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A NEW ANALYTICAL METHOD FOR ECLIPSE ENTRY/EXIT POSITIONS DETERMINATION CONSIDERING A CONICAL SHADOW AND AN OBLATE EARTH SURFACE

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Satellite eclipse determination is one of the most important tasks to be analyzed in the preliminary design of a planetary space mission. Indeed, the duration of the eclipse is a driver for the sizing of the batteries that must be used when solar energy is not available. Many analytical and numerical algorithms based on sev-eral assumptions exist both for the simple determination of the satellite state (i.e., umbra/penumbra/sunlight) and for the definition of the entry/exit anomalies delimiting the umbra and penumbra regions. This paper wants to define a new analytical procedure for the determination of the entry/exit anomalies of a satellite inside a conical shadow generated by the Earth surface modelled as an ob-late ellipsoid of rotation. The methodology is tested for different orbit scenarios and is compared with state-of-the art algorithms to check both the effectiveness of the results and the computational performance.

INTRODUCTION

Eclipses have always represented one of the most studied celestial phenomena due to their spectacularity. The prediction of the occurrence of an eclipse is not crucial when the shadowing times are small and repeating with large periods like for the Sun eclipses. However, the analysis of the eclipse periods becomes relevant when satellites orbiting the Earth (or another celestial body) are considered because most of them are based on solar energy and the eclipse period is important for the sizing of the batteries that should replace the solar power during that phase.

There are many works dealing with the analysis of the eclipses in literature. The first reference can be found in Escobal, who defines an analytical procedure to determine the true anomalies corresponding to the entry point and exit point from an eclipse in the framework of classic Keplerian elements assuming the Earth's surface as a perfect sphere and a cylindrical shadow.[1] These assumptions simplify the modelling from a three dimensional problem to a planar one due to the spherical symmetry and it results in a quartic equation where the unknown is the cosine of the entry/exit true anomalies. Escobal suggests also a procedure to handle the same problem modelling the Earth's surface as an oblate ellipsoid of rotation, but this time the algorithm is numerical, iterative and it gives the possibility to define the penumbra region starting from user-made assumptions related to the amplitude of the region itself.

Vallado presents a numerical shadow analysis for both cylindrical and conical cases starting from the same assumptions and shadow function developed by Escobal and solving the quartic polyno-

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mial equation in the true anomaly with a Newton-Raphson numerical scheme, and so no further modelling is added to the solution of the problem.[2] Fixler introduces an analytical procedure to determine the umbra and penumbra regions assuming a conical shadow and a spherical Earth resulting in another transcendental equation to be solved numerically.[3] However, this method is based on projection of the Sun position vector onto the satellite orbital plane which modifies slightly the real geometry of the problem. Kraft solves the same problem arriving at the results given by Fixler by a different derivation.[4] Ortiz Longo et al. extended the methods developed by Fixler and Kraft to a perturbed environment taking into account the effect of J_2 orbital perturbation.[5] Montenbruck and Gill used a spherical Earth conical shadow model based on the angular separation and diameters of the Sun and the Earth.[6] A more complete analysis has been carried out by Dreher considering also the effect of the atmospheric refraction on the light rays, but the modelling is always based on an unperturbed environment and a spherical Earth.[7] Vokrouhlicky et al. proposed the concept of ”osculating spherical Earth” to account for the errors introduced by assuming a spherical model with respect to an oblate Earth's surface.[8] Adhya et al. are the first ones introducing an analytical procedure for the eclipse phenomenon modelling the Earth's surface as an oblate ellipsoid of rotation.[9] Their methodology can only be used to state if the satellite is in light, penumbra or umbra region and it is limited for low-Earth orbiting satellites. However, a numerical investigation by Vokrouhlicky et al. has shown that the oblateness of the Earth did not bring significant differences compared to a spherical Earth for the LEO satellites.[10] On the contrary, Woodburn showed that the cylindral assumption which neglects totally the penumbra region has important consequences for the precise numerical integration of orbit trajectories depending on the numerical scheme and the definition of the boundaries to be used for the occulted region.[11]

In this paper, an analytical general procedure to determine the entry and exit points from the penumbra and umbra region is derived modelling the Earth's surface as an oblate ellipsoid of rotation and assuming a conical shadow. The methodology can be applied for all the elliptical orbits because it is based on pure geometrical considerations. The algorithm needs as input variables the inertial position vector of the satellite and of the Sun at a given epoch and results in the exact inertial coordinates of the osculating orbit entry and exit points for the penumbra and umbra region. The main assumptions used for the derivation of the formulations are the following:

- The Earth is modelled as an oblate ellipsoid of rotation with semi-major axis and semi-minor axis equal respectively to the Earth equatorial and polar radii.
- All perturbing forces are neglected.
- The Sun is modelled as a perfect sphere.
- The refraction of the light rays caused by the Earth's atmosphere is neglected.
- The shadow generated by the Earth is a perfect cone, that is all the light rays connecting the Sun and the Earth intersect in the same point which is the vertex of the cone.

The last assumption is the only one introducing an approximation into the modelling of the problem. However, it can be considered a good assumption because the maximum deviation between all the intersection of the light rays with the shadow axis is relatively small with respect to the astronomical distances involved in the problem.

MODELLING

In this section all the basic mathematical transformations that will be used for the derivation of the final equation are described.

