ON THE STABILITY OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH ROBIN BOUNDARY CONDITIONS ON RIEMANNIAN MANIFOLDS

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Abstract. We investigate existence and nonexistence of stationary stable nonconstant solutions, i.e. patterns, of semilinear parabolic problems in bounded domains of Riemannian manifolds satisfying Robin boundary conditions. These problems arise in several models in applications, in particular in Mathematical Biology. We point out the role both of the nonlinearity and of geometric objects such as the Ricci curvature of the manifold, the second fundamental form of the boundary of the domain and its mean curvature. Special attention is devoted to surfaces of revolution and to spherically symmetric manifolds, where we prove refined results.

1. Introduction

In this paper we study stability and instability of solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} + \alpha u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here Ω is a smooth domain in a Riemannian manifold (M, g), Δ is the Laplace-Beltrami operator on M, ν is the outer normal to $\partial\Omega$ with respect to M and $\alpha \in \mathbb{R}$ is an arbitrary fixed number. A solution of problem (1.1) may be regarded as a stationary solution of the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \Omega \times (0, T) \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{in } \partial\Omega \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{1.2}$$

Stable solutions are used in mathematical models of pattern formation. They are often called *patterns* and have attracted much attention in the literature (see e.g. [2], [3], [4], [9], [12], [14], [16], [19], [21], [22], [25]). Applications of problem (1.1) to mathematical biology are found e.g. in [5], [15], [24]. A biochemical process on surfaces of revolutions is described and analyzed in [22]. In most papers it is assumed that the boundary is impermeable (that is $\alpha = 0$). However it is reasonable to consider also the case of a flux that is proportional to the solution (see e.g. [5], [15], [24]). This motivates our choice of Robin boundary conditions with $\alpha \in \mathbb{R}$.

We recall (see e.g. [10]) that a solution of problem (1.1) is *asymptotically stable* if the smallest eigenvalue λ_1 of the linearized problem

$$\begin{cases} \Delta\phi + f'(u)\phi + \lambda\phi = 0 & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial \nu} + \alpha\phi = 0 & \text{in } \partial\Omega \end{cases} \tag{1.3}$$

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is positive, it is *unstable* if λ_1 is negative and *neutrally stable* if $\lambda_1 = 0$. Asymptotically stable solutions U of problem (1.1) have the property that they attract for large time all solutions of (1.2) which initially are sufficiently close to U (see e.g. [10]) while the latter are repelled from the unstable ones. If $\lambda_1 = 0$ both situations may occur.

The sign of λ_1 is also crucial in the calculus of variations. Indeed, let $\{x_i\}_1^m$ be a system of local coordinates and let g_{ij} be the corresponding metric tensor of M. Its inverse will be denoted by g^{ij}. Furthermore we have $g(\nabla u, \nabla \phi) = g^{ij} u_{x_i} \phi_{x_j}$ and $|\nabla u|^2 = g(\nabla u, \nabla u)$. Here and in the sequel we shall use the Einstein summation convention. The solutions of (1.1) are related to the following *energy* functional

$$\mathcal{E}(u, \Omega) = \int_\Omega |\nabla u|^2 \, d\mu + \alpha \oint_{\partial\Omega} u^2 \, dS - 2 \int_\Omega F(u) \, d\mu,$$

where $d\mu$ is the Riemannian volume element, dS is the surface element of $\partial\Omega$, and $F'(u) = f(u)$. If u is a solution of (1.1) then the Fréchet derivative of $\mathcal{E}$ vanishes at u, more precisely

$$\frac{1}{2}\dot{\mathcal{E}}(u, \Omega) = \int_\Omega g(\nabla u, \nabla \phi) \, d\mu + \alpha \oint_{\partial\Omega} u\phi \, dS - \int_\Omega f(u)\phi \, d\mu = 0 \quad \text{for all } \phi \in W^{1,2}(\Omega).$$

The second derivative of $\mathcal{E}$ at u is

$$\frac{1}{2}\ddot{\mathcal{E}}(u, \Omega) = \int_\Omega |\nabla \phi|^2 \, d\mu + \alpha \oint_{\partial\Omega} \phi^2 \, dS - \int_\Omega f'(u)\phi^2 \, d\mu.$$

If λ_1 is positive, then $\mathcal{E}(u, \Omega)$ is a local minimum whereas if λ_1 is negative u is a saddle point.

The study of stability of the solutions of (1.1) has a long history. First results were obtained by Hudjaev [11] and Keller and Cohen [13] for problems in $\mathbb{R}^n$, while the stability and the uniqueness of solutions to (1.1) in the case of weighted concave nonlinearities and positive solutions were studied in [3].

Casten and Holland [4], and also Matano [16], observed that for problems with Neumann boundary conditions ($\alpha = 0$) in a convex domain in $\mathbb{R}^n$, all nonconstant solutions are unstable. This result was generalized to problems on manifolds by Jimbo [12] and by Bandle, Punzo and Tesei [2] (see also [21]). Jimbo proved that all non-stationary solutions are unstable if the Ricci curvature of M and the second fundamental form of the boundary are positive. The aim of this paper is to study the stability of solutions to problems with Robin boundary conditions in bounded domains, both in $\mathbb{R}^m$ and, more in general, in Riemannian manifolds. In this case the conditions on the boundary, which imply instability, depend also on the nonlinearity f.

To give an idea of our results, let Ω be a domain in $\mathbb{R}^m$, let κ_i, $i = 1, ..m-1$, be the principal curvatures of $\partial\Omega$ and denote by $H = -(m-1)^{-1} \sum_1^{m-1} \kappa_i$ the mean curvature of $\partial\Omega$. By Theorem 4.4, we have that if $\alpha + (m-1)H + \frac{f(u)}{\alpha u} < 0$ on $\partial\Omega$ and if an additional assumption involving the second fundamental form of $\partial\Omega$ and α is satisfied, then every non trivial solution is unstable. Note that this condition in case $m = 2$ reduces to $\alpha - H > 0$. For non positive α in particular it implies that $\partial\Omega$ is convex.

As a counterpart we can derive from this type of considerations estimates for stable solutions. It should be pointed out that no assumption on the sign of α or on the solution is made. In the case of Riemannian manifolds a similar result holds under the additional assumption that the Ricci curvature is nonnegative. However for surfaces of revolution or for problems on spherically symmetric manifolds we can allow the Ricci curvature to be negative provided it satisfies a suitable bound from below, see Sections 5 and 6. Furthermore we construct by means of arguments developed in [2] and in [25] a counterexample which shows that this bound is sharp .

The discussion of nonexistence of stable solutions is based on the variational characterization of λ_1, on the well-known Bochner-Weitzenböck formula and on a, to our knowledge, new result on the decomposition of the normal derivative of $|\nabla w(x)|^2$ on $\partial\Omega$ where w satisfies $\partial_\nu w = -\alpha w$ on $\partial\Omega$, for some $\alpha \in \mathbb{R}$, see Theorem 3.4. Similar formulas are known in the literature for special cases and have been extensively used by L.E. Payne and his collaborators to obtain estimates for the solutions of boundary value problems see e.g. [23].

The particular case of Neumann boundary conditions $\alpha = 0$ was studied in [16] for $\Omega \subset \mathbb{R}^m$ and in [2] for $\Omega \subset M$. The case of a general $\alpha \in \mathbb{R}$ involves a deeper rather technical analysis based on the method of moving frames, see Section 3.

The paper is organized as follows. At first in Section 2, we deal with the case of domains of $\mathbb{R}^2$ in order to illustrate the flavor of the technique. Section 3 is an introduction to the geometrical notions and tools needed in the sequel. Section 4 is concerned with the general result on instability on Riemannian manifolds, while Section 5 is devoted to the case of surfaces of revolution where sharper results are obtained. The last section contains some applications to specific manifolds.

2. Domains in the plane

In order to get a better insight and because the arguments are elementary we treat first the case where Ω is a domain in the plane.

Suppose that the boundary $\partial\Omega$ is represented by the curve $s \mapsto x(s) := (x_1(s), x_2(s))$, with $s \in [0, l]$, where s is the arc-length. We assume that $x(s)$ is positively oriented, and that the curve is sufficiently smooth, so that the differential equation (1.1) holds up to the boundary. The outer normal to $\partial\Omega$ will be denoted by ν. In a neighborhood of the boundary a point $x \in \Omega$ is given by

$$x(\rho, s) = x(s) - \rho\nu(s). \tag{2.1}$$

Here (ρ, s) are called *normal coordinates*. By the Frenet formula $\dot{\nu} = \kappa\dot{x}$ where κ is the curvature of $\partial\Omega$. The metric g can be written as $g = (1-\rho\kappa)^2 ds^2 + d\rho^2$; we thus have

$$g_{ij} = \begin{pmatrix} (1-\rho\kappa)^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} (1-\rho\kappa)^{-2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{g}^2 = \det(\mathrm{g_{ij}}) = (1-\rho\kappa)^2,$$

where g_{ij} are the components of g and g^{ij} are the components of its inverse. Moreover, the volume element dx is $dx = (1-\rho\kappa) ds d\rho$, while for a sufficiently regular function u the gradient and the Laplacian (see also the next section) are, respectively, given by

$$\begin{aligned} |\nabla u|^2 &= \frac{u_s^2}{(1-\rho\kappa)^2} + u_\rho^2, \\ \Delta u &= \frac{1}{1-\rho\kappa}\left(\frac{u_s}{1-\rho\kappa}\right)_s + \frac{1}{1-\rho\kappa}((1-\rho\kappa)u_\rho)_\rho. \end{aligned} \tag{2.2}$$

Consider now the original problem (1.1) and the corresponding eigenvalue problem (1.3). The smallest eigenvalue is characterized by the Rayleigh principle

$$\lambda_1 = \min_{W^{1,2}(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx - \int_\Omega f'(u)v^2\,dx + \alpha\oint_{\partial\Omega} v^2\,ds}{\int_\Omega v^2\,dx}. \tag{2.3}$$

Following Casten and Holland [4], we choose $v = u_{x_i}$ as a test function in (2.3). Then by the Gauss theorem

$$\begin{aligned} \lambda_1 \int_\Omega u_{x_i}^2\,dx &\le \int_\Omega |\nabla u_{x_i}|^2\,dx + \alpha\oint_{\partial\Omega} u_{x_i}^2\,ds - \int_\Omega f'(u)u_{x_i}^2\,dx \\ &= -\int_\Omega u_{x_i}\Delta u_{x_i}\,dx + \frac{1}{2}\oint_{\partial\Omega}\partial_\nu u_{x_i}^2\,ds + \alpha\oint_{\partial\Omega} u_{x_i}^2\,ds - \int_\Omega f'(u)u_{x_i}^2\,dx. \end{aligned}$$

Replacing Δu_{x_i} by $-f'(u)u_{x_i}$ and summing over i we get

$$\lambda_1 \int_\Omega |\nabla u|^2\,dx \le \frac{1}{2}\oint_{\partial\Omega}\partial_\nu|\nabla u|^2\,ds + \alpha\oint_{\partial\Omega}|\nabla u|^2\,ds. \tag{2.4}$$

Next we compute the first integral on the right-hand side of (2.4). From (2.2) we conclude that on $\partial\Omega$ (that is $\rho = 0$ in (2.1))

$$\frac{1}{2}\partial_\nu |\nabla u|^2 = -\frac{1}{2}\partial_\rho |\nabla u|^2 = -u_s u_{s\rho} - \kappa u_s^2 - u_\rho u_{\rho\rho}.$$

Keeping in mind the boundary condition $u_\rho = \alpha u$ and from (2.2) and (1.1) the relation

$$u_{\rho\rho} - \kappa u_\rho + u_{ss} + f(u) = 0 \text{ on } \partial\Omega,$$

we obtain

$$\frac{1}{2}\partial_\nu |\nabla u|^2 = -(\alpha+\kappa)u_s^2 - \kappa\alpha^2 u^2 + \alpha u u_{ss} + \alpha u f(u). \tag{2.5}$$

Since

$$\alpha \oint_{\partial\Omega} u u_{ss}\, dS = -\alpha \oint_{\partial\Omega} u_s^2\, ds$$

we finally get

$$\lambda_1 \int_\Omega |\nabla u|^2\, dx \le \oint_{\partial\Omega} \alpha^2 u^2 \left[\alpha - \kappa + \frac{f(u)}{\alpha u}\right] - (\alpha+\kappa)u_s^2\, ds. \tag{2.6}$$

Theorem 2.1. *Assume $\alpha \neq 0$. If on $\partial\Omega$ the conditions*

$$(\mathbf{C0}) \qquad \oint_{\partial\Omega} \alpha^2 u^2 \left[\alpha - \kappa + \frac{f(u)}{\alpha u}\right] < 0$$

$$(\mathbf{C1}) \qquad \alpha + \kappa_{\min} \ge 0.$$

are satisfied then $\lambda_1 < 0$ and u is unstable.

