Stochastic linear Model Predictive Control with chance constraints – A review

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In the past ten years many Stochastic Model Predictive Control (SMPC) algorithms have been developed for systems subject to stochastic disturbances and model uncertainties. These methods are motivated by many application fields where a-priori knowledge of the stochastic distribution of the uncertainties is available, some degree of constraint violation is allowed, and nominal operation should be defined as close as possible to the operational constraints for economic/optimality reasons. However, despite the large number of methods nowadays available, a general framework has not been proposed yet to classify the available alternatives. For these reasons, in this paper the main ideas underlying SMPC are first presented and different classifications of the available methods are proposed in terms of the dynamic characteristics of the system under control, the performance index to be minimized, the meaning and management of the probabilistic (chance) constraints adopted, and their feasibility and convergence properties. Focus is placed on methods developed for linear systems. In the first part of the paper, all these issues are considered, also with the help of a simple worked example. Then, in the second part, four algorithms representative of the most popular approaches to SMPC are briefly described and their main characteristics are discussed.

Keywords: Model Predictive Control, Stochastic systems, Probabilistic constraints, Analytic approximation methods, Scenario generation methods

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1. Introduction

Model Predictive Control (MPC) is nowadays a standard in many industrial contexts, see e.g., [1], due to its ability to cope with complex control problems and to the availability of theoretical results guaranteeing feasibility and stability properties, see [2]. These reasons have motivated the many efforts devoted to develop MPC algorithms robust with respect to unknown, but bounded disturbances or model uncertainties, see for example [3,4]. However, deterministic robust MPC algorithms can suffer from some problems. Indeed, feasibility, convergence, and stability are usually achieved by resorting, implicitly or explicitly, to a worst-case analysis. In fact, they do not consider the possible a-priori knowledge of the statistical properties of the disturbances, i.e., their distribution function which can be assumed to be available in many problems. Therefore, robust approaches may lead to very conservative schemes, even in cases where some degree of constraint violation is allowed and where, for economic/optimality reasons, nominal operation should be defined as close as possible to the operational constraints.

To overcome these limitations and to tackle those applications where to violate constraints (or to be as close as possible to some operational limits) is somehow rewarded in terms of cost, an emerging field of research concerns the design of innovative Stochastic Model Predictive Control (SMPC) algorithms. SMPC schemes are aimed at exploiting the stochastic nature of the uncertainty and, when available, its statistical description. In this framework, hard constraints on the system variables are reformulated as stochastic ones, allowing the controlled system to violate them in prescribed probabilistic terms. In this scenario, it is possible to consider also unbounded disturbances and/or uncertainties, for example in case they are characterized by a Gaussian distribution.

The use of chance constrained optimization and stochastic MPC has been considered as a promising solution in a number of application domains, such as water reservoir management [5–7], temperature and HVAC control in buildings [8–17], process control [18–22], power production, management, and supply in systems with renewable energy sources [23–34], cellular networks management [35], driver steering, scheduling, and energy management in vehicles [36–44], path planning and formation control [45–48], air traffic control [49], inventory control and supply chain management [50–52], resource allocation [53], portfolio optimization and finance [54–59]. For a detailed review on application of chance constrained programming see [60].

In spite of the large number of potential applications and the already available theoretical results, in the authors' knowledge SMPC has not been yet applied to real plants or systems. The reason of this can be ascribed, for example, to the fact that SMPC solutions are relatively recent and their reliability in real case studies has not yet been assessed. For example, the characterization of the noise properties, fundamental in many SMPC methods, is a crucial point that, in non-ideal conditions, may carry about reliability and robustness issues. Also, many tough challenges emerge in this setup related to the development of methods with guaranteed stability and feasibility properties, together with the possibility of enforcing hard constraints on the input variables.

The classification of the many available SMPC algorithms can be quite difficult due to the large variety of problem formulations and solutions. For example, design methods have been developed for linear or nonlinear, discrete-time or continuous-time systems, with additive, multiplicative or parametric uncertainties, finite or infinite horizon cost functions, polytopic, quadratic or more complex probabilistic constraints. In our opinion, however, most of the available methods share some common basic methodologies and definitions, and it is of great importance to unravel their main common features and differences in a clear and unambiguous way. To do so we will consider, when necessary, the simplest scenario consisting of linear systems with additive noise and individual linear constraints in which, by the way, many of the available methods have been devised.

Therefore, the first goal of this paper is to present the most widely used problem formulations, with particular emphasis on the definition of state and control constraints in probabilistic terms, on the cost function to be minimized, and on the structure of the adopted control law. Also, we propose a classification of the available methods based on the system's assumptions, the adopted MPC formulation, and their feasibility and convergence properties. In Section 2 we introduce the general definitions of stochastic systems and probabilistic-type constraints commonly adopted in the SMPC framework. From Section 3 on, attention is placed on linear systems and on chance constraints. In particular, in Section 3, the scope is to introduce and compare together, in an analytical and comprehensive fashion, the different design choices and approaches, including the different control strategies. We will highlight how these basic choices directly impact on the constraint formulation, on the definition of the cost function, and on the resulting feasibility and convergence properties. In Section 4, focusing on discrete-time linear systems with additive stochastic noise, the main ideas and properties of four paradigmatic algorithms are described. These algorithms can be considered, in our opinion, as representative of different classes of SMPC methods currently available. In this part focus will be placed on how the main ingredients (introduced in general in Section 3) are used to devise such methods and on what are the main properties that originate from these choices, rather than comparing the performance of the single algorithms. In fact, each method basically allows for many different implementations and design choices trading, e.g., optimality (or the possibility to satisfy constraints in a tight way) with computational complexity, or simplicity of design and implementation with guaranteed properties. A section of conclusions, including some general final remarks, comparisons on major implementation issues, and future research directions closes the paper.

1.1. Motivating example

Before continuing, we introduce a motivating example which will be also used in the next section to exemplify some key concepts concerning stochastic systems chance constraints. Consider the following simplified problem: we aim to control the temperature T, assumed to be uniform, inside a room by acting on the heating power Q. The control goal is to maintain T above a prescribed value T_M, suitably chosen to guarantee the comfort level, with the minimum power consumption. The temperature $T_{ex} < T_M$ of the external environment is subject to stochastic variations and any fluctuation of T_{ex} induces variations of T. In addition, the door or the window of the room can be opened for short time intervals. The effect of these actions can be modeled as an impulse-type cooling power Q_c whose effect is to decrease T. The constraint $T > T_M$ can be occasionally violated, since pointwise peaks of the temperature do not cause significant discomfort, for this reason a robust "worst-case" formulation of the control problem can be too conservative, leading to an unnecessary and costly solution. In this case, in addition, even an excessive heat can cause discomfort on the room occupants making it mandatory to be as close as possible to T_M. Rather, it is better to formulate an optimization problem where the constraints on the temperature are stated in probabilistic terms. In addition, it is worth mentioning that the Model Predictive Control approach is the most appropriate design method also in view of the possibility to benefit from the possible a-priori knowledge of future and predictable disturbance variations of T_{ex} and Q_c.

2. System under control and constraints

2.1. The model

The system under control is assumed to be described by the following discrete-time, nonlinear model

$$x_{t+1} = f(x_t, u_t, w_t) \tag{1}$$

In (1) $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathcal{W} \subseteq \mathbb{R}^{n_w}$ are the state, input, and stochastic noise vectors, respectively. Regarding the noise term w_t, several different assumptions may be required. In some works it is assumed that the noise support is bounded and convex, while some approaches discard this assumption; also, some notable works allow for general probability distributions (provided that they are known or that samples can be obtained out of them), while other papers assume specific noise probability density functions, e.g., the Gaussian one; some methods require just the knowledge of the moments (e.g., mean and variance). Finally, in general the terms w_t are assumed to be independent and identically distributed.[1]

The state and input vectors must satisfy, at least ideally, a set of constraints described in very general terms by the inequalities

$$g(x_t, u_t) \le 0 \tag{2}$$

where the function $g(\cdot) : \mathbb{R}^{n \times m} \to \mathbb{R}^r$, can take different forms, as specified in the following. In fact, depending on the noise characteristics, these constraints can be formulated as "hard constraints", i.e., constraint that must be satisfied deterministically, or as "stochastic constraints" to denote that a partial violation is allowed to consider that their deterministic fulfillment can be too tight or even impossible due to the presence of the stochastic noise.

Many SMPC algorithms have been developed with specific reference to linear systems with additive or multiplicative model uncertainty. In the case of additive uncertainty, the adopted model is

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t \tag{3}$$

where the noise term can represent a real disturbance acting on the system or an unmodeled dynamics.

Systems with multiplicative uncertainty are described by the model

$$x_{t+1} = Ax_t + B_u u_t + \sum_{j=1}^{q} [A_j x_t + B_j u_t] w_{jt} \tag{4}$$

and their use is widely popular in specific application fields, such as in financial applications, see e.g., [54,59], where stock prices and the portfolio wealth dynamics are represented as in (4).

In view of the stochastic nature of the noise w, the inclusion of hard constraints (2) in the problem formulation can lead to infeasibility. For instance, if the uncertainty acting on the system has an unbounded support, i.e., the set $\mathcal{W}$ is unbounded as in the case of a Gaussian noise, there is no way to ensure hard constraints on the state variable. Moreover, even if the uncertainty is bounded, the worst-case solution required to satisfy hard constraints might be too conservative, and performance of the solution could be improved by resorting to a stochastic reformulation of (2).