Rotation and Translation

The first mathematical transformation that is used for the derivation of the model is the rotation and translation of the Cartesian equation of the cone from its reference frame x, y, z to another generic reference system $\tilde{x}, \tilde{y}, \tilde{z}$. The generic Cartesian of a cone in a generic reference frame x, y, z is defined as follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{1}$$

where a, b represent the semi-major and semi-minor axis of the ellipse obtained cutting the indefinite conical surface with the plane $z = c$. This equation defines a right cone having the vertex in the origin of the reference frame and the circular base obtained cutting the cone with a plane normal to the z axis is an ellipse of semi-major axis and semi-minor axis equal to a and b, respectively. The first task is to express Eq. (1) another reference system x', y', z' whose origin is coincident with x, y, z reference frame and parallel to the final $\tilde{x}, \tilde{y}, \tilde{z}$ reference frame. Such transformation is a pure rotation that can be described mathematically by using a the rotation matrix $\mathbf{R}$ which aligns the reference frame x', y', z' to the reference system x, y, z:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{2}$$

The rotation matrix is obtained projecting the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ aligned with the x', y', z' reference frame onto the x, y, z reference system.

$$R = \begin{bmatrix} i_x & i_y & i_z \\ j_x & j_y & j_z \\ k_x & k_y & k_z \end{bmatrix} \tag{3}$$

Developing the matricial operations in Eq. (1), the transformation equations to obtain the rotation are obtained:

$$\begin{cases} x = R_{11}x' + R_{12}y' + R_{13}z' \\ y = R_{21}x' + R_{22}y' + R_{23}z' \\ z = R_{31}x' + R_{32}y' + R_{33}z' \end{cases} \tag{4}$$

It is possible to replacing Eq. (4) in place of the variables x, y, z in Eq. (1) to obtain the conical surface in terms of the x', y', z' reference frame. After some mathematical manipulation the following expression is derived:

$$\begin{aligned} &(b^2c^2R_{11}^2 + a^2c^2R_{21}^2 - a^2b^2R_{31}^2)x'^2 + (b^2c^2R_{12}^2 + a^2c^2R_{22}^2 - a^2b^2R_{32}^2)y'^2 + (b^2c^2R_{13}^2 + \\ &a^2c^2R_{23}^2 - a^2b^2R_{33}^2)z'^2 + 2(b^2c^2R_{11}R_{12} + a^2c^2R_{21}R_{22} - a^2b^2R_{31}R_{32})x'y' + 2(b^2c^2R_{11}R_{13} \\ &+ a^2c^2R_{21}R_{23} - a^2b^2R_{31}R_{33})x'z' + 2(b^2c^2R_{12}R_{13} + a^2c^2R_{22}R_{23} - a^2b^2R_{32}R_{33})y'z' = 0 \end{aligned}$$

Eq. () can be rewritten in an easier way introducing the following coefficients which are constant numbers once the rotation matrix and cone geometrical parameters are defined:

$$\begin{aligned}
&\tilde{A}x'^2 + \tilde{B}y'^2 + \tilde{C}z'^2 + 2\tilde{D}x'y' + 2\tilde{E}x'z' + 2\tilde{F}y'z' = 0 \\
&\tilde{A} = b^2c^2R_{11}^2 + a^2c^2R_{21}^2 - a^2b^2R_{31}^2 \\
&\tilde{B} = b^2c^2R_{12}^2 + a^2c^2R_{22}^2 - a^2b^2R_{32}^2 \\
&\tilde{C} = b^2c^2R_{13}^2 + a^2c^2R_{23}^2 - a^2b^2R_{33}^2 \\
&\tilde{D} = b^2c^2R_{11}R_{12} + a^2c^2R_{21}R_{22} - a^2b^2R_{31}R_{32} \\
&\tilde{E} = b^2c^2R_{11}R_{13} + a^2c^2R_{21}R_{23} - a^2b^2R_{31}R_{33} \\
&\tilde{F} = b^2c^2R_{12}R_{13} + a^2c^2R_{22}R_{23} - a^2b^2R_{32}R_{33}
\end{aligned} \tag{5}$$

After rotating the rotation, Eq. 1 is expressed in the reference frame x', y', z' which is parallel to final one $\tilde{x}, \tilde{y}, \tilde{z}$, but with a different origin. It is necessary to perform a translation transformation to align the origin of the two reference systems. The translation equations can be derived considering a general position vector defined in the final shifted reference frame, $\tilde{\mathbf{r}}$, and the same position vector defined in original frame, $\tilde{\mathbf{r}'}$. The transformation requires the position vector of the shifted reference frame with respect to the original one,$\tilde{\mathbf{r}}_{\mathbf{V}}$. In the general problem, the position vector of the vertex of the cone is defined with respect to the inertial reference system. Therefore, it is better to express the traslation equations considering the position vector of the origin of the x', y', z' with respect to the shifted one.

$$\tilde{\mathbf{r}} = \tilde{\mathbf{r}}_{\mathbf{V}} + \mathbf{r}' \tag{6}$$

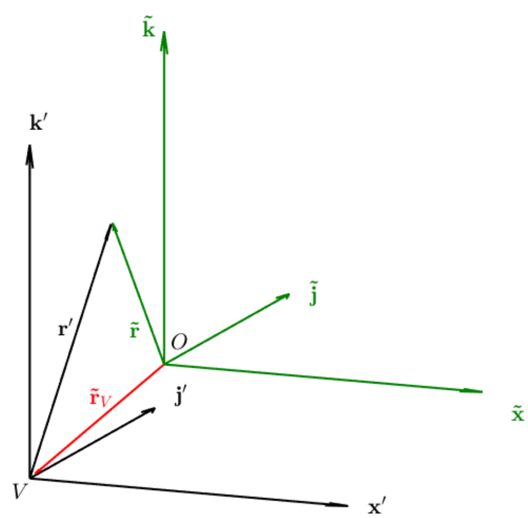

Figure 1. General traslation problem from one reference frame to another one.

