ON THE DIRICHLET AND SERRIN PROBLEMS FOR THE INHOMOGENEOUS INFINITY LAPLACIAN IN CONVEX DOMAINS: REGULARITY AND GEOMETRIC RESULTS

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ABSTRACT. Given an open bounded subset Ω of $\mathbb{R}^n$, which is convex and satisfies an interior sphere condition, we consider the pde $-\Delta_\infty u = 1$ in Ω, subject to the homogeneous boundary condition $u = 0$ on $\partial\Omega$. We prove that the unique solution to this Dirichlet problem is power-concave (precisely, 3/4 concave) and it is of class $C^1(\Omega)$. We then investigate the overdetermined Serrin-type problem, formerly considered in [10], obtained by adding the extra boundary condition $|\nabla u| = a$ on $\partial\Omega$; by using a suitable P-function we prove that, if Ω satisfies the same assumptions as above and in addition contains a ball with touches $\partial\Omega$ at two diametral points, then the existence of a solution to this Serrin-type problem implies that necessarily the cut locus and the high ridge of Ω coincide. In turn, in dimension $n = 2$, this entails that Ω must be a stadium-like domain, and in particular it must be a ball in case its boundary is of class C^2.

1. Introduction

1.1. **Setting of the problem.** The infinity Laplacian is the differential operator defined for smooth functions u by

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u\,.$$

It was firstly discovered by Aronsson in the sixties [3], and afterwards a fundamental contribution came by Jensen [29], who proved the well-posedness of the Dirichlet problem $\Delta_\infty u = 0$ in Ω with $u = g$ on $\partial\Omega$, for every boundary datum $g \in C(\partial\Omega)$. (Here and in general when dealing with the infinity Laplacian, solutions must be intended in the viscosity sense, as the operator is not in divergence form). Moreover, Jensen proved that u is characterized by the variational property of being a so-called *absolute minimizing Lipschitz extension of* g, meaning that it minimizes the L^∞ norm of the gradient on every set $A \subset\subset \Omega$, among all functions which have the same trace on ∂A. In particular, this property justifies the name “infinity Laplacian”; a general existence theory of Calculus of Variations in the sup-norm and related Aronsson–Euler type equations has been later developed by Barron, Jensen and Wang [6].
An excellent paper reviewing of the state of the art on problems involving the infinity-Laplacian up to 2004 is [4]. In the last decade these problems have raised an increasing interest in the pde community, stimulated also by their connections with tug-of-war games (see e.g. [34]), and further progresses have been made in both existence and regularity theory.

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Concerning advances in the existence theory, a notable contribution has to be ascribed to Lu and Wang, who proved in particular the well-posedness of the Dirichlet problem

$$(1)\qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

see [32], where the Authors deal also with the case of non-constant source terms with constant sign; more general source terms have been recently considered in [7].

Concerning regularity matters, the mostly investigated case is the one of infinity harmonic functions: they have been proved to be differentiable in any space dimension n by Evans and Smart [20], whereas their $C^{1,\alpha}$ regularity (which is the optimal one expected) has been proved only for $n = 2$ by Evans and Savin [19], and remains a major open problem in higher dimensions. Recently, the everywhere differentiability property in any dimension n has been extended by Lindgren [31] to a class of inhomogeneous Dirichlet problems including (1) (see also [37] for the same kind of result for some Aronsson-type equations). At present, no C^1 type regularity result is available to the best of our knowledge for the Dirichlet problem (1).

A new investigation direction in this field has been suggested by Buttazzo and Kawohl in the pioneering paper [10], where they started the study of the following overdetermined problem:

$$(2)\qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = a & \text{on } \partial\Omega\,. \end{cases}$$

The analogous problem with the classical Laplacian in place of the infinity Laplacian was studied by Serrin, who proved the seminal symmetry result stating that existence of a solution implies that Ω is a ball [36]. For its mathematical beauty and the elegance of its proof, which is based on the moving planes method by Alexandrov, Serrin result has become a masterpiece in pde's. It has originated a huge amount of literature, including alternative proofs and many generalizations, about the cases when the Laplacian is replaced by a possibly degenerate elliptic operator and when the elliptic problem is stated on an exterior domain, or on a ring-shaped domain, or on a domain with not smooth boundary. Since it is impossible to give here an exhaustive bibliography on overdetermined boundary value problems, we limit ourselves to quote the papers [8, 9, 21, 22, 23, 24, 30, 38], where many further relevant references can be found.

