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Calculus of Variations — *Elastic-brittle reinforcement of flexural structures*, by Francesco Maddalena, Danilo Percivale and Franco Tomarelli, communicated on 12 November 2021.

Dedicated to Claudio Baiocchi with sore heart for the loss of a teacher and friend

Abstract. — This note provides a variational description of the mechanical effects of flexural stiffening of a 2D plate glued to an elastic-brittle or an elastic-plastic reinforcement. The reinforcement is assumed to be linear elastic outside possible free plastic yield lines or free crack. Explicit Euler equations and a compliance identity are shown for the reinforcement of a 1D beam.

Key words: Calculus of variations, free discontinuity, variational inequality, adhesion, elastic clamped plate, crack, plastic yielding, flexural structures, coating, reinforcement

Mathematics Subject Classification: 49J45, 74K30, 74K35, 74R10

1. Introduction

The main purpose of the present paper relies in studying the mechanisms ruling stress transfer between material structures described by strongly different constitutive properties.

We study some mechanical problems related to the behavior of two thin interacting flexural structures through variational techniques. The different constitutive nature of the structures makes the problem nontrivial: while one structure remains indefinitely in the elastic range, the other one can develop singularities like plastic yelding or craks. We refer to the first one as *plate*, whose displacement field will be denoted by v_p, we refer to the second one as *reinforcement*, whose displacement field will be denoted by v_r, and we consider several functionals of the kind

$$\Lambda(K, v_r, v_p) = \mathscr{H}^1(K \cap \overline{\Omega}) + \sigma \int_K |[Dv_r]| \, d\mathscr{H}^1 + \int_{\Omega \setminus K} (\eta |D^2 v_r|^2 - f_r v_r) \, d\mathbf{x} + \mu \int_\Omega |v_r - v_p|^2 \, d\mathbf{x} + \int_\Omega (\gamma |D^2 v_p|^2 - f_p v_p) \, d\mathbf{x} \tag{1.1}$$

dependent on competing triplets (K, v_r, v_p). Here $\Omega \subset \mathbb{R}^2$ is given together with the loads f_r, f_p, the nonnegative constitutive parameters σ, η, μ, γ and a suitable Dirichlet boundary condition, shared by both displacements v_r and v_p. $\mathscr{H}^1$

denotes the 1-dimensional Hausdorff measure. Concerning competing triplets (K, v_r, v_p), the set K is closed, the function v_r is smooth outside K while v_p is smooth on Ω. Structural reinforcements are extensively employed in manufactured engineering systems, ranging from the traditional field of composite structures to the more recent applications in microelectronic devices and nano-reinforced composites ([39]). Indeed, external bonding of plates is a method of strengthening which involves an additional adhering reinforcement to a structural element ([37]). The adhesive is needed to transfer the stresses between the two elements. This technique is aimed to reduce deflection, hence confine crack and plastic yielding location in order to increase load carrying capacity (see e.g. [4, 20, 21, 26, 33, 40]), as well as predicting the behavior of paint coating layers ([38, 39]).

In a typical reinforcement system, represented by a brittle structure ([24]) bonded to a more compliant substrate, cracking and debonding instabilities (delamination) of the brittle element may appear under the action of external data that may be ruled by external loading, temperature change or even residual thermal stresses. The occurrence of plastic yielding, cracking and loss of adhesion (or delamination) constitute the main failure modes of reinforcements, so a great deal of work has been done over past decades to apply fracture mechanics in description of behavior and the influence of cracks nucleated on or near an interface between two dissimilar materials and a number of papers have been published on the problem (see, for instance, [22, 25, 26]). The crucial questions, when studying the possible failure of bonded structures, rely in understanding crack or yielding nucleation and crack propagation in presence of debonding or delamination of the constituent materials. This issue, precisely the role played by bonding layer in the formation of singularities, has not been yet investigated in spite of its influence in the mechanical behavior of such structural systems. Therefore, in our opinion, an appropriate description of such problems should incorporate all these *strongly nonlinear* effects in a mathematical theory which is able to detect qualitative and quantitative features of the underlying physics ([31, 32]).

To this aim we propose in the present paper a variational approach in which debonding and possible singular states arise as minimizers of suitable energy functionals.

In the previous works [27–30], we have studied the adhesion interaction of linear and nonlinear elastic structures by focusing on the influence of different constitutive choices for the adhesive material, while in [30] we have investigated the occurrence of global collapse and the interplay of cracking and debonding for a couple of plane elastic sheets.

Here reinforcements are modelled in the framework of Kirchhoff-Love theory, while the addition of a fracture energy term according to the Griffith theory allows to capture crack formation. This program is achieved by exploiting the techniques developed in the study of second order variational problems with free discontinuity ([7–12, 13, 14, 34, 35]).

Precisely we aim to describe the mechanical effects of bending a 2-dimensional stiff substrate fixed at the boundary, shortly called clamped plate from now on, which is glued to an elastic-plastic-brittle reinforcement. The reinforcement is assumed to be linear elastic outside possible free plastic yield lines or free 1D crack. The adhesive interaction between the structures is modelled through an energetic contribution whose density is the square of the modulus of difference of the displacement; concerning this assumption we recall that, if one assumes the q-power of the modulus of difference, with $0 < q < 1$, then the only stable configurations of the system are the completely detached or completely glued ones, even in the case of a flat plate as it was proved in Section 2.2 of [27].

The phenomenon is modeled as a variational problem which allows free discontinuity and free gradient discontinuity for the reinforcement. We assume that both configurations of the plate and its reinforcement are described as graphs referred to the coordinates in the horizontal plane, undergo vertical displacements and are subject to Dirichlet boundary conditions. We describe the details in these three main cases.

(1) *Hard-device reinforcement:* The structure consists in the gluing of a plate with a reinforcement; the plate undergoes a prescribed configuration (described by a given displacement) and consequently acts on the reinforcement through the adhesive layer; the reinforcement behaves as a piece-wise Kirchhoff-Love plate since it can develop lower-dimensional singularities of two kinds, plastic yielding (free gradient-discontinuity) and/or crack (free discontinuity).
(2) *Strengthening reinforcement:* The structure consists in the gluing of two objects, which are still labeled as plate and reinforcement; the plate behaves like a Kirchhoff-Love plate, whose unstressed flat reference configuration is horizontal, under the action of a given transverse (vertical) load f, while the plate displacement acts on the reinforcement through the adhesive layer; the reinforcement behaves as a piece-wise Kirchhoff-Love plate since it can develop lower-dimensional singularities of two kinds, plastic yielding and/or crack. We denote the admissible vertical displacement of the reinforcement by v_r and the admissible vertical displacement of the plate by v_p.
(3) *Elastic-plastic reinforcement:* As in case (2), but without crack and with a refinement of the yielding energy along the a priori unknown plastic yield lines.

As far as we know in structural mechanics literature there are few studies ([8]) of the interplay of plastic-yielding or fracture with bending bulk energy: here we aim to study this coupling in terms of integral functionals with free discontinuity and free gradient discontinuity by methods introduced in calculus of variations (see [2, 15–17, 19]).

In Section 3 we discuss the analogous one-dimensional case, say elastic-brittle reinforcement of flexural beams (Theorems 3.1, 3.2, 3.3). In Sections 2, 4 and 5 we examine details of the clamped plate reinforcement (hard-device and strengthening) showing the existence of energy minimizing solutions (Theorems 2.1, 2.2, 2.3). In the 1D case (beam reinforcement) we provide explicit Euler equations (Proposition 3.9), transmission conditions at free-discontinuity set and compliance identities fulfilled by minimizers (Proposition 3.10). Lack of convexity in these functionals may lead to non uniqueness of minimizers ([1, 5, 6]). However we show uniqueness and hence smoothness in case of small loads: see Theorem 3.4, Remark 3.5. A more detailed analysis of non uniqueness phenomenon is postponed in a forthcoming article.

In present paper we omit the consideration of a unilateral constraint forcing non-interpenetration the plate and its reinforcement: such constraint leads to technical difficulties and substantial problems for existence of strong solutions in the 2D case of the plate, since Euler equations are replaced by variational inequalities ([3]); this issue is postponed to a subsequent paper (see Remark 5.2). Here we only start the analysis of solutions for the 1D cases of the beam reinforced by a hard device, obtaining a variational inequality coupled with free discontinuity (Propositions 3.7 and 3.12), and of the strengthening reinforced beam, obtaining quasi-variational inequalities coupled with free discontinuity (Propositions 3.8 and 3.13).

2. Statement of the problem and main results

Assume

$$\Omega \subset\subset \Omega_p \subset\subset \mathbb{R}^2 \quad \text{are bounded and connected } C^2 \text{ open sets} \tag{2.1}$$

$$w \in C^2(\overline{\Omega_p}), \tag{2.2}$$

where Ω_p represents the horizontal reference configuration of the plate and Ω represents the horizontal reference configuration of the reinforcement and $w : \Omega_p \mapsto \mathbb{R}$ prescribes the Dirichlet datum of the clamped plate.

In the case of hard-device reinforcement $v : \Omega_p \mapsto \mathbb{R}$ denotes the generic admissible vertical displacement of the reinforcement, while the vertical displacement w of the plate is prescribed: since the reinforcing structure has to accomplish a prescribed configuration w, this amounts to deal with a pre-strained state of the material competing with other energetic terms. In the case of strengthening reinforcement, $v_r : \Omega_p \mapsto \mathbb{R}$ denotes the generic admissible vertical displacement of the reinforcement, while the generic admissible vertical displacement of the plate is denoted by v_p where $v_p : \Omega_p \mapsto \mathbb{R}$, where v_p is subject to a Dirichlet-type boundary condition, prescribed on $\Omega_p \backslash \Omega$. We face the three cases mentioned in the Introduction, by studying the minimization of suitable energy functionals:

- the energy E associated to a hard-device reinforcement, which is dependent on pairs (K, v) where K denotes the damaged region of the reinforcement and v its transversal displacement;
- the energy F associated to a strengthening reinforcement, which is dependent on triplets (K, v_r, v_p) where K still denotes the damaged region of the reinforcement (free discontinuity and free gradient-discontinuity) while v_r and v_p denote respectively the transversal displacements of reinforcement and plate;
- the energy G associated to an elastic-plastic reinforcement, which is dependent on triplets (K, v_r, v_p) as above, but v_r may undergo only free gradient-discontinuity on K.

We state some results related to minimization of these energies; their proofs are postponed in Sections 4 and 5. All functions under exam are real-valued.

THEOREM 2.1 (Hard-device reinforcement). *Assume* (2.1), (2.2) *and*

$$\eta > 0, \quad \mu > 0, \quad f \in L^4(\Omega), \tag{2.3}$$

then there exists a pair (Z, u) *minimizing*

$$E(K, v) = \mathscr{H}^1(K \cap \overline{\Omega}) + \int_{\Omega \setminus K} (\eta |D^2 v|^2 - fv)\, d\mathbf{x} + \mu \int_{\Omega} |v - w|^2\, d\mathbf{x} \tag{2.4}$$

over essential admissible pairs (K, v), *say pairs s.t.*

$$\begin{cases} K \text{ is the smallest closed subset of } \mathbb{R}^2 \text{ s.t.} \\ v \in C^2(\Omega_p \setminus K), \quad v \equiv w \text{ a.e. in } \Omega_p \setminus \overline{\Omega}. \end{cases} \tag{2.5}$$

Moreover, $Z \cap \Omega_p = Z \cap \overline{\Omega}$ *is an* $(\mathscr{H}^1, 1)$ *rectifiable set and* $E(Z, u) < +\infty$.

Here and in the sequel we denote by $\mathscr{H}^1$ the 1-dimensional Hausdorff measure.

If $(Z, u) \in \operatorname{argmin} E$, say (Z, u) is an optimal pair among the ones fulfilling (2.5), then Z represents the damaged zone of the reinforcement Ω, say the 1D set where either plastic yielding or fracture occur, and u is the related transverse displacement of the reinforcement.

THEOREM 2.2 (Strengthening reinforcement). *Assume* (2.1), (2.2), (2.3) *and*

$$f_r \in L^4(\Omega), \quad f_p \in L^2(\Omega), \quad \gamma > 0. \tag{2.6}$$

Then there exists a triplet $(Z, u_r, u_p) := (Z, U)$ *minimizing*

$$F(K, v_r, v_p) = F(K, V) := \mathscr{H}^1(K \cap \overline{\Omega}) + \eta \int_{\Omega \setminus K} (|D^2 v_r|^2 - f_r v_r)\, d\mathbf{x} + \mu \int_{\Omega} |v_r - v_p|^2\, d\mathbf{x} + \int_{\Omega} (\gamma |D^2 v_p|^2 - f_p v_p)\, d\mathbf{x} \tag{2.7}$$

over essential admissible triplets (K, v_r, v_p), *say triplets s.t.*

$$\begin{cases} v_p \in H^2(\Omega_p), \quad v_p \equiv w \text{ a.e. in } \Omega_p \setminus \overline{\Omega}, \\ K \text{ is the smallest closed subset of } \mathbb{R}^2 \text{ s.t.} \\ v_r \in C^2(\Omega_p \setminus K), \quad v_r \equiv w \text{ a.e. in } \Omega_p \setminus \overline{\Omega}. \end{cases} \tag{2.8}$$

Moreover, $Z \cap \Omega_p = Z \cap \overline{\Omega}$ *is an* $(\mathscr{H}^1, 1)$ *rectifiable set and* $F(Z, u_r, u_p) < +\infty$.