2.2. Constraints

Before introducing how constraints are formally enforced in a Stochastic Model Predictive Control framework, it is of interest to focus on the desired properties obtainable by a control scheme in practice. To do so, we first recall the motivating example shortly introduced in Section 1. The main peculiarity of this example (as well as of a number of other applications of SMPC) is that constraints and control objective are partially conflicting. While the cost function aims to penalize the energy consumption (in this respect, the aim is to keep the temperature as low as possible), the constraints must be formalized in order to guarantee a sufficient level of comfort (heat level) to the occupants, i.e., the temperature should not go below a given value. To do so, in a (deterministic) robust control framework we can enforce this constraint with a given level of conservativity; however, it should be necessary to reduce such conservativity as much as possible for a twofold reason, both economic and comfort-wise (i.e., the temperature must not also exceed a given level), and so we trade conservativity by allowing the lower threshold to be violated with a given rate. It is worth noting that, in this example, the term *rate* may denote the frequency of threshold violation *in time*. On the other hand, another requirement can be stated as follows: when the window/door of the room gets opened (i.e., when the state of the system is reset), the control system must be able to counteract this event by limiting the probability of violating the lower temperature constraint during the transient phase, i.e., we consider the probability of violation as its rate of occurrence with respect to all possible noise realizations.

More in general, consider the constraint $g(x(t), u(t)) \le 0$, to be properly recast in a probabilistic setting. As discussed, e.g., in [61–64], the former requirement can be formulated as the requirement that the expected number of samples over a given time horizon (say of length $N_{horizon}$) at which (2) is violated cannot exceed N_{max}. Mathematically, this translates into

$$\mathbb{E}\left\{ \frac{1}{N_{horizon}} \sum_{t=0}^{N_{horizon}-1} \mathbf{1}_g(x_t, u_t) \right\} < \frac{N_{\max}}{N_{horizon}} \tag{5}$$

where $\mathbf{1}_g$ is the indicator function, i.e., $\mathbf{1}_g(x_t, u_t) = 1$ if $g(x_t, u_t) > 0$ and $\mathbf{1}_g(x_t, u_t) = 0$ otherwise, and $\mathbb{E}\{\mathbf{1}_g(x_t, u_t)\} = \mathcal{P}\{g(x_t, u_t) > 0\}$, where $\mathcal{P}\{\varphi\}$ denotes the probability of φ. In [61–63] (5) is enforced by including the probabilistic (or chance) constraint

$$\mathcal{P}\{g(x_t, u_t) \le 0\} \ge 1 - p \tag{6}$$

where p is a design parameter to be tuned to obtain a trade-off between performance and constraint violation. The second

[1] Concerning the latter point, this assumption can be easily relaxed to the case of non-white processes, provided that they are generated according to a dynamic model fed by white noise, e.g., the well know ARMA models. This extension can be carried out simply by incorporating the noise model into the system's equations.

requirement, related to the point-wise in time allowed rate of violation, e.g., during the transient phase, can also be naturally cast in the form of (6).

Other types of stochastic constraints can be encountered in the literature, e.g.,

- Expectation constraints $\mathbb{E}[g(x, u)] \leq 0$
- Integrated chance constraints $\int_0^{\infty} \mathcal{P}\{g(x, u) \geq s\} ds \leq p$

The use of expectation constraints amounts to ensure that the constraints are satisfied on average for the considered problem. This kind of constraints is used is [65], where it is required that the expected value of quadratic functions of state and input variables respects given bounds. In this way, however, the number of occurred violations is not controlled, at least explicitly.

Finally, integrated chance constraints [66], are a useful tool to express the idea of constraint violation in a more quantitative way. Roughly speaking, in this formulation the constraint violation is allowed with high probability if the amount by which it is violated is small enough.

For a more detailed explanation of the constraint models and a clear analysis of their effects on the problem, the reader is referred to, e.g., [67].

In this paper focus will be placed on the first class of constraints, i.e., the probabilistic ones (6). In general, when $g(x, u)$ is a vector, for example when the goal is it to express the probability that the state and/or the control are inside a certain set, the constraint is called "joint chance constraint". Otherwise, if $g(x, u)$ is a scalar function, the constraint is addressed as "independent chance constraint".[2] Even if the joint representation is more natural, its exact tractable representation may not exist, despite the convexity of the (deterministic) original constraint. This is because the mere evaluation of the constraint requires the computation of a multivariate integral, which is known to become prohibitive in high dimensions. On the other hand, as it will be clearer in the next section, some types (e.g., linear) of individual constraints can be easily accounted for in an analytical way.

For this reason joint chance constraints, especially in the analytical framework, are commonly approximated to obtain a tractable and convex expression. A clear overview of the problem can be found for example in [69,70] and in the references therein. Besides the use of confidence ellipsoids or sampling techniques, the simplest way to work with a joint chance constraint is, however, to approximate it by splitting the overall set into a sequence of individual chance constraints, whose probability sums up to the original one, as described in [46]. In a more formal way, if $g(x, u) = [g_1(x, u), \ldots, g_r(x, u)]^T$, the constraint can be rewritten as

$$\mathcal{P}\{g(x, u) \leq 0\} = \mathcal{P}\left\{\bigwedge_{i=1}^{r} g_i(x, u) \leq 0\right\} \geq 1 - p \rightarrow \mathcal{P}\left\{\bigvee_{i=1}^{r} g_i(x, u) > 0\right\} \leq p$$

[2] Note that this includes the case in which the number of individual chance constraints is greater than one, each possibly imposed with a different rate of violation, as in [68].

and applying the Boole's inequality to the latter expression

$$\mathcal{P}\left\{\bigvee_{i=1}^{r} g_i(x, u) > 0\right\} \leq \sum_{i=1}^{r} \mathcal{P}\{g_i(x, u) > 0\} \rightarrow \mathcal{P}\{g_i(x, u) \leq 0\} \geq 1 - p_i, \quad \sum_{i=1}^{r} p_i = p$$

Due to the fact that $\sum_{i=1}^{r} p_i = p$, the above equation clearly gives a conservative approximation of the original constraint. Based on that, an easy choice can be to equally subdivide the overall risk p setting $p_i = p/r$, $i = 1, \ldots, r$. However, if a less conservative solution is required, the approximation can be reduced including the values p_i as free variables in the optimization problem. This iterative *risk allocation* technique is discussed in [71] for the case of Gaussian uncertainty.

To clarify on the conservativity induced by the reformulation of joint chance constraints into a number of individual chance constraints, consider two independent scalar Gaussian variables $x \sim \mathcal{N}(\bar{x}, 1)$ and $y \sim \mathcal{N}(\bar{y}, 1)$. We aim to define the set of all pairs $(\bar{x}, \bar{y})$ so that the following joint chance constraint is satisfied

$$\mathcal{P}\left\{\begin{bmatrix} x \\ y \end{bmatrix} \geq 0\right\} \geq 1 - p$$

Following the procedure reported above, we formulate the above joint constraint as the pair

$$\mathcal{P}\{x \geq 0\} \geq 1 - p_x, \quad \mathcal{P}\{y \geq 0\} \geq 1 - p_y$$

where $p_x + p_y = p$.

Focusing on the feasible regions of both the original joint chance constrained problem and the simplified one, results are shown in Fig. 1 for $p = 0.5$. In particular, the region corresponding to a single choice of the pair (p_x, p_y) is rectangular and results much smaller than the feasible region of the original problem. Moreover, even using a dynamic risk allocation procedure, i.e., allowing the parameters p_x and p_y to vary (provided that $p_x + p_y = p$), the resulting feasibility region results more conservative than the one obtained using joint chance constraint.

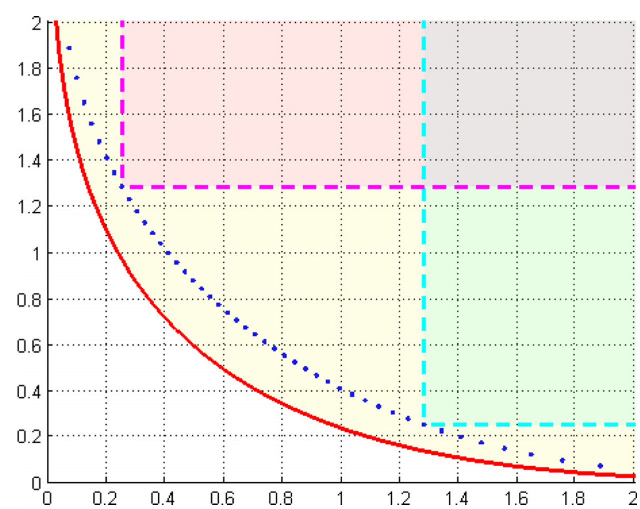

Fig. 1. Feasible region for the joint chance constraint for $p = 0.5$ (yellow region) and for the pair of individual chance constraints corresponding to two different choices of pairs (p_x, p_y). Magenta region: $(p_x, p_y) = (0.4, 0.1)$; cyan region: $(p_x, p_y) = (0.1, 0.4)$. The blue dots define the feasibility region that can be obtained using a risk allocation technique. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.3. Motivating example

The motivating example introduced in Section 1 in a qualitative fashion will be now developed, for later use and analytical analysis in Section 3. Specifically, we need to introduce the mathematical model and how chance constraints can be formulated in this case. The model used, for simplicity, is the discrete-time scalar system

$$x_{t+1} = ax_t + bu_t + w_t \tag{7}$$

where the state x_t coincides with the room temperature T and the control u_t denotes the heating power Q. The noise w_t, induced by the deviation of the external temperature T_{ex} with respect to its nominal value, is white, with zero mean and variance W. To allow comparisons with deterministic robust control methods we assume that w_t is bounded, i.e., $w_t \in [-\bar{w}, \bar{w}]$, $\bar{w} > 0$. For generality purposes, all parameters (i.e., a, b, W, and $\bar{w}$) are not specified. We just assume that $0 < a < 1$. The temperature constraint, in its deterministic version, is

$$x_t \geq T_M \tag{8}$$

which is formulated as the following probabilistic constraint.

$$\mathcal{P}\{x_t \leq T_M\} \geq 1 - p \tag{9}$$

Finally, the role of the impulse-type cooling power input Q_c is to reset the initial condition of x_t.