Now what happens for problem (2) is that *all* the methods known in the literature to deal with overdetermined boundary value problems completely fail. There are several deep reasons which may be addressed for this fact, among which the high degeneracy of the operator, the failure of a strong maximum principle and the lack of regularity results for solutions to the Dirichlet problem. Actually, until now only a highly simplified version of problem (2) has been successfully investigated: it consists in studying for which domains Ω the unique solution $\overline{u}$ to the Dirichlet problem (1) depends only on the distance $d_{\partial\Omega}$ from the boundary of Ω. Functions depending only on $d_{\partial\Omega}$ are called *web functions*, since if Ω is a polygon their level lines look like a spider web; for a short history of web functions, and an example of their application in variational problems, see [18]. If $\overline{u}$ is a web function, it has a constant normal derivative on $\partial\Omega$, and hence it solves (2); on the other hand, it is clear that problem (2) might well have solutions which are *not* web functions.

A necessary and sufficient condition for $\overline{u}$ being a web function is the concidence between the cut locus and the high ridge of Ω (for their definition see the end of this Introduction).

Let us emphasize that this geometric phenomenon has been firstly discovered by Buttazzo and Kawohl in [10]. Afterwards, the same result has been proved in [15] under milder regularity assumptions. A complete characterization of sets satisfying such geometric condition in two space dimensions has been provided in [17], as parallel neighborhoods of $C^{1,1}$ one-dimensional manifolds (in particular, they do not need to be balls, unless they are asked in addition to be simply connected and of class C^2).
This paper can be framed into the above described state of the art (see also [16] for a review), and deals with the following two mutually related topics:

(i) *About the Dirichlet problem* (1): Does its unique solution enjoy stronger regularity than everywhere differentiability? This question is relevant in connection with the study of the overdetermined problem (2), but it has an autonomous interest since, as mentioned above, even the C^1 regularity of infinity harmonic functions is still object of investigation in dimension higher than 2.

(ii) *About the Serrin-type problem* (2): If one does not work within the restricted class of web-functions, which kind of geometric information on Ω can be inferred from the existence of a solution? Is the coincidence of cut locus and high ridge still a necessary condition? The fact that one can no longer reduce the problem to an ODE for a function depending on $d_{\partial\Omega}$ increases dramatically the difficulty level, and some completely new approach is needed with respect to the methods employed in [10] and [15].

1.2. **Outline of the results.** As a first step, in Section 2 we prove a power-concavity result for the solution $\overline{u}$ to problem (1): precisely we prove that it is 3/4 concave, provided the domain Ω is convex and satisfies an interior sphere condition (see Theorem 1). This result, which is obtained by the convex envelope method introduced by Alvarez, Lasry and Lions in [1], yields as a crucial by-product that, under the same assumptions on Ω, the solution $\overline{u}$ is locally semiconcave (see Corollary 2). We remark that a similar power-concavity property has been proved by Sakaguchi in [35] in the case of the p-Laplace operator.
In Section 3 we exploit the local semiconcavity of the solution $\overline{u}$ in order to obtain its C^1 regularity (see Theorem 10). Incidentally, we provide an alternative proof of the differentiability of $\overline{u}$ which works in convex domains and is completely different with respect to the one given in [31]. Actually, the main ingredient of our approach is a new estimate holding for locally semiconcave functions near singular points (see Theorem 8): we use this estimate within a contradiction argument, in order to construct ad hoc viscosity test functions for problem (1), which allows to conclude that $\overline{u}$ cannot have singular points.
In Section 4 we introduce the P-function given by

$$P(x) := \frac{|\nabla \overline{u}|^4}{4} + \overline{u}\,.$$

Note that, thanks to the C^1 regularity result obtained for $\overline{u}$, the function P is continuous in Ω. The idea is that the existence of a solution to the overdetermined problem (2) (or equivalently the constancy of $|\nabla \overline{u}|$ over the boundary) might imply that P is constant on the whole of Ω. Actually, should the function P be constant on the whole of Ω, one would obtain immediately the information that cut locus and high ridge of Ω coincide (see Proposition 13). In order to investigate the possible constancy of P, we study its behaviour along the steepest ascent lines of $\overline{u}$, intended as trajectories of the Cauchy problem

$$\begin{cases} \dot\gamma(t) = \nabla \overline{u}(\gamma(t)) \\ \gamma(0) = x \in \overline{\Omega}\,. \end{cases}$$