If $(Z, U) = (Z, u_r, u_p)$ is an optimal triplet among the ones fulfilling (2.8), say it is an essential admissible pair $(Z, U) \in \operatorname{argmin} F$, then Z represents the damaged zone of the reinforcement Ω, say the 1D set where either plastic yielding or fracture occur, and u_r, u_p respectively are the related displacement of the reinforcement and the plate.

Theorem 2.3 (Elastic-plastic reinforcement of flexural plate). *Assume* (2.1), (2.2), (2.3), $\sigma > 0$, $f_r \in L^s(\Omega)$, $s > 2$, *and* $f_p \in L^2(\Omega)$. *Then there is a triplet* $(Z, u_r, u_p) = (Z, U)$ *minimizing*

$$G(K, v_r, v_p) = G(K, V) := \mathscr{H}^1(K \cap \overline{\Omega}) + \sigma \int_{K \cap \overline{\Omega}} |[Dv_r]| \, d\mathscr{H}^1 + \int_{\Omega \setminus K} (\eta |D^2 v_r|^2 - f_r v_r) \, d\mathbf{x} + \mu \int_{\Omega} |v_r - v_p|^2 \, d\mathbf{x} + \int_{\Omega} (\gamma |D^2 v_p|^2 - f_p v_p) \, d\mathbf{x} \tag{2.9}$$

over elastic-plastic essential admissible triplets (K, v_r, v_p), *say triplets s.t.*

$$\begin{cases} K \text{ is the smallest closed subset of } \mathbb{R}^2 \text{ s.t.} \\ v_r \in C^0(\Omega_p) \cap C^2(\Omega_p \setminus K), \quad v_r = v_p \equiv w \text{ a.e. in } \Omega_p \setminus \overline{\Omega}. \end{cases} \tag{2.10}$$

Moreover, $Z \cap \Omega_p = Z \cap \overline{\Omega}$ *is an* $(\mathscr{H}^1, 1)$ *rectifiable set and* $G(Z, u_r, u_p) < +\infty$.

Theorem 2.3 describes a situation where crack is a priori excluded, while elastic deformation is present together with possible damage due to plastic yielding on the one-dimensional subset K: the free gradient-discontinuity set.

In the subsequent analysis we shall use the following notation for the various contribution to the total mechanical energy:

$$F_r(K, v_r) := \mathscr{H}^1(K \cap \overline{\Omega}) + \int_{\Omega \setminus K} (\eta |D^2 v_r|^2 - f_r v_r) \, d\mathbf{x}, \tag{2.11}$$

$$M(v_r - v_p) := \mu \int_{\Omega} |v_r - v_p|^2 \, d\mathbf{x}, \tag{2.12}$$

$$F_p(v_p) := \int_{\Omega} (\gamma |D^2 v_p|^2 - f_p v_p) \, d\mathbf{x}. \tag{2.13}$$

Hence

$$E(K, v) = F_r(K, v) + M(v - w), \quad f_r = f, \qquad \text{with domain (2.5)}, \tag{2.14}$$

$$F(K, v_r, v_p) = F_r(K, v_r) + M(v_r - v_p) + F_p(v_p), \qquad \text{with domain (2.8)}, \tag{2.15}$$

$$G(K, v_r, v_p) = F(K, v_r, v_p) + \sigma \textstyle\int_{K \cap \overline{\Omega}} |[Dv_r]| \, d\mathscr{H}^1, \qquad \text{with domain (2.10)} \tag{2.16}$$

where F_r represents the potential energy of the reinforcement under the Griffith assumption on the fracture energy, M represents the adhesive interaction energy (dependent on the slip $|v_r - v_p|$ between the plate and the reinforcement) and F_p represents the elastic energy of the Kirchhoff-Love plate under the action of a transverse dead load f.

REMARK 2.4. We emphasize that, when minimizing (2.7), the Dirichlet datum turns out to be forced on the plate ($v_p = w$ on $\partial\Omega$) since $v_p - w \in H_0^2(\Omega)$ and $H^2(\Omega_p) \subset C^0(\Omega_p)$, while the Dirichlet datum is prescribed by penalization on the reinforcement (through $v_r = w$ a.e. $\Omega_p \backslash \overline{\Omega}$). Hence the damage of the reinforcement may develop also at the boundary: if this is the case then $\mathscr{H}^1(K \cap \partial\Omega) > 0$.

In any case: $K \subset \overline{\Omega}$; K is the closure of the set where either v_r or ∇v_r is not continuous; $w \in C^2$ and $v_r = w$ in $\Omega_p \backslash \overline{\Omega}$.

REMARK 2.5. The notions of essential pair or triplet in (2.5), (2.8), (2.10) select those pairs or triplets which are cleansed of every artifact that does not affect the functional value and are good representatives in equivalence classes of admissible displacements. These classes allow highly irregular displacement function v for v_r for the reinforcement: see Remarks 2.3–2.5 and Lemmas 2.6, 2.7 in [17] for comparison with Definition 2.1 in [15] of admissible triplets in the context of image segmentation and/or image inpainting. Minimization among admissible triplets (as defined in [14]) would be equivalent to minimization among essential admissible triplets.

REMARK 2.6. The more general case where $|D^2 v_p|^2$ is replaced by $Q(D^2 v_p)$, with Q positive definite quadratic form, leads to claims similar to the ones we prove here (in Theorems 2.2 and 2.3) without any change in the proofs.

REMARK 2.7. The present paper deals with the Dirichlet boundary condition for both reinforcement and plate: explicitly the reinforcement acts on the whole plate Ω_p and sticks perfectly to it outside $\overline{\Omega}$.

Nevertheless, the study of Neumann boundary condition for the reinforcement, still keeping the Dirichlet condition w on the plate (this boundary conditions correspond to a structure where the reinforcement is present only on the proper subset Ω of the plate Ω_p), can be easily recovered by the present analysis with minor changes: by considering admissible displacements for the reinforcement defined only in the smaller domain reference set Ω and replacing (2.5), (2.8), (2.10) respectively by

$$K \text{ is the smallest closed subset of } \mathbb{R}^2 \text{ s.t. } v \in C^2(\Omega\backslash K); \tag{2.17}$$

$$v_p - w \in H_0^2(\Omega_p),\ K \text{ smallest closed subset of } \mathbb{R}^2 \text{ s.t. } v_r \in C^2(\Omega\backslash K); \tag{2.18}$$

$$v_p - w \in H_0^2(\Omega_p),\ K \text{ smallest closed subset of } \mathbb{R}^2 : v_r \in C^2(\Omega\backslash K) \cap C^0(\Omega). \tag{2.19}$$

All the claims in Theorem 2.1 and Theorem 2 still hold true under these different admissible classes of pairs and triplets. The only change to be made in the proofs amounts to refer to [11] instead of [14], to perform the analysis of partial regularity for weak minimizers.

3. One-dimensional analysis: reinforcement of flexural beams

We study the 1D case, namely the hard-device reinforcement and strengthening reinforcement of a clamped beam, in order to make explicit some properties of minimizers like compliance identity, Euler equations, issues related to uniqueness and possible addition of the unilateral constraint describing the non-interpenetration of beam and reinforcement. The displacement of the clamped beam is modeled by a function of one variable which is free in the interval $[-1,+1]$ while it must coincide with a given function outside $[-1,+1]$ to take into account of boundary conditions. We consider possibly different weights for energy dissipation when crack or crease do appear: the constants α and β introduced below.

We consider real valued functions defined on bounded intervals, and set

$$\Omega=(-1,1), \quad \Omega_p=(-2,2), \quad w \in C^2(-2,+2), \tag{3.1}$$

moreover, concerning the notation, $\dot{v}$ denotes the absolutely continuous part of the distributional derivative v' of v, $\ddot{v}$ denotes the absolutely continuous part of $(\dot{v})'$, S_v denotes the set of discontinuity points of v, $S_{\dot{v}}$ denotes the set of discontinuity points of $\dot{v}$ and v^-, v^+ denote respectively the left and right limit of v. Since we will consider admissible only piece-wise H^2 functions $v:(-2,2)\to\mathbb{R}$ fulfilling $v=w$ in $(-2,-1)\cup(1,2)$, we have $(S_v\cup S_{\dot{v}})\subset[-1,1]$ for them all. Here $H^2(a,b)$ denotes the usual Sobolev space of real-valued functions $v\in L^2(a,b)$ s.t. $v',v''\in L^2(a,b)$.

We emphasize that the beam may develop singularities also at both clamped endpoints ±1: namely, it may undergo crack discontinuity (if $S_v\cap\{\pm1\}$ is nonempty) or plastic-yield bending (if $S_{\dot{v}}\cap\{\pm1\}$ is nonempty).

After labeling by $\sharp$ the counting measure, we denote by

$$J(v)=\alpha\sharp(S_v)+\beta\sharp(S_{\dot{v}}\backslash S_v) \tag{3.2}$$

the whole energy associated to damage of the reinforcement: in this one-dimensional setting we allow different release energy for crack and crease, respectively α and β. In the one-dimensional setting the functionals E, F and G are replaced respectively by E_1, F_1 and G_1 defined below: we emphasize that for them all the strong and weak formulation of related free discontinuity problems coincide in the one-dimensional case, since finite energy entails that only a finite number of discontinuity points is allowed by finite energy, hence only piece-wise regular functions have finite energy.

The total energy for hard-device reinforcement of a clamped beam is given by functional E_1:

$$E_1(v)=J(v)+\int_{-1}^{1}(\eta|\ddot{v}|^2-fv)\,dx+\mu\int_{-1}^{1}|v-w|^2\,dx; \tag{3.3}$$

functional E_1 has to be minimized among the admissible functions v such that

$$v \in \mathbf{H}^2(-2,2) := \{v : (-2,2) \to \mathbb{R}, \text{ s.t. } v \text{ is piece-wise } H^2\}, \tag{3.4}$$

$$v = w \quad \text{a.e. } (-2,-1) \cup (1,2). \tag{3.5}$$

The total energy for strengthening reinforcement of a clamped beam is given by functional F_1:

$$F_1(v_r, v_p) = F_1(V) := J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 - f_r v_r)\, dx + \mu \int_{-1}^{1} |v_r - v_p|^2\, dx + \int_{-1}^{1} (\gamma|v_p''|^2 - f_p v_p)\, dx. \tag{3.6}$$

Functional F_1 has to be minimized among the admissible pairs V such that

$$V = (v_r, v_p) \in \mathbf{H}^2(-2,2) \times H^2(-2,2) \quad \text{with} \tag{3.7}$$

$$v_r = v_p = w \quad \text{in } (-2,-1) \cup (1,2). \tag{3.8}$$

The total energy for strengthening reinforcement of an elastic-plastic clamped beam is given by functional G_1:

$$G_1(v_r, v_p) = G_1(V) := \beta\sharp(S_{\dot{v}_r}) + \sigma \sum_{S_{\dot{v}_r}} |[\dot{v}_r]| + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 - f_r v_r)\, dx + \mu \int_{-1}^{1} |v_r - v_p|^2\, dx + \int_{-1}^{1} (\gamma|v_p''|^2 - f_p v_p)\, dx. \tag{3.9}$$

Functional G_1 has to be minimized among the admissible pairs V fulfilling

$$V = (v_r, v_p) \in (C^0([-2,2]) \cap \mathbf{H}^2(-2,2)) \times H^2(-2,2), \tag{3.10}$$

$$v_r = v_p = w \quad \text{in } (-2,-1) \cup (1,2). \tag{3.11}$$

Concerning respectively (3.4), (3.7), (3.10), we recall that in all cases the finiteness of total energy implies respectively $\sharp(S_u) < +\infty$, $\sharp(S_{u_r} \cup (S_{\dot{u}_r})) < +\infty$ and $S_{u_r} = \emptyset$ with $\sharp(S_{\dot{u}_r}) < +\infty$, hence u and u_r, are made by finitely many H^2 pieces.

THEOREM 3.1. *Assume* (3.1), (3.2), (3.3), $\eta > 0$, $\mu > 0$, $f \in L^2(-1,1)$ *and*

$$0 < \beta \leq \alpha \leq 2\beta, \tag{3.12}$$

then the functional E_1 *defined by* (3.3) *achieves a finite minimum over functions* v *fulfilling conditions* (3.4), (3.5).