A better representation of the traslation transformation is shown in Figure 1. Extending the vectorial equation and taking into account that the final aim is to express everything in the $\tilde{x}, \tilde{y}, \tilde{z}$ reference frame, the following relations are obtained:

$$\begin{cases} x' = \tilde{x} - \tilde{x}_v \\ y' = \tilde{y} - \tilde{y}_v \\ z' = \tilde{z} - \tilde{z}_v \end{cases} \tag{7}$$

Is is possible to get the final expression of the generic conical surface defined in Eq. (1) in a new different reference frame replacing the formulas defined in Eq. (7) inside Eq. (5):

$$\begin{aligned} &\tilde{A}\tilde{x}^2 + \tilde{B}\tilde{y}^2 + \tilde{C}\tilde{z}^2 + 2\tilde{D}\tilde{x}\tilde{y} + 2\tilde{E}\tilde{x}\tilde{z} + 2\tilde{F}\tilde{y}\tilde{z} - 2(\tilde{A}\tilde{x}_v + \tilde{D}\tilde{y}_v + \tilde{E}\tilde{z}_v)\tilde{x} \\ &- 2(\tilde{D}\tilde{x}_v + \tilde{B}\tilde{y}_v + \tilde{F}\tilde{z}_v)\tilde{y} - 2(\tilde{E}\tilde{x}_v + \tilde{F}\tilde{y}_v + \tilde{C}\tilde{z}_v)\tilde{z} + (\tilde{A}\tilde{x}_v^2 + \tilde{B}\tilde{y}_v^2 + \tilde{C}\tilde{z}_v^2 \\ &+ 2\tilde{D}\tilde{x}_v\tilde{y}_v + 2\tilde{E}\tilde{x}_v\tilde{z}_v + 2\tilde{F}\tilde{y}_v\tilde{z}_v) = 0 \end{aligned} \tag{8}$$

Again, it is possible to introduce new variables which are constant coefficients once the position vector of the conical surface vertex is known. This way Eq. (8) can be rewritten as follows:

$$\begin{aligned} &\tilde{A}\tilde{x}^2 + \tilde{B}\tilde{y}^2 + \tilde{C}\tilde{z}^2 + 2\tilde{D}\tilde{x}\tilde{y} + 2\tilde{E}\tilde{x}\tilde{z} + 2\tilde{F}\tilde{y}\tilde{z} + 2\tilde{H}\tilde{x} + 2\tilde{I}\tilde{y} + 2\tilde{J}\tilde{z} + \tilde{G} = 0 \\ &\tilde{G} = \tilde{A}\tilde{x}_v^2 + \tilde{B}\tilde{y}_v^2 + \tilde{C}\tilde{z}_v^2 + 2\tilde{D}\tilde{x}_v\tilde{y}_v + 2\tilde{E}\tilde{x}_v\tilde{z}_v + 2\tilde{F}\tilde{y}_v\tilde{z}_v \\ &\tilde{H} = -(\tilde{A}\tilde{x}_v + \tilde{D}\tilde{y}_v + \tilde{E}\tilde{z}_v) \\ &\tilde{I} = -(\tilde{D}\tilde{x}_v + \tilde{B}\tilde{y}_v + \tilde{F}\tilde{z}_v) \\ &\tilde{J} = -(\tilde{E}\tilde{x}_v + \tilde{F}\tilde{y}_v + \tilde{C}\tilde{z}_v) \end{aligned} \tag{9}$$

The expression obtained in Eq. (9) represents the analytical expression of a cylindrical conical surface characterised by a specific semi-major axis, semi-minor axis and height defined in a generic roto-translated reference system.

Intersection with the Orbit

The next topic to be discussed regards the intersection of the mathematical expression of the generic conical surface representing the eclipse shadow with an elliptical orbit. The intersection of the conical surfaces with the elliptical orbit requires that both the two mathematical expressions are defined with respect to the same reference frame. In the previous paragraph Eq. (9) represents the Cartesian equation of a cone that can be defined in whatever reference frame once the rotation matrix and position vector of the cone vertex are defined. The mathematical expression of an elliptical orbit can be retrieved starting from the canonic equation of an ellipse.

$$\frac{\bar{x}^2}{\tilde{a}^2} + \frac{\bar{y}^2}{\tilde{b}^2} = 1 \tag{10}$$

where $\tilde{a}$ and $\tilde{b}$ are the semi-major axis and semi-minor axis of the ellipse, respectively. The canonic equation of the ellipse is defined with respect a reference frame centered in the center of the ellipse. Therefore, Eq. (10) is not describing the mathematical expression of an orbit which requires the Earth or the celestial body to be the origin of the reference frame. The analytical expression of the elliptical orbit can be obtained applying a simple translation of the origin of the reference frame knowing that the distance between the the ellipse center and the celestial body is equal to the focal distance, $\bar{\mathbf{c}}$:

$$\tilde{\mathbf{c}} = [-\sqrt{\tilde{a}^2 - \tilde{b}^2}, 0, 0] \tag{11}$$

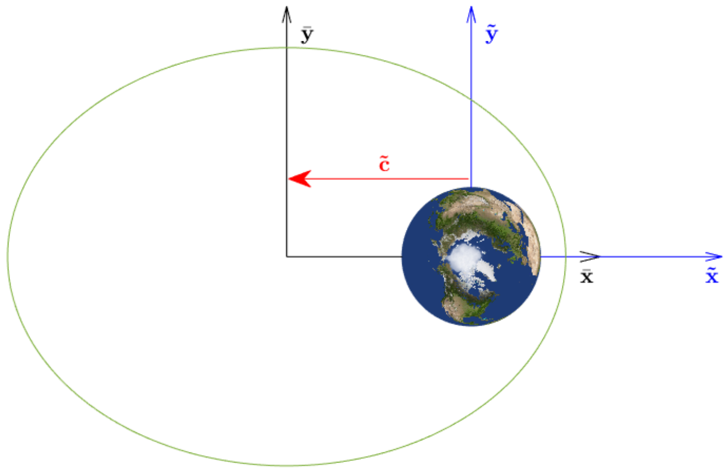

Figure 2. Traslation from ellipse centered to focal reference frame.