Indeed, it is easy to see that the map $t \mapsto P(\gamma(t))$ has vanishing first order derivative almost everywhere (see Lemma 15) and that, should the function P be constant along a trajectory γ, one could immediately compute the solution along it (see Proposition 14). Unfortunately, the constancy of P along a trajectory cannot be deduced from the vanishing property of the first derivative, because $\overline{u}$ is not known to be $C^{1,1}$ (and actually it cannot expected to be so, see below), so that the map $P \circ \gamma$ is *not* absolutely continuous.
Nevertheless, we manage to get some control on the properties of trajectories, and to infer some information on the *global* behavior of P on Ω. The approach we adopt consists in constructing the unique forward gradient flow associated with $\overline{u}$ (which can be done thanks to its semiconcavity, see Lemma 18), and then approximating it by the sequence of gradient flows associated with the supremum convolutions of $\overline{u}$ (which enjoy $C^{1,1}$ regularity and, by the "magical properties" of their superjets, turn out to be sub-solutions to the pde, see Lemma 19). By this way, in Theorem 16, we obtain the crucial estimates

$$\min_{\partial\Omega} \frac{|\nabla \overline{u}|^4}{4} \leq P(x) \leq \max_{\overline{\Omega}} \overline{u} \qquad \forall x \in \overline{\Omega}\,.$$

These bounds can be used to infer some information both on the geometry of domains on which problem (2) admits a solution, and on the regularity of the solution to problem (1). This is done respectively in the last two sections of the paper. In Section 5 we prove that the existence of a solution to problem (2) entails the coincidence of cut locus and high ridge provided the domain Ω is convex and contains an inner ball which touches $\partial\Omega$ at two diametral points (see Theorem 22). For instance, this excludes existence of a solution to problem (2) when Ω is an ellipse. It is our belief that that both the convexity and the "diametral touching ball" conditions are not necessary for the validity of the result, but by now proving it in full generality remains an open problem. We wish to emphasize that combining Theorem 22 with the results proved in our previous paper [17] reveals an interesting phenomenon which seems to be completely new in the field of overdetermined problems, and more generally in the interplay between geometry and pde's: convex domains where problem (2) admits a solution may obey or not symmetry according to the regularity of their boundary; more precisely, in dimension $n = 2$, they must be spherical as soon as they are of class C^2, but may be nonspherical (precisely stadium-like domains) if they do not enjoy such regularity (see Corollary 23). This seems somehow to reflect the fact that regularity properties for the solution to the pde finer than C^1 are a delicate stuff. Such properties are discussed in the final Section 6 where, via the use of the P-function, we show that the expected optimal regularity of $\overline{u}$ is $C^{1,\alpha}$ with $\alpha \leq 1/3$ (for the precise statements see Propositions 26 and 27).

1.3. **Some preliminary notions.** Let us specify what we mean by a *solution* to problems (1) and (2). For convenience of the reader, let us first remind the definition of viscosity sub- and super-solutions. Recall first that second order sub-jet (resp. super-jet), $J^{2,-}_{\Omega} u(x_0)$ (resp. $J^{2,+}_{\Omega} u(x_0)$), of a function $u \in C(\overline{\Omega})$ at a point $x_0 \in \Omega$, is by definition the set of pairs $(p, A) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\rm sym}$ such that

$$u(y) \geq (\leq)\ u(x_0) + \langle p,\, y - x_0\rangle + \frac{1}{2}\,\langle A(y - x_0),\, y - x_0\rangle + o(|y - x_0|^2) \qquad \text{as } y \to x_0,\ y \in \Omega\,,$$

Then, following [14], a viscosity subsolution to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$(3) \quad -\Delta_\infty \varphi(x_0) - 1 \le 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0,$$

or equivalently

$$(4) \qquad -\langle Xp, p\rangle - 1 \le 0 \quad \forall (p, X) \in J^{2,+}_\Omega u(x_0)\,.$$