Proof. After noticing that

$$\int_{-1}^{1} (\mu|v - w|^2 - fv)\,dx = \int_{-1}^{1} \mu(v - (w + f/(2\mu)))^2\,dx - \int_{-1}^{1} (fw + f^2/(4\mu))\,dx \tag{3.13}$$

where the last summand on right-hand side is a constant, we deduce that the functional E_1 is bounded from below since all terms are nonnegative, except such constant. Thus the claim follows by choosing $g = w + f/(2\mu) \in L^2(-1,1)$ in the result of [23]. □

Theorem 3.2. *Assume* (3.1)*,* (3.2)*,* (3.6)*,* (3.12)*,* $\eta > 0$*,* $\mu > 0$*,* $\gamma > 0$*,* $f_r, f_p \in L^2(-1,1)$*. Then the functional* F_1 *defined by* (3.6) *achieves a finite minimum over functions v fulfilling conditions* (3.7)*,* (3.8)*.*

Proof. The only novelty with respect to Theorem 3.1 consists in the addition of the functional $\int_{-1}^{1} (\gamma|v_p''|^2 - f_p v_p - f_r v_r)\,dx$ and adhesive interaction $\mu \int_{-1}^{1} |v_r - v_p|^2\,dx$ coupling v_r and v_p.

In case of functional F_1 the identity (3.13) reads as follows

$$\int_{-1}^{1} (\mu|v_r - v_p|^2 - f_r v_r)\,dx = \int_{-1}^{1} \mu(v_r - (v_p + f_r/(2\mu)))^2\,dx - \int_{-1}^{1} (f_r v_p + f_r^2/(4\mu))\,dx \tag{3.14}$$

where the last summand is not a priori bounded from below, unless we show an a priori bound on $\|v_p\|_{L^2(-1,1)}$, moreover we have to check that minimizing sequences are not made by pair sequences $((v_r)_n, (v_p)_n)$ balancing $\mu\|(v_p)_n\|^2_{L^2(-1,1)} \to +\infty$ together with $\int_{-1}^{1} (\gamma|v_p''|^2 - fv_p)\,dx \to -\infty$.

This is prevented by the subsequent estimate from below (3.16), obtained by use of Young inequality and the fact that $v_p - w \in H_0^2(-1,1)$, here C_P denotes the best Poincaré constant fulfilling

$$\|v\|^2_{L^2(-1,1)} \le C_P \|v''\|^2_{L^2(-1,1)} \quad \forall v \in H_0^2(-1,1), \tag{3.15}$$

and we denote shortly $\|\cdot\|^2$ in place of $\|\cdot\|^2_{L^2(-1,1)}$:

$$F_1(v_r, v_p) = J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \mu|v_r - v_p|^2 + \gamma|v_p''|^2)\,dx - \int_{-1}^{1} (f_r v_r + f_p v_p)\,dx \tag{3.16}$$

$$
\begin{aligned}
&= J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \mu|v_r - v_p|^2 + \gamma|v_p''|^2)\,dx \\
&\quad - \int_{-1}^{1} f_r(v_r - v_p)\,dx - \int_{-1}^{1} (f_r + f_p)(v_p - w)\,dx \\
&\quad - \int_{-1}^{1} (f_r + f_p)w\,dx \\
&\geq J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \mu|v_r - v_p|^2 + \gamma|v_p''|^2)\,dx \\
&\quad - \frac{1}{4\mu}\|f_r\|^2 - \mu\|v_r - v_p\|^2 - \sqrt{C_P}\|f_r + f_p\|\,\|(v_p - w)''\| \\
&\quad - \|f_r + f_p\|\,\|w\| \\
&\geq J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \gamma|v_p''|^2)\,dx \\
&\quad - \frac{1}{4\mu}\|f_r\|^2 - \sqrt{C_P}\|f_r + f_p\|\,\|(v_p - w)''\| - \|f_r + f_p\|\,\|w\| \\
&\geq J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \gamma|v_p''|^2)\,dx - \sqrt{C_P}\|f_r + f_p\|\,\|v_p''\| \\
&\quad - \frac{1}{4\mu}\|f_r\|^2 - \sqrt{C_P}\|f_r + f_p\|\,\|w''\| - \|f_r + f_p\|\,\|w\| \\
&\geq J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + \gamma|v_p''|^2)\,dx - \frac{\gamma}{2}\|v_p''\|^2 - \frac{C_P}{2\gamma}\|f_r + f_p\|^2 \\
&\quad - \frac{1}{4\mu}\|f_r\|^2 - \sqrt{C_P}\|f_r + f_p\|\,\|w''\| - \|f_r + f_p\|\,\|w\| \\
&\geq J(v_r) + \int_{-1}^{1} (\eta|\ddot{v}_r|^2 + (\gamma/2)|v_p''|^2)\,dx - C(\mu, \gamma, f_r, f_p, C_P).
\end{aligned}
$$

Then we can fix a minimizing sequence $((v_r)_h, (v_p)_h)$ for F_1, and get boundedness of $\|(v_p)_h\|_{H^2(-1,1)}$, thanks to $F_1(w, w) < +\infty$, (3.16) and the Poincaré inequality (3.15). There is $u_p \in L^2$ such that we can extract a subsequence, without relabeling, fulfilling $(v_p)_h \to u_p$ weakly in H^2 and strongly in L^2, with $\|(v_p)_h''\|_{L^2} \to \ell \in \mathbb{R}$. By (3.14) and (3.16) also $\|(v_r)_h - (v_p)_h\|_{L^2}$ and $\|(v_r)_h\|_{L^2}$ are bounded: by extracting again, without relabeling, $(v_r)_h \to u_r$ weakly in L^2. We write

$$
\begin{aligned}
F_1((v_r)_h, u_p) &= F_1((v_r)_h, (v_p)_h) + (F_1((v_r)_h, (v_p)_k) - F_1((v_r)_h, (v_p)_h)) \\
&\quad + (F_1((v_r)_h, u_p) - F_1((v_r)_h, (v_p)_k)) \\
&= F((v_r)_h, (v_p)_h) + A(h, k) + B(h, k).
\end{aligned} \tag{3.17}
$$

By lower semicontinuity of the functional $v \mapsto \mu \int_{-1}^{1} |v - v_p|^2 \, dx + \int_{-1}^{1} (\gamma |v''|^2 - f_p v) \, dx$ we get $\liminf_k B(h,k) \le 0$. Moreover, $\forall \varepsilon > 0 \; \exists h_\varepsilon$: for every $h, k > h_\varepsilon$

$$|A(h,k)| \le \eta(\|(v_p)_k''\|^2 - \|(v_p)_h''\|^2) + \|f_p\|(\|(v_p)_k - (v_p)_h\|) + \mu \int_{-1}^{1} |(v_p)_k - (v_p)_h| \, |2(v_r)_h - ((v_p)_k + (v_p)_h)| \, dx < \varepsilon.$$

By evaluating on both sides of (3.17) first $\liminf_k$, then $\liminf_h$, we obtain that also $((v_r)_h, u_p)$ is a minimizing sequence for the functional F_1 which is lower semicontinuous, or equivalently $(v_r)_h$ is a minimizing sequence for functional E_1 with datum u_p. Then (u_r, u_p) belongs to $\operatorname{argmin} F_1$. □

Theorem 3.3. *Assume* (3.1), (3.2), (3.9), $\beta \ge 0$, $\sigma > 0$, $\eta > 0$, $\mu > 0$, $\gamma > 0$, *and* f_r, f_p *belong to* $L^2(-1,1)$. *Then the functional* G_1 *defined by* (3.9) *achieves a finite minimum over pairs* (v_r, v_p) *fulfilling Dirichlet condition* (3.8) *and*

$$V = (v_r, v_p) \in (C^0(-2,2) \cap \mathbf{H}^2(-2,2)) \times H^2(-2,2). \tag{3.18}$$

Proof. Notice that $v_r \in C^0$ entails $S_{v_r} = \emptyset$. We set $I(v_r) = \beta \sharp(S_{\dot v_r}) + \sigma \sum_{S_{\dot v_r}} |[\dot v_r]|$. By arguing as like as in the derivation of (3.16) (the only difference consists in replacing $J(v_r)$ with $I(v_r)$), we get

$$G_1(v_r, v_p) \ge I(v_r) + \int_{-1}^{1} (\eta |\ddot v_r|^2 + (\gamma/2)|v_p''|^2) \, dx - C(\mu, \gamma, f_r, f_p, C_P). \tag{3.19}$$

If $\beta > 0$, we conclude by arguing as in the proof of Theorem 3.2 that a minimizing sequence has a subsequence converging to a minimum. When $\beta = 0$, after finding an optimal u_p again by the argument in the proof of Theorem 3.2, we exploit Theorem 2.1 of [35] to find the related optimal u_r.

We emphasize that the safe load condition assumed in [35] is unnecessary here thanks to adhesion term $\int_{-1}^{1} |v_r - v_p|^2 \, dx$, providing boundedness from below by (3.19). □

Next result shows that, provided the load and Dirichlet datum are suitably small, the strengthening reinforcement of the clamped beam ($\operatorname{argmin} F_1$) has a unique solution (u_r, u_p) where v_r has neither crack nor hinges, say $S_{u_r} \cup S_{\dot u_r} = \emptyset$.

Theorem 3.4. *In addition to assumptions of Theorem* 3.2 *we assume*

$$\frac{1}{4\mu} \|f_r\|^2_{L^2(-1,1)} + \frac{C_\Omega}{2\gamma} \|f_r + f_p\|^2_{L^2(-1,1)} + \gamma \|w''\|^2_{L^2(-1,1)} + \int_{-1}^{1} (f_r + f_p) w \, dx < \beta - M, \tag{3.20}$$

where $-\infty < M = \min\{F_1(v_r, v_p) : v_r - w \in H_0^2(-1,1), v_p - w \in H_0^2(-1,1)\} < +\infty$ *and* C_Ω *is the Poincaré constant in* (3.15).

Then there is a unique minimizer (u_r, u_p) *of* F_1 *over* $\mathbf{H}^2 \times H^2$ *and such minimizer fulfils* $S_{u_r} \cup S_{\dot{u}_r} = \emptyset$, *thus* $u_r - w \in H_0^2(-2,2)$.

PROOF. We denote shortly $\|\cdot\|^2$ in place of $\|\cdot\|^2_{L^2(-1,1)}$. First we note that

$$\begin{aligned}&\min\{F_1(v_r, v_p) : v_r - w \in H_0^2(-1,1), v_p - w \in H_0^2(-1,1)\}\\ &\quad\le F_1(w,w) = (\eta+\gamma)\|w''\|^2 - \int_{-1}^{1}(f_r+f_p)w\,dx;\end{aligned}$$

moreover, thanks to (3.16), F_1 is bounded from below and coercive hence its infimum M is attained and is a finite minimum.

Assume by contradiction that a minimizer (u_r, u_p) of F_1 on $\mathbf{H}^2 \times H^2$ has $S_{u_r} \cup S_{\dot{u}_r} \neq \emptyset$, we deduce $\alpha\sharp(S_{u_r}) + \beta\sharp(S_{u_r}\backslash S_{\dot{u}_r}) \ge \beta$, hence, exploiting Poincaré inequality (3.15) and assumption (3.20) we get

$$\begin{aligned}&\int_{-1}^{1} f_r u_r + \int_{-1}^{1} f_p u_p - \int_{-1}^{1}(f_r+f_p)w\\
&\quad= \int_{-1}^{1} f_r(u_r-u_p) + \int_{-1}^{1}(f_r+f_p)(u_p-w)\\
&\quad\le \frac{1}{4\mu}\|f_r\|^2 + \mu\|u_r-u_p\|^2 + \frac{C_\Omega}{2\gamma}\|f_r+f_p\|^2 + \frac{\gamma}{2C_\Omega}\|u_p-w\|^2\\
&\quad\le \frac{1}{4\mu}\|f_r\|^2 + \mu\|u_r-u_p\|^2 + \frac{C_\Omega}{2\gamma}\|f_r+f_p\|^2 + \frac{\gamma}{2}\|u_p''-w''\|^2\\
&\quad\le \frac{1}{4\mu}\|f_r\|^2 + \frac{C_\Omega}{2\gamma}\|f_r+f_p\|^2 + \gamma\|w''\|^2 + \eta\|\ddot{u}_r\|^2 + \mu\|u_r-u_p\|^2 + \gamma\|u_p''\|^2\\
&\quad< \beta - M - \int_{-1}^{1}(f_r+f_p)w + \eta\|\ddot{u}_r\|^2 + \mu\|u_r-u_p\|^2 + \gamma\|u_p''\|^2\\
&\quad\le \alpha\sharp(S_{u_r}) + \beta\sharp(S_{u_r}\backslash S_{u_r}) + \eta\|\ddot{u}_r\|^2 + \mu\|u_r-u_p\|^2 + \gamma\|u_p''\|^2\\
&\qquad - M - \int_{-1}^{1}(f_r+f_p)w,\end{aligned}$$

say, an inequality contradicting minimality of (u_r, u_p): $M < F_1(u_r, u_p)$.

