - \alpha) = (1 - \alpha)\mathbb{I}\{v_i^\top\xi_i \geq p_i\}$, as desired. Since $\mathbb{E}_{\tilde{\xi}_i\sim\gamma}\mathbb{I}\{v_i^\top\tilde{\xi}_i \geq [p_i - \epsilon]_+\} \geq (1 - \alpha)\mathbb{I}\{v_i^\top\xi_i \geq p_i\}$ for every $v_i \in [0, 1]^d$, by taking the expectation over vectors $v \sim \mathcal{V}$ we obtain $\mathbb{E}_{V\sim\mathcal{V}}\mathbb{E}_{\tilde{\xi}_i\sim\gamma}\mathbb{I}\{V_i\tilde{\xi}_i \geq [p_i - \epsilon]_+\} \geq (1 - \alpha)\mathbb{E}_{V\sim\mathcal{V}}\mathbb{I}\{V_i\xi_i \geq p_i\}$, which fulfills the condition in Definition 3 and proves the result. □

**Lemma 8.** *Given a feasible solution to LP (7), it is possible to recover a pair $(\phi, f)$ that provides the seller with an expected revenue equal to the value of the solution.*

**Proof.** We define the set of signals for buyer $i \in \mathcal{N}$ as $\mathcal{S}_i := \Xi_i^q \times P^b$. Then, we set $\phi : \Theta \to \Delta_{\mathcal{S}}$ so that, for every $\theta \in \Theta$ and $s \in \mathcal{S}$, it holds $\phi_\theta(s) = \frac{y_{\theta,\xi,p}}{\mu_\theta}$, where the pair $(\xi, p)$ with $\xi = (\xi_1, \ldots, \xi_n) \in \Xi^q$ and $p \in \mathcal{P}^b$ is such that $(\xi_i, p_i) = s_i$ for each $i \in \mathcal{N}$. Moreover, we set $f_i(s_i) = p_i$ for every buyer $i \in \mathcal{N}$ and signal $s_i = (\xi_i, p_i) \in \mathcal{S}_i$. First, we show that $\phi$ is well defined, that is, for every state of nature $\theta \in \Theta$, it holds

$$\sum_{s\in\mathcal{S}}\phi_\theta(s)=\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b}\frac{y_{\theta,\xi,p}}{\mu_\theta}=\sum_{\xi_1\in\mathcal{S}_1}\sum_{p_1\in P^b}\frac{\xi_1(\theta)t_{1,\xi_1,p_1}}{\mu_\theta}=\sum_{\xi_1\in\Xi_1^q}\frac{\xi_1(\theta)\gamma_{1,\xi_1}}{\mu_\theta}=\frac{\mu_\theta}{\mu_\theta}=1,$$

where we use Constraints (7b) in the second equality, Constraints (7c) in the third one, and Constraints (7d) in the last one. Next, we show that, for any $\xi \in \Xi^q$ and $p \in \mathcal{P}^b$, it holds $\textsc{Rev}(\mathcal{V}, f(s), \xi_s) = \textsc{Rev}(\mathcal{V}, p, \xi)$, where the signal profile $s \in \mathcal{S}$ is such that $s_i = (\xi_i, p_i)$ for every $i \in \mathcal{N}$. Clearly, the prices coincide, namely $f(s) = p$. Thus, it is sufficient to prove that each signal $s_i = (\xi_i, p_i)$ induces posterior $\xi_i$ for buyer $i \in \mathcal{N}$. For every $\theta \in \Theta$, it holds

$$\mu_\theta\sum_{s'\in\mathcal{S}:s'_i=s_i}\phi_\theta(s')=\sum_{\xi'\in\Xi^q,p'\in\mathcal{P}^b:(\xi'_i,p'_i)=s_i}y_{\theta,\xi',p'}=\xi_i(\theta)t_{i,\xi_i,p_i}.$$

Hence, for every $\theta \in \Theta$,

$$\xi_{i,s_i}(\theta)=\frac{\mu_\theta\phi_{i,\theta}(s_i)}{\sum_{\theta'\in\Theta}\mu_{\theta'}\phi_{i,\theta'}(s_i)}=\frac{\mu_\theta\sum_{s'\in\mathcal{S}:s'_i=s_i}\phi_\theta(s')}{\sum_{\theta'\in\Theta}\mu_{\theta'}\sum_{s'\in\mathcal{S}:s'_i=s_i}\phi_{\theta'}(s')}=\frac{\xi_i(\theta)t_{i,\xi_i,p_i}}{t_{i,\xi_i,p_i}}=\xi_i(\theta).$$

Thus, the seller's expected revenue for the pair $(\phi, f)$ is

$$\begin{aligned}\sum_{s\in\mathcal{S}}\sum_{\theta\in\Theta}\mu_\theta\phi_\theta(s)\textsc{Rev}(\mathcal{V},f(s),\xi_s)&=\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b}\sum_{\theta\in\Theta}\mu_\theta\frac{y_{\theta,\xi,p}}{\mu_\theta}\textsc{Rev}(\mathcal{V},p,\xi)\\&=\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b}y_{\theta,\xi,p}\textsc{Rev}(\mathcal{V},p,\xi),\end{aligned}$$

which proves the lemma. □

**Lemma 9.** *For every $\eta > 0$, there exist $b(\eta), q(\eta) \in \mathbb{N}_{>0}$ such that LP (7) has optimal value at least $OPT - \eta$.*