Similarly, a viscosity super-solution to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$(5) \quad -\Delta_\infty \varphi(x_0) - 1 \ge 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0,$$

or equivalently

$$(6) \qquad -\langle Xp, p\rangle - 1 \ge 0 \quad \forall (p, X) \in J^{2,-}_\Omega u(x_0)\,.$$

By a viscosity solution to the equation $-\Delta_\infty u - 1 = 0$ we mean a function $u \in C(\overline{\Omega})$ which is both a viscosity sub-solution and a viscosity super-solution on Ω.
By a solution to problem (1), we mean a function $u \in C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and u is a viscosity solution to $-\Delta_\infty u = 1$ in Ω.
By saying that a the overdetermined boundary value problem (2) admits a solution, we mean that the following regularity hypothesis is fulfilled

(hu) the unique viscosity solution u to problem (1) satisfies

$$\exists\, \delta > 0 \ :\ u \text{ is of class } C^1 \text{ on } \{x \in \overline{\Omega} \ :\ \mathrm{dist}(x, \partial\Omega) < \delta\}\,,$$

and that $|\nabla u| = a$ on $\partial\Omega$.
Finally, let us introduce some definitions related to the distance function to the boundary of Ω, which will be denoted by $d_{\partial\Omega}$. We let $\Sigma(\Omega)$ be the set of points in Ω where $d_{\partial\Omega}$ is not differentiable, and we call *cut locus* and *high ridge* the sets given respectively by

$$(7) \qquad \overline{\Sigma}(\Omega) := \text{the closure of } \Sigma(\Omega) \text{ in } \overline{\Omega}$$

$$(8) \qquad \mathrm{M}(\Omega) := \text{the set where } d_{\partial\Omega}(x) = \rho_\Omega := \max\nolimits_{\overline{\Omega}} d_{\partial\Omega}\,.$$

Moreover, we denote by ϕ_Ω the web-function defined on Ω by

$$(9) \qquad \phi_\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_{\partial\Omega}(x))^{4/3}\right], \qquad \text{where } c_0 := 3^{4/3}/4\,.$$

2. Power-concavity and semiconcavity of solutions

Throughout the paper, Ω denotes a nonempty open bounded subset of $\mathbb{R}^n$.
Most of our results will be proved under the following additional hypothesis (which however will be specified in each statement):

($h\Omega$) Ω is convex and satisfies an interior sphere condition.

Theorem 1. *Assume* ($h\Omega$)*, and let* u *be the solution to problem* (1)*. Then* $u^{3/4}$ *is concave in* Ω*.*

Before proving Theorem 1, we observe that it readily implies the following semiconcavity result. We recall that $u : \Omega \to \mathbb{R}$ is called *semiconcave (with constant* C*) in* Ω if

$$u(\lambda x + (1-\lambda)y) \ge \lambda u(x) + (1-\lambda)u(y) - C\frac{\lambda(1-\lambda)}{2}\,|x-y|^2 \qquad \forall [x, y] \subset \Omega \ \text{ and } \ \forall \lambda \in [0, 1]\,.$$

We say that u is *locally semiconcave in* Ω if it is semiconcave on compact subsets of Ω.

Corollary 2. *Assume* $(h\Omega)$*, and let* u *be the solution to problem* (1)*. Then* u *is locally semiconcave in* Ω*.*

Proof. Given $\epsilon > 0$, we claim that u is semiconcave with constant $C_\epsilon := 4\epsilon^{-1/2}M_\epsilon^2/9$ in the set $U_\epsilon := \{x \in \Omega : \ u(x) \geq \epsilon\}$, where M_ϵ is the Lipschitz constant of $w := u^{3/4}$ on the compact set U_ϵ. Namely, the function $\psi(t) := t^{4/3} - 2\epsilon^{-1/2}t^2/9$ is concave in $[\epsilon^{3/4}, +\infty)$. Then the inequality $\psi\big(\lambda w(x) + (1-\lambda)w(y)\big) \geq \lambda\psi(w(x)) + (1-\lambda)\psi(w(y))$ entails

$$w(\lambda x + (1-\lambda)y)^{4/3} \geq \lambda w(x)^{4/3} + (1-\lambda)w(y)^{4/3} - \frac{2\epsilon^{-1/2}}{9}\lambda(1-\lambda)|w(x) - w(y)|^2$$

for every $[x, y] \subset U_\epsilon$ and $\lambda \in [0, 1]$. On the other hand, we have $|w(x) - w(y)| \leq M_\epsilon|x - y|$, hence we obtain