3. Model Predictive Control formulations, properties, and review of the literature

Once the system model has been chosen and the state and control variables have been properly reformulated as discussed in the previous section, the MPC optimization problem can be stated by defining a suitable cost function together with additional constraints which can be considered to achieve recursive feasibility and stability properties. Then, the specific algorithms can be developed according to different approaches.

It is possible to roughly cluster the different stochastic MPC methods nowadays available in two main classes, based on the approach used for solving the underlying chance constrained optimization problem [60]: the first one, i.e., the so-called [60] *analytic approximation* methods (referred in [72] as *probabilistic approximation* methods), is based on the reformulation of probabilistic-type constraints and of the cost function in terms of variables whose behavior can be characterized in deterministic terms, e.g., mean values and variances, to be included in the MPC formulation. The second class of approaches relies on the *randomized*, or *scenario generation* methods, i.e., on the on-line random generation of a sufficient number of noise realizations, and on the solution to a suitable constrained optimization problem. The main features of these methods are described in this section.

3.1. Control strategies

In the following, to simplify the setup as much as possible, focus will be placed on linear systems of the type (3) with additive zero-mean white noise with bounded support.

In view of the superposition principle, it is always possible, at time t, to write the future evolution of the state variable as $x_{t+i|t} = \bar{x}_{t+i} + e_{t+i}$, with $\bar{x}_t = x_t$, $e_t = 0$, and where $\bar{x}_{t+i}$ evolves independently of the noise w_{t+i}, while e_{t+i} depends (linearly) just on the evolution of the exogenous variable w_{t+i}. Letting $\mathbf{w}_t = \begin{bmatrix} w_t^T & \cdots & w_{t+N-1}^T \end{bmatrix}^T \in \mathcal{W}^N$, it is possible to write

$$e_{t+i} = E_i \mathbf{w}_t \tag{10}$$

where E_i is a suitable matrix representing the effect of the noise on the uncertainty of the evolution of the state variable, and in turn on the reliability of the prediction given by $\bar{x}_{t+i}$. In the following we will show how E_i can assume different values depending on the adopted control strategy. More specifically, we will describe how E_i depends on the adopted control law and its degrees of freedom, similarly to the discussion given in [73].

3.1.1. Open loop control

Some approaches (e.g., [47,71,74]) require that, at time t, the candidate control sequence $u_t, \ldots, u_{t+N-1}$ to be applied to the system (3) is computed as a result of the optimization problem and is independent of $\mathbf{w}_t$. Therefore it is deterministically defined as a function of the current state x_t, i.e., we write $u_{t+i} = \bar{u}_{t+i|t}$. Therefore, the input sequence is defined as $\mathbf{u}_t = \begin{bmatrix} \bar{u}_{t|t}^T & \cdots & \bar{u}_{t+N-1|t}^T \end{bmatrix}^T$. The evolution of the deterministic state $\bar{x}_{t+i}$ is described by

$$\bar{x}_{k+1} = A\bar{x}_k + B_u \bar{u}_{k|t} \tag{11}$$

for $k = t, \ldots, t + N - 1$, while the "open loop" evolution of perturbed component of the state variable is

$$e_{k+1} = Ae_k + B_w w_k$$

for $k = t, \ldots, t + N - 1$. In this case, it follows that the matrix E_i in (10) corresponds to the ith row-block of the matrix

$$\begin{bmatrix} B_w & 0 & \cdots & 0 \\ AB_w & B_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_w & A^{N-2}B_w & \cdots & B_w \end{bmatrix}$$

It is clear that, in this case, the variance of e_{t+i} (i.e., the uncertainty on the evolution of the state variable) evolves in an uncontrolled fashion. Especially in case the system is unstable, this approach has significant drawbacks, since it may induce serious feasibility problems.

3.1.2. Disturbance feedback control

This approach is employed in different works, see, e.g., [64,75–77]. In this case, the input sequence $\mathbf{u}_t$ is defined as a function of $\mathbf{w}_t$. The most common choice corresponds to the affine case, where it is set

$$\mathbf{u}_t = \mathbf{c}_{t|t} + \Theta_{t|t} \mathbf{w}_t \tag{12}$$

where both $\mathbf{c}_{t|t}$ and $\Theta_{t|t}$ are the results to the optimization procedure defined by the adopted SMPC formulation, computed at time t, of the degrees of freedom $\mathbf{c}_t = \begin{bmatrix} c_t^T & \cdots & c_{t+N-1}^T \end{bmatrix}^T$ and

$$\Theta_t = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \theta_{t+1,t} & 0 & 0 & \cdots & 0 \\ \theta_{t+2,t} & \theta_{t+2,t+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{t+N-1,t} & \theta_{t+N-1,t+1} & \theta_{t+N-1,t+2} & \cdots & 0 \end{bmatrix} \tag{13}$$

Note that, for causality reasons, Θ_t is set to be a lower-block triangular matrix with zero diagonal blocks.

Also in this case, the evolution of the deterministic state $\bar{x}_{t+i}$ is described by (11) for $k = t, \ldots, t + N - 1$, where $\bar{u}_{k|t} = H_{t-k+1}\mathbf{c}_{t|t}$ and $H_i \in \mathbb{R}^{n \times nN}$ is the matrix selecting the ith vector element from $\mathbf{c}_{t|t}$. The evolution of the perturbed component of the state variable is

$$e_{k+1} = Ae_k + B_u H_i \Theta_{t|t} \mathbf{w}_t + B_w w_k$$

for $k = t, \ldots, t+N-1$. In this case, it follows that the matrix E_i corresponds to the ith block-row of the matrix

$$\begin{bmatrix} B_w & 0 & \cdots & 0 \\ AB_w & B_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_w & A^{N-2}B_w & \cdots & B_w \end{bmatrix} + \begin{bmatrix} B_u & 0 & \cdots & 0 \\ AB_u & B_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_u & A^{N-2}B_u & \cdots & B_u \end{bmatrix} \Theta_{t|t}$$

From the latter it is clear that, at the optimization level, the choice of $\Theta_{t|t}$ can greatly reduce the effect of the noise sequence $\mathbf{w}_t$ on e_{t+i}, and eventually its variance.

3.1.3. State feedback control

State feedback approaches include, e.g., [61,62,78–82]. In this case, the input variable u_{t+i} is defined as a function of x_{t+i}. Slightly different versions of state feedback control laws have been proposed in the literature. More specifically

- in [65,81] it is set

  $$u_{t+i} = \bar{u}_{t+i|t} + K_{t+i|t}(x_{t+i} - \bar{x}_{t+i}) \tag{14}$$

  where $\bar{x}_k$ evolves according to (11) for $k = t, \ldots, t+N-1$. In this case, the results of the optimization problem are the sequences $K_{t+i|t}$, $\bar{u}_{t+i|t}$, for $i = 0, \ldots, N-1$. The evolution of perturbed component of the state variable is

  $$e_{k+1} = (A + B_u K_{k|t})e_k + B_w w_k$$

  for $k = t, \ldots, t+N-1$. Defining $\Phi_k = A + B_u K_{k|t}$, it follows that the matrix E_i corresponds to the ith block-row of the matrix

  $$\begin{bmatrix} B_w & 0 & \cdots & 0 \\ \Phi_{t+1}B_w & B_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{t+N}\cdots\Phi_{t+1}B_w & \Phi_{t+N}\cdots\Phi_{t+2}B_w & \cdots & B_w \end{bmatrix} \tag{15}$$

  As it is discussed in [83], it is possible to obtain an equivalent disturbance feedback formulation, with fewer degrees of freedom with respect to the ones in matrix Θ_k. It is worth mentioning, however, that both formulations result in convex problems, see e.g., [65].
- in, e.g., [78,79,80,84], the control law is

  $$u_{t+i} = \bar{u}_{t+i|t} + Kx_{t+i} \tag{16}$$

  Here the result of the optimization problem is the sequence $\bar{u}_{t+i|t}$, for $i = 0, \ldots, N-1$. The evolution of the deterministic state $\bar{x}_{t+i}$ is described by

  $$\bar{x}_{k+1} = \Phi\bar{x}_k + B_u\bar{u}_{k|t} \tag{17}$$

  for $k = t, \ldots, t+N-1$, where $\Phi = A + B_u K$ and the matrix E_i corresponds to the ith block-row of the matrix (15), where $\Phi_k = \Phi$ for all k.