Uniqueness of minimizer over $\mathbf{H}^2 \times H^2$ with Dirichlet datum w follows by uniqueness over $H^2 \times H^2$. □

REMARK 3.5. By analogous computations to the ones in the last proof, we obtain that the inequality

$$\frac{1}{4\mu}\|f\|^2_{L^2(-1,1)} + \int_{-1}^{1} fw\,dx < \beta - \widetilde{M}$$

entails uniqueness and H^2 regularity for minimizer of E_1 in $\mathbf{H}^2$ with Dirichlet boundary condition w, where

$$-\infty < \widetilde{M} = \min\{E_1(v) : v - w \in H_0^2(-1,1)\} \leq E_1(w) < +\infty.$$

We show the analysis of E_1 under the addition of the unilateral constraint

$$v \geq w \quad \text{on } [-1,+1]. \tag{3.21}$$

Concerning notation, from now on we set $v^+(x) = \lim_{t\to x_+} v(t)$, $v^-(x) = \lim_{t\to x_-} v(t)$.

Remark 3.6. Actually, the constraint (3.21) has to be understood as a pointwise everywhere weak inequality, since it refers to functions $v \in \mathbf{H}^2(-2,2)$: explicitly, $v(x) \geq w(x)$ at $x \in [-1,+1]\backslash S_v$; $v^+(x) \geq w(x)$ at $x \in S_v \cup \{-1\}$; $v^-(x) \geq w(x)$ at $x \in S_v \cup \{+1\}$.

Thus, the contact set $\{x \in [-2,2] : v^+(x) = w(x) \text{ or } v^-(x) = w(x)\}$ is a closed set for every v fulfilling (3.21); the complement in $(-1,1)$ of the contact set is an open set.

Actually, the inequality (3.21) prevents interpenetration and refers to a reinforcement placed above: this conventional choice is made here in order to have agreement with the usual formulation of variational inequalities ([3]).

Theorem 3.7 (Hard device with unilateral constraint). *Assume* (3.1), (3.2), (3.3), (3.12) $\eta > 0$, $\mu > 0$ *and* $f \in L^2(-1,1)$. *Then the functional* E_1 *achieves a finite minimum over pairs* (v_r, v_p) *fulfilling conditions* (3.4), (3.5) *together with the unilateral constraint* (3.21).

Proof. The proof can be achieved by exact repetition of the argument in the proof of Theorem 3.1 for the unconstrained case: both unilateral constraint $v \geq w$ on $[-1,+1]$ and Dirichlet condition $v = w$ a.e. on $(-2,-1) \cup (1,2)$ affect neither the compactness, nor the lower semicontinuity properties of E_1; moreover the a.e. convergence preserves the constraint in the limit of minimizing sequences. □

Theorem 3.8 (Reinforcement with unilateral constraint). *Assume* (3.1), (3.2), (3.6), (3.12), $\eta > 0$, $\mu > 0$, $\gamma > 0$, $f_r, f_p \in L^2(-1,1)$. *Then the functional* F_1 *achieves a finite minimum over pairs* (v_r, v_p) *fulfilling the conditions* (3.7) *and* (3.8) *together with the unilateral constraint* (*corresponding to a reinforcement placed above the plate*)

$$v_r \geq v_p \quad \textit{on } [-1,+1]. \tag{3.22}$$

Also the constraint (3.22) has to be understood as a pointwise everywhere weak inequality, in the sense of Remark 3.6, as like as (3.21) but here with v_p replacing w: thus, the admissible pairs belong to the convex set

$$\begin{aligned}\mathbf{K} := \{(v_r, v_p) \in \mathbf{H}^2(-2,2) \times H^2(-2,2) : v_r \geq v_p \text{ on } [-1,1],\\ v_r = v_p = w \text{ on } (-2,-1) \cup (1,2)\}.\end{aligned}$$

PROOF OF THEOREM 3.8. The proof can be achieved by exact repetition of the argument in the proof of Theorem 3.2 for the unconstrained case: both unilateral constraint $v_r \geq v_p$ a.e. on $(-1, +1)$ and Dirichlet condition $v_r = v_p = w$ a.e. on $(-2, -1) \cup (1, 2)$ affect neither the compactness, nor the lower semicontinuity properties of F_1; moreover a.e. convergence preserves the constraint in the limit of minimizing sequences. □

By performing all the admissible variations of minimizers for E_1 and F_1 without the unilateral constraint, we can deduce the necessary conditions for minimality listed below in Propositions 3.9 and 3.10.

PROPOSITION 3.9 (Euler equations for functional E_1). *Every* $u \in \operatorname{argmin} E_1$ *fulfils*

$$\eta u'''' + \mu(u - w) = f/2 \quad \text{in } (-1, 1)\backslash(S_u \cup S_{\dot{u}}), \tag{3.23}$$

$$\ddot{u}^+ = \ddot{u}^- = \dddot{u}^+ = \dddot{u}^- = 0 \quad \text{in } S_u\backslash\{\pm 1\}, \tag{3.24}$$

$$\ddot{u}^+ = \ddot{u}^- = [\dddot{u}] = 0 \quad \text{on } S_{\dot{u}}\backslash(S_u \cup \{\pm 1\}), \tag{3.25}$$

$$\ddot{u} \in H^2(-1, 1) \quad \text{and} \quad \eta(\ddot{u})'' + \mu(u - w) = f/2 \quad \text{on } \mathscr{D}'(-1, 1), \tag{3.26}$$

$$\begin{cases} \ddot{u}^+(-1) = \dddot{u}^+(-1) = 0 & \text{if } -1 \in S_u\backslash S_{\dot{u}}, \\ \ddot{u}^-(+1) = \dddot{u}^-(+1) = 0 & \text{if } +1 \in S_u\backslash S_{\dot{u}}, \end{cases} \tag{3.27}$$

$$\begin{cases} \ddot{u}^+(-1) = 0 & \text{if } -1 \in S_{\dot{u}}\backslash S_u, \\ \ddot{u}^-(+1) = 0 & \text{if } +1 \in S_{\dot{u}}\backslash S_u. \end{cases} \tag{3.28}$$

When $\alpha = \beta$ *the conditions* (3.24), (3.25), (3.27), (3.28) *altogether are improved as follows:*

$$\text{if } \alpha = \beta \quad \text{then} \quad \ddot{u}^+ = \ddot{u}^- = \dddot{u}^+ = \dddot{u}^- = 0 \quad \text{on } (S_u \cup S_{\dot{u}}). \tag{3.29}$$

(Euler equations for functional F_1) *Every* $(u_r, u_p) \in \operatorname{argmin} F_1$ *fulfils*

$$\eta u_r'''' + \mu(u_r - u_p) = f_r/2 \quad (-1, 1)\backslash(S_{u_r} \cup S_{\dot{u}_r}), \tag{3.30}$$

$$\gamma u_p'''' + \mu(u_p - u_r) = f_p/2 \quad \text{in } \mathscr{D}'(-1, 1), \tag{3.31}$$

$$\eta u_r'''' + \gamma u_p'''' = (f_r + f_p)/2 \quad (-1, 1)\backslash(S_{u_r} \cup S_{\dot{u}_r}), \tag{3.32}$$

$$\ddot{u}_r^+ = \ddot{u}_r^- = \dddot{u}_r^+ = \dddot{u}_r^- = 0 \quad S_{u_r}\backslash\{\pm 1\}, \tag{3.33}$$

$$\ddot{u}_r^+ = \ddot{u}_r^- = [\dddot{u}_r] = 0 \quad S_{\dot{u}_r}\backslash(S_{u_r} \cup \{\pm 1\}), \tag{3.34}$$

hence $\ddot{u}_r \in H^2(-1, 1)$ *and*

$$\eta(\ddot{u}_r)'' + \mu(u_r - u_p) = f_r/2 \quad \text{in } \mathscr{D}'(-1, 1), \tag{3.35}$$

$$\eta(\ddot{u}_r)'' + \gamma u_p'''' = (f_r + f_p)/2 \quad \text{in } \mathscr{D}'(-1, 1), \tag{3.36}$$

$$\begin{cases} \ddot{u}_r^+(-1) = \dddot{u}_r^+(-1) = 0 & \text{if } -1 \in S_{u_r} \backslash S_{\dot{u}_r}, \\ \ddot{u}_r^-(+1) = \dddot{u}_r^-(+1) = 0 & \text{if } +1 \in S_{u_r} \backslash S_{\dot{u}_r}, \end{cases} \tag{3.37}$$

$$\begin{cases} \ddot{u}_r^+(-1) = 0 & \text{if } -1 \in S_{\dot{u}_r} \backslash S_{u_r}, \\ \ddot{u}_r^-(+1) = 0 & \text{if } +1 \in S_{\dot{u}_r} \backslash S_{u_r}. \end{cases} \tag{3.38}$$

So, (3.35) *and* (3.36), *together give*

$$2\eta(\ddot{u}_r)'' + \mu(u_r - u_p) + \gamma u_p'''' = f_r + (1/2) f_p \quad \text{in } \mathscr{D}'(-1,1). \tag{3.39}$$

When $\alpha = \beta$ *the conditions* (3.33), (3.34), (3.37), (3.38) *altogether are improved as follows:*

$$\text{if } \alpha = \beta \quad \text{then} \quad \ddot{u}_r^+ = \ddot{u}_r^- = \dddot{u}_r^+ = \dddot{u}_r^- = 0 \quad \text{on } (S_{u_r} \cup S_{\dot{u}_r}). \tag{3.40}$$

Eventually we deduce the following compliance identities.

Proposition 3.10. Compliance identity for functional E_1:
Assume $w(-1) = w(1) = w'(-1) = w'(1) = 0$. *Then any* $u \in \operatorname{argmin} E_1$ *fulfils*

$$E_1(u) = J(u) + \mu \int_{-1}^{1} (w^2 - wu)\, dx - \frac{1}{2} \int_{-1}^{1} fu\, dx. \tag{3.41}$$

If boundary conditions are nonhomogeneous then the right-and side of compliance (3.41) *has to be added with the correction* $+\eta[\ddot{u}\dot{u} - \dddot{u}u]_{-1}^{+1}$, *where* $[z]_{-1}^{1} = z^-(1) - z^+(-1)$. *Notice that (due to* (3.27), (3.28), (3.29)) *some of the four terms in the correction may be null if one endpoint or the other belongs to* $S_{\dot{u}} \cup S_u$.

Compliance identity for functional F_1:
Assume $w(-1) = w(1) = w'(-1) = w'(1) = 0$, *then any* $(u_r, u_p) \in \operatorname{argmin} F_1$ *fulfils*

$$F_1(u_r, u_p) = J(u_r) - \mu \int_{-1}^{1} (u_r - u_p)^2\, dx - \frac{1}{2} \int_{-1}^{1} f_r u_r\, dx - \frac{1}{2} \int_{-1}^{1} f_p u_p\, dx. \tag{3.42}$$

If boundary conditions are nonhomogeneous then the right-and side of compliance (3.42) *has to be added with the correction* $+[\gamma(u_p'' w' - u_p''' w) + \eta(\ddot{u}_r \dot{u} - \dddot{u}_r u)]_{-1}^{+1}$, *where* $[z]_{-1}^{1} = z^-(1) - z^+(-1)$.

Remark 3.11. Due to (3.3) and (3.41), any $u \in \operatorname{argmin} E_1$ fulfils

$$-\frac{1}{2} \int_{-1}^{1} (2\mu(u - w) + f) u\, dx = \int_{-1}^{1} (\eta |\ddot{u}|^2 - fu)\, dx. \tag{3.43}$$

This equality has a simple mechanical interpretation: despite the presence of the jump term, if $\mathscr{L}(u)$ denotes the sum of the work done by the dead force f and by the adhesion force $2\mu(u - w)$ and $\mathscr{E}(u)$ denotes the elastic energy on the undamaged region (both evaluated on a minimizer u), then (3.43) reads as $\mathscr{L} = -2\mathscr{E}$, say the usual compliance identity which occurs in absence of discontinuities.

PROOF OF PROPOSITION 3.9 (Euler equations for E_1 and F_1). Let u be a minimizer of E_1 among $v \in \mathbf{H}^2 := \mathbf{H}^2(-2,2)$ fulfilling (3.4) and (3.5). For any $v \in \mathbf{H}^2$ we set $[\![v]\!] = v^+ - v^-$ where v^-, v^+ denote respectively the left and right values of v on S_v.

We introduce the localized version of functional E_1: given w, α, β, we set, for any v in $\mathbf{H}^2$ and any Borel set $A \subset [-1,1]$,

$$(3.44) \qquad E_1(v,A) = \int_A (\eta|\ddot{v}^2| + \mu|v-w|^2)\,dx + \alpha\sharp(S_v \cap A) + \beta\sharp((S_{\dot{v}} \backslash S_v) \cap A).$$

Step 1 (*Green's formula*) – Assume: $u \in \operatorname{argmin} E_1$.

Since $J(u) \le E_1(u) < +\infty$, the set $S_u \cup S_{\dot{u}}$ is finite and contained in $[-1,1]$; $u \in H^4(I)$ for every interval $I \subset (-2,2)\backslash\{S_u \cup S_{\dot{u}}\}$.