**Proof.** We show that, given a revenue-maximizing pair $(\phi, f)$ (with seller's revenue $OPT$), we can recover an optimal solution to LP (7) whose value is at least $OPT - \eta$ when the LP is instantiated with suitable constants $b(\eta) \in \mathbb{N}_{>0}$ and $q(\eta) \in \mathbb{N}_{>0}$ (depending on the approximation level $\eta$). Let $\alpha = \epsilon = \frac{\eta}{3}$, $b = \lceil \frac{3}{\eta} \rceil$, and $q = \frac{32 \log \frac{4}{\alpha}}{\epsilon^2}$. Recalling that $\xi_{i,s_i} \in \Delta_\Theta$ denotes buyer $i$'s posterior induced by signal $s_i \in \mathcal{S}_i$, we let $\gamma^{s_i} \in \Delta_{\Xi_i^q}$ be the probability distribution over $q$-uniform posteriors obtained by decomposing $\xi_{i,s_i}$ according to Theorem 4. By Lemma 7 and Theorem 4, it follows that, for every $p_i \in [0,1]$ and $\theta \in \Theta$,

$$\sum_{\xi_i \in \Xi_i^q} \gamma^{s_i}(\xi_i)\xi_i(\theta)\Pr_{v_i \sim \mathcal{V}_i}\left\{v_i^\top \xi_i \ge [p_i - \epsilon]_+\right\} \ge \xi_{i,s_i}(\theta)(1-\alpha)\Pr_{v_i \sim \mathcal{V}_i}\left\{v_i^\top \xi_i \ge p_i\right\}. \tag{F.1}$$

For every signal profile $s \in \mathcal{S}$, we define a non-Bayesian posted price auction in which the distributions of buyers' valuations are $\mathcal{V}^s = \{\mathcal{V}_i^s\}_{i \in \mathcal{N}}$, where each $\mathcal{V}_i^s$ is such that a valuation $v_i \sim \mathcal{V}_i^s$ is obtained by first sampling $\tilde{v}_i \sim \mathcal{V}_i$ and then letting $v_i = \tilde{v}_i^\top \xi_{i,s_i}$. Moreover, we let $p^s \in [0,1]^n$ be a price vector for the seller in such non-Bayesian auction, with $p_i^s \ge \textsc{Rev}_{>i}(\mathcal{V}^s, p^s)$ for every $i \in \mathcal{N}$. By Lemma 1, such a vector always exists. Finally, given $p^s$, we let $\hat{p}^s \in [0,1]^n$ be such that each price $\hat{p}_i^s$ is the greatest price $p_i \in P^b$ (among discretized prices) satisfying the inequality $p_i \le [p_i^s - \epsilon]_+$; formally,

$$p_i^s = \max\left\{p_i \in P^b \mid p_i \le [p_i^s - \epsilon]_+\right\}.$$

Next, we define the optimal solution to LP (7) that we need to prove the result:

- $\gamma_{i,\xi_i} = \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i)\gamma^{s_i}(\xi_i)$ for every $i \in \mathcal{N}$ and $\xi_i \in \Xi_i^q$.
- $t_{i,\xi_i,p_i} = \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i)\gamma^{s_i}(\xi_i)\mathbb{I}\left\{p_i = \hat{p}_i^s\right\}$ for every $i \in \mathcal{N}$, $\xi_i \in \Xi_i^q$, and $p_i \in P^b$.
- $y_{\theta,\xi,p} = \sum_{s \in \mathcal{S}} \mu_\theta \phi_\theta(s) \prod_{i \in \mathcal{N}} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi_{i,s_i}(\theta)}\mathbb{I}\{p_i = \hat{p}_i^s\}$ for every $\theta \in \Theta$, $\xi \in \Xi^q$, and $p \in \mathcal{P}^b$.

The next step is to show that, for every signal profile $s \in \mathcal{S}$, the seller's expected revenue obtained by decomposing each signal $s_i$ according to Theorem 4 is "close" to the one for $s$. Formally, we show that, for every $s \in \mathcal{S}$ and $\theta \in \Theta$,

$$\sum_{\xi \in \Xi^q} \prod_{i \in \mathcal{N}} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi_{i,s_i}(\theta)} \textsc{Rev}(\mathcal{V}, \hat{p}^s, \xi) \ge \textsc{Rev}(\mathcal{V}, f(s), \xi_s) - \left(\alpha + \epsilon + \frac{1}{b}\right). \tag{F.2}$$

In order to do so, we relate the LHS of Equation (F.2) to the seller' revenue in a non-Bayesian posted price auction. In particular, we show that it is equivalent to the seller's revenue when employing price vector $\hat{p}^s$ in the auction defined by the distributions of buyers' valuations $\hat{\mathcal{V}}^{s,\theta} = \{\hat{\mathcal{V}}_i^{s,\theta}\}_{i \in \mathcal{N}}$, where each $\hat{\mathcal{V}}_i^{s,\theta}$ is such that a valuation $v_i \sim \hat{\mathcal{V}}_i^{s,\theta}$ is defined as $v_i = \tilde{v}_i^\top \tilde{\xi}_i$ with $\tilde{v}_i \sim \mathcal{V}_i$ and $\tilde{\xi}_i \in \Delta_\Theta$ sampled from a distribution such that $\Pr\left\{\tilde{\xi}_i = \xi_i\right\} = \frac{\xi_i(\theta)}{\xi_{i,s_i}(\theta)}\gamma^{s_i}(\xi_i)$. Notice that each $\hat{\mathcal{V}}_i^{s,\theta}$ is well defined, since $\mathcal{V}_i$ is by definition a probability distribution and $\sum_{\xi_i \in \Xi_i^q} \frac{\xi_i(\theta)}{\xi_{i,s_i}(\theta)}\gamma^{s_i}(\xi_i) = 1$ by Theorem 4, defining a probability distribution over the posteriors. Moreover, it is easy to check that valuations sampled from distributions $\hat{\mathcal{V}}_i^{s,\theta}$ are independent among each other. Finally, $\textsc{Rev}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s)$ is equal to the LHS of Equation (F.2), since, by an inductive argument, for every $i \in \mathcal{N}$, it holds

$$\sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \prod_{j \in \mathcal{N}} \frac{\xi'_j(\theta)\gamma^{s_j}(\xi'_j)}{\xi_{j,s_j}(\theta)} = \frac{\xi_i(\theta)}{\xi_{i,s_i}(\theta)}\gamma^{s_i}(\xi_i),$$

where the equality comes from the fact that, for every $j \in \mathcal{N}$, it is the case that

$$\sum_{\xi_j \in \Xi_j^q} \frac{\xi_j(\theta)\gamma^{s_j}(\xi_j)}{\xi_{j,s_j}(\theta)} = 1. \tag{F.3}$$