In both cases, it is clear that a proper choice of the control gain (which is an optimization variable in [81] or a design parameter in [79]) can reduce the effect of the noise sequence $\mathbf{w}_t$ on e_{t+i}, and eventually its variance. This results in a larger feasibility region with respect to the case of open loop solutions, especially when the system is unstable.

3.2. Reformulation of state constraints

In this section, to simplify the setup as much as possible, individual linear state chance constraints will be considered, while the possible presence of deterministic or stochastic constraints on the control variable u will be thoroughly discussed in the sequel. Assume that, at time t, the goal is to impose the following state constraint in the next $i \geq 1$ prediction steps

$$\mathcal{P}\{g^T x_{t+i|t} \leq h\} \geq 1 - p \tag{18}$$

where $g \in R^n$ and $x_{t|t} = x_t$. As discussed in Section 3.1, in view of (10) we can write (18) as

$$\mathcal{P}\{g^T(\bar{x}_{t+i} + E_i\mathbf{w}_t) \leq h\} \geq 1 - p \tag{19}$$

and two main approaches are available for enforcing (19). First we discuss how this issue is approached by analytic approximation methods, then the scenario generation ones will be considered.

3.2.1. Analytic reformulation of state constraints

In general, in an analytic framework, we can guarantee (19) by verifying the following constraint

$$g^T\bar{x}_{t+i} \leq h - q_i(1-p) \tag{20}$$

where we can characterize the constraint tightening level $q_i(1-p)$, as discussed below, based on the available information regarding the noise sequence $\mathbf{w}_t$, its bounds, and its properties.

- To not allow constraint violations is equivalent to impose (19) with $p = 0$. This corresponds to the worst-case tightening adopted in the deterministic framework, and amounts to setting

  $$q_i(1) = \max_{\mathbf{w}_t \in \mathcal{W}^N} g^T E_i \mathbf{w}_t \tag{21}$$

  In this case, (20) may admit a solution only under the assumption that $\mathcal{W}$ is bounded.
- In case a non-zero probability of violation is allowed, it is possible to use the knowledge on the distribution of the noise, if available. Following, for example, the approach proposed in [80,85], the term $q_i(1-p)$ can be computed as

  $$q_i(1-p) = \underset{q}{argmin}\, q, \quad s.t. \quad \mathcal{P}\{g^T E_i \mathbf{w}_t \leq q\} = 1 - p \tag{22}$$

  In general, the previous expression cannot be computed analytically since it involves the evaluation of a multivariate convolution integral. It is then necessary to approximate it numerically, for example by discretizing the distributions of w or using sample-based approaches. However, if the value of p and the shape of the constraints are fixed in the problem, this computation can be performed off-line only once and with an arbitrary precision.
- Under the assumption that the noise is unbounded, Gaussian with expected value $\bar{w} = 0$ and covariance matrix W, also the variable e_{t+i} is Gaussian and the exact constraint tightening can be computed analytically (e.g., in [47,71,82]). Specifically,

  $$q_i(1-p) = \sqrt{g^T E_i \mathbf{W} E_i^T g}\,\mathcal{N}^{-1}(1-p) \tag{23}$$

  where $\mathbf{W} = \text{diag}(W, \ldots, W)$ and where $\mathcal{N}$ is the cumulative probability function of a Gaussian variable with zero mean and unitary variance.
- Finally, if the distribution of the noise is not specified, but its expected value $\bar{w}$ and covariance matrix W are known, it is possible to resort to the Chebyshev inequality [86]. Recalling that (without loss of generality) we are now considering $\bar{w} = 0$, it is

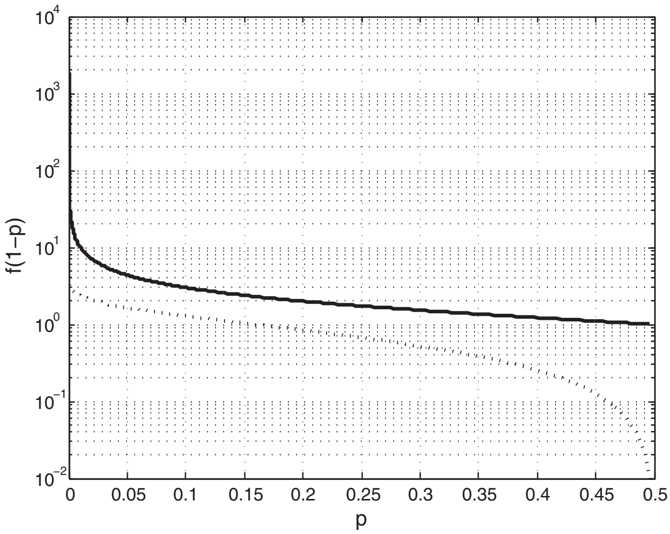

Fig. 2. Plots of $f(1-p)=\sqrt{(1-p)/p}$ (solid line) and $f(1-p)=\mathcal{N}^{-1}(1-p)$ (dotted line).

possible to use the approach proposed for example in [81], and provide a bound for $q_i(1-p)$ in (22), i.e.,

$$q_i(1-p) \le \sqrt{g^T E_i \mathbf{W} E_i^T g}\sqrt{\frac{1-p}{p}} \tag{24}$$

Note that (23) and (24) take the same general form, i.e.,

$$q_i(1-p) \le \sqrt{g^T E_i \mathbf{W} E_i^T g} f(1-p) \tag{25}$$

where $f(1-p)=\mathcal{N}^{-1}(1-p)$ for Gaussian variables and $f(1-p)=\sqrt{\frac{1-p}{p}}$ for unknown distributions. The choice (24) is conservative: in Fig. 2 (taken from [82]) we show the comparison between the two cases.

From this discussion is now clear that the choice of the matrix E_i (which in turn depends on the adopted control strategy, see Section 3.1) greatly affects the size of $q_i(1-p)$, which can be minimized to possibly enhance the feasibility properties of the SMPC-based control scheme.

3.2.2. *Analytic reformulation for the motivating example*

Assuming again that $e_t=0$, it is first of interest to compute the equation describing the evolution of the noise component e_{t+k}, $k=0, 1, \ldots$ when the different control strategies described in 3.1 are used. According to the open loop control strategy

$$e_{t+k+1} = a e_{t+k} + w_{t+k} \tag{26}$$

From this expression we compute that $\mathrm{Var}\{e_{t+k}\} = \frac{1-a^{2k}}{1-a^2} W$ for all $k \ge 0$ and, at the same time, $e_{t+k} \in [-\frac{1-a^k}{1-a}\bar{w}, \frac{1-a^k}{1-a}\bar{w}]$.

On the other hand, concerning the closed loop control strategies described in Sections 3.1.2 and 3.1.3 with a proper parameter setting (i.e., $\theta_{k|k-1}=-a/b$, $\theta_{k|k-i}=0$ for all $k=t+1, \ldots, t+N-1$ and for all $i>1$ in (13), and $K_{k|t}=K=-a/b$ for all $k \ge 0$ in (14) and (16)) we obtain that, for all $k \ge t$

$$e_{k+1} = w_k \tag{27}$$

which implies that $\mathrm{Var}\{e_{t+k}\}=W$ and that $e_t \in [-\bar{w}, \bar{w}]$ for all $k \ge 0$.

Neglecting the recursive feasibility issues for simplicity, to enforce (8) it is enough to guarantee that, for all $k \ge 1$

$$\bar{x}_{t+k} + e_{t+k} \ge T_M$$

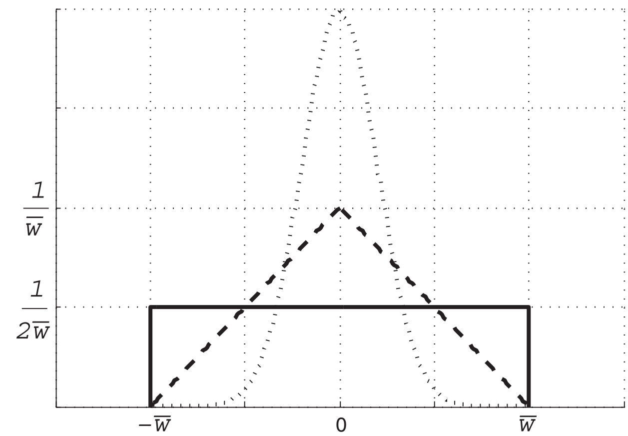

Fig. 3. Distributions: uniform (case A, solid line), triangular (case B, dashed line), truncated Gaussian (case C, dotted line).

for all possible realizations of w_{t+k}. In case of the open loop strategy, this is equivalent to enforce the conservative constraint

$$\bar{x}_{t+k} \ge T_M + \frac{1-a^k}{1-a}\bar{w} \tag{28}$$

while, in case of the closed loop strategies (12), (14) and (16), we should set

$$\bar{x}_{t+k} \ge T_M + \bar{w} \tag{29}$$

On the other hand, to enforce (9), in case of the open loop strategy we must verify that

$$\bar{x}_{t+k} \ge T_M + \sqrt{\frac{1-a^{2k}}{1-a^2} W} f(1-p) \tag{30}$$

while, in case of the closed loop strategy, we should set

$$\bar{x}_{t+k} \ge T_M + \sqrt{W} f(1-p) \tag{31}$$

A remark is due at this point. From the comparison of the constraint (28) with (29), as well as of (30) with (31), it is apparent that the use of a feedback control strategy is desirable for reducing the conservativity of the robust constraints, both in the deterministic and in the stochastic framework, since $\frac{1-a^k}{1-a} > 1$ and $\sqrt{\frac{1-a^{2k}}{1-a^2}} > 1$, respectively.