From now on we label $t_0 = -1$ and $t_{T+1} = 1$ and t_j, for $j = 1,\dots,\mathsf{T}$, the (possibly empty) finite ordered set $(S_u \cup S_{\dot{u}}) \cap (-1,+1)$. Then, integrating by parts, the next identity is achieved for every $\varphi \in \mathbf{H}^2$

$$(3.45) \qquad \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \ddot{u}\ddot{\varphi}\,dx = \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} u''\varphi''\,dx$$

$$= \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} u''''\varphi\,dx + \sum_{l=1}^{\mathsf{T}} \big((-\dddot{u}^-(t_{l+1})\varphi^-(t_{l+1}) + \dddot{u}^+(t_l)\varphi^+(t_l)) + (\ddot{u}^-(t_{l+1})\dot{\varphi}^-(t_{l+1}) - \ddot{u}^+(t_l)\dot{\varphi}^+(t_l))\big)$$

$$+ \big(-\dddot{u}^-(t_1)\varphi^-(t_1) + \ddot{u}^-(t_1)\dot{\varphi}^-(t_1) + \dddot{u}^+(t_T)\varphi^+(t_T) - \ddot{u}^+(t_T)\dot{\varphi}^+(t_T)\big) + \big(\dddot{u}^+(-1)\varphi^+(-1) - \ddot{u}^+(-1)\dot{\varphi}^+(-1) - \dddot{u}^-(1)\varphi^-(1) + \ddot{u}^-(1)\dot{\varphi}^-(1)\big).$$

Step 2 – At first we show that each minimizer u solves the fourth order elliptic equation (3.23) on the interior of $(-1,1)\backslash(S_u \cup S_{\dot{u}})$, by performing smooth variations. For every open set $A \subset\subset (-1,1)\backslash(S_u \cup S_{\dot{u}})$, for every $\varepsilon \in \mathbb{R}$ and for every $\varphi \in C_0^\infty(A)$ we have

$$0 \le E_1(u+\varepsilon\varphi, A) - E_1(u,A)$$
$$= 2\varepsilon\Big(\eta\int_A u''\varphi''\,dx + \mu\int_A (u-w)\varphi\,dx - \int_A \frac{f}{2}\varphi\,dx\Big) + o(\varepsilon)$$

where $o(\varepsilon)$ is an infinitesimal of higher order than ε. Hence

$$\eta\int_A u''\varphi''\,dx = \int_A (f/2 - \mu(u-w))\varphi\,dx \quad \forall\varphi \in C_0^\infty(A).$$

Then (3.23) follows integrating by parts with Green's formula (3.45).

Now we seek the Euler conditions at inner discontinuity points and at clamped endpoints.

Step 3 – We prove necessary conditions (3.24) for extremality on S_u and necessary conditions (3.27) for extremality at endpoints when they do belong to $S_u \backslash S_{\dot u}$.

Choose $\varphi \in \mathbf{H}^2 \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathsf{T}$, $\mathrm{spt}(\varphi) \subset A$, where A is a Borel subset of $[-1, 1]$ with $(S_{\dot u} \backslash S_u) \cap A = \emptyset$. Then for every $\varepsilon \in \mathbb{R}$ we have

$$(S_{u+\varepsilon\varphi} \cup S_{\dot u + \varepsilon\dot\varphi}) \cap A \subset S_u \cap A.$$

By (3.45) we have

$$\begin{aligned}
0 &\le E_1(u + \varepsilon\varphi, A) - E_1(u, A) \\
&= \alpha(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)) + \beta\sharp((S_{\dot\varphi} \backslash S_{u+\varepsilon\varphi}) \cap A) \\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \Big(\eta u''\varphi'' + \mu(u - w)\varphi - \frac{f}{2}\varphi\Big)\, dx\Big) + o(\varepsilon) \\
&= \alpha(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)) + \beta\sharp((S_{\dot\varphi} \backslash S_{u+\varepsilon\varphi}) \cap A) \\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \Big(\eta u''''\varphi + \mu(u - w)\varphi - \frac{f}{2}\varphi\Big)\, dx \\
&\quad + \dddot u^+(-1)\varphi^+(-1) - \ddot u^+(-1)\dot\varphi^+(-1) - \dddot u^-(1)\varphi^-(1) + \ddot u^-(1)\dot\varphi^-(1) \\
&\quad + \eta \sum_{(S_u \cap A) \backslash \{\pm 1\}} ([\![\dddot u\varphi]\!] - [\![\ddot u\dot\varphi]\!])\Big) + o(\varepsilon).
\end{aligned}$$

Up to a finite set of possible values of ε entailing cancellation of discontinuity, we have $S_{u+\varepsilon\varphi} \cap A = S_u \cap A$. Then by discarding such values we can choose arbitrarily small ε satisfying

$$\sharp((S_{\dot\varphi} \backslash S_{u+\varepsilon\varphi}) \cap A) = \sharp((S_{\dot\varphi} \backslash S_u) \cap A) = 0.$$

By taking into account (3.23) and the arbitrariness of the two traces of φ and $\dot\varphi$ on the two sides of points in S_u, for small ε, we can choose φ with $\varphi^\pm = 0$, and $\dot\varphi^+ = 0$ together with $\dot\varphi^-$ arbitrary, or viceversa to get $\ddot u^\pm = 0$ on $S_u \backslash \{\pm 1\}$.

Similarly, we obtain $\dddot u^\pm = 0$ on $S_u \backslash \{\pm 1\}$ by choosing $\dot\varphi^\pm = 0$, and $\varphi^+ = 0$ together with φ^- arbitrary or vice-versa. So (3.24) is proved.

If some clamped endpoint (-1 and/or $+1$) belong to S_u, then (3.27) is obtained as above, but taking into account that $\varphi \equiv 0$ outside $[-1, 1]$.

Step 4 – We prove the necessary condition (3.25) for extremality on the set $S_{\dot u} \backslash (S_u \cup \{\pm 1\})$:

$$\ddot u^\pm = 0 \quad \text{in } S_{\dot u} \backslash (S_u \cup \{\pm 1\}), \tag{3.46}$$

$$[\![\dddot u]\!] = 0 \quad \text{in } S_{\dot u} \backslash (S_u \cup \{\pm 1\}). \tag{3.47}$$

Let $\varphi \in \mathbf{H}^2 \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathsf{T}$, $\mathrm{spt}(\varphi) \subset A$, with A Borel subset of $(-1, 1)$ and $S_\varphi = \emptyset = (S_u \backslash S_{\dot u}) \cap A$. Then, up to a finite set of possible values of ε entailing cancelation of $\dot u$ discontinuity, we can choose ε arbitrarily small such that

$$(S_{u+\varepsilon\varphi} \cup S_{\dot u+\varepsilon\dot\varphi}) \cap A = S_{\dot u+\varepsilon\dot\varphi} \cap A = S_{\dot u}.$$

Moreover, by Green's formula (3.45):

$$\begin{aligned}
0 &\le E_1(u+\varepsilon\varphi, A) - E_1(u, A)\\
&\le \beta(\sharp(S_{\dot u+\varepsilon\dot\varphi} \cap A) - \sharp(S_{\dot u} \cap A))\\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \Big(\eta u''\varphi'' + \mu(u-w)\varphi - \frac{f}{2}\varphi\Big)\,dx\Big) + o(\varepsilon)\\
&= 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \Big(\eta u''''\varphi\,dx + \mu(u-w)\varphi - \frac{f}{2}\varphi\Big)\,dx\\
&\quad + \ddot u^+(-1)\varphi^+(-1) - \ddot u^+(-1)\dot\varphi^+(-1) - \ddot u^-(1)\varphi^-(1) + \ddot u^-(1)\dot\varphi^-(1)\\
&\quad + \eta \sum_{(S_{\dot u}\cap A)\backslash\{\pm 1\}} ([\![+\ddot u\varphi]\!] - [\![\ddot u\dot\varphi]\!])\Big) + o(\varepsilon).
\end{aligned}$$

By taking into account (3.23), for small ε and by the arbitrariness of φ and of the two traces of $\dot\varphi$ on the two sides of $S_{\dot u}$, we can choose φ with $\varphi^\pm = 0$, and arbitrary $\dot\varphi^+ = \dot\varphi^-$, to get (3.46).

On the other hand, by choosing $\dot\varphi^\pm = 0$ together with arbitrary φ and taking into account that $[\![\varphi]\!] = 0$, we obtain (3.47).

Then (3.25) follow from (3.46) and (3.47).

Step 5 – The analysis of minimizers at $(S_{\dot u} \backslash S_u) \cap \{\pm 1\}$ can be done exactly in the same way as in Step 5, but taking into account that $u = w$ and $\varphi = 0$ on $[-2, -1] \cup [1, 2]$, thus obtaining (3.27) and (3.28).

Step 6 – (3.26) is a straightforward consequence of (3.23)–(3.25).

Step 7 – Eventually, under the additional condition $\alpha = \beta$, we prove the refinement (3.29) of (3.24), (3.25), (3.27), (3.28) on $(S_{\dot u} \cup S_u)$ for every minimizer u.

We are left only to show that

(3.48) if $\alpha = \beta$ then: $1 \in S_{\dot u}\backslash S_u \Rightarrow \ddot u^-(1) = 0$; $-1 \in S_{\dot u}\backslash S_u \Rightarrow \ddot u^+(-1) = 0$.

Fix a Borel set A s.t. $A \subset\subset (-2, 2)$, $S_u \cap A = \emptyset \ne S_{\dot u} \cap A$.

Let $\varphi \in \mathbf{H}^2 \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathsf{T}$ and

$$S_{\dot u} \cap A = S_\varphi \cap A \quad \text{and} \quad S_u \cap A = S_{\dot\varphi} \cap A = \emptyset.$$

Then, for every value of $\varepsilon \in \mathbb{R}$ we have $S_{u+\varepsilon\varphi} \cap A = S_\varphi \cap A$ and

$$(S_{u+\varepsilon\varphi} \cup S_{(\dot u+\varepsilon\dot\varphi)}) \cap A = S_{\dot u} \cap A.$$

By (3.45), (3.23), (3.24) and (3.25) we have

$$\begin{aligned}
0 &\le E_1(u+\varepsilon\varphi, A) - E_1(u, A) \\
&= \alpha\sharp(S_{u+\varepsilon\varphi} \cap A) + \beta(\sharp((S_{(\dot u+\varepsilon\dot\varphi)} \backslash S_{u+\varepsilon\varphi}) \cap A) - \beta\sharp(S_{\dot u} \cap A)) \\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_l+1} \Big(\eta u''\varphi'' + \mu(u-w)\varphi - \frac{f}{2}\varphi\Big)\, dx\Big) + o(\varepsilon) \\
&= \alpha\sharp(S_\varphi \cap A) + \beta\sharp((S_{\dot u}\backslash S_\varphi) \cap A) - \beta\sharp(S_{\dot u} \cap A) \\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_l+1} \Big(\eta u''''\varphi + \mu(u-w)\varphi - \frac{f}{2}\varphi\Big)\, dx + \eta \sum_{(S_{\dot u}\cap A)\backslash\{\pm 1\}} (\llbracket \ddot u \varphi \rrbracket - \llbracket \ddot u \dot\varphi \rrbracket) \\
&\quad + \eta(\ddot u^+(-1)\varphi^+(-1) - \ddot u^+(-1)\dot\varphi^+(-1) - \ddot u^-(1)\varphi^-(1) + \ddot u^-(1)\dot\varphi^-(1))\Big) + o(\varepsilon) \\
&= \alpha\sharp(S_\varphi \cap A) - \beta\sharp(S_{\dot u} \cap A) + 2\eta\varepsilon \sum_{(S_{\dot u}\cap A)\backslash\{\pm 1\}} \llbracket \ddot u \varphi \rrbracket + o(\varepsilon).
\end{aligned}$$

Since $S_\varphi \cap A = S_{\dot u} \cap A$, when $\alpha > \beta$ then the inequality is fulfilled for ε small enough, hence we do not obtain further information (recall that the necessary condition for semicontinuity $\alpha \ge \beta$ is always assumed). On the other hand, when $\alpha = \beta$, we get

$$\begin{aligned}
0 &\le E_1(u+\varepsilon\varphi, A) - E_1(u, A) \\
&= 2\eta\varepsilon\Big(\sum_{S_{\dot u}\cap A} \llbracket \ddot u \varphi \rrbracket + \ddot u^+(-1)\varphi^+(-1) - \ddot u^-(1)\varphi^-(1)\Big) + o(\varepsilon).
\end{aligned}$$

So the coefficient of 2ε must vanish, and by the arbitrariness of the two traces of φ at points in $S_{\dot u} \cap A$, of the right trace at -1 and of the left trace at $+1$, taking into account that $\varphi \equiv 0$ outside $[-1, 1]$ we get (3.48).