Observe that

$$(\mathbf{C2}) \qquad \alpha - \kappa_{\min} + \frac{f(u)}{\alpha u} < 0$$

implies (**C0**).

Remark 2.2. If $\partial\Omega$ is a level line of u, then $u_s = 0$ on $\partial\Omega$ and hence Theorem 2.1 remains true only under the condition (**C0**).

If $\partial\Omega$ consists of several closed curves Γ_i, $i = 1, \ldots, k$, then condition (**C2**) can be replaced by $\alpha - \min_{\Gamma_i}\kappa + \max_{\Gamma_i}\frac{f(u)}{\alpha u} < 0$ for $i = 1, \ldots, k$.

Example 2.3. Let Ω be an annulus with the radii $r_0 < R$ and $u = u(r)$ be radial. Then u is unstable if

$$\left[r_0\alpha + 1 + \frac{r_0 f(u(r_0))}{\alpha u(r_0)}\right] u^2(r_0) + \left[R\alpha - 1 + \frac{R f(u(R))}{\alpha u(R)}\right] u^2(R) < 0.$$

Note that inequality (2.6) contains also information on stable solutions. In fact for a stable solution λ_1 is positive and thus

$$0 \le \oint_{\partial\Omega} \left\{\alpha^2 u^2 \left[\alpha - \kappa + \frac{f(u)}{\alpha u}\right] - (\alpha+\kappa)u_s^2\right\} ds.$$

If (**C1**) holds then the expression $\alpha - \kappa + \frac{f(u)}{\alpha u}$ must be positive somewhere. For instance if u is the first eigenfunction of $\Delta u + \lambda u = 0$ in Ω, $\frac{\partial u}{\partial \nu} + \alpha u = 0$ on $\partial\Omega$ and $\alpha > 0$ then

$$\lambda_1 \ge \alpha\kappa_{\min} - \alpha^2.$$

3. Some useful tools from Riemannian geometry

In this section we collect some notions and results from Riemannian Geometry following [18] and [1]. Moreover we prove a decomposition theorem (see Theorem 3.4 below) for the normal derivative of the squared norm of the gradient of an arbitrary smooth function satisfying Robin boundary conditions.

3.1. **Basics on the method of moving frames.** Let (M, g) be a Riemannian manifold of dimension m with metric g. Let $p \in M$ and let (U, φ) be a *local chart* such that $p \in U$. Denote by $x^1, \dots, x^m$, $m = \dim M$ the coordinate functions on U. Then, at any $q \in U$ we have

$$g = g_{ij}\, dx^i \otimes dx^j \tag{3.1}$$

where dx^i denotes the differential of the function x^i and g_{ij} are the (local) components of the metric defined by $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. In equation (3.1) and throughout this section we adopt the Einstein summation convention over repeated indices. Applying in q the Gram-Schmidt orthonormalization process we can find linear combinations of the 1-forms dx^i which we will call θ^i for $i = 1, \dots, m$. Then (3.1) takes the form

$$g = \delta_{ij}\theta^i \otimes \theta^j, \tag{3.2}$$

where δ_{ij} is the Kronecker symbol. Since, as q varies in U, the previous process gives rise to coefficients that are C^∞ functions of q, the set of 1-forms $\{\theta^i\}$ defines an orthonormal system on U for the metric g, i.e. a (local) *orthonormal coframe*. It is usual to write

$$g = \sum_{i=1}^{m} (\theta^i)^2,$$

instead of (3.2). We also define the (local) *dual orthonormal frame* $\{e_i\}$, for $i = 1, \dots, m$, as the set of vector fields on U satisfying

$$\theta^j(e_i) = \delta_i^j, \tag{3.3}$$

where δ_i^j is the Kronecker symbol. We have the following

Proposition 3.1. *Let $\{\theta^i\}$ be a local orthonormal coframe defined on the open set $U \subset M$; then on U there exist unique 1-forms $\{\theta^i_j\}$, for $i, j = 1 \dots, m$, such that*

$$d\theta^i = -\theta^i_j \wedge \theta^j \tag{3.4}$$

and

$$\theta^i_j + \theta^j_i = 0 \tag{3.5}$$

The forms θ^i_j are called the *Levi-Civita connections forms* associated to the orthonormal coframe $\{\theta^i\}$, while equation (3.4) is called the *first structure equation*.

The *curvature forms* $\{\Theta^i_j\}$ are associated to the orthonormal coframe $\{\theta^i\}$ through the *second structure equation*

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j.$$

Because of (3.5) it follows immediately that

$$\Theta^i_j + \Theta^j_i = 0. \tag{3.6}$$

Using the basis $\{\theta^i \wedge \theta^j\}$, for $1 \le i < j \le m$, of the space of skew-symmetric 2-forms $\Lambda^2(U)$ on the open set U, we may write

$$\Theta^i_j = \frac{1}{2} R^i_{jkt}\theta^k \wedge \theta^t \tag{3.7}$$

for some coefficients $R^i_{jkt} \in C^\infty(U)$ satisfying

$$R^i_{jkt} + R^i_{jtk} = 0. \tag{3.8}$$

These are the coefficients of the $(1, 3)$–version of the *Riemann curvature tensor* which we denote by R. More precisely, in this local orthonormal frame we have

$$R^i_{jkt} = \Theta^i_j(e_k, e_t) = (d\theta^i_j + \theta^i_k \wedge \theta^k_j)(e_k, e_t) = g(R(e_k, e_t)e_j, e_i),$$

so that its components are

$$R = R^i_{jkt}\theta^k \otimes \theta^t \otimes \theta^j \otimes e_i.$$

Note that (3.6) implies

$$R^i_{jkt} + R^j_{ikt} = 0. \tag{3.9}$$

The $(0,4)$–version of R is defined by $\mathrm{Riem}(X,Y,Z,W) = g(R(Z,W)Y,X)$, so that its local coefficients R_{ijkt} satisfy

$$R_{ijkt} = \mathrm{Riem}(e_i, e_j, e_k, e_t) = g(R(e_k, e_t)e_j, e_i) = R^i_{jkt}$$

and thus in the local orthonormal frame

$$\mathrm{Riem} = R_{ijkt}\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t. \tag{3.10}$$

For further details, we refer to [1]. The *Ricci tensor* is the symmetric $(0,2)$–tensor obtained from (3.10) by tracing either with respect to i and k or, equivalently, with respect to j and t. Thus

$$\mathrm{Ric} = R_{ij}\theta^i \otimes \theta^j$$

with

$$R_{ij} = R_{itjt} = R_{titj}.$$

Now let $u \in C^\infty(M)$; for the differential of u, du, we can write

$$du = u_i\theta^i \tag{3.11}$$

for some smooth coefficients u_i; the *Hessian* of u is then defined as the $(0,2)$ tensor field $\mathrm{Hess}(u) = \nabla du$ of components u_{ij} given by

$$u_{ij}\theta^j = du_i - u_k\theta^k_i, \tag{3.12}$$

that is

$$\mathrm{Hess}(u) = u_{ij}\theta^j \otimes \theta^i. \tag{3.13}$$

Here and in what follows ∇ is the (unique) Levi-Civita connection associated to the metric g. It is easy to prove that

$$u_{ij} = u_{ji}, \tag{3.14}$$

so that $\mathrm{Hess}(u)$ is a symmetric tensor. In global notation we have, for all smooth vector fields X, Y on M,

$$\mathrm{Hess}(u)(X,Y) = (\nabla du)(X,Y) = Y(X(u)) - (\nabla_Y X)(u) = X(Y(u)) - (\nabla_X Y)(u);$$

it is also possible to show that, equivalently,

$$\mathrm{Hess}(u)(X,Y) = \frac{1}{2}(\mathcal{L}_{\nabla u}g)(X,Y), \tag{3.15}$$

where $\mathcal{L}_{\nabla u}g$ is the *Lie derivative* of the metric g in the direction of ∇u. With respect to local coordinates $\{x^i\}$, $i = 1, \ldots, m$, the Hessian is given by

$$(\mathrm{Hess}(u))_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_l} - \Gamma^k_{il}\frac{\partial u}{\partial x_k},$$

where Γ^k_{ij} are the Christoffel symbols, defined as usual as

$$\Gamma^k_{ij}\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} = \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}.$$

In the moving frame formalism the squared norm $|\mathrm{Hess}(u)|^2$ is given by $u_{ij}u_{ij}$, while in coordinates we have

$$|\mathrm{Hess}(u)|^2 = g^{ik}g^{jl}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial u}{\partial x_k}\right)\left(\frac{\partial^2 u}{\partial x_k \partial x_l} - \Gamma^s_{kl}\frac{\partial u}{\partial x_s}\right).$$

The *Laplacian* of u is, by definition, the trace of the Hessian, (more precisely, of the $(1,1)$ version of the Hessian), that is,

$$\Delta u = \mathrm{Tr}(\mathrm{Hess}(u)) = u_{ii}.$$

The *gradient* of a function $u : M \to \mathbb{R}$ relative to the metric of M, ∇u, is the vector dual to the 1-form du, that is

$$g(\nabla u, X) = du(X) = X(u).$$

for all smooth vector fields X on M. In a local orthonormal frame we have $\nabla u = u^i e_i = u_i e_i$, so that $|\nabla u|^2 = u_i u_i$, while in local coordinates we have

$$|\nabla u|^2 = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

The *divergence* of a vector field $V = V^i e_i$ on M is given by the trace of ∇V, the covariant derivative of V; since $\nabla V = \left(dV^i + V^j \theta^i_j\right) \otimes e_i = V^i_j \theta^j \otimes e_i$ we have

$$\operatorname{div} V = V^i_i,$$

while in local coordinates

$$\operatorname{div} V = \frac{\partial V^i}{\partial x_i} + V^k \Gamma^i_{ki}.$$

Note that the Laplacian of the function u is the divergence of its gradient, that is

$$\Delta u = \operatorname{div}(\nabla u).$$

In local coordinates it has the form

$$\Delta f = \mathfrak{g}^{-1} \frac{\partial}{\partial x_i} \left(\mathfrak{g}\, g^{ij} \frac{\partial f}{\partial x_j} \right), \text{ where } \mathfrak{g} = \sqrt{\det(\mathrm{g_{ij}})}.$$

The third derivatives of u are defined by

$$u_{ijk}\theta^k = du_{ij} - u_{kj}\theta^k_i - u_{ik}\theta^k_j. \tag{3.16}$$

Note that taking the covariant derivative of (3.14) we have

$$u_{ijk} = u_{jik}. \tag{3.17}$$

The commutation rule of the last two indices is given by

$$u_{ijk} = u_{ikj} + u_t R^t_{ijk} = u_{ikj} + u_t R_{tijk}. \tag{3.18}$$

We state here the classical Bochner-Weitzenböck formula, see e.g. [18], which will play a crucial role in our investigations.