Looking at Figure 2, the translation relations between the two reference systems are obtained considering:

$$\begin{cases} \bar{x} = \tilde{x} - \tilde{c} \\ \bar{y} = \tilde{y} \\ \bar{z} = \tilde{z} \end{cases} \tag{12}$$

The analytical equation of the orbit is derived replacing Eq. (12) in Eq. (10):

$$\tilde{b}^2(\tilde{x}^2 + \tilde{c}^2 - 2\tilde{x}\tilde{c}) + \tilde{a}^2\tilde{y}^2 = \tilde{a}^2\tilde{b}^2 \tag{13}$$

The new reference frame obtained is centered in the celestial body and has the x-axis aligned with the apse lines direction and the z-axis normal to the ellipse plane. Therefore, Eq. (13) represents the analytical expression of the elliptical orbit defined in the perifocal reference system usually denoted as $\hat{\mathbf{e}}, \hat{\mathbf{p}}, \hat{\mathbf{h}}$. Therefore, it is convenient to choose to express also the conical surface in the perifocal reference frame so that the intersection between the two curves is possible. First of all, the ellipse is defined in the plane $\tilde{x}\tilde{y}$. This means that only the portion of the conical surface defined in that plane will eventually intersect the ellipse orbit. The expression of the curve associated to the conical surface in the orbital plane is obtained by imposing in Eq. (9) that $\tilde{z} = 0$.

$$\tilde{A}\tilde{x}^2 + \tilde{B}\tilde{y}^2 + 2\tilde{D}\tilde{x}\tilde{y} + 2\tilde{H}\tilde{x} + 2\tilde{I}\tilde{y} + \tilde{G} = 0 \tag{14}$$

At this point, the intersection with the elliptical orbit is carried out expressing in an explicit way the variable $\bar{y}$ in Eq. (13):

$$\tilde{y} = \pm \frac{\tilde{b}}{\tilde{a}} \sqrt{\tilde{b}^2 - \tilde{x}^2 + 2\tilde{x}\tilde{c}} \tag{15}$$

Replacing Eq. (15) in Eq. (14) leads to the following expression.

$$\tilde{A}\tilde{x}^2 + \tilde{B}\frac{\tilde{b}}{\tilde{a}}(\tilde{b}^2 - \tilde{x}^2 + 2\tilde{x}\tilde{c}) \pm 2\tilde{D}\tilde{x}\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{b}^2 - \tilde{x}^2 + 2\tilde{x}\tilde{c}} + 2\tilde{H}\tilde{x} \pm 2\tilde{I}\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{b}^2 - \tilde{x}^2 + 2\tilde{x}\tilde{c}} + \tilde{G} = 0 \tag{16}$$

It is convenient to move all the terms containing the square root at the right-hand side so that it is possible to put the square root as common factor.

$$\begin{aligned}
&\bar{A}\tilde{x}^2 + \bar{B}\tilde{x} + \bar{C} = \mp 2\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{b}^2 - \tilde{x}^2 + 2\tilde{x}\tilde{c}}\left(\tilde{D}\tilde{x} + \tilde{I}\right)\\
&\bar{A} = \left(\tilde{A} - \tilde{B}\frac{\tilde{b}^2}{\tilde{a}^2}\right)\\
&\bar{B} = 2\left(\tilde{B}\frac{\tilde{b}^2}{\tilde{a}^2}\tilde{c} + \tilde{H}\right)\\
&\bar{C} = \left(\tilde{G} + \tilde{B}\frac{\tilde{b}^2}{\tilde{a}^2}\right)
\end{aligned} \tag{17}$$

In order to solve for the variable x which represents the unknown of the problem and corresponds to the abscissa of the entry/exit point in the perifocal reference frame, both sides of Eq. (17) are squared. In this way, the ambiguity given by the two halves of the ellipse is removed. The final equation will be a quartic equation in the unknown x.

$$\begin{aligned}
&C_1\tilde{x}^4 + C_2\tilde{x}^3 + C_3\tilde{x}^2 + C_4\tilde{x} + C_5 = 0\\
&C_1 = \bar{A}^2 + 4\tilde{D}^2\frac{\tilde{b}^2}{\tilde{a}^2}\\
&C_2 = 2\left(\bar{A}\bar{B} + 4\tilde{c}\tilde{D}^2\frac{\tilde{b}^2}{\tilde{a}^2} + 4\tilde{D}\tilde{I}\frac{\tilde{b}^2}{\tilde{a}^2}\right)\\
&C_3 = \bar{B}^2 + 2\bar{A}\bar{C} - 4\tilde{D}^2\frac{\tilde{b}^4}{\tilde{a}^2} + 4\tilde{I}^2\frac{\tilde{b}^2}{\tilde{a}^2} + 16\tilde{c}\tilde{D}\tilde{I}\frac{\tilde{b}^2}{\tilde{a}^2}\\
&C_4 = 2\bar{B}\bar{C} - 8\tilde{D}\tilde{I}\frac{\tilde{b}^4}{\tilde{a}^2} + 8\tilde{c}\tilde{I}^2\frac{\tilde{b}^2}{\tilde{a}^2}\\
&C_5 = \bar{C}^2 - 4\tilde{I}^2\frac{\tilde{b}^4}{\tilde{a}^2}
\end{aligned} \tag{18}$$

Solutions Decision Making

In the previous paragraph Eq. (18) has been derived to compute the abscissa of the entry/exit points generated by a conical shadow. This quartic equation can be solved analytically using Cardan's procedure. Three possible scenarios can occur:

- 4 real solutions

- 2 real solutions and 2 complex solutions
- no real solutions and 4 complex solutions