Let also notice that, in the auction defined above, the probability with which a buyer $i \in \mathcal{N}$ has a valuation greater than or equal to $p_i^s$ is $\sum_{\xi_i \in \Xi_i^q} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi_{i,s_i}(\theta)}\Pr_{\tilde{v}_i \sim \mathcal{V}_i}(\tilde{v}_i^\top \xi_i \ge p_i^s) \ge (1-\alpha)\Pr_{\tilde{v}_i \sim \mathcal{V}_i}(\tilde{v}_i^\top \xi_{i,s_i} \ge p_i^s)$, where the inequality holds by Equation (F.1). First, we compare the seller's revenue in the two non-Bayesian, namely $\textsc{Rev}(\mathcal{V}^s, p^s)$ and $\textsc{Rev}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s)$. In particular, we show by induction that $\textsc{Rev}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s) \ge \textsc{Rev}(\mathcal{V}^s, p^s) - \alpha - \epsilon - \frac{1}{b}$. Let $\textsc{Rev}_{\ge i}(\mathcal{V}, p)$ be the seller's expected revenue for $p$ in the auction restricted to all buyers $j \in \mathcal{N} : j \ge i$. The base case is

$$\begin{aligned}\textsc{Rev}_{\ge n}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s) &= \hat{p}_n^s \Pr_{v_n \sim \hat{\mathcal{V}}_n^{s,\theta}}\left\{v_n \ge \hat{p}_n^s\right\} \\ &\ge \hat{p}_n^s(1-\alpha)\Pr_{v_n \sim \mathcal{V}_n^s}\left\{v_n \ge p_n^s\right\}\end{aligned}$$

$$\geq p_n^s \Pr_{v_n \sim \mathcal{V}_n^s}\left\{v_n \geq p_n^s\right\} - \epsilon - \alpha - \frac{1}{b}$$
$$= \textsc{Rev}_{\geq n}(\mathcal{V}^s, p^s) - \epsilon - \alpha - \frac{1}{b}.$$

By induction, let us assume that the condition holds for $i+1$, then

$$\begin{aligned}
\textsc{Rev}_{\geq i}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s) &= \hat{p}_i^s \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\} + \left(1 - \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\}\right) \textsc{Rev}_{>i}(\hat{\mathcal{V}}^{s,\theta}, \hat{p}^s) \\
&\geq \left(p_i^s - \epsilon - \frac{1}{b}\right) \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\} \\
&\qquad + \left(1 - \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\}\right) \left(\textsc{Rev}_{>i}(\mathcal{V}^s, p^s) - \epsilon - \alpha - \frac{1}{b}\right) \\
&= p_i^s \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\} \\
&\qquad + \left(1 - \Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\}\right) (\textsc{Rev}_{>i}(\mathcal{V}^s, p^s) - \alpha) - \epsilon - \frac{1}{b} \\
&\geq p_i^s (1-\alpha) \Pr_{v_i \sim \mathcal{V}_i^s}\left\{v_i \geq p_i^s\right\} \\
&\qquad + \left[1 - (1-\alpha)\Pr_{v_i \sim \mathcal{V}_i^s}\left\{v_i \geq p_i^s\right\}\right] \left(\textsc{Rev}_{>i}(\mathcal{V}^s, p^s) - \alpha\right) - \epsilon - \frac{1}{b} \\
&\geq \textsc{Rev}_{\geq i}(\mathcal{V}^s, p^s) - \epsilon - \alpha - \frac{1}{b},
\end{aligned}$$

where the second to last inequality follows from $p_i^s \geq \textsc{Rev}_{>i}(\mathcal{V}^s, p^s)$ and $\Pr_{v_i \sim \hat{\mathcal{V}}_i^{s,\theta}}\left\{v_i \geq \hat{p}_i^s\right\} \geq (1-\alpha)\Pr_{v_i \sim \mathcal{V}_i^s}\left\{v_i \geq p_i^s\right\}$. Hence, Equation (F.2) is readily proved, as follows

$$\begin{aligned}
\sum_{\xi \in \Xi^q} \prod_{i \in \mathcal{N}} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi_{i,s_i}(\theta)} \textsc{Rev}(\mathcal{V}, \hat{p}^s, \xi) &\geq \textsc{Rev}(\mathcal{V}^s, p^s) - \left(\alpha + \epsilon + \frac{1}{b}\right) \\
&\geq \textsc{Rev}(\mathcal{V}, f(s), \xi_s) - \left(\alpha + \epsilon + \frac{1}{b}\right),
\end{aligned}$$

Now, we are ready to bound the objective of LP (7), as follows:

$$\begin{aligned}
&\sum_{\theta \in \Theta} \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b} y_{\theta,\xi,p} \textsc{Rev}(\mathcal{V}, p, \xi) \\
&\qquad = \sum_{\theta \in \Theta} \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b} \sum_{s \in \mathcal{S}} \mu_\theta \phi_\theta(s) \prod_{i \in \mathcal{N}} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi_{s_i}(\theta)} \mathbb{I}\{\hat{p}_i^s = p_i\} \textsc{Rev}(\mathcal{V}, p, \xi) \\
&\qquad = \sum_{s \in \mathcal{S}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(s) \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b s} \prod_{i \in \mathcal{N}} \frac{\xi_i(\theta)\gamma^{s_i}(\xi_i)}{\xi^{s_i}(\theta)} \mathbb{I}\{\hat{p}_i^s = p_i\} \textsc{Rev}(\mathcal{V}, p, \xi) \\
&\qquad \geq \sum_{s \in \mathcal{S}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(s) \left[\textsc{Rev}(\mathcal{V}, f(s), \xi_s) - \left(\alpha + \epsilon + \frac{1}{b}\right)\right] \\
&\qquad \geq OPT - \left(\alpha + \epsilon + \frac{1}{b}\right) \geq OPT - \eta.
\end{aligned}$$