Lemma 3.2. *For all $u \in C^3(M)$ we have*

$$\frac{1}{2}\Delta |\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ric}\,(\nabla u, \nabla u) + g\,(\nabla \Delta u, \nabla u)\,. \tag{3.19}$$

Example 3.3. Let $M \subset \mathbb{R}^2$ be a simply connected surface and let Ω be a C^2 domain on M. It is well-known that M can be mapped conformally into $\mathbb{R}^2$. In this case the coordinates (x_1, x_2) are isothermal and the corresponding metric tensor is $g_{ij} = p^2 \delta_{ij}$. The differential operators then become

$$\Delta = p^{-2}\Delta_R \quad \text{and} \quad |\nabla|^2 = p^{-2}|\nabla_R|^2,$$

where Δ_R and ∇_R are the Laplacian and the gradient in $\mathbb{R}^2$. In this case we have

$$\begin{aligned} p^2 \frac{1}{2}\Delta |\nabla u|^2 &= u_{ik}u_{ik}p^{-2} + u_i u_{ikk} p^{-2} + 2u_i u_{ik}(p^{-2})_k + \frac{1}{2}|\nabla_R u|^2 \Delta p^{-2} \\ &= u_{ik}u_{ik}p^{-2} + 2u_i u_{ik}(p^{-2})_k - u_i (p^{-2})_i \Delta_R u + u_i \left(\frac{\Delta_R u}{p^2}\right)_i + \frac{1}{2}|\nabla_R u|^2 \Delta_R p^{-2}. \end{aligned}$$

Here the subscripts denote differentiation with respect to x_i. Set for short $f := \log p$. Then $(p^{-2})_k = -\frac{2}{p^2} f_k$. Thus in two-dimensions

$$\begin{aligned}&u_{ik}u_{ik}p^{-2} + 2u_i u_{ik}(p^{-2})_k - u_i(p^{-2})_i \Delta_R u\\&= p^{-2}[(u_{11}^2 + 2u_{12}^2 + u_{22}^2) + 2(u_1 f_1 - u_2 f_2)(u_{22} - u_{11}) - 4(u_1 f_2 + u_2 f_1)u_{12})]\\&= p^{-2}\left[(u_{11} - u_1 f_1 + u_2 f_2)^2 + (u_{22} + f_1 u_1 - u_2 f_2)^2 + 2(u_{12} - u_1 f_2 - u_2 f_1)^2\right]\\&- 2p^{-2}(u_1^2 + u_2^2)(f_1^2 + f_2^2).\end{aligned}$$

Moreover

$$\frac{1}{2}\Delta_R p^{-2} = -\frac{1}{p^2}\Delta_R f + \frac{2}{p^2}|\nabla_R f|^2.$$

Note that $K = -p^{-2}\Delta_R f$ is the Gaussian curvature of M. Consequently

$$\begin{aligned}\frac{1}{2}\Delta|\nabla u|^2 = &\underbrace{p^{-4}\left[(u_{11} - u_1 f_1 + u_2 f_2)^2 + (u_{22} + f_1 u_1 - u_2 f_2)^2 + 2(u_{12} - u_1 f_2 - u_2 f_1)^2\right]}_{|\text{Hess(u)}|^2}\\&+ p^{-2}K|\nabla_R u|^2 + p^{-2}u_i(\Delta u)_i.\end{aligned}$$

3.2. **Immersed submanifolds.** Let (N, g_N) and M be respectively a Riemannian manifold and a manifold of dimensions n and m, with $m \leq n$. Let $f : M \to N$ be an *immersion* and let $g = f^* g_N$ be the metric induced on M by f where f^* denotes the pullback. If g_M is a given Riemannian metric on M and $f : M \to N$ is an immersion we say that f is an *isometric immersion* if $g_M = g = f^* g_N$.

Let $\mathcal{V} \subset N$ be an open set, and let $p \in f^{-1}(\mathcal{V})$. By reducing $\mathcal{V}$ we can assume that the connected component $\mathcal{U}$ of $f^{-1}(\mathcal{V})$ with p is an embedded submanifold in the domain of a local flat chart.

We fix the following indices convention:

$$1 \leq i, j, k, \ldots \leq m, \quad m + 1 \leq \alpha, \beta, \gamma, \ldots \leq n, \quad 1 \leq a, b, c, \ldots \leq n.$$

By means of the Gram-Schmidt procedure we can construct an orthonormal frame $\{E_a\}$ in a neighborhood of $f(\mathcal{U})$ such that $\{E_i\}$ is a basis for $Tf(\mathcal{U})$. We call this frame a *Darboux frame along* f, and we write $\{e_i\}$ for the basis of the tangent space at $\mathcal{U}$ such that $f_* e_i = E_i$ (where $f_* e_i$ is the pushforward of e_i by the map f). The dual $\{\theta^a\}$ of a Darboux coframe is called a *Darboux coframe along* f. The definition of a Darboux (co)frame is equivalent to say that the vectors $\{E_i\}$ (locally) span $f_* TM$, the image of TM through f in TN, while the vectors $\{E_\alpha\}$ are orthogonal to $f_* TM$ and span in fact the *normal bundle* $TM^\perp$, that is the set of (local) vector fields in N that are orthogonal to $f_* TM$. A consequence of the choice of a Darboux frame is that

$$f^* \theta^\alpha = 0, \tag{3.20}$$

where $f^* \theta^\alpha$ is the pullback of θ^α by the map f. Indeed, for every i, $(f^* \theta^\alpha)(e_i) = \theta^\alpha(f_* e_i) = \theta^\alpha(E_i) = 0$.

Let now $\{\theta_b^a\}$ be the Levi-Civita connection forms of N relative to $\{\theta^a\}$. Pulling-back on M the first structure equation of N, and using the properties of the pullback we have

$$f^*(d\theta^a) = d(f^* \theta^a) = -f^*\left(\theta_b^a \wedge \theta^b\right) = -(f^* \theta_b^a) \wedge \left(f^* \theta^b\right).$$

By (3.20) we obtain in particular that

$$d\left(f^* \theta^i\right) = -\left(f^* \theta_j^i\right) \wedge \left(f^* \theta^j\right); \tag{3.21}$$

moreover we obviously have

$$f^*\left(\theta_j^i\right) + f^*\left(\theta_i^j\right) = 0.$$

Thus from the uniqueness, see Proposition 3.1, we deduce that $\left\{f^* \theta_j^i\right\}$ are the Levi-Civita connection forms of M.

Since the pullback commutes with exterior differentiation and wedge product we shall omit from now on the pullback. From the context the reader should be able to distinguish between forms or tensors.

Then equation (3.20) becomes

$$\theta^\alpha = 0 \text{ on } M \tag{3.22}$$

and on M we have

$$\theta^i_j + \theta^j_i = 0 \tag{3.23}$$

and

$$d\theta^i = -\theta^i_j \wedge \theta^j . \tag{3.24}$$

To obtain further information we differentiate (3.22), use (3.24) and (3.22) again to obtain

$$0 = d\theta^\alpha = -\theta^\alpha_i \wedge \theta^i - \theta^\alpha_\beta \wedge \theta^\beta = -\theta^\alpha_i \wedge \theta^i . \tag{3.25}$$

Hence a simple computation shows that there exist locally smooth functions h^α_{ij} such that

$$\theta^\alpha_i = h^\alpha_{ij}\theta^j \tag{3.26}$$

and

$$h^\alpha_{ij} = h^\alpha_{ji}. \tag{3.27}$$

It can be shown that the h^α_{ij}'s are the coefficients of the *second fundamental tensor* $\mathrm{II} : TM \times TM \to TM^\perp$ of the immersion. II is a $(1,2)$-tensor *along* f or, equivalently, a section of $T^*M \otimes T^*M \otimes TM^\perp$ (where $TM^\perp$ is considered as a subset of the *pullback bundle* f^*TN) which in the present setting is defined by

$$\mathrm{II} = h^\alpha_{ij}\theta^i \otimes \theta^j \otimes E_\alpha . \tag{3.28}$$

One can also verify that by (3.27) II is defined globally and that it is symmetric. The *mean curvature vector field* is given by its normalized trace, that is

$$\mathbf{H} = \frac{1}{m}\mathrm{tr}(\mathrm{II}) = \frac{1}{m} h^\alpha_{ii} E_\alpha .$$

If ν is a unit normal vector field the *mean curvature in the direction of* ν is

$$h^\nu = g\left(\mathbf{H}, \nu\right)_N .$$

If $m+1 = n$ and both the hyper surface M and N are orientable, we can choose Darboux frames along f preserving orientations, i.e. such that $\theta^1 \wedge \cdots \wedge \theta^{m+1}$ and $\theta^1 \wedge \cdots \wedge \theta^m$ give the correct orientations. More precisely the vector field E_{m+1} dual to θ^{m+1} on N is, when restricted to M, a global normal vector field on M. We shall call it ν. Furthermore note that in this case in local coordinates one has

$$h_{ij} = -g(\nabla_{e_i}\nu, e_j) \qquad \text{for } \ i, j = 1, \ldots, m,$$

which in global notation can be expressed as

$$\mathrm{II}(X,Y) = -g(\nabla_X \nu, Y) \quad \text{for any } \ X, Y \in TM . \tag{3.29}$$

The mean curvature in the direction of ν is called the *mean curvature of the immersed hypersurface* and denoted by H. Observe that, according to our sign convention, the mean curvature of the sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$, with respect to the outer normal $\frac{\partial}{\partial r}$, is -1. Note that, with respect to the notation used in Section 2, if we consider $\partial\Omega$, the boundary of a regular domain $\Omega \subset \mathbb{R}^2$, we have $H = -\kappa$.

3.3. **A decomposition theorem.**

Theorem 3.4. *Let (M, g) be an m-dimensional Riemannian manifold and let $\Omega \subset M$ be a compact, orientable domain with boundary $\partial\Omega$. Let II and H denote respectively the second fundamental tensor and the mean curvature of the embedding $\partial\Omega \hookrightarrow \Omega$ in the direction of the outward unit normal vector field ν. Let $w \in C^3(\overline{\Omega})$; if w satisfies*

$$\frac{\partial w}{\partial \nu} = g\left(\nabla w, \nu\right) = -\alpha w \quad \text{on } \partial\Omega \tag{3.30}$$

for some $\alpha \in \mathbb{R}$, *then*

$$\frac{1}{2}\frac{\partial}{\partial \nu}|\nabla w|^2 = \mathrm{II}\big(\widetilde{\nabla} w, \widetilde{\nabla} w\big) - \alpha|\widetilde{\nabla} w|^2 - \alpha w \operatorname{Hess}(w)(\nu,\nu) \quad \text{on } \partial\Omega, \tag{3.31}$$

where $\widetilde{\nabla} w = \nabla w - g(\nabla w, \nu)\,\nu$ *is the tangential gradient with respect to* $\partial\Omega$.