In case of zero real solutions, the conical shadow is not intersecting the elliptical orbit, and so the satellite cannot be in eclipse. In case of two or four real solutions, there are four and eight possible points that are candidates for umbra and penumbra entry/exit points. Indeed, each real abscissa obtained from Eq. (18) can correspond only to a point on the upper-half or lower-half of the elliptical orbit. The ambiguity can be canceled computing for each abscissa the point of the upper-half or lower-half of the ellipse using Eq. (15) and replacing the points in Eq. (14). If the results is not exactly zero, but a small value under a certain treshold the point belongs to the conical surface and it represents a real candidate for being an entry/exit point for the eclipse. If the condition is not satisfied the point will not lie on the conical surface and it will be discarded. This represents a first criterion that should be applied to filter for the wrong points and select the real candidates. A second ambiguity is generated because the conical surface can intersect the orbit both in the shadow and sunlight side. This time the ambiguity can be solved considering that the shadow must always be on the opposite side of the Sun. This physical condition can be exploited to write a mathematical constraint for the selection of the real entry/exit eclipse points.

$$\mathbf{r}_{\oplus-\odot} \cdot \mathbf{r}_P < 0 \tag{19}$$

with $\mathbf{r}_{\oplus-\odot}$ representing the position vector of the Sun with respect to the Earth considered as origin of the Geocentric Equatorial Reference frame and $\mathbf{r}_P$ the generic position vector connecting the Earth with the candidate entry/exit points on the elliptical orbit. The entry/exit points will lie in the opposite plane with respect to the Sun so that the dot product of the two vectors should be negative. The last step for the derivation of the final solution is to decide which of the two points represent the entry and exit. The simplest way is to convert the two points into anomalies which represent the true anomalies and verifying which is the entrance and exit anomaly.

$$\nu_P = \arccos\left(\frac{x_P}{r_P}\right) \tag{20}$$

A picture of the ambiguities generated by the intersection of the conical surface with the elliptical orbit is presented in Figure 3.

ORBITAL SCENARIO

In the previous section, the analytical procedure to derive a quartic equation resulting in the intersections of a conical shadow with an elliptical orbit has been carried out. This section points out how to compute all the coefficients needed to solve for the intersection points.

Intersection Eclipse with the Oblate Earth

The input required for the algorithm are three:

- The position vector of the Sun, $\mathbf{r}_{\odot}$
- The position vector of the satellite, $\mathbf{r}_{\mathbf{sat}}$

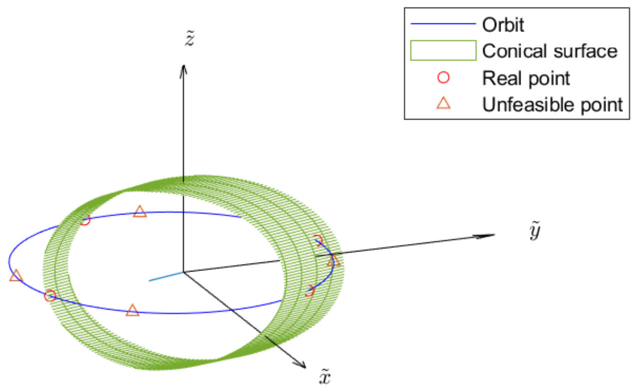

Figure 3. Intersection of the conical surface with the elliptical orbit.

- The geometrical parameters of the occulting body.

The first step is to define the ellipse representing the base of the conical shadow. This cone is delimited by the Earth surface so that the elliptical base is given by the intersection of the plane normal to the cone axis and the oblate ellipsoid associated to the Earth surface. It is natural to assume that the axis of the conical shadow is parallel to the position vector of the Sun, $\mathbf{r}_{\odot}$. The Cartesian equation of the plane normal to this direction is:

$$n_x x + n_y y + n_z z = 0 \tag{21}$$

where n_x, n_y, n_z represent the components of the unit vector associated to $\mathbf{r}_{\odot}$. The second equation that is required is the one modelling the Earth surface as an oblate ellipsoid of rotation. This equation requires the geometrical parameters of the occulting body to be known.

$$\frac{x^2 + y^2}{R_{eq}^2} + \frac{z^2}{R_{pol}^2} = 1 \tag{22}$$

with R_{eq} and R_{pol} defining the equatorial and polar radii, respectively. The semi-major axis and semi-minor axis of the ellipse resulting from the intersection of the plane normal to the conical axis and the oblate ellipsoide can be computed as follows [cita paper]:

$$\begin{aligned} a &= R_{eq}\sqrt{1 - \frac{d^2}{R_{eq}^2(1 - E_{\oplus}^2 n_z^2)}} \\ b &= R_{pol}\frac{\sqrt{1 - d^2/R_{eq}^2 - E_{\oplus}^2 n_z^2}}{1 - E_{\oplus}^2 n_z^2} \end{aligned} \tag{23}$$

where d is the distance between the plane and the origin of the reference system and it is equal to zero because this point belongs to the axis of the conical shadow, and $E_\oplus$ is the eccentricity of the occulting body. Eq.(23) provides the values of a and b to be used for the definition of the Cartesian equation of the conical surface described by Eq. (1).

Characterisation of Cone Vertex and Rotation Matrix

The last geometrical parameter that is required to define the conical surface is the cone height that is assumed to be coincident with the position vector of the cone vertex connecting the Earth center with the cone vertex, $\mathbf{r}_v$. The following assumption is considered to determine the vertex of the cone:
The ellipse resulting from the intersection of the plane normal to the conical axis and passing through the Sun center can be obtained from the ellipse representing the conical shadow basis delimited by the occulting body through an homothetic transformation.