We conclude the proof showing that the defined solution is feasible for LP (7). First, it proves that, for every $i \in \mathcal{N}$ and $\theta \in \Theta$,

$$\begin{aligned}
\sum_{\xi_i \in \Xi_i^q} \xi_i(\theta)\gamma_{i,\xi_i} &= \sum_{\xi_i \in \Xi_i^q} \xi_i(\theta) \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \gamma^{s_i}(\xi_i) \\
&= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \sum_{\xi_i \in \Xi_i^q} \xi_i(\theta)\gamma^{s_i}(\xi_i) \\
&= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \xi_{i,s_i}(\theta)
\end{aligned}$$

$$= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \frac{\mu_\theta \phi_{i,\theta}(s_i)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i)}$$

$$= \sum_{s_i \in \mathcal{S}_i} \mu_\theta \phi_\theta(s_i) = \mu_\theta.$$

Moreover, for every $i \in \mathcal{N}$ and $\xi_i \in \Xi_i^q$, it holds

$$\sum_{p_i \in P^b} t_{i,\xi_i,p_i} = \sum_{p_i \in P^b} \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i) \gamma^{s_i}(\xi_i) \mathbb{I}\{p_i = \hat{p}_i^s\} =$$

$$\sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i) \gamma^{s_i}(\xi_i) \sum_{p_i \in P^b} \mathbb{I}\{p_i = \hat{p}_i^s\} =$$

$$\sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta \phi_{i,\theta}(s_i) \gamma^{s_i}(\xi_i) = \gamma_{i,\xi_i}.$$

Finally, for every $\theta \in \Theta$, $i \in \mathcal{N}$, $\xi_i \in \Xi_i^q$, and $p_i \in P^b$, it holds

$$\sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\xi',p'} = \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} \sum_{s \in \mathcal{S}} \mu_\theta \phi_\theta(s) \prod_{j \in \mathcal{N}} \frac{\xi'_j(\theta)}{\xi_{j,s_j}(\theta)} \gamma^{s_j}(\xi'_j) \mathbb{I}\{p_j = \hat{p}_j^s\}$$

$$= \sum_{s \in \mathcal{S}} \mu_\theta \phi_\theta(s) \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} \prod_{j \in \mathcal{N}} \frac{\xi'_j(\theta)}{\xi_{j,s_j}(\theta)} \gamma^{s_j}(\xi'_j) \mathbb{I}\{p_j = \hat{p}_j^s\}$$

$$= \sum_{s \in \mathcal{S}} \mu_\theta \phi_\theta(s) \frac{\xi_i(\theta)}{\xi_{s_i}(\theta)} \gamma^{s_i}(\xi_i) \mathbb{I}\{p_i = \hat{p}_i^s\} \qquad \text{(From Equation (F.3))}$$

$$= \sum_{s_i \in \mathcal{S}_i} \mu_\theta \phi_{i,\theta}(s_i) \frac{\xi_i(\theta)}{\xi_{s_i}(\theta)} \gamma^{s_i}(\xi_i) \mathbb{I}\{p_i = \hat{p}_i^s\}$$

$$= \xi_i(\theta) \sum_{s_i \in \mathcal{S}} \mu_\theta \phi_{i,\theta}(s_i) \frac{\sum_{\theta' \in \Theta} \phi_{i,\theta'}(s_i)}{\mu_\theta \phi_{i,\theta}(s_i)} \gamma^{s_i}(\xi_i) \mathbb{I}\{p_i = \hat{p}_i^s\}$$

$$= \xi_i(\theta) \sum_{s_i \in \mathcal{S}} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \gamma^{s_i}(\xi_i) \mathbb{I}\{p_i = \hat{p}_i^s\} = \xi_i(\theta) t_{i,\xi_i,p_i}.$$

This concludes the proof. □

**Lemma 10.** *For any $\delta > 0$, there exists a dynamic programming algorithm (Algorithm 2) that provides a $\delta$-approximation (in the additive sense) to* MAX-LINREV. *Moreover, the algorithm runs in time polynomial in the size of the input and $\frac{1}{\delta}$.*

**Proof.** The algorithm is described in Algorithm 2. It works in polynomial time since the matrix $M$ has $n|A| = O(\frac{1}{\delta} n^3)$ entries and each entry is computed in polynomial time. This proves the second part of the statement.

In the following, we denote with $\text{Rev}_{\geq i}(\mathcal{V}, p, \xi)$ the seller's expected revenue in the Bayesian posted price auction when they select price vector $p \in \mathcal{P}^b$ and the buyers' posteriors are specified by the tuple $\xi = (\xi_1, \dots, \xi_n) \in \Xi^q$.