Proof. Let $\{e_A\} = \{e_1, \ldots, e_{m-1}, e_m = \nu\}$ be a Darboux frame along $\partial M \hookrightarrow M$. Set

$$H_{ij} = g\left(\mathrm{II}(e_i, e_j), \nu\right),$$

so that

$$H = \frac{1}{m-1} H_{kk},$$

where $1 \le i, j, k \le m-1$. By definition of the covariant derivative we have

$$dw_m = w_{mB}\theta^B + w_i\theta^i_m,$$

thus

$$w_{mB}\theta^B = w_i\theta^m_i + dw_m.$$

Pulling back the previous relation to $\partial\Omega$ and using (3.30) we deduce

$$w_{mi} = H_{ij}w_j - \alpha w_i \quad \text{on } \partial\Omega, \tag{3.32}$$

which implies

$$w_i w_{mi} = H_{ij}w_i w_j - \alpha w_i w_i \quad \text{on } \partial\Omega. \tag{3.33}$$

We now have

$$\frac{1}{2}\left(|\nabla w|^2\right)_A = w_B w_{BA} = w_i w_{iA} + w_m w_{mA} = w_i w_{iA} - \alpha w w_{mA} \quad \text{on } \partial\Omega,$$

thus

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla w|^2 = \frac{1}{2}\langle \nabla|\nabla w|^2, \nu\rangle = w_i w_{im} + w_m w_{mm} = w_i w_{im} - \alpha w w_{mm} \quad \text{on } \partial\Omega. \tag{3.34}$$

Combining (3.33) and (3.34) we get the desired result.

We conclude the section by recalling a relation between the Laplace–Beltrami operator Δ of the manifold (M, g) acting on a smooth function w defined in a neighborhood of the boundary $\partial\Omega$ and the Laplace-Beltrami operator $\widetilde{\Delta}$ of the manifold $\partial\Omega$, acting on the trace of the function w on $\partial\Omega$. Let H be the mean curvature of $\partial\Omega$ and $\widetilde{\Delta}$ be the Laplace–Beltrami operator of the manifold $\partial\Omega$ endowed with the metric induced by the embedding $\partial\Omega \hookrightarrow M$. Then on has, see e.g. [18]

$$\Delta w = \widetilde{\Delta} w - (m-1)H\frac{\partial w}{\partial \nu} + \operatorname{Hess}(w)(\nu,\nu). \tag{3.35}$$

Example 3.5. Let Ω be a domain on a two-dimensional surface as in Example 3.3. We consider as before its conformal projection onto the plane. We shall use the same notation as in Section 2 and Example 3.3. In this case we have $\partial/\partial\nu = -p^{-1}\partial\rho$ and the expression (3.34) reads as

$$\frac{\partial}{\partial\nu}|\nabla u|^2 = -p^{-3}|\nabla_R u|^2_\rho - p^{-1}|\nabla_R u|^2 (p^{-2})_\rho$$

Keeping in mind that $p^{-1}(\kappa - \partial_\rho \log p) = \kappa_g$ is the geodesic curvature of $\partial\Omega$ we find by the arguments in Section 2

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial\nu}|\nabla u|^2 &= p^{-3}(-u_s u_{s\rho} - \kappa u_s^2 - u_\rho u_{\rho\rho}) + p^{-4}p_\rho(u_\rho^2 + u_s^2) \\ &= -p^{-3}(u_s u_{s\rho} + u_\rho u_{\rho\rho}) - p^{-2}\kappa_g u_s^2 + p^{-4}p_\rho u_\rho^2. \end{aligned} \tag{3.36}$$

Taking into account the boundary condition $u_\rho = \alpha u$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla u|^2 = -\frac{u_s^2}{p^2}(\kappa_g + \alpha) + \frac{\alpha u}{p^2}\left[u_\rho(\log p)_\rho - u_s(\log p)_s - u_{\rho\rho}\right].$$

Remark 3.6. Similar results to Theorem 3.4 in the special case where $M = \mathbb{R}^m$ were used by L.E. Payne and his collaborators in their study of a priori bounds for elliptic problems. This theorem provides a tool to determine the points where the *P-function* takes its maximum. A survey of these results is found in [23].

4. Instability results on Riemannian manifolds

Let $\Omega \subset (M, g)$ be a smooth bounded domain and let $u : \Omega \to \mathbb{R}$ be a solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where ν denotes the outer normal unit vector at $\partial\Omega$ and $f \in C^1$. Define

$$\lambda_1 := \inf_{\phi \in H^1(\Omega),\, \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2\, d\mu - \int_\Omega f'(u)\phi^2\, d\mu + \alpha \int_{\partial\Omega} \phi^2\, d\sigma}{\int_\Omega \phi^2\, d\mu}. \tag{4.2}$$

We note that by standard elliptic theory λ_1 is achieved by a function $\phi_1 \in H^1(\Omega)$ which is a positive solution of (1.3).

Note that the case $\alpha = 0$ corresponds to the problem of homogeneous Neumann boundary conditions which has already been studied in [2].

We first consider the case of constant solutions to problem (4.1). It follows immediately from the boundary conditions that $u \equiv 0$ is the only possibility. The equation implies that $f(0) = 0$. Let

$$\Lambda_1 := \inf_{\phi \in H^1(\Omega),\, \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2\, d\mu + \alpha \int_{\partial\Omega} \phi^2\, d\sigma}{\int_\Omega \phi^2\, d\mu} \tag{4.3}$$

be the smallest eigenvalue of

$$\begin{cases} -\Delta \varphi_1 = \Lambda_1 \varphi_1 & \text{in } \Omega, \\ \frac{\partial \varphi_1}{\partial \nu} + \alpha \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

By (4.2) we have immediately

Proposition 4.1. *Let $\alpha \neq 0$ and let u be a constant solution of problem* (4.1)*. Then $u \equiv 0$ and $f(0) = 0$. Moreover, for Λ_1 defined in* (4.3)*,*

- i) *if $f'(0) > \Lambda_1$ then $u \equiv 0$ is unstable,*
- i) *if $f'(0) < \Lambda_1$ then $u \equiv 0$ is asymptotically stable.*

Remark 4.2. If we use $\phi \equiv 1 \in H^1(\Omega)$ as a test function in the definition (4.3) of Λ_1 we get the upper bound

$$\Lambda_1 \leq \alpha \frac{|\partial\Omega|}{|\Omega|},$$

where

$$|\Omega| := \int_\Omega d\mu, \qquad |\partial\Omega| := \int_{\partial\Omega} d\sigma.$$

This together with Proposition 4.1 implies that the solution $u \equiv 0$ is unstable if $f'(0) > \alpha \frac{|\partial\Omega|}{|\Omega|}$

We now discuss the stability of non constant solutions to problem (4.1).

Proposition 4.3. *Let $u \in C^3(\overline{\Omega})$ be a solution of* (4.1)*, with $f \in C^1$. Then*

$$\begin{aligned} \lambda_1 \int_\Omega |\nabla u|^2\, d\mu \quad \leq \quad & \int_{\partial\Omega} \Big(\mathrm{II}(\widetilde{\nabla} u, \widetilde{\nabla} u) - \alpha |\widetilde{\nabla} u|^2 + \alpha^3 u^2 + \alpha u f(u) + \alpha^2 (m-1) u^2 H \Big)\, d\sigma \\ & - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u)\, d\mu. \end{aligned} \tag{4.5}$$

Proof of Proposition 4.3. If we introduce in the variational characterization (4.2) of λ_1 the test function $|\nabla u|^2$ we get

$$\lambda_1 \int_\Omega |\nabla u|^2 \, d\mu \le \int_\Omega |\nabla|\nabla u||^2 \, d\mu - \int_\Omega f'(u)|\nabla u|^2 \, d\mu + \alpha \int_{\partial\Omega} |\nabla u|^2 \, d\sigma. \tag{4.6}$$

If we apply the Bochner-Weitzenböck formula (3.19) to the solution u of (4.1) and use the divergence theorem we obtain

$$\begin{aligned} \int_\Omega |\mathrm{Hess}(u)|^2 \, d\mu &= \frac{1}{2} \int_\Omega \Delta|\nabla u|^2 \, d\mu - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu - \int_\Omega g\,(\nabla \Delta u, \nabla u) \; d\mu \\ &= \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial \nu}|\nabla u|^2 \, d\sigma - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu + \int_\Omega f'(u)|\nabla u|^2 \, d\mu. \end{aligned} \tag{4.7}$$

The first integral at the right-hand side of (4.6) can be estimated by means of the inequality

$$|\nabla|\nabla u||^2 \le |\mathrm{Hess}(u)|^2.$$

This result follows immediately from Schwarz's inequality. Indeed if we use a local orthonormal frame (see Section 3.1) then

$$\frac{u_{ik} u_k u_{ji} u_i}{u_s u_s} \le u_{ik} u_{ik}$$

which is the desired result, see also for instance [2, formula (3.6)]. The Hessian is then replaced by the expression in (4.7) and inserted in (4.6). This leads to the inequality

$$\lambda_1 \int_\Omega |\nabla u|^2 \, d\mu \le \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial \nu}|\nabla u|^2 \, d\sigma - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu + \alpha \int_{\partial\Omega} |\nabla u|^2 \, d\sigma. \tag{4.8}$$

Now by (3.31) and (3.35), taking into account that $\frac{\partial u}{\partial \nu} = -\alpha u$ on $\partial\Omega$, we have

$$\frac{1}{2} \frac{\partial}{\partial \nu}|\nabla u|^2 = \mathrm{II}\big(\widetilde{\nabla} u, \widetilde{\nabla} u\big) - \alpha|\widetilde{\nabla} u|^2 + \alpha u \widetilde{\Delta} u + \alpha u f(u) + \alpha^2 (m-1) u^2 H$$

on $\partial\Omega$. Integrating over $\partial\Omega$ and substituting into (4.8) we deduce

$$\begin{aligned} \lambda_1 \int_\Omega |\nabla u|^2 \, d\mu &\le \int_{\partial\Omega} \Big(\mathrm{II}\big(\widetilde{\nabla} u, \widetilde{\nabla} u\big) - \alpha|\widetilde{\nabla} u|^2 + \alpha u \widetilde{\Delta} u + \alpha u f(u) + \alpha^2 (m-1) u^2 H\Big) \, d\sigma \\ &\quad - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu + \alpha \int_{\partial\Omega} |\nabla u|^2 \, d\sigma. \end{aligned} \tag{4.9}$$

Since $\partial\Omega$ is a manifold without boundary we have by the divergence theorem

$$\int_{\partial\Omega} u \widetilde{\Delta} u \, d\sigma = - \int_{\partial\Omega} |\widetilde{\nabla} u|^2 \, d\sigma.$$

On $\partial\Omega$ there holds

$$|\nabla u|^2 = |\widetilde{\nabla} u|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 = |\widetilde{\nabla} u|^2 + \alpha^2 u^2.$$

Substitution into (4.9) leads to

$$\begin{aligned} \lambda_1 \int_\Omega |\nabla u|^2 \, d\mu &\le \int_{\partial\Omega} \Big(\mathrm{II}\big(\widetilde{\nabla} u, \widetilde{\nabla} u\big) + \alpha|\nabla u|^2 - 2\alpha|\widetilde{\nabla} u|^2 + \alpha u f(u) + \alpha^2 (m-1) u^2 H\Big) \, d\sigma \\ &\quad - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu \\ &= \int_{\partial\Omega} \Big(\mathrm{II}\big(\widetilde{\nabla} u, \widetilde{\nabla} u\big) - \alpha|\widetilde{\nabla} u|^2 + \alpha^3 u^2 + \alpha u f(u) + \alpha^2 (m-1) u^2 H\Big) \, d\sigma \\ &\quad - \int_\Omega \mathrm{Ric}\,(\nabla u, \nabla u) \, d\mu \end{aligned}$$

which completes the proof.