Step 8 – We make explicit all the details for E_1 only, since the proof of Euler equations for F_1 is identical. In fact, $F_1(v_r, v_p) - E_1(v_r) = \int_{-1}^1 (\gamma|\ddot v_p|^2 - f v_p)$ is a classical integral functional: so the analysis of any minimizer $U = (u_r, u_p)$ of F_1 can be done by performing all the admissible variations separately for u_r and u_p. □

Proof of Proposition 3.10 (compliance identities). Assume $u \in \operatorname{argmin} E_1$ and label $t_0 = -1$ and $t_{T+1} = 1$ and t_j, for $j = 1, \ldots, \mathsf{T}$, the (possibly empty) finite ordered set $(S_u \cup S_{\dot u}) \cup (-1, +1)$.

Then, by (3.23)–(3.28), integrating by parts on the intervals $[t_j, t_{j+1}]$ we get

$$\eta \int_{-1}^{1} |\ddot{u}|^2\,dx = \eta \sum_{j=0}^{T} \int_{t_j}^{t_{j+1}} |\ddot{u}|^2\,dx$$
$$= -\eta \int_{-1}^{1} (\ddot{u})'\dot{u}\,dx + \eta[\ddot{u}\dot{u}]_{-1}^{+1}$$
$$= \eta \int_{-1}^{1} (\ddot{u})''u\,dx + \eta[\ddot{u}\dot{u} - \dddot{u}u]_{-1}^{+1}$$
$$= \mu \int_{-1}^{1} (w-u)u\,dx + \int_{-1}^{1} (f/2)u\,dx + \eta[\ddot{u}\dot{u} - \dddot{u}u]_{-1}^{+1},$$

here above and in the sequel, the notation $[z]_{-1}^{+1}$ stands for $z^-(1) - z^+(-1)$. Hence

$$E_1(u) = \eta \int_{-1}^{1} |\ddot{u}|^2\,dx - \int_{-1}^{1} fu\,dx + \mu \int_{-1}^{1} |u-w|^2\,dx + J(u)$$
$$= J(u) + \mu \int_{-1}^{1} ((w-u)u + (u-w)^2)\,dx + (-1+1/2)\int_{-1}^{1} fu\,dx$$
$$+ \eta(-\ddot{u}^+(-1)\dot{u}(-1) + \ddot{u}^-(1)\dot{u}(1) + \dddot{u}^+(-1)u(-1) - \dddot{u}^-(1)u(1)).$$

Now assume $(u_r, u_p) \in \operatorname{argmin} F_1$ and label t_j as above.

By taking into account (3.30)–(3.38) and performing integrations by parts, we get

$$\eta \int_{-1}^{1} |\ddot{u}_r|^2\,dx = \eta \int_{-1}^{1} (\ddot{u}_r)''u_r\,dx$$
$$= -\mu \int_{-1}^{1} (u_r - u_p)u_r\,dx + \int_{-1}^{1} (f_r/2)u_r\,dx + \eta[\ddot{u}_r\dot{u}_r - \dddot{u}_r u_r]_{-1}^{+1}.$$

Performing two integrations by parts and taking into account (3.31), we get

$$\gamma \int_{-1}^{1} |u_p''|^2\,dx = \gamma \int_{-1}^{1} u_p''''u_p\,dx + \gamma[u_p''u_p' - u_p'''u_p]_{-1}^{1}$$
$$= \mu \int_{-1}^{1} (u_r - u_p)u_p\,dx + \frac{1}{2}\int_{-1}^{1} f_p u_p\,dx + \gamma[u_p''u_p' - u_p'''u_p]_{-1}^{+1}.$$

Then for any $(u_r, u_p) \in \operatorname{argmin} F_1$ we obtain

$$F_1(u_r, u_p) = J(u_r) + (-1+1/2)\Big(\int_{-1}^{1} f_r u_r\,dx + \int_{-1}^{1} f_p u_p\,dx\Big) - \mu \int_{-1}^{1} (u_r - u_p)^2\,dx$$
$$+ [\eta(\ddot{u}_r\dot{u}_r - \dddot{u}_r u_r) + \gamma(u_p''w' - u_p'''w)]_{-1}^{1}. \qquad \square$$

Eventually we consider the additional constraint of non-interpenetration of beam and reinforcement, assuming that the reinforcement is above the beam.

Proposition 3.12 (Variational conditions for the minimizers of E_1 under unilateral constraint). *Every minimizer u of E_1 over the closed convex set*

$$K := \{v \in H^2(-2,2) : v \geq w \text{ on } [-1,1],\ v = w \text{ on } (-2,-1) \cup (1,2)\} \tag{3.49}$$

(*see Remark* 3.6 *about the pointwise everywhere meaning the unilateral constraint*) *fulfils the variational inequality*

$$\begin{cases} u \in K : \\ \displaystyle\int_{(-1,1)\backslash(S_u \cup S_{\dot u})} (\eta u'''' + \mu(u-w) - f/2)(u-v) \leq 0 \quad \forall v \in K, \end{cases} \tag{3.50}$$

together with the bilateral conditions at the free discontinuity and free-gradient discontinuity set where the contact does not play a role:

$$\ddot u^+ = \ddot u^- = 0 \quad \text{on } (S_u \backslash \{\pm 1\}) \cap \{u^+ > w\} \cap \{u^- > w\}, \tag{3.51}$$

$$\dddot u^+ = \dddot u^- = 0 \quad \text{on } (S_u \backslash \{\pm 1\}) \cap \{u^+ > w\} \cap \{u^- > w\}, \tag{3.52}$$

$$\ddot u^+ = \ddot u^- = [\dddot u] = 0 \quad \text{on } (S_{\dot u} \backslash (S_u \cup \{\pm 1\})) \cap \{u^+ > w\} \cap \{u^- > w\}, \tag{3.53}$$

$$\begin{aligned} &\ddot u \in H^2((-1,1)\backslash\{u^+ = w \text{ or } u^- = w\}) \quad \text{and} \\ &\eta(\ddot u)'' + \mu(u-w) = f/2 \quad \text{in } \mathscr{D}'((-1,1)\backslash\{u^+ = w \text{ or } u^- = w\}), \end{aligned} \tag{3.54}$$

$$\begin{cases} \ddot u^+(-1) = \dddot u^+(-1) = 0 & \text{if } -1 \notin S_{\dot u} \backslash S_u \text{ and } u^+(-1) > w(-1), \\ \ddot u^-(+1) = \dddot u^-(+1) = 0 & \text{if } +1 \notin S_{\dot u} \backslash S_u \text{ and } u^-(+1) > w(+1), \end{cases} \tag{3.55}$$

$$\begin{cases} \ddot u^+(-1) = 0 & \text{if } -1 \in S_{\dot u} \backslash S_u \text{ and } u^+(-1) > w(-1), \\ \ddot u^-(+1) = 0 & \text{if } +1 \in S_{\dot u} \backslash S_u \text{ and } u^-(+1) > w(+1), \end{cases} \tag{3.56}$$

jump condition $[\dddot u^+] = 0$ *in* (3.53) *can be improved when* $\alpha = \beta$, *hence*

$$\text{if } \alpha = \beta \text{ then } \begin{cases} \dddot u^+ = \dddot u^- = 0 \quad \text{on } ((S_u \cup S_{\dot u}) \backslash \{\pm 1\}) \cap \{u^+ > w^+\} \\ \qquad\qquad\qquad \cap \{u^- > w\}, \\ \dddot u^+(-1) = 0 \quad \text{if } -1 \in S_u \cup S_{\dot u} \text{ and } u^+(-1) > w(-1), \\ \dddot u^-(+1) = 0 \quad \text{if} +1 \in S_u \cup S_{\dot u} \text{ and } u^-(+1) > w(+1), \end{cases} \tag{3.57}$$

and in addition the unilateral conditions at the free discontinuity and free-gradient discontinuity sets of u where the contact with the obstacle plays a role:

$$\ddot u^+ \geq 0 \quad \text{on } ((S_u \cup S_{\dot u}) \backslash \{+1\}) \cap \{u^+ = w\}, \tag{3.58}$$

$$\ddot u^- \geq 0 \quad \text{on } ((S_u \cup S_{\dot u}) \backslash \{-1\}) \cap \{u^- = w\}. \tag{3.59}$$

No condition on $\dddot u^\pm$ is present on $S_u \cup S_{\dot u}$.

PROOF. The proof repeats the first 7 steps of Proposition 3.9 proof, but achieves less information since there is a strictly smaller set of admissible variations.

Step 1 is fully recovered thus, here we can exploit Green's formula (3.45).

We repeat Steps 2–7, by performing all the admissible variations of u which are of the kind $u + \varepsilon(v - u)$, with $\varepsilon \in [0, 1]$ and $v \in K$: for comparison, here $\varphi = v - u$.

As in Step 3 for the case of non constrained competitors, for $\varphi = v - u$ belonging to $\mathbf{H}^2 \cap C^2([t_l, t_{l+1}])$, $l = 0, \ldots, \mathsf{T}$, with $\mathrm{spt}(\varphi) \subset A$ Borel subset of $[-1, 1]$ and $(S_{\dot u} \backslash S_u) \cap A = \emptyset$ we still get $(S_{u+\varepsilon\varphi} \cup S_{\dot u+\varepsilon\dot\varphi}) \cap A \subset S_u \cap A$ and

$$
\begin{aligned}
(3.60)\quad 0 &\le E_1(u+\varepsilon\varphi, A) - E_1(u, A)\\
&= \alpha(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)) + \beta\sharp((S_{\dot\varphi} \backslash S_{u+\varepsilon\varphi}) \cap A)\\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (\eta u''\varphi'' + \mu(u-w)\varphi - (f/2)\varphi)\,dx\Big) + o(\varepsilon)\\
&= \alpha(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)) + \beta\sharp((S_{\dot\varphi} \backslash S_{u+\varepsilon\varphi}) \cap A)\\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (\eta u''''\varphi + \mu(u-w)\varphi - (f/2)\varphi)\,dx\\
&\quad + \ddot u^+(-1)\varphi^+(-1) - \ddot u^+(-1)\dot\varphi^+(-1) - \ddot u^-(1)\varphi^-(1) + \ddot u^-(1)\dot\varphi^-(1)\\
&\quad + \eta \sum_{(S_u \cap A)\backslash\{\pm 1\}} ([\![\ddot u\varphi]\!] - [\![\ddot u\dot\varphi]\!])\Big) + o(\varepsilon).
\end{aligned}
$$

As in Step 4, let $\varphi \in \mathbf{H}^2 \cap C^2([t_l, t_{l+1}])$, $l = 0, \ldots, \mathsf{T}$, $\mathrm{spt}(\varphi) \subset A$, with A Borel subset of $(-1, 1)$ and $S_\varphi = \emptyset = (S_u \backslash S_{\dot u}) \cap A$. Then, up to a finite set of possible values of ε entailing cancelation of $\dot u$ discontinuity, we can choose ε arbitrarily small such that $(S_{u+\varepsilon\varphi} \cup S_{\dot u+\varepsilon\dot\varphi}) \cap A = S_{\dot u+\varepsilon\dot\varphi} \cap A = S_{\dot u}$; thus, by Green's formula (3.45)

$$
\begin{aligned}
(3.61)\quad 0 &\le E_1(u+\varepsilon\varphi, A) - E_1(u, A)\\
&\le \beta(\sharp(S_{\dot u+\varepsilon\dot\varphi} \cap A) - \sharp(S_{\dot u} \cap A))\\
&\quad + 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (\eta u''\varphi'' + \mu(u-w)\varphi - (f/2)\varphi)\,dx\Big) + o(\varepsilon)\\
&= 2\varepsilon\Big(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (\eta u'''' + \mu(u-w) - (f/2))\varphi\,dx\\
&\quad + \ddot u^+(-1)\varphi^+(-1) - \ddot u^+(-1)\dot\varphi^+(-1) - \ddot u^-(1)\varphi^-(1) + \ddot u^-(1)\dot\varphi^-(1)\\
&\quad + \eta \sum_{(S_{\dot u} \cap A)\backslash\{\pm 1\}} ([\![+\ddot u\varphi]\!] - [\![\ddot u\dot\varphi]\!])\Big) + o(\varepsilon).
\end{aligned}
$$

In all cases now $\varphi = v - u$ with $v \in K$.

By all choices of open sets A and $v \in K$ fulfilling $\operatorname{spt}(v-u) \subset A \subset\subset (-1,1)\backslash (S_u \cup S_{\dot u})$ we get,

$$\int_{(-1,1)\backslash(S_u\cup S_{\dot u})} (\eta u'''' + \mu(u-w) - f/2)(v-u)\,dx$$
$$= \int_{(-1,1)\backslash(S_u\cup S_{\dot u})} (\eta u''(v-u)'' + \mu(u-w) - f/2)(v-u)\,dx \geq 0$$

say (3.50). Then, by inserting (3.50) in (3.60), (3.61), we single out the conditions at every point of singular set.

Outside the contact set $\{u^+ = w\} \cup \{u^- = w\}$ we can repeat the discussion made in the proof of Proposition 3.9, since $\varphi^{\pm} = (v-u)^{\pm}$ and $\dot\varphi^{\pm} = (\dot v - \dot u)^{\pm}$ are allowed to achieve both positive and negative values outside the contact set.