From the previous results, it is also possible to verify that the deterministic constraint can be more conservative than the stochastic one, at the price of allowing for a probability of violation that verifies

$$f(1-p) \le \frac{\frac{1-a^k}{1-a}\bar{w}}{\sqrt{\frac{1-a^{2k}}{1-a^2} W}} \tag{32}$$

in case the open loop strategy is adopted, or

$$f(1-p) \le \frac{\bar{w}}{\sqrt{W}} \tag{33}$$

for the closed-loop strategy. Consider now the three distributions depicted in Fig. 3 with $W=\bar{w}^2/\alpha$, with (A) $\alpha=3$ for uniform distribution; (B) $\alpha=18$ for triangular distribution; (C) $\alpha=25$ for truncated Gaussian distribution. Assume, for analytical simplicity, that $f(1-p)$ is chosen according to the Cantelli–Chebyshev inequality, i.e., as in (24). It is therefore possible to verify (32) if, for all $k \ge 1$

$$p \ge \frac{\frac{1-a^{2k}}{1-a^2}}{\left(\frac{1-a^k}{1-a}\right)^2 \alpha + \frac{1-a^{2k}}{1-a^2}} \tag{34}$$

A sufficient condition guaranteeing that (34) holds for all $k \geq 1$ is

$$p > \frac{1}{1+\alpha} \tag{35}$$

Also we guarantee (33) provided that (35) holds. Therefore, even if the conservative Cantelli–Chebyshev inequality is used, we obtain less conservative bounds with respect to the deterministic (worst-case) strategy by allowing a minimal probability of constraint violation equal to A) $p = 1/4$ for the uniform distribution; B) $p = 1/19$ for the triangular distribution; (C) $p = 1/26$ for the truncated Gaussian distribution. These results show that in the three cases (A), (B), and (C), even small values of p lead to weakened constraints with respect to the deterministic case. Note also that, although formally truncated, the distribution in case (C) can be well approximated with a non-truncated Gaussian distribution: if this information were available, one could use $f(1-p) = \mathcal{N}^{-1}(1-p)$ for constraint tightening, and in this case (35) would be replaced by $p \geq 1 - \mathcal{N}(\sqrt{\alpha}) < 3 \times 10^{-7}$.

3.2.3. *Reformulation of state constraints in a scenario generation framework*

In scenario generation stochastic MPC methods, constraints are verified for a finite number of sampled deterministic constraints, computed on the basis of proper extractions of the uncertainty. The number of the extracted samples are carefully selected to ensure a prescribed level of constraint violation.

Specifically, recalling (10), the probabilistic constraint (18) is guaranteed by enforcing, at any time instant i along the prediction horizon, the deterministic ones

$$g^T(\bar{x}_{t+i} + E_i \mathbf{w}_t^{[i,k_i]}) \leq h \tag{36}$$

for all $k_i = 1, \ldots, N_{s,i}$, where $\mathbf{w}_t^{[i,k_i]}$ is the k_ith noise realization relative to the ith constraint, and where it is assumed that $\mathbf{w}_t^{[i,k]}$ is independent of $\mathbf{w}_t^{[j,h]}$ for all $i \neq j$ or $h \neq k$.

The number $N_{s,i}$ of noise realizations must be carefully chosen. More in detail, denote by d_i the number of variables involved in the optimization problem referred to the ith prediction. The probability that $g^T x_{t+i} > h$ is smaller than p with a given confidence level β is (see e.g., [63,87])

$$\sum_{k=1}^{d_i-1} \binom{N_{s,i}}{k} p^k (1-p)^{N_{s,i}-k} \geq \beta \tag{37}$$

The value of $N_{s,i}$ can thus be chosen so as to satisfy

$$N_{s,i} \geq \frac{d_i + 1 + \ln(1/\beta) + \sqrt{2(d_i+1)\ln(1/\beta)}}{p} \tag{38}$$

It may happen, especially when the required number of samples, $N_{s,i}$ is low, that the solution to the problem is too conservative due to "unlucky" extractions of the uncertainty used to reformulate the constraint. To avoid this situation, it is possible to implement iterative sample-removal algorithms, which are able to detect the more stringent constraints to be ignored. In this case, the expression in (37) is not valid anymore and a different procedure, described for example in [63,88] is required to compute both the number of samples $N_{s,i}$ and the number of constraints $N_{r,i}$ to be removed given the probability p and the confidence β. Of course, due to the removed samples, the number of extractions that are needed to approximate the chance constraint correctly is increased. An advantage of using this removal technique is that it is easier to reach the level of relaxation required in the problem, while the main drawback is related to the increase of the computational complexity.

Finally note that Eq. (36) applies when open loop predictions are performed at time t. In this case, in principle, to enforce (18) for each considered future instant $i = 1, \ldots, N$, one should extract a number $N_{s,i}$ of independent realizations, resulting in a total number of realizations required at time t of $\sum_{i=1}^{N} N_{s,i}$. Recall that the number of independent optimization variables d_i (and hence $N_{s,i}$) for the single constraint increases with i. This, however, results unnecessary in a closed-loop receding-horizon control setting, as thoroughly discussed and proved in [63], for more details see Section 4.4.

3.3. *Constraints on input variables*

Contrarily to the case of state constraints, there is no general consensus in the SMPC literature on input constraints and saturations. More specifically, the issue is whether to reformulate input constraints as probabilistic ones (as it is generally done in the case of state constraints) or not. In a practical context, indeed, it is desirable that the input variables lie in specified ranges; however, this requirement is not always compatible with the adopted control strategy and with recursive feasibility requirements.

Considering open loop control strategies, since the input sequence $\bar{u}_{t+i}$ is deterministically defined at time t, hard bounds on it could be enforced regardless of the entity of the disturbance (both in the bounded and in the unbounded case). On the other hand, when state feedback and disturbance feedback policies are adopted, the values taken by $u_{t+1}, \ldots, u_{t+N-1}$ depend on $w_t, \ldots, w_{t+N-2}$, and therefore saturation constraints can be imposed at time t at most only on u_t in case the noise is unbounded. Of course, if the noise support is bounded, hard constraints can be enforced on the whole input trajectory at the price of applying robust worst-case arguments. Consistently with this discussion, hard constraints are assumed on the input variables in two cases: (i) when the noise is bounded, e.g., [64,79]; (ii) when recursive feasibility requirements are relaxed, e.g., [63,75]. A notable exception is discussed in [89], where a nonlinear (output) feedback policy is adopted; in this case, the use of bounded nonlinear functions of $y_{t+i} - \hat{y}_{t+i}$ (where y is the system output and $\hat{y}$ is the system output nominal prediction), together with the assumptions that the system is stable and that state constraints are absent, allows hard bounds on inputs to be enforced at any time instant and to guarantee recursive feasibility. In [81] constraints on input variables must be formulated in general as probabilistic ones, while some exceptions are discussed in [82], e.g., when the system is stable and we reduce the degrees of freedom on $K_t, \ldots, K_{t+N-1}$.

3.4. *Cost function*

In a deterministic framework, the MPC cost function J_t^N is usually selected to weight, over a finite number of steps defined by the prediction horizon N, a stage cost $l(x, u)$, plus a cost related to the state at the end of the horizon, $l_f(x)$. Since in a stochastic framework, the state and possibly the control variables are random processes, J_t^N is itself a random variable that depends on the uncertainty affecting the system. This makes the derivation of the cost function to be minimized in a probabilistic setup arbitrary, to some extent. Some of the possible choices taken in the literature are listed below.

- The most commonly used cost function in a stochastic framework is the following

$$\bar{J}_t = \mathbb{E}\left[\sum_{i=t}^{t+N-1} l(x_i, u_i) + l_f(x_{t+N})\right] \tag{39}$$

  where the expectation is taken over the distribution of the disturbance. In the analytic framework, e.g., [65,81], the cost function above can be reformulated as a function of the mean value and the variance of the system variables by selecting

$l(x,u) = \|x\|_Q^2 + \|u\|_R^2$ and $l_f(x) = \|x\|_P^2$. Then, defining $\mathbb{E}[x_i] = \bar{x}_i$, $\mathbb{E}[u_i] = \bar{u}_i$, X_i=var(x_i), and U_i=var(u_i), we can write

$$\mathbb{E}[l(x_i,u_i)] = \mathbb{E}[\|x_i\|_Q^2 + \|u_i\|_R^2] = \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 + \mathrm{tr}(QX_i + RU_i) \tag{40a}$$

$$\mathbb{E}[l_f(x_{t+N})] = \mathbb{E}[\|x_{t+N}\|_P^2] = \|\bar{x}_{t+N}\|_P^2 + \mathrm{tr}(PX_{t+N}) \tag{40b}$$