We are now in position to state our main result.

Theorem 4.4. *Let $u \in C^3(\overline{\Omega})$ be a solution of* (4.1) *with $f \in C^1$. Assume that* $\mathrm{Ric} \geq 0$ *in Ω and that for every $p \in \partial\Omega$ the quadratic form* $\mathrm{II} - \alpha\widetilde{g}$ *on the tangent space $T_p(\partial\Omega)$, where $\widetilde{g}$ is the restriction of the metric g on $T_p(\partial\Omega)$, is nonpositive definite. If in addition*

$$\int_{\partial\Omega} \alpha^3 u^2 + \alpha u f(u) + \alpha^2(m-1)u^2 H \, d\sigma < 0, \tag{4.10}$$

then u is unstable.

Proof of Theorem 4.4. Using (4.5) it is immediate to see that under our assumptions λ_1 as defined in formula (4.2) is strictly negative, so that u is an unstable solution of (4.1). □

Next we extend this result to the case where condition (4.10) is relaxed relaxed.

Theorem 4.5. *Assume that all assumptions of Theorem 4.4 hold except for condition* (4.10) *which is replaced by*

$$\int_{\partial\Omega} \alpha^3 u^2 + \alpha u f(u) + \alpha^2(m-1)u^2 H \, d\sigma \leq 0. \tag{4.11}$$

Suppose moreover that $u \not\equiv 0$ and

(i) *$\alpha > 0$, or*

(ii) *$\alpha < 0$ and u does not change sign, i.e. either $u \geq 0$ or $u \leq 0$ on $\overline{\Omega}$.*

Then u is unstable.

Proof of Theorem 4.5. We want to show that under our assumptions λ_1 as defined in formula (4.2) is strictly negative, so that u is an unstable solution of (4.1).

We first note that u cannot be constant on Ω because the only constant solution of (4.1) is $u \equiv 0$. Thus, since $|\nabla u| \not\equiv 0$ in Ω it follows immediately from (4.5) and our assumptions that $\lambda_1 \leq 0$. Let $\alpha > 0$ and suppose that $\lambda_1 = 0$. Then $|\nabla u|$ is a minimizer of the Rayleigh quotient given in (4.2). Hence $|\nabla u|$ is a nontrivial eigenfunction associated to the eigenvalue $\lambda_1 = 0$. By the strong maximum principle we must have $|\nabla u| > 0$ in Ω, so that u does not have any critical point in Ω. Since $\overline{\Omega}$ is compact, u must achieve its absolute maximum over $\overline{\Omega}$ at a point $p \in \partial\Omega$ and its absolute minimum at a point $q \in \partial\Omega$. By the Robin boundary conditions and since $\alpha > 0$ we have

$$u(p) = -\frac{1}{\alpha}\frac{\partial u}{\partial \nu}(p) \leq 0, \qquad u(q) = -\frac{1}{\alpha}\frac{\partial u}{\partial \nu}(q) \geq 0,$$

so that for every $x \in \overline{\Omega}$ we have

$$0 \leq u(q) \leq u(x) \leq u(p) \leq 0,$$

which contradicts our assumption $u \not\equiv 0$. Then $\lambda_1 < 0$, and hence u in unstable.

Assume now $\alpha < 0$ and that $u \geq 0$ in $\overline{\Omega}$. If we assume by contradiction that $\lambda_1 = 0$, arguing as above we see that u must achieve its absolute minimum over $\overline{\Omega}$ at a point $q \in \partial\Omega$, where there holds

$$0 \leq u(q) = -\frac{1}{\alpha}\frac{\partial u}{\partial \nu}(q) \leq 0.$$

Hence we see that

$$u(q) = \frac{\partial u}{\partial \nu}(q) = 0.$$

Since $q \in \partial\Omega$ is a minimum point for u, all tangential derivatives of u must vanish at q, so that $\widetilde{\nabla} u(q) = 0$. We conclude that $|\nabla u|(q) = 0$, and hence q is an absolute minimum point for $|\nabla u|$. Since $|\nabla u|$ is an eigenfunction of problem (4.1) associated to the eigenvalue $\lambda_1 = 0$ and since by the strong maximum principle we have $|\nabla u| > 0$ in Ω, we conclude by the Hopf lemma that

$$\frac{\partial}{\partial \nu}|\nabla u|(q) < 0. \tag{4.12}$$

On the other hand, by the Robin boundary condition in (4.4), we must have

$$\frac{\partial}{\partial \nu}|\nabla u|(q) = -\alpha|\nabla u|(q) = 0,$$

which contradicts (4.12). Thus we have $\lambda_1 < 0$ and u is unstable.

The case that $\alpha < 0$ and $u \leq 0$ in $\overline{\Omega}$ can be treated in similar way and the proof will thus be omitted. □

Remark 4.6. Note that the condition $\mathrm{II} - \alpha\widetilde{g} \leq 0$ immediately implies that $H = \frac{1}{m-1}\operatorname{Tr}(\mathrm{II}) \leq \alpha$ on $\partial\Omega$. Thus, under the above assumptions, condition (4.11) is automatically satisfied if

1) $\alpha > 0$ and $tf(t) \leq -\alpha^2 m t^2$ for every $t \in \mathbb{R}$, or
2) $\alpha < 0$ and $tf(t) \geq -\alpha^2 m t^2$ for every $t \in \mathbb{R}$.

Remark 4.7. Notice that if $\widetilde{\nabla} u = 0$ on $\partial\Omega$, i.e. if u is constant on each connected component of $\partial\Omega$, then the hypothesis

$$\mathrm{II} - \alpha\widetilde{g} \leq 0 \quad \text{in } T_p(\partial\Omega) \text{ for every } p \in \partial\Omega$$

can be dropped in both Theorems 4.4 and 4.5 .

Example 4.8. If M is a two-dimensional manifold as in Example 3.3 then the conditions of Theorem 4.4 (see also Theorem 4.5) become in view of the computation in Examples 3.3 and 3.5

$$\Delta \log p \leq 0 \text{ in } \Omega, \tag{4.13}$$

$$\kappa_g + \alpha \geq 0 \text{ on } \partial\Omega, \tag{4.14}$$

$$\alpha - \kappa_g + \frac{f(u)}{\alpha u} < 0 \text{ on } \partial\Omega. \tag{4.15}$$

Observe that the conditions (4.14) and (4.15) coincide with (**C1**) and (**C2**) in Section 2. Condition (4.13) is satisfied for a sphere.

Remark 4.9. If $\mathrm{Ric} \geq 0$ in Ω and $\mathrm{II} - \alpha\widetilde{g} \leq 0$ on $\partial\Omega$ we obtain estimates for stable solutions. In this case $\lambda_1 \geq 0$. Consequently

$$\frac{f(u)}{\alpha u} \geq -[\alpha + (m-1)H] \text{ somewhere on } \partial\Omega.$$

Consider the following two examples.

1. Let $f(u) = \lambda_1 u$, so that u is a solution of $\Delta u + \lambda_1 u = 0$ in Ω, $\frac{\partial}{\partial\nu}u + \alpha u = 0$ on $\partial\Omega$ with $\alpha > 0$. Then
$$\lambda_1 \geq -[\alpha^2 + (m-1)H_{\max}\alpha].$$
2. Let $f(u) = -c^2 u + |u|^{p-1}u$, so that u is a solution of $\Delta u - c^2 u + |u|^{p-1}u = 0$ in Ω, $\frac{\partial}{\partial\nu}u + \alpha u = 0$ on $\partial\Omega$ with $p > 1$, $\alpha > 0$. Then
$$\max_{\Omega} |u|^{p-1} \geq -[\alpha^2 + (m-1)H_{\max}\alpha] + c^2.$$

We conclude the section with a Barta type inequality that gives a sufficient condition for stability and which will be used in Section 5.

Lemma 4.10. *Let v be a stationary solution of problem (4.1) in Ω. Let there exist a function $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $w > 0$ in $\overline{\Omega}$ and*

$$\begin{cases} \Delta w + f'(v)w < 0 & \text{in } \Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha w \geq 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then v is asymptotically stable.

Proof. Let λ_1 be the smallest eigenvalue of (1.3) and let φ_1 be the corresponding eigenfunction. We have

$$\begin{aligned} 0 \;&>\; \int_\Omega \varphi_1\{\Delta w \,+\, f'(v)w\}\,dV \;=\; \int_\Omega w\,\{\Delta\varphi_1 + f'(v)\varphi_1\}\,d\mu + \int_{\partial\Omega}\left\{\varphi_1\frac{\partial w}{\partial\nu} - w\frac{\partial\varphi_1}{\partial\nu}\right\}d\sigma \\ &\geq\; -\lambda_1\int_\Omega w\varphi_1\,d\mu + \alpha\int_{\partial\Omega}(w\varphi_1 - \varphi_1 w)\,dv \;=\; -\lambda_1\int_\Omega w\varphi_1\,d\sigma. \end{aligned}$$

Therefore $\lambda_1 > 0$, thus the conclusion follows.

5. Surfaces of revolution in $\mathbb{R}^3$

A *surface of revolution* S_ψ *in* $\mathbb{R}^3$ is obtained by rotating around the z-axis a simple, regular plane curve $r \to (\psi(r), \chi(r))$ ($r \in I \equiv [r_1, r_2]$; $r_1 < r_2$) with $\psi > 0$ in (r_1, r_2). Therefore it admits a parametrization of the form

$$\begin{cases} x = \psi(r)\cos\theta \\ y = \psi(r)\sin\theta, \qquad\qquad \big((r,\theta)\in[r_1,r_2]\times[0,2\pi)\big) \\ z = \chi(r). \end{cases} \tag{5.1}$$

We can always assume that $(\psi')^2 + (\chi')^2 = 1$ in I. Moreover, we suppose that $\psi(r_1) > 0, \psi(r_2) > 0$, thus

$$\partial S_\psi \;=\; \big\{\big(\psi(r)\cos\theta, \psi(r)\sin\theta, \chi(r)\big)\,|\, r\in\{r_1,r_2\}, \theta\in[0,2\pi)\big\}\,.$$

A surface of revolution S_ψ in $\mathbb{R}^3$ (with parameterization (5.1)) is a 2-dimensional Riemannian manifold with metric

$$ds^2 = dr^2 + \psi^2(r)d\theta^2\,.$$

In the coordinates (r,θ) ($r \in (r_1, r_2), \theta \in (0, 2\pi)$) the Laplace-Beltrami operator on S_ψ is expressed as

$$\Delta u \;=\; \frac{\partial^2 u}{\partial r^2} \;+\; \frac{\psi'}{\psi}\frac{\partial u}{\partial r} \;+\; \frac{1}{\psi^2}\frac{\partial^2 u}{\partial\theta^2}\,. \tag{5.2}$$

A direct calculation shows that the Ricci (Gaussian) curvature of S_ψ is

$$R(r) = -\frac{\psi''(r)}{\psi(r)} \qquad (r\in(r_1,r_2))\,. \tag{5.3}$$

Observe that it does not depend on the direction X, nor on the angle θ. This is in accordance with the fact that on 2-dimensional surfaces the Ricci curvature is independent of the direction and coincides with the Gaussian curvature. Let us also point out for further references that the quantity $\dfrac{\psi'}{\psi}$ represents the *geodesic curvature* k_g of the parallel circles $r = constant$ on S_ψ.