Let $S(i,a) := \{(\xi, p) \in \Xi^q \times \mathcal{P}^b \mid \sum_{j \geq i} w_{j,\xi_j,p_j} \geq a\}$ for every $i \in \mathcal{N}$ and $a \in A$. Moreover, for every $a' \in A$, let $\bar{S}(i,a,a') = \{(\xi, p) \in \Xi^q \times \mathcal{P}^b \mid w_{i,\xi_i,p_i} \geq a' \wedge \sum_{j > i} w_{j,\xi_j,p_j} \geq a - a'\}$. First, we prove by induction that $M(i, a - \frac{n-i}{c}) \geq \max_{(\xi,p) \in S(i,a)} \text{Rev}(\mathcal{V}, p, \xi)$ for every $i \in \mathcal{N}$ and $a \in A$. For $i = n$, the condition trivially holds by Line 5. For $i < n$,

$$\max_{(\xi,p) \in S(i,a)} \text{Rev}_{\geq i}(\mathcal{V}, p, \xi) = \max_{a' \in [0,1]} \max_{(\xi,p) \in \bar{S}(i,a,a')} \text{Rev}_{\geq i}(\mathcal{V}, p, \xi)$$

$$= \max_{a' \in [0,1]} \max_{(\xi,p) \in \bar{S}(i,a,a')} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \xi_i \geq p_i\} + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \xi_i \geq p_i\}\right) \text{Rev}_{\geq i+1}(\mathcal{V}, p, \xi)$$

$$\leq \max_{a' \in A} \max_{(\xi,p) \in \bar{S}(i,a-\frac{1}{c},a')} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \xi_i \geq p_i\} + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \xi_i \geq p_i\}\right) \text{Rev}_{\geq i+1}(\mathcal{V}, p, \xi)$$

$$= \max_{a' \in A} \max_{\xi_i \in \Xi_i^q, p_i \in P^b : w_{i,\xi_i,p_i} \geq a'} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \xi_i \geq p_i\}$$

$$
\begin{aligned}
&\qquad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i}\{v_i^\top \xi_i \geq p_i\}\right) \max_{(\xi,p)\in S(i+1,a-a'-\frac{1}{c})} \textsc{Rev}_{\geq i+1}(\mathcal{V},p,\xi)\\
&\leq \max_{a'\in A} \max_{\xi_i \in \Xi_i^q, p_i \in P^b : w_{i,\xi_i,p_i} \geq a'} p_i \Pr_{v_i \sim \mathcal{V}_i}\{v_i^\top \xi_i \geq p_i\}\\
&\qquad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i}\{v_i^\top \xi_i \geq p_i\}\right) M\left(i+1, a - a' - \frac{1}{c} - \frac{n-i-1}{c}\right)\\
&= \max_{a'\in A, \xi \in \Xi^q, p \in \mathcal{P}^b : w_{i,\xi_i,p_i} + a' \geq a - \frac{n-i}{c}} p_i \Pr_{v_i \sim \mathcal{V}_i}\{v_i^\top \xi_i \geq p_i\}\\
&\qquad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i}\{v_i^\top \xi_i \geq p_i\}\right) M(i+1, a')\\
&= M\left(i, a - \frac{n-i}{c}\right).
\end{aligned}
$$

In conclusion, let $OPT_{\textsc{Rev}}$ the revenue term in the value of an optimal solution to MAX-LINREV, while $a \in [0,n]$ is the sum of the linear components in such optimal solution (the second term in the value of the solution). Let $a^*$ be the greatest element in $A$ such that $a^* \leq a - \frac{n-1}{c}$. Notice that $a^* \geq a - \frac{n}{c}$. Moreover, we have $M(1,a^*) \geq \max_{(\xi,p)\in S(i,a)} \textsc{Rev}(\mathcal{V},p,\xi) = OPT_{\textsc{Rev}}$.[17] Hence, there exists a solution with value $M(1,a^*) + a^* \geq OPT_{\textsc{Rev}} + a - \frac{n}{c} = OPT - \frac{n}{c} \geq OPT - \delta$, concluding the proof. □

**Lemma 11.** *LP (7) and LP (8) have the same optimal value. Moreover, given a feasible solution to LP (8), it is possible compute in polynomial time a feasible solution to LP (7) with a greater or equal value.*

**Proof.** To show the equivalence between the two LPs, it is sufficient to show that, given a feasible solution to LP (8), we can construct a solution to LP (7) with a greater or equal value. Let $(y,t,\gamma)$ be a solution to LP (8). For every $i \in \mathcal{N}$, $\xi_i \in \Xi_i^q$, $p_i \in P^b$, let $\delta_{i,\xi_i,p_i} := \xi_i(\theta) t_{i,\xi_i,p_i} - \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\xi',p'}$. Moreover, let $\iota = \mu_\theta - \sum_{\xi \in \Xi^q, p \in \mathcal{P}^b} y_{\theta,\xi,p}$. First, we show that $\sum_{\xi_i \in \Xi_i^q, p_i \in P^b} \delta_{i,\xi_i,p_i} = \iota$ for every $i \in \mathcal{N}$. For each $i \in \mathcal{N}$, it holds

$$
\begin{aligned}
\sum_{\xi_i \in \Xi_i^q} \sum_{p_i \in P^b} \delta_{i,\xi_i,p_i} &= \sum_{\xi_i \in \Xi_i^q} \sum_{p_i \in P^b} \left[ \xi_i(\theta) t_{i,\xi_i,p_i} - \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\xi',p'} \right]\\
&= \sum_{\xi_i \in \Xi_i^q} \xi_i(\theta) \gamma_{i,\xi_i} - \sum_{\xi' \in \Xi^q} \sum_{p' \in \mathcal{P}^b} y_{\theta,\xi',p'}\\
&= \mu_\theta - \sum_{\xi \in \Xi^q} \sum_{p \in \mathcal{P}^b} y_{\theta,\xi,p} = \iota.
\end{aligned}
$$