Even if this assumption is true just in specific cases, it can be numerically proved that this assumption is good and is not affecting the accuracy of the results provided by the algorithm. Using the previous assumption, it is possible to compute the points where the sun rays generating the conical surface depart from the intersection ellipse between the Sun surface and the plane normal to the conical axis.

$$\begin{aligned} \mathbf{r}_{up_\odot} &= \mathbf{r}_\odot + R_\odot \hat{\mathbf{l}} \\ \mathbf{r}_{down_\odot} &= \mathbf{r}_\odot - R_\odot \hat{\mathbf{l}} \end{aligned} \tag{24}$$

Other two points on the Earth surface are required to define the equations of the sun rays generating the conical shadow. These points can be defined similarly using the following equations:

$$\begin{aligned} \mathbf{r}_{up_\oplus} &= R_\oplus \hat{\mathbf{l}} \\ \mathbf{r}_{down_\oplus} &= -R_\oplus \hat{\mathbf{l}} \end{aligned} \tag{25}$$

where $\hat{\mathbf{l}}$ identifies the apse line direction associated to the ellipse obtained from the intersection of the oblate ellipsoid and the plane normal to conical shadow and can be derived as follows:

$$\hat{\mathbf{l}} = \frac{1}{\sqrt{n_x^2 + n_y^2}}[-n_y, n_x, 0] \tag{26}$$

The vertex of the cone is computed considering the intersection of the two sun rays connecting the apsidal points of the two ellipses defined on the Sun surface and Earth surface.

$$\begin{aligned} \mathbf{m}_{up} &= \mathbf{r}_{up_\odot} - \mathbf{r}_{up_\oplus} \\ \mathbf{m}_{down} &= \mathbf{r}_{down_\odot} - \mathbf{r}_{down_\oplus} \\ t_{int} &= \frac{x_{down_\oplus} - x_{up_\oplus}}{m_{x_{up}} - m_{x_{down}}} \\ \mathbf{r}_v &= \mathbf{r}_{up_\oplus} + t_{int}\mathbf{m}_{up} \end{aligned} \tag{27}$$

The magnitude of the vertex position vector, $\mathbf{r}_v$ is the last geometrical quantity needed to define the Cartesian equation of the conical surface, c.

$$c = \|\mathbf{r}_v\| \tag{28}$$

There is not a unique way to define the vertex of the conical shadow depending on the way the couple of points delimiting the sun rays are defined. An alternative approach to compute the position vector of the cone vertex is to consider the sun rays as the tangent lines to the ellipse obtained from the intersection of the occulting body and the conical axis passing through the points on the Sun surface defined in Eq. (24). The equations to derive the geometrical parameters of the tangent lines are described hereafter. The first step is to intersect the general equation of a line with the analytical equation of the ellipse considered.

$$\begin{cases} y - y_p = m(x - x_p) \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases} \tag{29}$$

where x_p and y_p are the coordinates of the position vectors defined by Eq. (24) rotated in the intersection ellipse reference frame. Indeed, the canonical equation of the ellipse is valid only in a reference frame with the origin coincident with the center of the ellipse, the x-axis aligned with the apse line direction, the y-axis aligned with the ellipse semi-minor axis and the z-axis aligned with the axis of the cone. The rotation matrix moving from the geocentric equatorial inertial frame to the ellipse local reference frame can be obtained considering the expressions of the unit vectors defined in the inertial reference frame.

$$\begin{aligned} \hat{\mathbf{e}} &= \frac{1}{\sqrt{n_x^2 + n_y^2}}[-n_y, n_x, 0] \\ \hat{\mathbf{p}} &= \frac{1}{\sqrt{n_x^2 + n_y^2}}[n_x n_z, n_y n_z, -(n_x^2 + n_y^2)] \\ \hat{\mathbf{n}} &= \frac{\mathbf{r}_\odot}{|\mathbf{r}_\odot|} \end{aligned} \tag{30}$$

The rotation matrix moving from the geocentric inertial reference frame to the local ellipse reference frame is obtained as follows:

$$\mathbf{R}_{in \to loc} = \begin{bmatrix} e_x & e_y & e_z \\ p_x & p_y & p_z \\ n_x & n_y & n_z \end{bmatrix} \tag{31}$$

Replacing the general expression of a line in the canonical equation of the ellipse defines a second-order equation in the unknown x.

$$\begin{cases} y = mx + (y_p - m x_p) \\ (b^2 + a^2 m^2) x^2 - 2a^2 m (m x_p - y_p) x + a^2 \left[(m x_p - y_p)^2 - b^2 \right] = 0 \end{cases} \tag{32}$$

The second-order equation can admit only two equal solutions if the condition of tangency is applied. This means that the discriminant associated to the second-order equation is equal to 0. This condition defines a new second-order equation where the unknown is angular coefficient of the tangent lines, m.

$$a^2 b^2 (a^2 - x_p^2) m^2 + 2a^2 b^2 x_p y_p m + a^2 b^2 (b^2 - y_p^2) = 0 \tag{33}$$

The solution to Eq. (33) is:

$$m = \frac{-x_p y_p \pm \sqrt{a^2 y_p^2 + b^2 x_p^2 - a^2 b^2}}{a^2 - x_p^2} \tag{34}$$

At this point it is possible to replace Eq. (34) in Eq. (33) to derive the coordinates of the tangent points.

$$x_t = \frac{a^2 m(m x_p - y_p)}{b^2 + a^2 m^2} \qquad y_t = m x_t + (y_p - m x_p) \tag{35}$$

The last step is to rotate from the local ellipse reference frame to the inertial reference frame the tangent points so that Eq. (27) can be applied to get the position vector of the cone vertex.

$$\mathbf{r}_{up\oplus} = \mathbf{R}_{in\to loc}^T \begin{bmatrix} x_{up}^t \\ y_{up}^t \\ 0 \end{bmatrix} \qquad \mathbf{r}_{down\oplus} = \mathbf{R}_{in\to loc}^T \begin{bmatrix} x_{down}^t \\ y_{down}^t \\ 0 \end{bmatrix} \tag{36}$$

The different ways to compute the cone vertex position vertex arise from the assumptions which are modifying the real path and geometry of the sun rays which are not creating a perfect conical surface. However, a numerical analysis where different couple of points are used to generate the cone vertex shows that the maximum error in the definition of the magnitude of the cone vertex position vector is in the order of 1000 km corresponding to a relative error less than 1%. This small error justifies the assumption used for the derivation of the algorithm and why the results are not considerably affected by the selection of a specific way to compute the vertex of the cone.