Up to a finite set of possible values of ε entailing cancellation of discontinuity, we have $S_{u+\varepsilon\varphi} \cap A = S_u \cap A$. Then by discarding such values we can choose arbitrarily small ε satisfying

$$\sharp((S_{\dot\varphi}\backslash S_{u+\varepsilon\varphi}) \cap A) = \sharp((S_{\dot\varphi}\backslash S_u) \cap A) = 0.$$

By taking into account (3.23) and the arbitrariness of the two traces of φ and $\dot\varphi$ on the two sides of points in S_u, for small ε, we can choose φ with $\varphi^{\pm} = 0$, and $\dot\varphi^+ = 0$ together with $\dot\varphi^-$ arbitrary, or viceversa to get $\ddot u^{\pm} = 0$ on $S_u \backslash \{\pm 1\}$.

Similarly, we obtain $\dddot u^{\pm} = 0$ on $S_u\backslash\{\pm 1\}$ by choosing $\dot\varphi^{\pm} = 0$, and $\varphi^+ = 0$ together with φ^- arbitrary or vice-versa. So (3.51) is proved.

If some clamped endpoint (-1 and/or $+1$) belong to S_u, then (3.55) is obtained as above, but taking into account that $\varphi \equiv 0$ outside $[-1,1]$.

Summarizing, we obtain (3.52), (3.54), (3.55), (3.56), hence (3.53) and (3.57), by the same argument of Steps 3–7.

On the contact set $\{u = w\}$ we can repeat again the discussion made in the proof of Proposition 3.9, but here the coefficient of 2ε in (3.61) must be only nonnegative, thus we get inequalities in place of equalities. Moreover, since $\dot\varphi^+ = (\dot v - \dot u)^+$ is allowed to achieve only positive values and $\dot\varphi^- = (\dot v - \dot u)^-$ is allowed to achieve only negative values, whereas left and right values have always opposite sign, we deduce (3.58), (3.59). On the other hand, on the contact set $\varphi^{\pm} = (v-u)^{\pm}$ is always null; therefore, we get no condition on every term whose multiplier is φ^+ or φ^-. □

PROPOSITION 3.13 (Variational conditions for the minimizers of F_1 under unilateral constraint). *Every minimizing pair* (u_r, u_p) *of* F_1 *over the convex set*

$$\mathbf{K} := \{(v_r, v_p) \in \mathbf{H}^2(-2,2) \times H^2(-2,2) : v_r \geq v_p \ on\ [-1,1],$$
$$v_r = v_p = w \ on\ (-2,-1) \cup (1,2)\}$$

fulfils the quasi-variational inequalities

$$(3.62)\quad \begin{cases} u_r \in \mathbf{H}^2(-2,2) : v_r \geq v_p \text{ on } [-1,1], \\ \quad v_r = v_p = w \text{ on } (-2,-1) \cup (1,2) \text{ and} \\ \displaystyle\int_{(-1,1)\setminus(S_{u_r}\cup S_{\dot{u}_r})} (\eta u_r'''' + \mu(u_r - u_p) - f_r/2)(u_r - v) \leq 0 \quad \forall v : (u_r, v) \in \mathbf{K}, \end{cases}$$

$$(3.63)\quad \begin{cases} u_p \in H^2(-2,2) : v_p \leq v_r \text{ on } [-1,1], \\ \quad v_r = v_p = w \text{ on } (-2,-1) \cup (1,2) \text{ and} \\ \displaystyle\int_{(-1,1)\setminus(S_{u_r}\cup S_{\dot{u}_r})} (\gamma u_p'''' + \mu(u_p - u_r) - f_p/2)(u_p - z) \leq 0 \quad \forall z : (z, u_p) \in \mathbf{K}, \end{cases}$$

together with the standard bilateral conditions (*say* (3.51)–(3.56) *with* u_r, u_p *replacing respectively* u, w) *at the free discontinuity and free-gradient discontinuity set where the contact does not play a role, and the unilateral conditions at the free discontinuity and free-gradient discontinuity set where the contact with the obstacle plays a role:*

$$(3.64)\qquad \ddot{u}_r^+ \geq 0 \quad \text{on } ((S_{u_r} \cup S_{\dot{u}_r})\setminus\{+1\}) \cap \{u_r^+ = u_p\},$$

$$(3.65)\qquad \ddot{u}_r^- \geq 0 \quad \text{on } ((S_{u_r} \cup S_{\dot{u}_r})\setminus\{-1\}) \cap \{u_r^- = u_p\}.$$

No condition on $\ddot{u}_r^\pm$ *is present on* $S_{u_r} \cup S_{\dot{u}_r}$.

PROOF. Repetition of the steps of the last proof provides the proof the claims about minimizers (u_r, u_p) of F_1 with unilateral implicit constraint, by performing all the admissible variations of u_r which are of the kind $u_r + \varepsilon(v - u_r)$, with $\varepsilon \in [0,1]$ and $(v, u_p) \in \mathbf{K}$ and $v_r + \varepsilon(z - v_r)$, with $\varepsilon \in [0,1]$ and $(u_r, z) \in \mathbf{K}$. □

REMARK 3.14. Euler equations for free minimizers of functional G_1, and quasi-variational inequalities for minimizers of functional G_1 under the non-interpenetration constraint $v_r \geq v_p$, can be deduced in pefect analogy to Propositions 3.9 and 3.13, by exploiting the Weierstrass-Erdmann corner conditions proved in [36].

4. HARD-DEVICE REINFORCEMENT OF FLEXURAL PLATE

In this section we deduce the existence statement in the case of hard-device reinforcement: minimization of functional E defined by (2.4).

PROOF OF THEOREM 2.1. After noticing that by

$$E(\emptyset, w) = \eta \|D^2 w\|^2_{L^2(\Omega)} - \int_\Omega f w \, d\mathbf{x} < +\infty,$$

the domain of E is not empty, and by

$$(4.1)\qquad \int_\Omega (\mu|v-w|^2 - fv)\,dx = \int_\Omega \mu(v-(w+f/(2\mu)))^2\,dx - \int_\Omega (fw+f^2/(4\mu))\,dx$$

where the last summand on the right-hand side is a constant, we have that the functional E is bounded from below since beside such constant all other terms are nonnegative.

The notion of essential admissible pairs, set by (2.5), selects ([18]) those pairs (K, v) which are cleansed of every spurious artifact that does not affect the functional value and are good representatives in equivalence classes of admissible pairs. This definition of admissible pair prevents diffused damage but allows to prove partial regularity of displacements v: free discontinuity (crack) and free gradient discontinuity (folds) are allowed in competing configurations of the structure.

Thus, the claims of present Theorem 2.1 follow from Theorem 2.3 in [11] and [14] about functional (2.2) defined therein, by setting $g = w + f/(2\mu)$, a datum which belongs to $L^4(\Omega)$ due to present assumptions. Precisely we can choose $\tilde{\Omega} = \Omega_p$, $\alpha = \beta = 1$; hence (2.3), (2.4), (2.5) and (2.20) of [14] are fulfilled thanks to the conditions (2.1), (2.3) assumed here. Moreover, $D^2 w \in L^\infty(A)$ for any open set s.t. $\Omega \subset\subset A \subset\subset \Omega_p$ and we have that the sets M, T_0, T_1 (as denoted in [14]) are empty, hence (2.6)–(2.11) of [14] hold true thanks to the assumption (2.2) made here. □

5. Strengthening reinforcement of flexural plate

In this section we deduce the existence statement in the case of strengthening reinforcement: minimization of functional F defined by (2.7).

To deal with the case of strengthening reinforcement we need a relaxed formulation of functional (2.7), as it is usual in the analysis of free discontinuity problems. We list standard notations (see [2, 11, 13, 17]): $B_\varrho(\mathbf{x})$ denotes the open ball $\{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}-\mathbf{x}| < \varrho\}$; $\mathscr{H}^1(A)$ and $|A|$ denote respectively, the 1-dimensional Hausdorff measure and the outer Lebesgue measure of a subset $A \subset \mathbb{R}^2$; for every Borel function $v : \Omega \to \mathbb{R}$ and $\mathbf{x} \in \Omega$, $z \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we set $z = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (notation for the *approximate limit* of v at $\mathbf{x}$) if, for every $g \in C^0(\overline{\mathbb{R}})$,

$$g(z) = \lim_{\varrho\to 0} |B_\varrho(\mathbf{0})|^{-1} \int_{B_\varrho(\mathbf{0})} g(v(\mathbf{x}+\boldsymbol{\xi}))\,d\boldsymbol{\xi};$$

the function $\tilde{v}(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ is called *representative* of v;

$$S_v = \left\{ \mathbf{x} \in \Omega : \nexists z \text{ such that } \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y}) = z \right\} \quad \text{is the } \textit{singular set} \text{ of } v.$$

A Borel function $v : \Omega \to \mathbb{R}$ is *approximately continuous at* $\mathbf{x} \in \Omega$ iff $v(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$.

As usual, Dv denotes the distributional gradient of v and $\nabla v(\mathbf{x})$ denotes the *approximate gradient* of v, say v is approximately differentiable at x if there exists a vector $\nabla v(\mathbf{x}) \in \mathbb{R}^2$ (the approximate gradient of v at $\mathbf{x}$) such that

$$\operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} \frac{|v(\mathbf{y}) - \tilde{v}(\mathbf{x}) - \nabla v(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})|}{|\mathbf{y} - \mathbf{x}|} = 0.$$

A function $u \in BV(\Omega)$ is approximately differentiable a.e., moreover for $\mathscr{H}^1$ almost every $\mathbf{x} \in S_u$ there exist $v(\mathbf{x}) \in \partial B_1$, $v_+(\mathbf{x}) \in \mathbb{R}$, $v_-(\mathbf{x}) \in \mathbb{R}$ with $v_+(\mathbf{x}) > v_-(\mathbf{x})$ such that

$$\lim_{\varrho\to 0} \varrho^{-n} \int_{\{\mathbf{y}\in B_\varrho;\, \mathbf{y}\cdot v(\mathbf{x})>0\}} |v(\mathbf{x}+\mathbf{y}) - v_+(\mathbf{x})|\, d\mathbf{y} = 0,$$
$$\lim_{\varrho\to 0} \varrho^{-n} \int_{\{\mathbf{y}\in B_\varrho;\, \mathbf{y}\cdot v(\mathbf{x})<0\}} |v(\mathbf{x}+\mathbf{y}) - v_-(\mathbf{x})|\, d\mathbf{y} = 0.$$

$SBV(\Omega)$ denotes the De Giorgi class of functions $v \in BV(\Omega)$ such that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, d\mathbf{x} + \int_{S_v} |v^+ - v^-|\, d\mathscr{H}^1.$$

We introduce:

$$SBV_{\mathrm{loc}}(\Omega) := \{v \in SBV(\Omega');\ \forall \Omega' \subset\subset \Omega\},$$

$$GSBV(\Omega) := \{v : \Omega \to \mathbb{R} \text{ Borel function};\ -k \vee v \wedge k \in SBV_{\mathrm{loc}}(\Omega)\ \forall k \in \mathbb{N}\}, \tag{5.1}$$

$$GSBV^2(\Omega) := \{v \in GSBV(\Omega),\ \nabla v \in (GSBV(\Omega))^2\}. \tag{5.2}$$

If $v \in GSBV(\Omega)$ then ∇v exists a.e., and for $v \in GSBV^2(\Omega)$ we set $\nabla^2 v = \nabla(\nabla v)$.

Eventually, we introduce the weak formulation $\mathscr{F}$ of functional F defined by (2.7):

$$\begin{cases} \mathscr{F}(v_r, v_p) = \mathscr{F}(V) := \mathscr{H}^1(S_{v_r}) + \eta \displaystyle\int_\Omega |\nabla^2 v_r|^2\, d\mathbf{x} + \mu \int_\Omega |v_r - v_p|^2\, d\mathbf{x} \\ \qquad\qquad + \displaystyle\int_\Omega (\gamma |D^2 v_p|^2 - f v_p)\, d\mathbf{x}, \\ \forall V = (v_r, v_p) \in \mathbb{X} := (GSBV^2(\Omega_p) \cap L^2(\Omega_p)) \times H^2(\Omega_p) \\ \quad \text{s.t. } v_r = v_p = w \text{ a.e. } \Omega_p \backslash \overline{\Omega}. \end{cases} \tag{5.3}$$

We emphasize that, since $v_p = w$ in $\Omega_p \backslash \overline{\Omega}$ and $w \in C^2(\Omega_p \backslash \overline{\Omega})$, we get $S_{v_p} \cup S_{\nabla v_p} = \emptyset$ and

$$\mathscr{F}(v_r, v_p) = \mathscr{F}_r(v_r) + M(v_r - v_p) + F_p(v_p) \tag{5.4}$$

where M, F_p and $\mathscr{F}_r$ are defined by (2.12), (2.13) and

$$\mathscr{F}_r(v_r) := \mathscr{H}^1(S_{v_r} \cup S_{\nabla v_r}) + \int_\Omega (\eta|\nabla^2 v_r|^2 - f_r v_r)\, d\mathbf{x}. \tag{5.5}$$

THEOREM 5.1. *Assume* (2.1), (2.2), (2.3) *and* (2.6). *Then the functional* $\mathscr{F}$ *achieves a finite minimum over* $\mathbb{X}$.