- An alternative to (39) can be found by resorting to the certainty equivalence principle (see, e.g., [63,77,90]). More specifically, we can define the cost function as the deterministic one

$$\hat{J}_t = \sum_{i=t}^{t+N-1} l(\hat{x}_i, u_i) + l_f(\hat{x}_{t+N}) \tag{41}$$

  where the nominal system trajectory $\hat{x}_i$ $i=t,\ldots,t+N$ is obtained using the update equation $\hat{x}_{i+1} = f(\hat{x}_i, u_i, \hat{w}_i)$ with initial condition $\hat{x}_t = x_t$, where $\hat{w}_i$, $i=t,\ldots,t+N-1$, is a nominal disturbance trajectory, e.g., defined as the expected value or the optimal predictor of w_i.
- In a scenario generation framework (see, e.g., [63,75,77,90]), a sampled average over N_s noise realizations can be considered at the place of (39), i.e.,

$$\bar{J}_t \simeq \frac{1}{N_s}\sum_{k=1}^{N_s} J_t^{[k]} \tag{42}$$

  where

$$J_t^{[k]} = \sum_{i=t}^{t+N-1} l(x_i^{[k]}, u_i) + l_f(x_{t+N}^{[k]}) \tag{43}$$

  and where, denoting by $w_i^{[k]}$ $i=t,\ldots,t+N-1$ the kth noise realization (with $k=1,\ldots,N_s$), the trajectory $x_i^{[k]}$ $i=t+1,\ldots,t+N$ is computed using the update equation $x_{i+1}^{[k]} = f(x_i^{[k]}, u_i, w_i^{[k]})$ with initial condition $x_t^{[k]} = x_t$, i.e., the measured state.
- In the context of scenario generation SMPC, a worst-case optimization procedure can be employed. For example, in [63,87] the following cost function is minimized

$$J_t^{\max} = \max_{k=1,\ldots,N_s} (J_t^{[k]}) \tag{44}$$

3.5. Feasibility and convergence properties

In the context of stochastic MPC the problem of guaranteeing recursive feasibility of the MPC optimization problem is still open. This issue relies on the fact that, to formulate the optimization problem at a given time t, the value of $\bar{x}_{t|t} = \mathbb{E}[x_t]$ is commonly defined as the conditional expected value $\mathbb{E}[x_t|t]$, i.e., with respect to the data available until time step t, i.e., $\bar{x}_{t|t} = x_t$ in a state-feedback framework. If the noise support is unbounded (as, e.g., in the Gaussian case), the value taken by w_{t-1} may take (even with small probability) unboundedly large values and therefore there may not exist a feasible control action capable of forcing the predicted values $\bar{x}_{t+i|t}$ (evolving as in (11)) to satisfy constraint (20) for all $i=1,\ldots,N$. For example, in [91] it is proved that, if the proposed MPC-related optimization problem is feasible at time t, then it will be feasible for all future time instants $t+1,\ldots,t+N$ with a given probability, thanks to the definition of probabilistic resolvability. We have identified three different types of solutions to overcome this problem in the current literature.

First, in some notable works (e.g., [63,74,87]), including the scenario-based methods, the feasibility of the optimization problem is assumed at each time step, and possibly enforced, e.g., by reformulating the problem constraints in a soft fashion. In other works (e.g., [61,78]), the alternative to the use of soft constraints is to define (thanks to the notion of invariance with probability p), in case of infeasibility, an alternative and feasible optimization problem. In this case, under suitable parameter tuning, it is proved that the original constraint on the time-rate of violations (5) can be met.

The second class of solutions (devised for both state feedback and disturbance feedback approaches, see e.g., [62,64,79]) has been devised under the assumption that the disturbances are bounded. They are based on imposing suitable mixed probabilistic/worst-case constraint tightening to equation (20) (or, in other words, by amplifying the term $q_i(1-p)$ for $i>1$) by accounting for bounded sets where the state can evolve due to the bounded noise affecting the state equation.

The third solution, proposed in [81,82] relies on the idea that a more flexible definition of $\bar{x}_{t|t}$ is allowed. Indeed it is possible to set $\bar{x}_{t|t} = \bar{x}_{t|t-1}$ where, e.g., $\bar{x}_{t|t-1} = \mathbb{E}[x_t|t-1]$ is the conditional probability of x_t obtained using data collected until time $t-1$. In this case $\mathrm{Var}[e_t] = \mathbb{E}[(x_t - \mathbb{E}[x_t|t-1])(x_t - \mathbb{E}[x_t|t-1])^T] \neq 0$ is currently available. In this way recursive feasibility is conserved, at the price of disregarding the current data x_t, from which it follows that the conditional probability $\mathcal{P}\{g(x_{t+k}, u_{t+k}) > 0|t-1\}$ can be verified instead of $\mathcal{P}\{g(x_{t+k}, u_{t+k}) > 0|t\}$. For more details see Section 4.3.

Concerning stability and convergence results, while in case of bounded deterministic disturbances practical stability (i.e., convergence in a neighborhood of the origin) can be established, in the stochastic framework mean square stability results are generally addressed, with the exception of [74], where convergence of the mean value of $x(k)$ to zero is proven. Indeed, in [79,84,85,89], and in general in case of additive noise, it can be proven that

$$\lim_{k\to\infty} \mathbb{E}[\|x_k\|^2] = \lim_{k\to\infty} (\|\mathbb{E}[x_k]\|^2 + \mathrm{var}(x_k)) \leq \text{const} \tag{45}$$

This means that the state of the system is driven to a neighborhood of the steady state condition (whose dimension depends on the amplitude of the input noise and on the adopted control policy).

On the other hand, when modeling uncertainties are described (e.g., for linear systems with multiplicative uncertianties [65,78,92]), point-wise convergence can be obtained, e.g., that $\lim_{k\to\infty} \mathbb{E}[\|x_k\|^2] = 0$.

Note that, for simplicity in the following section the property (45) will be denoted *mean square convergence* and the constant value will be possibly equal to zero.

3.6. Summarizing tables and comments

In this section we give a schematic overview of the many different approaches proposed in the recent literature on SMPC. Here we mostly focus on the more advanced methods and formulations of each considered approach: in this respect, it must not be considered as an exhaustive list of available references. First, in Table 1 we focus on the system assumptions required by the different algorithms. Secondly, in Table 2 we stress, for the available methods, how constraints are handled, and the information available/assumed on the disturbance acting on the system. Finally, in Table 3 we review the main theoretical properties guaranteed in the different algorithms.

In addition to the information summarized in Tables 1–3, the following comments are due to highlight the main characteristics of the approaches proposed so far.

- The main advantage of scenario generation approaches is that they are applicable to wide classes of systems (linear, nonlinear) affected by general disturbances (additive, multiplicative, parametric, bounded or unbounded) with constraints of general type on the inputs, states, and outputs, provided that the problem is convex in the optimization variables. Their drawbacks

Table 1
System assumptions.

		Uncertainty	
		Bounded	Unbounded
Nonlinear		[44]	[91,93–95]
Linear	Additive noise	[64,80,84,85]	[14,16,19,67,71,74–77,81,82,89,96–101]
	Multiplicative noise	[61,62,78,92,102]	[65,78]
	Parametric uncertainty	[87]	[63,103]

Table 2
Algorithms structure.

			State constraints type		
			Expected value	Joint CC	Independent CC
Analytical	Known PDF	All	[65,95]	×	[61,62,64,80,84,85,94,98,103]
		Gaussian	×	[19,71,91,97,101]	[99,100]
	Second order		×	[72]	[44,74,81,82]
Randomized (PDF, known or estimated)			×	[63,75–77,87,92,93]	×

Table 3
Properties.

		Convergence			
		In probability	Expected value	Mean square	No
Rec. feasibility	Yes	[87]	[102]	[61,62,73,80–82,84,85,89,92,96]	[44,64,76]
	No	×	×	[65,95,101]	[14,16,19,22,46,63,71,75,77,93,94,97–100,103]
	Probabilistic	×	[74]	[78]	[91]

come from the computational complexity in case a high number of samples is required, e.g., by sample-removal algorithms to reach the required probabilistic guarantees. This, as discussed more in details in Section 4.4, is not critical in many applications where the number of degrees of freedom involved in the constraints is limited. In addition, to the best of our knowledge, stability and recursive feasibility properties are not available at present, as it can be deduced from a joint analysis of Tables 2 and 3.

- In the wide class of analytic approximation methods, algorithms with recursive feasibility and convergence are already available both for bounded and unbounded noise. However, in order to rigorously reformulate costs and constraints in an analytical fashion, the most popular assumption is that the model is linear and affected by additive or multiplicative white noise. A notable exception is represented by the approaches that rely on polynomial chaos expansions, e.g., [94,101,103]. Thanks to the possibility of approximating general analytic functions using series of basis functions, these methods handle systems affected by parametric uncertainties; for example, continuous-time and discrete-time linear systems are considered in [103] and [101], respectively, while nonlinear discrete-time systems are addressed in [94].
- Very few algorithms, see [82,84,89], have been extended to deal with the output feedback case, which still represents a largely open issue.

4. Rationale and properties of some paradigmatic algorithms

In this section we present the main guidelines of four SMPC algorithms, selected to represent interesting classes of methods; specifically, we consider three analytic approximation-based methods for disturbances with bounded or unbounded support, and a general scenario generation approach.

4.1. Stochastic tube MPC

The stochastic tube MPC method is an analytic approximation scheme, proposed for linear systems affected by additive and/or multiplicative uncertainties, see e.g., [62,79,80,84]. Here we consider the approach presented in [80]. In this work the authors define an offline constraint tightening procedure that, thanks to the assumption that $w_t \in \mathcal{W} := \{w : |w| \le \alpha\}$ with $\alpha = [\alpha_1 \ \cdots \ \alpha_{n_w}]^T$, ensures the recursive feasibility of the algorithm while taking advantage of the probabilistic nature of the constraint. The noise terms w_t are also assumed to be independent and identically distributed. Also, the method presented in [80] allows for general but known noise distributions.