5.1. **Instability.** Let

$$\Omega := \{(\psi(r)\cos\theta, \psi(r)\sin\theta, \chi(r))\,|\,(r,\theta)\in[0,a]\times(0,2\pi]\}$$

be an annular domain on a surface of revolution S_ψ with parametrization (5.1) ($r_1 \leq 0 < a \leq r_2$). Note that $\partial\Omega$ is made of the two geodesic circles:

$$\begin{aligned} C_0 \;&:=\; \{(\psi(0)\cos\theta, \psi(0)\sin\theta, \chi(0))\,|\,\theta\in(0,2\pi]\}\,, \\ C_a \;&:=\; \{(\psi(a)\cos\theta, \psi(a)\sin\theta, \chi(a))\,|\,\theta\in(0,2\pi]\}\,. \end{aligned}$$

For the sake of simplicity we assume that $\chi'(0) > 0, \chi'(a) > 0$.

Let us start with a simple observation concerning non radial equilibrium solutions (see also [2], [22] for the case $\alpha = 0$).

Proposition 5.1. *Every equilibrium solution v of problem* (1.2)*, which depends on the angle θ, is unstable.*

Proof. By (5.2) v is a solution of

$$v_{rr} + \frac{\psi'}{\psi} v_r + \frac{1}{\psi^2} v_{\theta\theta} + f(v) = 0 \text{ in } (r_1, r_2) \times (0, 2\pi). \tag{5.4}$$

If we differentiate this equation with respect to θ we see that v_θ is an eigenfunction of (1.3) and that $\lambda = 0$ is the corresponding eigenvalue. The function v_θ changes sign and therefore it cannot be the eigenfunction corresponding to the lowest eigenvalue. Hence $\lambda_1 < 0$, which establishes the assertion.

From now on let $v(r)$ be a radial stationary solution. If we differentiate (5.4) with respect to r we get, setting $' := \frac{d}{dr}$,

$$\Delta v' + \left(\frac{\psi'}{\psi}\right)' v' + f'(v) v' = 0.$$

Multiplication by v' and integration over Ω yields

$$-\int_\Omega (v'')^2 \, dV + \int_\Omega f'(v)(v')^2 \, dV + \int_{C_a} v'v'' \, ds - \int_{C_0} v'v'' \, ds + \int_\Omega \left(\frac{\psi'}{\psi}\right)' (v')^2 \, dV = 0.$$

Hence

(5.5)

$$\lambda_1 \int_\Omega (v')^2 \, dV \le \int_\Omega \left(\frac{\psi'}{\psi}\right)' (v')^2 \, dV + L_a\{v'(a)v''(a) + \alpha[v'(a)]^2\} - L_0\{v'(0)v''(0) - \alpha[v'(0)]^2\},$$

where

$$L_0 := 2\pi\psi(0), \quad L_a := 2\pi\psi(a).$$

Note that for $p \in C_a$, $X \in T_p(C_a)$, one has $X = \gamma \frac{\partial}{\partial\theta}$ (for some $\gamma \in \mathbb{R}$) and thus (see, e.g., [20])

$$\mathrm{II}(X, X) = -\gamma^2 \psi(a)\psi'(a).$$

Hence

$$H \equiv H_a = -\frac{\psi'(a)}{\psi(a)}. \tag{5.6}$$

Similarly, for any $q \in C_0$ one has

$$H \equiv H_0 = \frac{\psi'(0)}{\psi(0)}. \tag{5.7}$$

Thus, also using (5.4),

(5.8)

$$\begin{aligned} &L_a\{v'(a)v''(a) + \alpha[v'(a)]^2\} - L_0\{v'(0)v''(0) - \alpha[v'(0)]^2\} \\ &\quad = L_a\left[H_a\alpha^2 v(a)^2 + \alpha v(a) f(v(a)) + \alpha^3 v(a)^2\right] + L_0\left[H_0\alpha^2 v(0)^2 + \alpha v(0) f(v(0)) + \alpha^3 v(0)^2\right] \end{aligned}$$

Therefore, we have the next result.

Theorem 5.2. *Suppose that v is a radial stationary solution of* (1.2)*. If*

$$-\left(\frac{\psi'}{\psi}\right)' = -\frac{\psi''}{\psi} + \left(\frac{\psi'}{\psi}\right)^2 \ge 0 \quad \text{in } (0, a), \tag{5.9}$$

and

$$L_a\left[H_a\alpha^2 v(a)^2 + \alpha v(a) f(v(a)) + \alpha^3 v(a)^2\right] + L_0\left[H_0\alpha^2 v(0)^2 + \alpha v(0) f(v(0)) + \alpha^3 v(0)^2\right] < 0, \tag{5.10}$$

then v is unstable.

The assumption (5.9) has a geometrical meaning in the sense that

$$\left(\frac{\psi'}{\psi}\right)'(r) = -R(r) - [\kappa_g(r)]^2 \qquad (r \in [r_1, r_2]),$$

where $\kappa_g(r)$ is the geodesic curvature of the circles r =const.

5.2. **Existence of stable solutions (patterns).** If the variational problem

$$\min_{v \in W^{1,2}(\Omega)} \mathcal{E}(v, \Omega)$$

is solvable then the minimizer is stable. Hence for positive α and large classes of nonlinearities this is often the case. For Neumann and Robin boundary conditions with negative α the minimum does in general not exist.

In this section we construct on surfaces for which condition (5.9) is violated a problem with negative α possessing a stable solutions satisfying the boundary condition (5.10).

Theorem 5.3. *If, for some $\widehat{R} \in (0, a)$,*

$$\left(\frac{\psi'}{\psi}\right)'(\widehat{R}) > 0\,,$$

then there exists $f \in C^1(\mathbb{R}), \alpha < 0$ such that problem (1.2) *admits a stationary asymptotically stable solution which satisfies* (5.10)*.*

In the proof we follow the arguments used in [2] for the case $\alpha = 0$ (see also [25] where a different differential operator is treated). Several modifications are needed to adapt those proofs to our problem; they are summarized in Remark 5.6.

Take $R_0 < R_1 < R_2 < R_3$ in a neighborhood of $\widehat{R}$. Since $\psi \in C^2(I)$, we can choose R_0 and R_3 such that

$$\left(\frac{\psi'}{\psi}\right)'(s) > 0 \quad \text{for any} \;\; s \in [R_0, R_3]\,. \tag{5.11}$$

Let $z_1 = z_1(s)$ be the solution of the Cauchy problem

$$\begin{cases} \left[\dfrac{(\psi z)'}{\psi}\right]' - Bz \,=\, 0 & \text{in} \;\; [0, R_1) \\ \\ z(0) = 0\,, \quad z'(0) = 1\,, \end{cases} \tag{5.12}$$

where

$$B > \overline{B} := \max_{[0,a]} \left| \left(\frac{\psi'}{\psi}\right)' \right|\,. \tag{5.13}$$

Similarly for any $\beta > 0$, let $z_2 = z_2(s)$ be the solution of the Cauchy problem

$$\begin{cases} \left[\dfrac{(\psi z)'}{\psi}\right]' - Bz \,=\, 0 & \text{in} \;\; (R_2, a] \\ \\ z(a) = \beta\,, \quad z'(a) = -1. \end{cases} \tag{5.14}$$

If necessary we shall write $z_1 = z_1(s, B)$, $z_2 = z_2(s, B, \beta)$ to stress the dependence of the solution on the parameters B and β.

Lemma 5.4. *The solution z_1 of problem* (5.12) *has the following properties:*
(i) $z_1 > 0$ in $(0, R_1)$;
(ii) $z_1(\cdot, B)$ is increasing in $[0, R_1)$ for any $B > \overline{B}$;
(iii) $z_1(r, \cdot)$ is increasing on $(\overline{B}, \infty)$ for any r in $(0, R_1)$;

(iv)
$$\lim_{B \to \infty} z_1(r, B) = \infty \quad \textit{for any} \;\; r \in (0, R_1)\,.$$

Similarly, for the solution z_2 of problem (5.13) *the following hold:*
(i′) $z_2 > \beta$ *in* (R_2, a)*;*
(ii′) $z_2(\cdot, B)$ *is decreasing in* (R_2, a) *for any* $B > \overline{B}$*;*
(iii′) $z_2(r, \cdot)$ *is increasing on* $(\overline{B}, \infty)$ *for any* $r \in (R_2, a)$*;*
(iv′)

$$\lim_{B\to\infty} z_2(r, B) = \infty \quad \textit{for any } r \in (R_2, a)\,.$$

Proof. The statements concerning z_1 have been shown in [2]. Let us show those concerning z_2.
(i') Assume that there exists $\widetilde{r} \in (R_2, a)$ such that

$$z_2(\widetilde{r}) = \beta, \;\; z_2(s) > \beta \quad \text{for any } \; s \in (\widetilde{r}, a)\,.$$

Then for some $\bar{r} \in (\widetilde{r}, a)$ we have

$$z_2(\bar{r}) = \max_{[\widetilde{r},a]} z_2 > \beta\,,\; z_2'(\bar{r}) = 0, \;\; z_1''(\bar{r}) \leq 0.$$

So,

$$\left[\frac{(\psi z_2)'}{\psi}\right]'(\bar{r}) \,-\, B z_2(\bar{r}) \;=\; z_2''(\bar{r}) + \frac{\psi'}{\psi} z_2'(\bar{r}) + \left[\left(\frac{\psi'}{\psi}\right)' - B\right] z_2(\bar{r}) < 0\,.$$

This contradicts the definition of z_2, hence the claim follows.

(ii') Suppose on the contrary that there exist $\widehat{r} \in (R_2, a)$ such that

$$z_2'(r) < 0 \quad \text{for any } \; r \in (\widehat{r}, a)\,, \qquad z_2'(\widehat{r}) = 0 \quad \Rightarrow \quad z_2''(\widehat{r}) \leq 0\,. \tag{5.15}$$

On the other hand, we have

$$z_2''(\widehat{r}) = -\frac{\psi'}{\psi} z_2'(\widehat{r}) - \left[\left(\frac{\psi'}{\psi}\right)' - B\right] z_2(\widehat{r}) > 0$$

since by (i) $z_2(\widehat{r}) > 0$. This is a contradiction, thus z_2 is increasing in (R_2, a).