The last variable to be defined to get all the coefficients needed to solve the quartic equation is the rotation matrix $\mathbf{R}$ to move from the conical surface reference frame to the perifocal reference frame. This matrix can be obtained combining two different rotation matrices. Indeed, the conical surface reference frame is parallel to the local ellipse reference frame but with a different origin of the reference system. Rotation matrices are not accounting for translation so that the rotation matrix defined in Eq. (31) allows to move from the inertial frame to the local ellipse or conical surface reference frame. It is well known in orbital mechanics how to compute the rotation matrix to move from the geocentric equatorial system to the perifocal reference frame using the orbital elements of a specific orbit.

$$\mathbf{R}_{in\to per} = \begin{bmatrix} \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega & -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega & \sin i\sin\Omega \\ \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega & -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega & -\sin i\cos\Omega \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \tag{37}$$

Combining the two rotation matrices it is possible to get the final rotation matrix which rotates the conical surface reference frame into the perifocal frame.

$$\mathbf{R} = \mathbf{R_{in\to per}}^T \mathbf{R_{in\to loc}} \tag{38}$$

In this way all the geometric quantities needed to define the coefficients of Eq. (5) are known and the analytical procedure to solve the quartic equation can be applied to solve for the x-coordinate of the umbra entry/exit points defined in the perifocal reference frame. The general solution to a quartic equation is provided in the Appendix.

RESULTS

In this section the proposed eclipse analytical algorithm is applied to different orbital scenarios for the determination of the entry/exit umbra and penumbra points. The same examples are solved also with state-of-art eclipse algorithms based on different assumptions to verify the correctness of the results.

Four different orbital scenarios are considered:

- Low-Earth slightly inclined orbit
- Low-Earth highly inclines orbit
- Geostationary orbit

All the simulations are carried out using a processor Intel® Core (TM) i7-9750H CPU @ 2.60 GHz. The orbital parameters assumed to identify the Sun position vector are summarized in Table 1:

Table 1. Sun state vector used for all the orbital scenarios

Variable	Numerical value
$r_\odot^x$	148979647.6842771 km
$r_\odot^y$	5289205.7023694 km
$r_\odot^z$	-1142.3033745 km
$v_\odot^x$	-0.561916547046 km/s
$v_\odot^y$	29.87780687 km/s
$v_\odot^z$	-0.002623168305249 km/s

Low-Earth slightly inclined orbit

The first orbital scenario deals with a satellite in a slightly inclined low-Earth orbit. The orbital parameters used for the solution of the exercise are reported in Table 2.

Table 2. Satellite initial orbital elements for scenario number 1

Variable	Numerical value
Semi-major axis	8000 km
Eccentricity	0.2
Inclination	5°
RAAN	60°
Pericenter anomaly	30°
True Anomaly	140°

The entry and exit points for satellite umbra are computed using different methods. The first one is of course the new proposed analytical approach. The second method is the classic Escobal cylindrical shadow model for the umbra determination. The third method is using Fixler approach

Table 3. Umbra entry and exit points for scenario number 1

	ν_{entry}	ν_{exit}
New Approach	8.2289°	136.8815°
Escobal Cylindrical	7.8676°	137.1116°
Fixler	8.2122°	136.8832 °
Escobal Conical	8.2210°	136.8821°

to define the same quantities. The fourth method is the iterative approach used to model the conical shadow taking into account for the Earth oblateness proposed always by Escobal. The results computed using the different methods are presented in Table 3.

The same considerations can be done for the entry and exit penumbra points. In this case the Escobal cylindrical model cannot be consider because the assumption of cylindrical shadow is not generating a penumbra region. The results presenting the different entry and exit anomalies for the penumbra region are shown in Table 4.

Table 4. Penumbra entry and exit points for scenario number 1

	ν_{entry}	ν_{exit}
New Approach	7.5336°	137.3448°
Fixler	7.5196°	137.3445 °
Escobal Conical	7.5253°	137.3454°

Low-Earth highly inclined orbit

The second orbital scenario considers a satellite in a highly inclined low-Earth orbit. The orbital parameters used for the solution of the exercise are reported in Table 5.

Table 5. Satellite initial orbital elements for scenario number 2

Variable	Numerical value
Semi-major axis	8000 km
Eccentricity	0.2
Inclination	56°
RAAN	60°
Pericenter anomaly	30°
True Anomaly	140°

The entry and exit points for satellite umbra computed using different methods are presented in Table 6.

The same considerations can be done for the entry and exit penumbra points. The results presenting the different entry and exit anomalies for the penumbra region are shown in Table 7.

Table 6. Umbra entry and exit points for scenario number 2

	ν_{entry}	ν_{exit}
New Approach	46.7973°	124.8715°
Escobal Cylindrical	44.5680°	125.5206°
Fixler	45.4431°	124.9756 °
Escobal Conical	46.7893°	124.8733°

Table 7. Penumbra entry and exit points for scenario number 2

	ν_{entry}	ν_{exit}
New Approach	45.0231°	125.9805°
Fixler	43.8634°	126.0686 °
Escobal Conical	45.0191°	125.9815°

Geosynchronous orbit

The third orbital scenario considers a satellite in a Geosynchronous orbit. The orbital parameters used for the solution of the exercise are reported in Table 8.

Table 8. Satellite initial orbital elements for scenario number 3

Variable	Numerical value
Semi-major axis	42164 km
Eccentricity	0
Inclination	0°
RAAN	60°
Pericenter anomaly	30°
True Anomaly	140°

The entry and exit points for satellite umbra computed using different methods are presented in Table 9.

Table 9. Umbra entry and exit points for scenario number 3

	ν_{entry}	ν_{exit}
New Approach	83.5979°	100.4686°
Escobal Cylindrical	83.3327°	100.7338°
Fixler	83.5967°	100.4698 °
Escobal Conical	83.5978°	100.4687°

The same considerations can be done for the entry and exit penumbra points. The results presenting the different entry and exit anomalies for the penumbra region are shown in Table 10.