The control scheme is based on a state feedback strategy of the type (16). In principle, transient probabilistic constraints are guaranteed as in (20), where $q_i(1-p)$ can be computed as in (22) or obtained, in a conservative way, using the Chebyshev inequality (24).

To ensure recursive feasibility, the solution presented in [80] takes advantage of the known bounds on the disturbance to implement a mixed stochastic/worst-case tightening procedure, which may include some conservativeness. More specifically, (20) is replaced by

$$g^T \bar{x}_{t+i} \le h - \beta_i, \quad i = 1, 2, \ldots \tag{46}$$

where $\beta_1 = q_1(1-p)$ and $\beta_i \ge q_i(1-p)$ for all $i \ge 2$. For details on the computation of the terms β_i see [80]. The approach proposed therein allows to compute $q_i(1-p)$ for general noise distributions by approximation of the distributions using discretization methods. The algorithm is implemented using the common dual mode prediction paradigm, where the above constraints are accounted for explicitly along the horizon defined by N and implicitly by means of a proper terminal constraint $\bar{x}_{t+N} \in \mathcal{S}_{\hat{N}}$ defined as

$$\mathcal{S}_{\hat{N}} = \left\{ \bar{x}_N : \begin{array}{l} g^T \Phi^l \bar{x}_N \le h - \beta_{N+l}, \quad l = 1, \ldots, \hat{N} \\ g^T \Phi^l \bar{x}_N \le h - \bar{\beta}, \quad l > \hat{N} \end{array} \right\} \tag{47}$$

where it is assumed that $\bar{u}_{t+i} = 0$ for all $i \geq N$ and $\hat{N}$ defines an additional prediction horizon. This allows to define an infinite-time expectation cost function of the type

$$J_t = \sum_{i=0}^{\infty} \mathbb{E}[x_{t+i}^T Q x_{t+i} + u_{t+i}^T R u_{t+i}] \tag{48}$$

which is minimized at each time step t. Note that, once all the terms required up to a desired numerical precision (i.e., β_i, $I = 1, \ldots, N + \hat{N}$) are computed offline (hence possibly requiring a significant computational effort), the optimization problem to be solved at each time instant is not more complex than a classical deterministic MPC, and the algorithm shares the same feasibility properties on the closed-loop operations in the robust case (for the detailed proof see [80]). In addition to recursive feasibility results and the satisfaction of chance constraints at all time instants, in [80] also quadratic convergence results are proved.

4.2. Stochastic MPC for controlling the average number of constraint violations

The analytic approximation approach presented in this section has been proposed in [64,104]. It applies to linear systems with bounded uncertainties, i.e., where the noise terms $w_t \in \mathcal{W}$ are independent and identically distributed (according to a general-type distribution), where $\mathcal{W}$ is a compact polyhedron. Recursive feasibility properties can be established and hard constraints on the input variable $u_t \in \mathbb{U}$ are allowed. The proposed approach relies on a disturbance feedback control strategy with average cost function of type (39), although in [64] it is shown that more general cases can be encompassed. The general approach is similar to the one discussed in Section 4.1; however, the mixed probabilistic/worst-case constraint tightenings are relaxed thanks to the idea that the sampled average of constraint violations must be limited, rather than its probabilistic counterpart.

In this section we describe a simplified version of the control scheme, inspired by [104] where, for simplicity, a single probabilistic constraint of the type (19) is considered and input constraints are neglected.

In this approach, denoting by υ_t the number of constraint violations occurred up to time t (with $\upsilon_t \leq t$), the constraint (20) for x_{t+1} is replaced by

$$g^T \bar{x}_{t+1} \leq h - q_1(1 - \beta_t) \tag{49}$$

where $\beta_t = \max(\min(p(t+1) - \upsilon_t, 1), p)$. Note that $\beta_t > p$ is equivalent to $\upsilon_t/t < p$, so that, if the average rate of violations υ_t/t occurred up to time t is smaller than the prescribed limit p, we can allow for a greater probability of constraint violation at the next time step. Also, at time $t+1$, the state must be included in a set $\mathcal{S}_{r_t}$, defined in such a way that, if $x_{t+1} \in \mathcal{S}_{r_t}$ then, for all possible realizations of the bounded noise sequence $\mathbf{w}_t, x_{t+r_t} \in \mathcal{S}$, where the set $\mathcal{S}$ is the so-called stochastic control invariant set. $\mathcal{S}$ is defined in such a way that if $x \in \mathcal{S}$ then there exists u such that $Ax + B_u u + B_w w \in \mathcal{S}$ for all $w \in \mathcal{W}$ and $g^T(Ax + B_u u) \leq h - q_1(1-p)$. Intuitively, the number r_t is computed as the number of steps ahead in which a constraint violation would allow to satisfy $\upsilon_{t+i}/(t+i) < p$ for all $i = 1, \ldots, r_t$. As discussed in [64], in our simplified setting $x_{t+1} \in \mathcal{S}_{r_t}$ is ensured if, for all $i = 1, \ldots, N - r_t$

$$g^T \bar{x}_{t+i} + \max_{\mathbf{w}_t \in \mathcal{W}^N} E_i e_{t+i} \leq h - q_1(1-p)$$

It is worth noting that in [64] a more general case is considered, where a forgetting factor is allowed in the computation of υ_t, as well as the use of a penalty function to quantify the distance from constraint violation (e.g., rather than just defining υ_t as the number of violations, υ_t can represent also how far the state has been from violating the bounds). Also, as already discussed, hard constraints on inputs are allowed.

Implementation details are well documented in [64]. Regarding the computational load of the method, the offline design complexity is basically given by the load required for computing of invariant sets while, as far as the online complexity is concerned, like other analytic approximation schemes, this scheme requires a load which is comparable with the one required by robust deterministic controllers designed according to the same selected control strategy.

Besides recursive feasibility results, in [64] it is proved that the average rate of violations tends to verify the required bounds (i.e., that $\upsilon_t/t \leq p$) with a well defined rate of convergence.

4.3. Probabilistic MPC

The approach proposed in [81,82], inspired by [105], falls in the category of analytic approximation methods, and has been developed for linear systems of type (3) with additive and possibly unbounded uncertainty w_t, assumed to be a zero mean white noise with variance W (the noise distribution can be very general, but less conservative results may be obtained in case of Gaussian noise). It encompasses the case where state constraints are in the form (19). In view of the unboundedness of the noise affecting the system, input constraints must also be formulated, in general, as probabilistic ones.

Here, for the sake of exposition, input constraints are neglected. The approach proposed in [81] lies on a state feedback policy of type (14). However, differently from the case considered in Section 3.1, here it is not necessarily set $\bar{x}_t = x_t$, but it is required, more in general, that

$$\bar{x}_t = \mathbb{E}[x_t] \tag{50}$$

Therefore, defining $e_t = x_t - \bar{x}_t$ as in Section 3.1, $\mathbb{E}[e_t] = 0$. Also, thanks to the fact that a state feedback strategy (14) is adopted, from (3) and (11), $e_{k+1} = (A + B_u K_k) e_k + B_w w_k$. In view of the fact that $\mathbb{E}[e_k] = 0$ for all $k \geq t$ this allows to explicitly describe the evolution of $X_k = \mathbb{E}[e_k e_k^T]$ as

$$X_{k+1} = (A + B_u K_k) X_k (A + B_u K_k)^T + B_w W B_w^T \tag{51}$$

Now, similarly to Section 3.2.1, the probabilistic constraint (19) is formulated as in (20) to be enforced for all the prediction horizon, i.e., for $i = t, \ldots, t+N-1$[3] with

$$q_i(1-p) = \sqrt{g^T X_{t+i} g} f(1-p)$$

The cost function to be minimized is the average cost (39). If the cost function is quadratic, thanks to (40), one can write $\bar{J}_t = J_t^m + J_t^v$, where

$$J_t^m = \sum_{i=t}^{t+N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 + \|\bar{x}_{t+N}\|_P^2 \tag{52a}$$

$$J_t^v = \sum_{i=t}^{t+N-1} tr\{(Q + K_i^T R K_i) X_i\} + tr\{P X_{t+N}\} \tag{52b}$$

Note that (52a) and (52b) depend solely on the nominal system (11) and on the variance in (51), respectively, while both the variance and the mean value appear in the constraint (20).

Recursive feasibility and mean square convergence results can be established in view of two ingredients:

[3] The constraints can be enforced also for a different horizon, i.e., for $i = t+1, \ldots, t+N-1$. Also in the latter case all guaranteed results can be obtained.