(iii') Let $B_2 > B_1 > \overline{B}$. Set $\zeta_1(r) := z_2(r; B_1), \zeta_2(r) := z_2(r; B_2)$. Then $w(r) := \zeta_1 - \zeta_2$ solves

$$\begin{cases} \left(\frac{(\psi w)'}{\psi}\right)' = B_1\zeta_1 - B_2\zeta_2 < B_2\zeta_1 - B_2\zeta_2 = B_2 w \quad \text{in } \; [R_2, a) \\ \\ w(a) = 0\,, \quad w'(a) = 0\,. \end{cases}$$

Therefore, w satisfies

$$\begin{cases} w'' + \frac{\psi'}{\psi} w' + \left\{\left(\frac{\psi'}{\psi}\right)' - B_2\right\} w < \; 0 \quad \text{in } \; [R_2, a) \\ \\ w(a) = 0\,, \quad w'(a) = 0\,. \end{cases}$$

Hence, it is easily seen that $w < 0$ in $[R_2, a)$, so

$$z_2(r, B_1) \leq z_2(r, B_2) \quad \text{for any } \; r \in [R_2, a)\,,$$

thus the claim follows.

(iv') Fix any $B_1 > \overline{B}$. Integrating the differential equation in (5.14) and using (ii') we get for any $r \in [R_2, a)$ and $B \geq B_1$:

$$\begin{aligned} z_2(r, B) &= \frac{1}{\psi(r)} \left\{ B \int_r^a \psi(\tau) \int_\tau^a z_2(t, B)\, dt d\tau + \beta\psi(a) + \left[1 + \frac{\psi'(a)}{\psi(a)}\beta\right] \int_r^a \psi(\tau) d\tau \right\} \geq \\ &\geq \frac{1}{\psi(r)} \left\{ B \int_r^a \psi(\tau) \int_\tau^a z_2(t, B_1)\, dt d\tau + \left[1 + \frac{\psi'(a)}{\psi(a)}\beta\right] \int_r^a \psi(\tau) d\tau \right\}\,. \end{aligned} \tag{5.16}$$

The claim follows by letting $B \to \infty$.

Define

$$z(r) := \begin{cases} z_1(r) & \text{if } r \in [0, R_1)\,, \\ \\ z_3(r) & \text{if } r \in [R_1, R_2]\,, \\ \\ z_2(r) & \text{if } r \in (R_2, a]\,; \end{cases} \tag{5.17}$$

here z_3 is any positive smooth function such that z is smooth at the points $r = R_1$, $r = R_2$. By its definition and Lemma 4.10-(i), the function z is smooth in $[0, a]$ and

$$z > 0 \quad \text{in } (0, a)\,, \qquad z(0) = 0,\; z(a) = \beta\,. \tag{5.18}$$

Clearly, z depends on the choice of the parameter β; to highlight this we write $z = z_\beta$, if it is needed.

Lemma 5.5. *Let $\beta > 0$, let the function $z = z_\beta$ be defined by* (5.17)*. Then there exists $f \in C^1(\mathbb{R})$ such that the function*

$$Z(r) := \int_0^r z(s)\, ds \qquad (r \in [0, a]) \tag{5.19}$$

is a stationary solution of problem (1.2)*, which satisfies* (5.10)*, provided*

$$\alpha = -\frac{\beta}{\int_0^a z(r) dr}\,.$$

Proof. Since $z > 0$ in $(0, a)$, the function $u = Z(r)$ is increasing in $(0, a)$. Denote by $r = Z^{-1}(u)$ the inverse function, then define

$$f(u) := \begin{cases} -Bu - 1 & \text{if } u \le 0\,, \\ \\ -\dfrac{d\big\{\psi[Z^{-1}(u)]z[Z^{-1}(u)]\big\}/du}{\psi[Z^{-1}(u)]\frac{d[(Z^{-1})(u)]}{du}} & \text{if } 0 < u < Z(a)\,, \\ \\ -Bu + BZ(a) + 1 - \beta\frac{\psi'(a)}{\psi(a)} & \text{if } u \ge Z(a)\,. \end{cases} \tag{5.20}$$

In order to guarantee that $f \in C^1(\mathbb{R})$ we have to prove that f is smooth at $u = 0$ and $u = Z(a)$. The smoothness at $u = Z(a)$ will follow, if we can show that

$$f(u) = -Bu + BZ(a) + 1 - \beta\frac{\psi'(a)}{\psi(a)} \quad \text{for any } u \in (Z(R_2), Z(a)]\,. \tag{5.21}$$

For that purpose, let us integrate the differential equation in (5.14) on (r, a) for any fixed $r \in (R_2, a)$. We obtain

$$\frac{(\psi z)'}{\psi}(r) = B[Z(r) - Z(a)] + z'(a) + \frac{\psi'(a)}{\psi(a)} z(a) \quad \text{for any } r \in (R_2, a)\,. \tag{5.22}$$

On the other hand, it is easily seen that

$$f[Z(r)] = -\frac{(\psi Z')'}{\psi}(r) = -\frac{(\psi z)'}{\psi}(r) \tag{5.23}$$

for any $r \in (0, a)$. Therefore, by (5.22)- (5.23) we have

$$f[Z(r)] = -BZ(r) + BZ(a) + 1 - \beta\frac{\psi'(a)}{\psi(a)} \quad \text{for any } r \in (R_2, a)\,. \tag{5.24}$$

Since Z is increasing, (5.24) holds for $u = Z(r)$ in $(Z(R_2), Z(a)]$. Similarly it is seen that

$$f(u) = -Bu - 1 \quad \text{for any } u \in [0, Z(R_1))\,, \tag{5.25}$$

which implies the smoothness of f at $u = 0$. Hence $f \in C^1(\mathbb{R})$.

Observe that due to (5.18) and our choice of α we have

$$Z'(0) + \alpha Z(0) = z(0) = 0, \quad Z'(a) + \alpha Z(a) = \beta + \alpha \int_0^a z(r)dr \ = \ 0.$$

Note that by (5.23), Z solves the differential equation in (1.2). Therefore, Z is a stationary solution to problem (1.2). Moreover, due to (5.8) it is easily seen that $Z \not\equiv 0$ satisfies (5.10). Then the conclusion follows. □

Now we are in position to prove Theorem 5.3.

Proof of Theorem 5.3. Let Z be the stationary non constant solution of problem (1.2) with the function f defined by (5.20) of Lemma 5.5. Define

$$w(r) := \begin{cases} z(r) - m_1 z(R_0)(r - R_1)^{3l} & \text{if } r \in [0, R_1)\,, \\ z(r) & \text{if } r \in [R_1, R_2]\,, \\ z(r) + m_2 z(R_3)(r - R_2)^{3l} & \text{if } r \in (R_2, a]\,, \end{cases} \tag{5.26}$$

with constants $m_1 \in (0,\infty), m_2 \in (0,\infty), l \in \mathbb{N}, l$ odd, that will be chosen later. Observe that $w > 0$ in $[0, a]$. Furthermore, recall that z, and hence w, depend on the parameter B in problems (5.12) and (5.14).

Without loss of generality, we can suppose $\chi' > 0$ in $(0, a)$ (see (5.1)). Therefore,

$$\frac{\partial w}{\partial \nu}(0) = -w'(0)\,, \quad \frac{\partial w}{\partial \nu}(a) = w'(a)\,.$$

Next we shall prove the following

Claim: There exist $m_1 > 0, m_2 > 0, l \in \mathbb{N}, l$ odd and $B > 0$ satisfying (5.13) such that

$$\begin{cases} \dfrac{(\psi w')'}{\psi} + f'(Z)w \ < \ 0 & \text{in } \ (0, a) \\ w'(0) - \alpha w(0) < 0\,, \ \ w'(a) + \alpha w(a) > 0\,. \end{cases} \tag{5.27}$$

In order to establish the first inequality in (5.27), we think of the interval $(0, a)$ as the disjoint union $(0, a) = (0, R_1) \cup [R_1, R_2] \cup (R_2, a)$. Recall that by definition $z = z_1$ in $(0, R_1)$ and $z = z_2$ in (R_2, a). Observe that for any $r \in (0, R_1) \cup (R_2, a)$ we have by (5.21) and (5.24)

$$f'(Z(r)) = -B$$

and by (5.12) and (5.14)

$$\frac{(\psi z')'}{\psi} \ - \ Bz = -\left(\frac{\psi'}{\psi}\right)' z.$$

This together with the definition of w yields for any $r \in (0, R_1)$

$$\begin{aligned} \frac{(\psi w')'}{\psi} \ + \ f'(Z)w &= \frac{(\psi w')'}{\psi} \ - Bw = -\left[\left(\frac{\psi'}{\psi}\right)' z\right](r) \\ &+ m_1 z(R_0)(R_1 - r)^{3l-2}\left[3l(3l-1) + 3l\left(\frac{\psi'}{\psi}\right)(r)(r - R_1) - B(r - R_1)^2\right]. \end{aligned} \tag{5.28}$$

Let us prove that the right-hand side of the above expression is negative in $(0, R_1) = (0, R_0) \cup [R_0, R_1)$. For this purpose observe that:

- in $(0, R_0)$ there holds

  $$-\left[\left(\frac{\psi'}{\psi}\right)' z\right](r) \le \left|\left(\frac{\psi'}{\psi}\right)' z\right|(r) \le \overline{B} z(R_0)$$

  (with $\overline{B}$ defined in (5.13)), since $z = z_1$ is increasing by Lemma 5.4-(ii);

- in $[R_0, R_1)$ we have
$$-\left[\left(\frac{\psi'}{\psi}\right)' z\right](r) \leq -\underline{B}z(r) \leq -\underline{B}z(R_0)\,,$$
where
$$\underline{B} := \min_{[R_0,R_1]} \left(\frac{\psi'}{\psi}\right)'.$$
Note that $\underline{B} > 0$ by (5.11); moreover, Lemma 5.4-(ii) has been used again.

By the above remarks, we have in $(0, R_0)$

$$(5.29)\qquad -\left[\left(\frac{\psi'}{\psi}\right)' z\right](r) + m_1 z(R_0)(R_1 - r)^{3l-2}\left[3l(3l-1) + 3l\left(\frac{\psi'}{\psi}\right)(r)(r - R_1) - B(r - R_1)^2\right] \leq$$
$$\leq z(R_0)\left\{\overline{B} + m_1(R_1 - R_0)^{3l-2}\left[3l(CR_1 + 3l - 1) - B(R_0 - R_1)^2\right]\right\},$$

if
$$B \geq \frac{3l(3l - 1 + CR_1)}{(R_1 - R_0)^2},$$
where
$$C := \max_{[0,a]}\left|\frac{\psi'}{\psi}\right|.$$
Similarly, in $[R_0, R_1)$ we have:

$$(5.30)\qquad -\left[\left(\frac{\psi'}{\psi}\right)' z\right](r) + m_1 z(R_0)(R_1 - r)^{3l-2}\left[3l(3l-1) + 3l\left(\frac{\psi'}{\psi}\right)(r)(r - R_1) - B(r - R_1)^2\right] \leq$$
$$\leq z(R_0)\left[-\underline{B} + 3l m_1 R_1^{3l-2}(3l - 1 + CR_1)\right].$$

It is easily seen that the right-hand sides of inequalities (5.29), (5.30) are both negative if we further require that
$$B \geq \frac{\overline{B} + 3l m_1 (R_1 - R_0)^{3l-2}(CR_1 + 3l - 1)}{m_1 (R_1 - R_0)^{3l}} \quad \text{and} \quad 0 < m_1 < \frac{\underline{B}}{3l R_1^{3l-2}(CR_1 + 3l - 1)}.$$
Then from (5.28) we obtain that

$$(5.31)\qquad \frac{(\psi w')'}{\psi} + f'(Z)w < 0 \quad \text{in } (0, R_1]\,.$$