Table 10. Penumbra entry and exit points for scenario number 3

	ν_{entry}	ν_{exit}
New Approach	83.0629°	101.0036°
Fixler	83.0639°	101.0026 °
Escobal Conical	83.0628°	101.0037°

CONCLUSIONS

In this paper a new analytical procedure for the determination of the entry and exit points of a generic satellite from the umbra and penumbra region is presented. The method is based on the modelling of a conical surface having as axis the position vector of the Sun with respect to the occulting body and an elliptical base given by the intersection of the plane normal to the cone axis and the occulting body surface modelled as an oblate ellipsoid of rotation. Such conical surface is rotated and traslated in the perifocal reference system so that it can be intersected with the analytical equation describing the orbit ellipse. The solutions are obtained analytically solving a quartic equation and discarding the wrong ones checking which points belong really to the conical surface and are on the opposite side with respect to the Sun. The proposed approach is applied to different relevant orbital scenarios showing that it is possible to get the same accuracy obtained using iterative and numerical methods.

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APPENDIX: QUARTIC EQUATION SOLUTIONS

In this appendix the general analytical solution to a quartic equation is presented. Such procedure is applied to Eq. (18) to get the solution representing the entry and exit points for the umbra and penumbra regions. The general form of the quartic equation is:

$$C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5 = 0 \tag{39}$$

The first step is to rewrite the quartic equation so that the first term has a unit coefficient:

$$\begin{aligned}
&x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta = 0 \\
&\alpha = C_2/C_1 \\
&\beta = C_3/C_1 \\
&\gamma = C_4/C_1 \\
&\delta = C_5/C_1
\end{aligned} \tag{40}$$

Define the variable h as:

$$h = -\frac{\alpha}{4} \tag{41}$$

Rewrite the quartic equation in a polynomial form that lack the x^3 term:

$$\begin{aligned}
&x^4 + Px^2 + Qx + R = 0 \\
&P = 6h^2 + 3\alpha h + \beta \\
&Q = 4h^3 + 3\alpha h^2 + 2\beta h + \gamma \\
&R = h^4 + \alpha h^3 + \beta h^2 + \gamma h + \delta
\end{aligned} \tag{42}$$

The solution to the equation depends on the Q parameter. Indeed, if $Q = 0$ the quartic equation becomes a biquadratic equation that can be solved by substitution of variables setting $y = x^2$. If the parameter Q is different from zero, the following cubic equation is solved:

$$\begin{aligned}
&t^3 + ut^2 + vt + w = 0 \\
&u = 2P \\
&v = P^2 - 4R \\
&w = -Q^2
\end{aligned} \tag{43}$$

The solution to the cubic equation is obtained applying Cardan's solution which write the cubic equation in a "depressed" form that lack the t^2 term and applying a substitution of variable.

$$\begin{aligned}
&Z^3 + mZ + n = 0 \\
&t = Z - \frac{u}{3} \\
&m = \frac{1}{3}\left(3v - u^2\right) \\
&n = \frac{1}{27}\left(2u^3 - 9uv + 27w\right)
\end{aligned} \tag{44}$$

A discriminant, Δ is introduced to take into account the number of real solutions and complex solutions associated with the cubic equation.

$$\Delta = \frac{m^3}{27} + \frac{n^2}{4} \tag{45}$$

The solution to the cubic equation are different according to the sign and value of the discriminant, Δ. If the discriminant is equal to 0 the cubic equation admits 3 real roots computed as follows:

$$\begin{aligned}
&Z_1 = 2\sqrt[3]{-\frac{n}{2}} \\
&Z_2 = \sqrt[3]{\frac{n}{2}} \\
&Z_3 = Z_2
\end{aligned} \tag{46}$$

If the discriminant is negative the cubic equation will also admit 3 real solution but that are computing using the following formulations:

$$
\begin{aligned}
E_0 &= 2\sqrt{-\frac{m}{3}} \\
\cos\phi &= -\frac{n}{2\sqrt{-\frac{m^3}{27}}} \\
\sin\phi &= \sqrt{1-\cos^2\phi} \\
\phi &= atan2(\sin\phi, \cos\phi) \\
Z_1 &= E_0 \cos\left(\frac{\phi}{3}\right) \\
Z_2 &= E_0 \cos\left(\frac{\phi}{3} + \frac{2}{3}\pi\right) \\
Z_3 &= E_0 \cos\left(\frac{\phi}{3} + \frac{4}{3}\pi\right)
\end{aligned} \tag{47}
$$

The last case is when the discriminant is positive. In this case the cubic equation admits one real solution and two complex solutions.

$$
\begin{aligned}
Z_1 &= \sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}} \\
Z_2 &= -\frac{1}{2}Z_1 + \frac{1}{2}\sqrt{-3}\left(\sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}}\right) \\
Z_3 &= -\frac{1}{2}Z_1 - \frac{1}{2}\sqrt{-3}\left(\sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}}\right)
\end{aligned} \tag{48}
$$

Once the depressed cubic equation is solved and the original roots t_1, t_2 and t_3 are obtained, the solutions of the quartic equation are computed as follows:

$$
\begin{aligned}
\xi &= \frac{1}{2}\left(P + max[t_1, t_2, t_3] - \frac{Q}{\sqrt{max[t_1, t_2, t_3]}}\right) \\
\zeta &= \frac{1}{2}\left(P + max[t_1, t_2, t_3] + \frac{Q}{\sqrt{max[t_1, t_2, t_3]}}\right) \\
k^2 &+ \sqrt{max[t_1, t_2, t_3]}k + \xi = 0 \\
k^2 &- \sqrt{max[t_1, t_2, t_3]}k + \zeta = 0 \\
x_i &= k_i + h
\end{aligned} \tag{49}
$$

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