- *Terminal constraints*, both in the mean value and in the variance, i.e.,

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_f \quad (53)$$

$$X_{t+N} \leq \bar{X} \quad (54)$$

The set $\bar{\mathbb{X}}_f$ is a positively invariant set for (11) controlled with a suitable auxiliary control law $u_t = \bar{K}x_t$, while $\bar{X}$ is the solution to $\bar{X} = (A + B\bar{K})\bar{X}(A + B\bar{K})^T + F\bar{W}F^T$, where $\bar{W} \geq W$. Also, the chance constraint (20) must be verified when the terminal conditions are fulfilled, i.e., $g^T\bar{x} \leq h - \sqrt{g^T\bar{X}g}\sqrt{\frac{1-p}{p}}$ for all $\bar{x} \in \bar{\mathbb{X}}_f$.
- *Initialization*. In [81], the initial condition for the pair $(\bar{x}_t, X_t)$ is free to take two different values, i.e., $(\bar{x}_t, X_t) = (x_t, 0)$ (denoted reset initialization strategy) or $(\bar{x}_t, X_t) = (\bar{x}_{t|t-1}, X_{t|t-1})$ (denoted nominal initialization strategy), where $(\bar{x}_{t|t}, X_{t|t-1})$ is given by the evolution of (11) and (51), starting from the optimal solution obtained by the related optimization problem solved at time $t-1$.

A remark is worth at this point. As shortly discussed in Section 3.6, the fact that the initial conditions for mean and variance are accounted for as free variables (where only two different solutions described above are possible) is fundamental to guarantee recursive feasibility. This can be done at the price of suitably characterizing the probabilistic constraints (18). In other approaches (such as all other methods described in this section, where it is set $\bar{x}_{t|t} = x_t$ for all t), for example, the probability constraints are implicitly conditioned to the data available at the current time t so that (18), for example, reads as

$$\mathcal{P}\{g^T x_{t+k} \geq 1|t\} \leq p$$

provided that feasibility holds. In the present approach, however, the two possible initializations imply different probability definitions. Specifically, if $(\bar{x}_t, X_t) = (x_t, 0)$, we implicitly enforce $\mathcal{P}\{g^T x_{t+1} \geq h|t\} \leq p$ while, if $(\bar{x}_t, X_t) = (\bar{x}_{t|t}, X_{t|t-1})$, we verify $\mathcal{P}\{g^T x_{t+1} \geq h|t-\tau\} \leq p$, where $t-\tau$ is the most recent past time step when the reset initialization strategy has been adopted. As a consequence, three different versions of the method can be devised, as described below.

(A) *Hybrid* scheme (i.e., the one described in [81]). Select the initialization strategy that, besides feasibility, guarantees the minimization of the cost function $\bar{J}_t$ at all t.
(B) *Nominal* scheme. For all time instants, select the nominal strategy, i.e., $(\bar{x}_t, X_t) = (\bar{x}_{t|t-1}, X_{t|t-1})$ for all $t > 0$.
(C) *Reset-based* scheme. For all time instants, select the reset strategy if feasible, otherwise select the nominal one.

Note that the nominal scheme allows to verify $\mathcal{P}\{g^T x_{t+1} \geq h|0\} \leq p$, i.e., the fulfillment of the "non-conditional" expectation constraint at each time instant. The recursive feasibility and the quadratic convergence properties of this approach can be proved along the lines of the hybrid one.

The third scheme guarantees that $\mathcal{P}\{g^T x_{t+1} \geq h|t\} \leq p$ is verified at all time instants when feasibility of the reset strategy is verified. The proof of its convergence could be provided, e.g., similarly to [61]. Note that both the nominal and the reset-based schemes allow for a clear probabilistic characterization of the method, at the price of suboptimal results in term of cost function minimization.

As far as the computational complexity required for the offline design phase is concerned, no robust invariant sets are required to be computed, but just solutions to standard matrix equations (e.g., Riccati/Lyapunov equations) making the design phase relatively lightweight and suitable for application to large systems. Regarding the required online computational load, note that it is possible to use linearized versions of (20), making the problem a quadratic one with linear constraints. Therefore, although the gain matrices K_k are free variables of the problem (besides the nominal input sequence), the computational complexity is limited. To further reduce the online computational burden, but at the price of obtaining suboptimal solutions, it is possible to discard the control gains as optimization variables and set $K_k = \bar{K}$ for all $k \geq 0$.

4.4. Scenario generation MPC

As previously discussed, the main idea of all the scenario generation techniques, see for example [63,76,87] and the references therein, is to take advantage of the possibility to draw samples of the uncertainty, or to use its past records or known realizations, to formulate a sample-based version of the control problem. In view of this, no particular assumption is made on the distribution and the support of the independent and identically distributed noise terms.

In this section we shortly review the scenario generation algorithm discussed in [63]. Since samples are generated of the disturbance realizations, the adopted control policy is the disturbance-feedback one, i.e., u_{t+i} is computed as in (12), and the goal is to minimize the expected cost function (39). The latter can be done, as discussed in Section 3.5, by minimizing the sampled cost (42). As highlighted in Section 3.5, other possible cost functions can be used in this framework.

The probabilistic constraints on state variables, similarly to (36), are derived along the lines of the general ideas presented in Section 3.2. However, as thoroughly discussed in [63], in a receding-horizon control framework, it is sufficient to extract a total number N_s of realizations, appearing in all future constraints at the same time

$$g^T(\bar{x}_{t+i} + E_i\mathbf{w}_t^{[k]}) \leq h \quad (55)$$

for all $k = 1, \ldots, N_s$. In particular, N_s is the number of realizations required for enforcing the probabilistic constraint (18) for $i = 1$. For example, $N_s = N_{s,1}$ as in (38) in case no constraint removal is performed. In addition, for further reducing the number of required realizations, in [63] it is shown that, in order to guarantee that the *average* (instead of the *point-wise*) probability of constraint violation is equal to p, $N_{s,1}$ in (38) can be replaced by

$$N_s \geq \frac{d_1 - p}{p}$$

Also in this case, the removal strategy can be adopted for reducing the possible conservativity of the results. For details, see again [63]. Concerning the scalability properties of the scheme, it is important to note that the number N_s of required realizations, in view of (38), does not depend on the number of state variables and of the uncertainty term, but indeed just on the number of degrees of freedom involved in the used constraints (i.e., the support rank). In view of this, it is possible to conclude that, for many applications, the use of scenario generation approaches can be viable from the computational point of view.

Other advantages of scenario generation methods are related to the fact that joint chance constraints can be naturally be enforced without including conservativity, and that no limitations on the type of probability distributions and support sets are imposed.

Interestingly, the proposed approach guarantees that the average-in-time constraint violation asymptotically fulfills the required rate expressed in (5) i.e., that $\lim_{N_{horizon}\to\infty} \sup \frac{1}{N_{horizon}} \sum_{t=1}^{N_{horizon}} \mathbf{1}_g(x_t, u_t) \leq p$.

5. Conclusions

The aim of this paper has been to review and propose possible classification criteria of the large number of SMPC algorithms nowadays available, often very different in their founding assumptions on the system under control, the characteristics of the noise, the adopted algorithmic solutions, and their properties in terms of stability and feasibility.

In particular, in Sections 2 and 3, we focused on the main possible design choices (or approaches) that can be taken (e.g., the use of different control parameterizations, the use of individual/joint constraints) and their possible implications. To simplify and exemplify this, we used simple examples, among which a motivating scalar case study to give a practical flavor of the possible implications (e.g., in terms of suboptimality, conservativity) of the different design choices.

In Section 4 some notable methods are discussed in details. From the analysis of the literature we can conclude that well defined methods are available, with their pros and cons. However, it is worth remarking that each method allows for many different design choices, each corresponding with a different implementation, and with specific performance and guaranteed properties, also as far as the computational online and offline cost are concerned. In view of this, our general approach has not been to compare the methods from a "top down perspective" (i.e., in the light of their properties, which are somehow case and implementation-dependent), but rather from a "bottom up" one, i.e., in the light on their ingredients and underlying assumptions.

In our opinion, the field of SMPC has not yet reached a full maturity, and there is room for many possible developments and improvements. Among them, the following ones seem to be of particular interest.

- Solutions based on analytic approximation methods are often (especially in case of general-type noise distributions and in case of joint constraints) characterized by a quite high level of conservativity, which reduces their potential benefits with respect to deterministic robust approaches. In addition, most of these techniques have been developed for linear systems, while their extension to the nonlinear case is still largely unexplored.
- Scenario generation algorithms have high potentialities for their application in a wide number of application fields, including nonlinear and systems affected by general-type noise. In some applications the required computational load may result more intensive than in case of analytic approximation methods, especially if the number of required scenarios grows. However, in many contexts the latter may result to be not significant, even in real-time applications. Open issues on this area concern the possibility of formally guaranteeing feasibility and stability properties.
- While, for deterministic systems, many distributed MPC algorithms have been developed (see, e.g., the comprehensive book [106]), their extension to the stochastic case is still a largely open problem. Specifically, some works have been recently devoted to this issue, with specific reference to distributed control of independent systems [107,108], while currently results on the application to interconnected systems are not available, with the exception of [109].
- SMPC methods could be efficiently used in multilevel control structures, where at the higher layer low frequency and deterministic methods are in charge of long-term predictions and control, while at the lower layer SMPC can be used at a higher frequency to compensate for the effect of stochastic noise as, e.g., in [31].
- Since classical methods for online estimation of the noise characteristics (e.g., their covariances) from the measurements are available, possible extensions of the SMPC schemes can be envisaged to make them adaptive and more reliable with respect to time-varying noise characteristics in a real-time framework.
- The potentialities of SMPC in many application fields have already been explored in a number of simulation studies; however its use in real world problems or in other applications is still limited or not fully explored, such as in process control or in the management and control of energy systems with renewable sources.

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