Next, we build a feasible solution $(\bar{y},t,\gamma)$ to LP (7) with $\bar{y}_{\theta,\xi,p} \geq y_{\theta,\xi,p}$ for all $\theta \in \Theta$, $\xi \in \Xi^q$, and $p \in \mathcal{P}^b$. In particular, we set $\bar{y}_{\theta,\xi,p} = y_{\theta,\xi,p} + \frac{\prod_{i\in\mathcal{N}} \delta_{i,\xi_i,p_i}}{\iota^{n-1}}$. Since $\delta_{i,\xi_i,p_i} \geq 0$ and $\iota \geq 0$ by the feasibility of $(y,t,\gamma)$, it holds that $\bar{y}_{\theta,\xi,p} \geq y_{\theta,\xi,p}$. Moreover, for each $i \in \mathcal{N}$, $\theta \in \Theta$, $\xi_i \in \Xi_i^q$, and $p_i \in P^b$, we have that

$$
\begin{aligned}
\sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} \bar{y}_{\theta,\xi',p'} &= \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\xi',p'} + \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} \frac{\prod_{j\in\mathcal{N}} \delta_{j,\xi_j,p_j}}{\iota^{n-1}}\\
&= \sum_{\xi' \in \Xi^q : \xi'_i = \xi_i} \sum_{p' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\xi,p'} + \delta_{i,\xi_i,p_i} = t_{i,\xi_i,p_i},
\end{aligned}
$$

where the second equality follows from $\sum_{\xi_j \in \Xi_j^q, p_j \in P^b} \delta_{j,\xi_j,p_j} = \iota$ for every $j \in \mathcal{N}$. Since $\textsc{Rev}(\mathcal{V},p,\xi) \geq 0$ for every $p \in \mathcal{P}^b$ and $\xi \in \Xi^q$, it follows that the value of $(\bar{y},t,\gamma)$ is greater than or equal to the value of $(\bar{y},t,\gamma)$. □

**Theorem 6.** *There exists an additive PTAS for computing a revenue-maximizing $(\phi, f)$ pair with private signaling.*

**Proof.** Our PTAS is described in Algorithm 3. Since we only have access to an oracle returning samples from distributions $\mathcal{V}$, our algorithm works with empirical distributions $\mathcal{V}^K$ built from $K$ i.i.d. samples, for a suitably-defined $K \in \mathbb{N}_{>0}$. The

[17] It is easy to see that, if $a < \frac{n-1}{c}$, then the equality holds for $a = 0$.

algorithm works with LP (8) for the values $b \in \mathbb{N}_{>0}$ and $q \in \mathbb{N}_{>0}$ defined in the following, finding an approximate solution to LP (8). Since LP (8) has an exponential number of variables, the algorithm works by applying the ellipsoid method to its dual formulation, as described in the following.

Let $a \in \mathbb{R}^{n\times|\Theta|}$, $w \in \mathbb{R}_{-}^{|\Theta|\times n\times|\Xi_i^q|\times|P^b|}$, and $c \in \mathbb{R}^{n\times|\Xi_i^q|}$. Then, the dual of LP (8) reads as follows.

$$\min_{a,w,c} \sum_{i\in\mathcal{N}}\sum_{\theta\in\Theta} \mu_\theta a_{i,\theta} \quad \text{s.t.} \tag{F.4a}$$

$$\sum_{i\in\mathcal{N}} -w_{\theta,i,\xi_i,p_i} \geq \textsc{Rev}(\mathcal{V}^K, p, \xi) \qquad \forall\theta\in\Theta, \forall\xi\in\Xi^q, \forall p\in\mathcal{P}^b \tag{F.4b}$$

$$\sum_{\theta\in\Theta} \xi_i(\theta) w_{\theta,i,\xi_i,p_i} + c_{i,\xi_i} \geq 0 \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b \tag{F.4c}$$

$$-c_{i,\xi_i} + \sum_{\theta\in\Theta} \xi_i(\theta) a_{i,\theta} \geq 0 \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q \tag{F.4d}$$

$$w_{\theta,i,\xi_i,p_i} \leq 0 \qquad \forall\theta\in\Theta, \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b. \tag{F.4e}$$

Notice that, by using the dual of LP (8) instead of that of LP (7), we get the additional constraint $w \leq 0$. LP (F.4) has a polynomial number of variables and a polynomial number of Constraints (F.4c), (F.4d), and (F.4e). Hence, to solve the LP using the ellipsoid method we need a separation oracle for Constraints (F.4b), which are exponentially many. Instead of an exact separation oracle, we use an approximate separation oracle that employs Algorithm 2 with a suitably-defined $\delta > 0$. We use a binary search scheme to find a value $\rho^\star \in [0,1]$ such that the dual problem with objective $\rho^\star$ is unfeasible, while the dual with objective $\rho^\star + \beta$ is *approximately* feasible, for some $\beta \geq 0$ defined in the following. The algorithm requires $\log(\beta)$ steps and, at each step, it works by determining, for a given value $\rho_3$, whether there exists a feasible solution for feasibility problem Ⓕ.

At each iteration of the bisection algorithm, the feasibility problem Ⓕ is solved via the ellipsoid method. To do so, we need a separation oracle. We use an approximate separation oracle that returns a violated constraint that will be defined in the following. The bisection procedure terminates when it determines a value $\rho^\star$ such that on Ⓕ the ellipsoid method returns unfeasible for $\rho^\star$, while returning feasible for $\rho^\star + \beta$. Finally, the algorithm solves a modified primal LP (F.7) with only the subset of variables $y$ in $H^*$, where $H^*$ is the set of violated constraints returned by the ellipsoid method applied on the unfeasible problem with objective $\rho^*$. From this solution, we can use Lemma 11 to find a solution to LP (7) with the same value and Lemma 8 to find a signaling scheme with the same seller's revenue as the value of the solution.

*Approximate separation oracle* Our separation oracle works as follows. Given a point $(a, w, c)$ in the dual space, we check if a constraint relative to the variables $t$ and $\gamma$ of the primal is violated. Since there are a polynomial number of these constraints, it can be done in polynomial time. If it is the case, we return that constraint. Otherwise, our idea is to use Algorithm 2 with a $\delta$ defined in the following to find if a constraint relative to variable $y$ is violated. We apply Algorithm 2, once for each possible state $\theta \in \Theta$. In the following, we assume that $\theta$ is fixed and we denote $w_{\theta,i,\xi_i,p_i}$ as $w_{i,\xi_i,p_i}$. Algorithm 2 needs values such that $w_{i,\xi_i,p_i} \in [0,1]$ for all $i \in \mathcal{N}$, $\xi_i \in \Xi_i^q$, and $p_i \in P^b$. We show that we can restrict the inputs to $w_{i,\xi_i,p_i} \in [-1,0]$.[18] By constraint $w \leq 0$, all $w_{i,\xi_i,p_i}$ are non-positive. Otherwise, this constraint is violated and would have been returned in the first step. Moreover, given a vector $w$, we give as input to the oracle a vector $\bar{w}$ such that $\bar{w}_{i,\xi_i,p_i} = -1$ whenever $w_{i,\xi_i,p_i} < -1$.