It is similarly seen that, for $m_2 > 0$ small enough and $B > 0$ sufficiently large,

$$(5.32)\qquad \frac{(\psi w')'}{\psi} + f'(Z)w < 0 \quad \text{in } (R_2, a)\,.$$

Now consider the interval $[R_1, R_2]$. Since Z is a stationary solution of problem (1.2), in $[R_1, R_2]$ there holds
$$Z'' + \frac{\psi'}{\psi}Z' + f(Z) = 0.$$
Deriving the above equality and recalling that $Z' = z$ we obtain
$$z'' + \frac{\psi'}{\psi}z' + f'(Z)z = -\left(\frac{\psi'}{\psi}\right)' z \quad \text{in } [R_1, R_2]\,.$$
The right-hand side of the above equality is negative in $[R_1, R_2]$ by inequality (5.11), thus

$$(5.33)\qquad \frac{(\psi w')'}{\psi} + f'(Z)w = \frac{(\psi z')'}{\psi} + f'(Z)z < 0 \quad \text{in } [R_1, R_2]\,.$$

From (5.32)-(5.33) we conclude that the first inequality of (5.27) is satisfied. It remains to prove the inequalities $-w'(0) + \alpha w(0) > 0$, $w'(a) + \alpha w(a) > 0$. For this purpose, note that in view of (5.16) we can infer that there exist two constants $C_0 > 0, B^* > \bar{B}$ such that

$$\int_0^a z(r; B)\, dr \geq C_0 B \quad \text{for any } B > B^* .$$

Thus,

$$\alpha = -\frac{\beta}{\int_0^a z(r)dr} > -\frac{\beta}{C_0 B} \quad \text{for any } B > B^* . \tag{5.34}$$

Hence, in view of (5.34), (iv) and (iv'), choosing $B > B^*$ large enough and $l > \frac{\beta}{3C_0 B} \max\{R_1, a - R_2\}$, we obtain

$$-w'(0) + \alpha w(0) = -1 + m_1 z(R_0) R_1^{3l-1}(3l + \alpha R_1) > 0 ,$$

and

$$w'(a) + \alpha w(a) = -1 + \alpha\beta + m_2 z(R_3)(a - R_2)^{3l-1}[3l + (a - R_2)\alpha] > 0 .$$

This completes the proof of the Claim. Observe that (5.10) is satisfied. Then by Lemmas 5.5 and 4.10 the function Z is a stable stationary solution of problem (1.2) with f given by (5.20). Then the conclusion follows. □

Remark 5.6. Note that the construction of z and f are similar to that in [2]. However, in [2] we had $\beta = 0$; instead now we need $\beta > 0$. Moreover, in the proof of the result in [2] analogous to Theorem 5.3, we had $l = 1$ in (5.26).

6. Further examples

6.1. **Spherically symmetric manifolds.** We start recalling some basic notions on spherically symmetric manifolds. Let M be a complete Riemannian manifold. Let us fix a point $o \in M$ and denote by $\mathrm{Cut}(o)$ the *cut locus* of o. For any $x \in M \setminus [\mathrm{Cut}(o) \cup \{o\}]$, one can define the *polar coordinates* with respect to o, see e.g. [8]. Namely, for any point $x \in M \setminus [\mathrm{Cut}(o) \cup \{o\}]$ there correspond a polar radius $r(x) := dist(x, o)$ and a polar angle $\theta \in \mathbb{S}^{m-1}$ such that the shortest geodesic from o to x starts at o with the direction θ in the tangent space $T_o M$. Since we can identify $T_o M$ with $\mathbb{R}^m$, θ can be regarded as a point of $\mathbb{S}^{m-1}$.

The Riemannian metric in $M \setminus [\mathrm{Cut}(o) \cup \{o\}]$ in polar coordinates reads

$$ds^2 = dr^2 + A_{ij}(r, \theta) d\theta^i d\theta^j ,$$

where $(\theta^1, \ldots, \theta^{m-1})$ are coordinates in $\mathbb{S}^{m-1}$ and (A_{ij}) is a positive definite matrix. It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r, \theta) \frac{\partial}{\partial r} + \Delta_{S_r} , \tag{6.1}$$

where $\mathcal{F}(r, \theta) := \frac{\partial}{\partial r}\big(\log \sqrt{A(r, \theta)}\big)$, $A(r, \theta) := \det(A_{ij}(r, \theta))$, Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B(o, r) \setminus \mathrm{Cut}(o)$.

M is a *manifold with a pole*, if it has a point $o \in M$ with $\mathrm{Cut}(o) = \emptyset$. The point o is called *pole* and the polar coordinates (r, θ) are defined in $M \setminus \{o\}$.

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$ds^2 = dr^2 + \phi^2(r) d\theta^2 , \tag{6.2}$$

where $d\theta^2$ is the standard metric in $\mathbb{S}^{m-1}$, and

$$\phi \in \mathcal{A} := \Big\{ f \in C^\infty((0, \infty)) \cap C^1([0, \infty)) : f'(0) = 1,\ f(0) = 0,\ f > 0 \text{ in } (0, \infty) \Big\}. \tag{6.3}$$

In this case, we write $M \equiv M_\phi$; furthermore, we have $\sqrt{A(r, \theta)} = \phi^{m-1}(r)$, so the boundary area of the geodesic sphere ∂S_R is computed by

$$S(R) = \omega_m \phi^{m-1}(R),$$

ω_m being the area of the unit sphere in $\mathbb{R}^m$. Also, the volume of the ball $B_R(o)$ is given by

$$\mathrm{Vol}(B_R(o)) = \int_0^R S(\xi)d\xi\,.$$

Moreover we have

$$\Delta = \frac{\partial^2}{\partial r^2} + (m-1)\frac{\phi'}{\phi}\frac{\partial}{\partial r} + \frac{1}{\phi^2}\Delta_{\mathbb{S}^{m-1}},$$

or equivalently

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\phi^2}\Delta_{\mathbb{S}^{m-1}}, \tag{6.4}$$

where $\Delta_{\mathbb{S}^{m-1}}$ is the Laplace-Beltrami operator in $\mathbb{S}^{m-1}$. Note that similarly to (5.6) and (5.7) one can compute the mean curvature of $\partial B_\rho(o)$ in the radial direction $\frac{\partial}{\partial r}$ as follows

$$H(r) := -\frac{\phi'(r)}{\phi(r)} \quad \text{for each } r>0\,. \tag{6.5}$$

Observe that for $\phi(r) = r$, $M = \mathbb{R}^m$, for $\phi(r) = \sinh r$, M is the $m-$dimensional hyperbolic space $\mathbb{H}^m$, while for $\phi(r) = \sin r$ $(r \in [0,\pi))$ we have the $m-$dimensional sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ (see [8]).

For any $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$, denote by $\mathrm{Ric}_o(x)$ the *Ricci curvature* at x in the direction $\frac{\partial}{\partial r}$. If $M \equiv M_\psi$ is a model manifold, then for any $x = (r,\theta) \in M \setminus \{o\}$

$$\mathrm{Ric}_o(x) = -(m-1)\frac{\phi''(r)}{\phi(r)}. \tag{6.6}$$

Now we discuss the stability of *radial* solutions of problem (1.2) with $\Omega := B_R(o) \setminus B_r(o) \subset M_\phi$ for each $0 < r < R$.

In view of (6.4) and (5.4), setting $S \equiv \psi$, the same results as in Section 5 hold. Indeed, we have the following theorem.

Theorem 6.1. *Let $\Omega := B_R(o) \setminus B_\rho(o) \subset M_\phi$ with $0<\rho<R$.*

(*i*) *Suppose that v is a radial stationary solution of* (1.2). *If*

$$-\left(\frac{\phi'}{\phi}\right)' = -\frac{\phi''}{\phi} + \left(\frac{\phi'}{\phi}\right)^2 \geq 0 \quad \text{in } (\rho, R)\,, \tag{6.7}$$

and

$$\begin{aligned} &\phi(R)\{v'(R)v''(R) + \alpha[v'(R)]^2\} - \phi(\rho)\{v'(\rho)v''(\rho) - \alpha[v'(\rho)]^2\} \\ &\quad = \phi(R)\big[\alpha^2(m-1)H(R)v(R)^2 + \alpha v(R)f(v(R)) + \alpha^3 v(R)^2\big] \\ &\qquad + \phi(\rho)\big[\alpha^2(m-1)H(\rho)v(\rho)^2 + \alpha v(\rho)f(v(\rho)) + \alpha^3 v(\rho)^2\big] < 0\,, \end{aligned} \tag{6.8}$$

then v is unstable.

(*ii*) *If for some $\widehat{R} \in (\rho, R)$*

$$\left(\frac{\phi'}{\phi}\right)'(\widehat{R}) > 0\,, \tag{6.9}$$

then there exists $f \in C^1(\mathbb{R}), \alpha < 0$ such that problem (1.2) *admits a stationary asymptotically stable solution which satisfies* (6.8).

Note that, in view of (6.5) and (6.6), the inequalities (6.7) and (6.9) have a geometrical meaning. Indeed, (6.7) is equivalent to the following requirement

$$\mathrm{Ric}_o(x) \geq -(m-1)[H(r)]^2 \quad \text{for any } x \equiv (r,\theta) \in \Omega\,,$$

and similarly for (6.9).

6.2. **Straight cylinder in $\mathbb{R}^3$.** A *straight cylinder* $\mathcal{C}$ in $\mathbb{R}^3$ is parameterized as follows

$$\begin{cases} x = \psi(t) \\ y = \chi(t) \qquad \big((t,s) \in [t_1,t_2] \times [s_1,s_2]\big) \\ z = s, \end{cases} \tag{6.10}$$

where $t \mapsto (\psi(t), \chi(t), 0)$ is a simple, regular, closed plane curve ($t \in [t_1, t_2]$; $t_1 < t_2$). We suppose that $[\psi'(t)]^2 + [\chi'(t)]^2 = 1$ for all $t \in [t_1, t_2]$. It is easily seen that, for all $p \in \mathcal{C}$, $X \in T_p\mathcal{C}$,

$$\mathrm{Ric}(X,X) \,=\, 0\,;$$

furthermore, since the second fundamental form of $\partial\mathcal{C}$ with respect to the embedding $\partial\mathcal{C} \hookrightarrow \mathcal{C}$ is identically zero, we also have that its mean curvature identically vanishes.

Note that

$$\Delta u(t,s) \,=\, u_{tt}(t,s) + u_{ss}(t,s)\,.$$

Then, by a similar argument to that of Proposition 5.1, one can see that any stable solution of problem (1.2) must depend only on the variable s.

Now, consider a solution $u = u(s)$ of problem (1.2). Thus, using the same notation as in Section 4, we have $\widetilde{\nabla} u = \frac{\partial}{\partial t} u = 0$. Hence, from the same arguments used in the proofs of Proposition 4.4 and of Theorem 4.5, we can infer that if

i) $\displaystyle\int_{\partial\mathcal{C}} \left[\alpha^3 u^2 + \alpha u f(u)\right] d\sigma < 0$, or

ii) $\alpha > 0$ and $\displaystyle\int_{\partial\mathcal{C}} \left[\alpha^3 u^2 + \alpha u f(u)\right] d\sigma \leq 0$, or

iii) $\alpha < 0$, $\displaystyle\int_{\partial\mathcal{C}} \left[\alpha^3 u^2 + \alpha u f(u)\right] d\sigma \leq 0$ and u does not change sign,

then u is not stable (see also Remark 4.7).

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