PROOF. First we notice that $\mathscr{F}$ has non empty domain: in fact (2.2) entails $S_w = S_{\nabla w} = \emptyset$ and

$$\mathscr{F}(w,w) < (\eta+\gamma)\|D^2 w\|^2_{L^2(\Omega)} - \int_\Omega (f_r + f_p) w\, d\mathbf{x} < +\infty. \tag{5.6}$$

We have the identity

$$\int_\Omega (\mu|v_r - v_p|^2 - f_r v)\, dx = \int_\Omega \mu(v_r - (v_p + f_r/(2\mu)))^2\, dx - \int_\Omega (f_r v_p + f_r^2/(4\mu))\, dx. \tag{5.7}$$

If K_Ω denotes the best Poincaré inequality constant in $H^2_0(\Omega)$, namely

$$\|v\|^2_{L^2(\Omega)} \le K_\Omega \|D^2 v\|^2_{L^2(\Omega)} \quad \forall v \in H^2_0(\Omega), \tag{5.8}$$

and, arguing as like as in (3.16) we get, for every $V = (v_r, v_p) \in \mathbb{X}$

$$F(v_r, v_p) \ge J(v_r) + \int_\Omega (\eta|\nabla^2 v_r|^2 + (\gamma/2)|D^2 v_p|^2)\, dx - C(\mu, \gamma, f_r, f_p, K_\Omega). \tag{5.9}$$

Then the functional F is bounded from below on its domain. Hence we can select a minimizing sequence $V_h = ((v_r)_h, (v_p)_h)$ for $\mathscr{F}$: $\lim_h \mathscr{F}(V_h) = \inf \mathscr{F} \in \mathbb{R}$.

Thanks to (5.6), we may suppose that

$$c \le \mathscr{F}(V_h) \le C := \mathscr{F}(w,w) < +\infty. \tag{5.10}$$

Summarizing $F_p((v_p)_h) \le C$, $F_r((v_r)_h) \le C$ and $(v_p)_h$ is bounded in $H^2(\Omega)$. Moreover, there is $u_p \in H^2(\Omega_p)$ such that, up to subsequences and without relabelling, $(v_p)_h$ is converging to u_p weakly in $H^2(\Omega)$ and strongly in $L^2(\Omega)$, and $D^2(v_p)_h \to D^2 u_p$ weakly in L^2. Hence $F(u_p) \le \liminf_h F((v_p)_h)$ and $M((v_r)_h - v_p) \le \liminf_h M((v_r)_h - (v_p)_h)$.

By using any fixed $(v_p)_h$ chosen from the sequence (which is bounded in H^2) as datum we find a minimizer, denoted by z_h, in $GSBV^2(\Omega_p) \cap L^2(\Omega_p)$ of

$$v \mapsto \mathscr{F}_r(v) + M(v - (v_p)_h)$$

since this problem is equivalent to the minimization of Blake & Zisserman functional for image segmentation with gray-level datum $g = (v_p)_h + f_r/(2\mu)$ and

Dirichlet boundary condition w, referring to notation of Theorem 3.1 in [14]. Then

$$(5.11)\qquad \mathscr{F}_r(z_h) + M(z_h - (v_p)_h) \le \mathscr{F}_r((v_r)_h) + M((v_r)_h - (v_p)_h) \quad \forall h.$$

Hence, by (5.4) and standard lower semicontinuity of F_p, the sequence of pairs $(z_h, (v_p)_h)$ is a minimizing sequence for $\mathscr{F}$ too. Moreover, by (5.10), (5.11) we get

$$\mathscr{F}_r(z_h) + M(z_h - (u_p)_h) \le C.$$

By compactness property of Theorem 8 in [10], there are $u_r \in GSBV^2(\Omega_p) \cap L^2(\Omega_p)$ and a subsequence z_h s.t., again by extracting without relabeling, $z_h \to u_r$ a.e., $z_h \rightharpoonup u_r$ L^2 and strongly in L^s, $1 \le s < 2$, $\nabla z_h \to \nabla u_r$ a.e., $\nabla^2 z_h \to \nabla^2 u_r$ a.e., $\nabla^2 z_h \to \nabla^2 z$ weakly in L^2 and by lower semicontinuity property of Theorem 10 in [10], we get

$$\mathscr{F}_r(u_r) + M(u_r - u_p) \le \liminf_h \mathscr{F}_r(z_h) + M(z_h - (v_p)_h) \le C.$$

Thus the pair (u_r, u_p) is a minimizer of relaxed functional $\mathscr{F}$. □

Proof of Theorem 2.2. Let $V = (u_r, u_p) \in \operatorname{argmin} \mathscr{F}$ (the existence of at least one such V is warranted by Theorem 5.1). Then u_p minimizes $z \mapsto F_p(z) + M(v_r - z)$ among $z \in H^2(\Omega_p)$ s.t. $z = w$ a.e. $\Omega_p \backslash \overline{\Omega}$. So, due to (2.2) and (2.1), $u_p \in C^2 \cap L^\infty(\Omega)$.

Moreover, if $V = (u_r, u_p) \in \operatorname{argmin} \mathscr{F}$, then $u_p \in H^4(\Omega)$ and, referring to (2.12) and (2.13), u_r minimizes $v \mapsto \mathscr{F}_r(v) + M(v - u_p)$ among $v \in GSBV^2(\Omega_p) \cap L^2(\Omega_p)$ s.t. $v = u_p = w$ a.e. $\Omega_p \backslash \overline{\Omega}$. Thus, exploiting the identity (5.7), by Theorem 2.2 of [14] with the choices $\alpha = \beta = 1$, $g = u_p + f_r/(2\mu) \in L^4(\Omega_p)$ and $M = T_0 = T_1 = \emptyset$, and setting $Z = \overline{S_{u_r} \cup S_{\nabla u_r}}$, we obtain that the triplet $(Z, \widetilde{u_r}, u_p)$ is an essential admissible triplet that minimizes F.

By applying the regularization argument detailed in [11, 14, 17] we obtain that $\widetilde{u_r} \in C^2(\Omega_p \backslash Z)$, where Z is the smallest closed subset of Ω_p containing the region where C^2 regularity of $\widetilde{u_r}$ is missing, and $\mathscr{H}^1(Z \backslash (S_{u_r} \cup S_{\nabla u_r}) = 0$. Eventually

$$\begin{aligned} \mathscr{F}_r(\widetilde{u_r}) + G(\widetilde{u_r} - u_p) + F_p(u_p) &\le \mathscr{F}_r(u_r) + G(u_r - u_p) + F_p(u_p) \\ &\le \liminf_h (\mathscr{F}_r(z_h) + G(z_h - u_h) + F_p(u_h)) = \inf_{\mathbb{X}} \mathscr{F} \end{aligned}$$

hence $\mathscr{F}(\widetilde{u_r}, u_p) = \min_{\mathbb{X}} \mathscr{F}$.

Summarizing $F(Z, \widetilde{u_r}, u_p) = \min\{F(K, v_r, v_p) : (K, v_r, v_p) \text{ admissible triplet}\}$. □

Remark 5.2. We emphasize that also the non-interpenetration between plate and reinforcement could be taken into account: e.g., adding the constraint $v \ge w$ a.e. Ω_p to the essential admissible pairs for hard-device reinforcement and adding the constraint $v_r \ge v_p$ a.e. Ω_p to the essential admissible triplets for strengthening reinforcement. Notice that here Remark 3.6 does not apply: competing functions are functions defined only almost everywhere, therefore the unilateral constraints act in the almost everywhere sense only. These unilateral constraints do not introduce any additional difficulty in the study of the weak formulations of both F and E, since inequalities are preserved by compactness properties of minimizing sequences. Therefore Theorem 5.1 holds true also under the additional constraint $v_r \geq v_p$.

But the subsequent step required to show Theorem 2.2, say the proof of partial regularity for weak minimizers, would be not straightforward.

For this reason in this short note we skip this substantial difficulty, postponing the analysis of the 2 dimensional problems with unilateral constraints to a forthcoming paper.

However, in 1 dimension the strong and weak formulation do coincide, so the analogous of Theorems 2.1 and 2.2 hold true with or without the non-interpenetration constraint for beams: we have taken into account these constraints in the one-dimensional case by Theorems 3.7, 3.8 and Propositions 3.12, 3.13.

6. Elastic-plastic reinforcement of flexural plate

In this section we deduce the existence statement in the case of strengthening reinforcement: minimization of functional G defined by (2.9).

Proof of Theorem 2.3. By $G(\emptyset, w, w) = (\eta + \gamma)\|D^2 w\|^2_{L^2(\Omega)} - \int_\Omega (f_r + f_p) w \, d\mathbf{x} < +\infty$, we know that the functional G has nonempty domain. Moreover, (5.9) warrants that the functional G is bounded from below.

The existence of a minimizer of G over *essential admissible triplets* (K, v_r, v_p), namely triplets fulfilling (2.10), can be achieved by repetition of the direct method approach with the techniques of [9].

Actually here, about minimization with respect to v_r, we have these differences with respect to [9]: presence of the additional coupling term $\mu \int_\Omega |v_r - v_p|^2 \, dx$; there are neither vanishing moments nor a safe load condition for the load f_r; last, there is a Dirichlet datum w at the boundary.

However vanishing moments and load f were exploited in [9] only to achieve the boundedness from below of the functional, whereas here such boundedness is already warranted by (5.9). Moreover, the additional term is a lower order perturbation, not affecting the existence of weak minimizers (thanks to the identity (5.7), still valid in present case), but requiring a technical correction in the proof of strong solutions by regularization of weak solutions.

Precisely, first step (existence of weak solutions) requires no change: we introduce the space SBH of Special Bounded hessian functions

$$SBH(\tilde{\Omega}) := \{ v \in H^{1,1}(\tilde{\Omega}) : Dv \in SBV(\Omega),\ v = w \text{ on } \tilde{\Omega} \backslash \Omega \},$$

here SBV is the space of bounded variation functions whose derivative has no Cantor part ([2]); then we set the weak formulation of functional G defined in (2.9), defined on $v \in SBH(\Omega)$:

$$\mathscr{G}(v_r, v_p) := \mathscr{H}^1(S_{Dv_r}) + \sigma \int_{S_{Dv_r}} |[Dv_r]|\, d\mathscr{H}^1 + \int_\Omega (\eta|\nabla^2 v_r|^2 - f_r v_r)\, d\mathbf{x} + \mu \int_\Omega |v_r - v_p|^2\, d\mathbf{x} + \int_\Omega (\gamma|D^2 v_p|^2 - f_p v_p)\, d\mathbf{x} \tag{6.1}$$

where $[z]$ denotes the jump of z and ∇z denotes the approximate gradient of z, say the absolutely continuous part of Dz.

The existence of a minimizing pair (u_r, u_p) for $\mathscr{G}$ follows by the same argument of present Theorem 5.1, taking into account of Theorem 2.9 in [9].

The proof of partial regularity in Ω for weak minimizers is achieved by exploiting blow-up and quasi-minimizers as in Theorem 4.15 in [9]: only Lemma 4.3 of [9] must be adapted as detailed below, to take into account of the additional gluing term.

Still by identity (5.7), the load and glue terms together are represented (up to the addition of a constant irrelevant in minimization) by $\mu \int_\Omega (v-h)^2\, dx$ where $h := u_p + f_r/(2\mu)$ and $h \in L^s$, $s > 2$: this contribution replaces here the term $-\int_\Omega gv$ of [9], however this does not affect regularization of weak solutions, since, setting $\mathscr{E}(v) = \mathscr{G}(v) - \mu \int_\Omega (v-h)^2$, every local minimizer of $\mathscr{G}$ is a local quasi minimizer of $\mathscr{E}$, due to the excess estimate (consequence of $u, v \in SBH(\Omega) \subset L^\infty(\Omega)$, $h \in L^s(\Omega)$, $s > 2$ and Hölder inequality):

$$\int_{B_\varrho(\mathbf{x})} ((v-h)^2 - (u-h)^2)\, dx = \int_{B_\varrho(\mathbf{x})} (u^2 - v^2 - h(v-u)) \le C\varrho^{2-2/s}$$

valid for $\overline{B_\varrho(\mathbf{x})} \subset \Omega$, $0 < \varrho < 1$ and $u, v \in SBH(\Omega)$ s.t. $v = u$ on $\Omega \backslash B_\varrho(\mathbf{x})$.

Partial regularity at the boundary under Dirichlet condition, can be achieved by the same argument of [14], taking into account of the simplifications due to the fact that here the competing functions are not only in $GSBV^2(\Omega)$, but they belong to $SBH(\Omega)$, hence they are globally continuous.

Summarizing a minimizing pair (u_r, u_p) of $\mathscr{G}$ leads to an essential minimizing triplet $(\overline{S_{\widetilde{Du_r}}}, u_r, u_p)$ of G. □

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