If for at least one state $\theta$ a violated constraint is found by Algorithm 2, we return that constraint, otherwise we return feasible. Our separation oracle has two properties. When it returns a violated constraint, the constraint is actually violated. In particular, if $\sum_{i\in\mathcal{N}} \bar{w}_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p, \xi) > 0$, then $\bar{w}_{\theta,i,\xi_i,p_i} > -1$ for every $i \in \mathcal{N}$, implying $\bar{w}_{\theta,i,\xi_i,p_i} = w_{\theta,i,\xi_i,p_i}$ and $\sum_{i\in\mathcal{N}} w_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p\xi) = \sum_{i\in\mathcal{N}} \bar{w}_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p, \xi) > 0$, Additionally, when the separation oracle returns feasible, then all the constraints relative to the variables $y$ are violated by at most $\delta$. Suppose by contradiction that a constraint for a triple $(\theta, \xi, p)$ is violated by more than $\delta$. Then, the separation oracle would have found $\theta^* \in \Theta$, $\xi^* \in \Xi^q$, and $p^* \in \mathcal{P}^b$ such that: $\sum_{i\in\mathcal{N}} \bar{w}_{\theta^*,i,\xi_i^*,p_i^*} + \textsc{Rev}(\mathcal{V}^K, p^*, \xi^*) \geq \sum_{i\in\mathcal{N}} \bar{w}_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p, \xi) - \delta \geq \sum_{i\in\mathcal{N}} w_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p, \xi) - \delta > 0$, and, thus, it would have returned this violated constraint.

*Approximation guarantee* The algorithm finds a $\rho^*$ such that the problem is unfeasible, *i.e.*, the value of $\rho_1$ when the algorithm terminates, and a value smaller than or equal to $\rho^* + \beta$ such that the ellipsoid method returns feasible, *i.e.*, the value of $\rho_2$ when the algorithm terminates. For each possible distribution of the samples $\mathcal{V}^K$, let $OPT^{\mathcal{V}^K}$ be the optimal value of LP (F.4). As a first step, we bound the value of $OPT^{\mathcal{V}^K}$. In particular, we show that $OPT^{\mathcal{V}^K} \leq \rho^* + \beta + \delta$. Since, the bisection algorithm returns that Ⓕ is feasible with objective $\rho^* + \beta$, it finds a solution $(a, w, c)$ such that all the

[18] It is easy to see that summing 1 to all the elements of the vector $w$ does not change the problem.

constraints regarding variables $t$ and $\gamma$ of the primal are satisfied and the approximate separation oracle did not find a violated constraint for the constraints regarding variables $y$. We show that $(a, w, c)$ is a solution to the following LP.

$$\sum_{i\in\mathcal{N}}\sum_{\theta\in\Theta}\mu_\theta a_{i,\theta} \le \rho^* + \beta \tag{F.5a}$$

$$\sum_{i\in\mathcal{N}} -w_{\theta,i,\xi_i,p_i} \ge \textsc{Rev}(\mathcal{V}^k, p, \xi) - \delta \qquad \forall\theta\in\Theta, \forall\xi\in\Xi^q, \forall p\in\mathcal{P}^b \tag{F.5b}$$

$$\sum_{\theta\in\Theta}\xi_i(\theta) w_{\theta,i,\xi_i,p_i} + c_{i,\xi_i} \ge 0 \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b \tag{F.5c}$$

$$-c_{i,\xi_i} + \sum_{\theta\in\Theta}\xi_i(\theta) a_{i,\theta} \ge 0 \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q \tag{F.5d}$$

$$w_{\theta,i,\xi_i,p_i} \le 0 \qquad \forall\theta\in\Theta, \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b. \tag{F.5e}$$

All the Constraints (F.5a), (F.5c), (F.5d), and (F.5e) are satisfied since the separation oracle checks them explicitly, while we have shown that, when the separation oracle return feasible, it holds $\sum_{i\in\mathcal{N}} w_{\theta,i,\xi_i,p_i} + \textsc{Rev}(\mathcal{V}^K, p, \xi) \le \delta$ for all $\theta\in\Theta, \xi\in\Xi_i^q$, and $p\in\mathcal{P}^b$, implying that all the Constraints (F.5b) are satisfied.

Then, by strong duality the value of the following LP is at most $\rho^* + \beta$.

$$\max_{y,t,\gamma} \sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y_{\theta,\xi,p}\left(\textsc{Rev}(\mathcal{V}^k, p, \xi) - \delta\right) \quad \text{s.t.} \tag{F.6a}$$

$$\xi_i(\theta) t_{i,\xi_i,p_i} \ge \sum_{\xi'\in\Xi^q:\xi'_i=\xi_i}\sum_{p'\in\mathcal{P}^b:p'_i=p_i} y_{\theta,\xi,p} \qquad \forall\theta\in\Theta, \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b \tag{F.6b}$$

$$\sum_{p\in P^b} t_{r,\xi_i,p} = \gamma_{i,\xi_i} \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q \tag{F.6c}$$

$$\sum_{\xi_i\in\Xi_i^q}\gamma_{i,\xi_i}\xi_i(\theta) = \mu_\theta \qquad \forall i\in\mathcal{N}, \forall\theta\in\Theta. \tag{F.6d}$$

Notice that any solution to LP (8) is also a feasible solution to the previous modified problem. Since in any feasible solution $\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q,p\in\mathcal{P}^b} y_{\theta,\xi,p} = 1$ and LP (F.6) has value at most $\rho^* + \beta$, then $OPT^{\mathcal{V}^K} \le \rho^* + \beta + \delta$.

Let $H^*$ be the set of constraints regarding variables $y$ returned by the ellipsoid method run with objective $\rho^\star$. Since the ellipsoid method with the approximate separation oracle returns unfeasible, by strong duality LP (8) with only the variables $y$ relative to constraints in $H^*$ has value at least $\rho^*$. Moreover, since the ellipsoid method guarantees that $H^*$ has polynomial size, the LP can be solved in polynomial time. Hence, solving the following LP, *i.e.*, the primal LP (8) with only the variables $y$ in $H^*$, we can find a solution with value at least $\rho^*$.

$$\max_{\gamma,t,y} \sum_{\theta\in\Theta}\sum_{(\xi,p):(\theta,\xi,p)\in H^*} y_{\theta,\xi,p}\textsc{Rev}(\mathcal{V}^K, p, \xi) \quad \text{s.t.} \tag{F.7a}$$

$$\xi_i(\theta) t_{i,\xi_i,p_i} \ge \sum_{\xi',p':(\theta,\xi',p')\in H^*:\xi'_i=\xi_i,p'_i=p_i,} y_{\theta,\xi',p'} \tag{F.7b}$$

$$\forall\theta\in\Theta, \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q, \forall p_i\in P^b \tag{F.7c}$$

$$\sum_{p_i\in P^b} t_{i,\xi_i,p_i} = \gamma_{i,\xi_i} \qquad \forall i\in\mathcal{N}, \forall\xi_i\in\Xi_i^q \tag{F.7d}$$

$$\sum_{\xi_i\in\Xi_i^q}\gamma_{i,\xi_i}\xi_i(\theta) = \mu_\theta \qquad \forall i\in\mathcal{N}, \forall\theta\in\Theta \tag{F.7e}$$

$$y_{\theta,\xi,p} \ge 0 \qquad \forall\theta\in\Theta, \forall\xi\in\Xi^q, \forall p\in\mathcal{P}^b. \tag{F.7f}$$

To conclude the proof, we show that replacing the distributions $\mathcal{V}$ with $\mathcal{V}^K$, the expected revenue decreases by a small amount. Let $y^{APX,\mathcal{V}^K}$ be the solution returned by the algorithm with distribution $\mathcal{V}^K$. Moreover, let $y^{OPT,\mathcal{V}^K}$ be the optimal solution to LP (8) with distributions $\mathcal{V}^K$ and $y^{OPT,\mathcal{V}}$ the optimal solution with distributions $\mathcal{V}$. Finally, let $OPT$ be the value of the optimal private signaling scheme with distributions $\mathcal{V}$.

Let $\epsilon$ be a constant defined in the following and $K = 8\log(2|\Xi^q||\mathcal{P}^b|/\epsilon)/\epsilon^2$. By an Hoeffding bound, for every $\xi\in\Xi^q$ and $p\in\mathcal{P}^b$, with probability at least $1 - e^{-2K/(\epsilon/4)^2} = 1 - |\Xi^q||\mathcal{P}^b|\epsilon/4$,

$$|\textsc{Rev}(\mathcal{V}, p, \xi)| - |\textsc{Rev}(\mathcal{V}^K, p, \xi)| \le \epsilon/4.$$

By the union bound, it implies that with probability at least $1-\epsilon/2$, $|\textsc{Rev}(\mathcal{V}, p, \xi)| - \textsc{Rev}(\mathcal{V}^K, p, \xi)| \le \epsilon/2$ for every $\xi \in \Xi^q$ and $p \in \mathcal{P}^b$. Then, with probability $1-\epsilon/2$,

$$\begin{aligned}
&\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{APX,\mathcal{V}^K}_{\theta,\xi,p}\textsc{Rev}(\mathcal{V},p,\xi) \ge \\
&\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{APX,\mathcal{V}^K}_{\theta,p,\xi}\textsc{Rev}(\mathcal{V}^K,p,\xi) - \epsilon/4 \ge \\
&\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{OPT,\mathcal{V}^K}_{\theta,p,\xi}\textsc{Rev}(\mathcal{V}^K,p,\xi) - \epsilon/4 - \delta - \beta \ge \\
&\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{OPT,\mathcal{V}}_{\theta,p,\xi}\textsc{Rev}(\mathcal{V}^K,p,\xi) - \epsilon/4 - \delta - \beta \ge \\
&\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{OPT,\mathcal{V}}_{\theta,p,\xi}\textsc{Rev}\mathcal{V},p,\xi) - \epsilon/2 - \delta - \beta \ge \\
&OPT - \epsilon/2 - \delta - \beta - \eta
\end{aligned}$$

Hence, with probability $1-\epsilon/2$, the solution has value at least $OPT - \epsilon/2 - \delta - \beta - \eta$ and

$$\mathbb{E}_{\mathcal{V}^K}\left[\sum_{\theta\in\Theta}\sum_{\xi\in\Xi^q}\sum_{p\in\mathcal{P}^b} y^{APX,\mathcal{V}^K}_{\theta,\xi,p} Rev(\mathcal{V},p,\xi)\right] \ge OPT - \epsilon/2 - \epsilon/2 - \delta - \beta - \eta = OPT - \epsilon - \delta - \beta - \eta,$$

where the expectation is on the sampling procedure.

To conclude the proof, to have an approximation error $\lambda$, we can set $b$ and $q$ such that the approximation error in Lemma 9 is $\eta = \lambda/4$ and $\epsilon = \delta = \beta = \lambda/4$. Finally, given an approximate solution to LP (F.7), we can use Lemma 11 to find a solution to LP (7) with greater or equal value, and we can employ Lemma 8 to recover a signaling scheme with the same revenue. □

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