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y) - C_\epsilon\frac{\lambda(1-\lambda)}{2}|x - y|^2,$$

i.e., u is semiconcave with semiconcavity constant C_ϵ in U_ϵ.

The remaining of this section is devoted to the proof of Theorem 1. We start with an elementary observation which will be exploited several times throughout the paper.

Remark 3. The viscosity solution u to problem (1) is strictly positive in Ω. Indeed, it is nonnegative by the comparison result proved in [32, Thm. 3]. Assume by contradiction that $u(x_0) = 0$ at some point $x_0 \in \Omega$. Then the function $\varphi \equiv 0$ touches u from below at x_0, and hence u cannot be a viscosity supersolution to the equation $-\Delta_\infty u = 1$ at x_0.

If u is the solution to (1), for every $\alpha \in (0, 1)$ the function $w := -u^\alpha$ (which is strictly negative in Ω by Remark 3) is a viscosity solution of

$$\begin{cases} -\Delta_\infty w - \frac{1-\alpha}{\alpha}\cdot\frac{1}{w}|\nabla w|^4 + \alpha^3(-w)^{3-3/\alpha} = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to prove Theorem 1, we are going to choose $\alpha = 3/4$ and show that, if w is a viscosity solution to

$$\begin{cases} -\Delta_\infty w - \frac{1}{w}\left[\frac{1}{3}|\nabla w|^4 + \left(\frac{3}{4}\right)^3\right] = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

then w is convex. To that aim, we adopt the convex envelope method introduced by Alvarez, Lasry and Lions in [1]. Following their notation, we denote by w_{**} the largest convex function below w. We first show that, for every $x \in \Omega$, in the characterization

$$w_{**}(x) = \inf\left\{\sum_{i=1}^k \lambda_i w(x_i) \ : \ x = \sum_{i=1}^k \lambda_i x_i\,,\ x_i \in \overline{\Omega}\,,\ \lambda_i > 0\,,\ \sum_{i=1}^k \lambda_i = 1\,,\ k \leq n+1\right\}$$

the infimum can be attained only at interior points $x_i \in \Omega$:

Lemma 4. *Assume* $(h\Omega)$*, and let* u *be the solution to problem* (1)*. Set* $w := -u^{3/4}$*. For a fixed* $x \in \Omega$*, let* $x_1, \dots, x_k \in \overline{\Omega}$*,* $\lambda_1, \dots, \lambda_k > 0$*, with* $\sum_{i=1}^k \lambda_i = 1$*, be such that*

$$x = \sum_{i=1}^k \lambda_i x_i\,, \quad w_{**}(x) = \sum_{i=1}^k \lambda_i w(x_i).$$

Then $x_1, \dots, x_k \in \Omega$*.*

Proof. Assume by contradiction that at least one of the x_i's, say x_1, belongs to $\partial\Omega$. Let $B_R(y) \subset \Omega$ be a ball such that $\partial B_R(y) \cap \partial\Omega = \{x_1\}$. Since $-\Delta_\infty u = 1$, by Lemma 2.2 in [13] the function $\tilde{u} := -u$ enjoys the property of comparison with cones from above according to Definition 2.3 in the same paper. Then, by Lemma 2.4 in [13], the function

$$r \mapsto \max_{x\in\partial B_r(y)} \frac{\tilde{u}(x) - \tilde{u}(y)}{r} = - \min_{x\in\partial B_r(y)} \frac{u(x) - u(y)}{r}$$

is monotone nondecreasing on the interval $(0, R)$. Namely, for all $r \in (0, R)$, there holds

$$\min_{x\in\partial B_r(y)} \frac{u(x) - u(y)}{|x-y|} \geq \min_{x\in\partial B_R(y)} \frac{u(x) - u(y)}{|x-y|} = -\frac{u(y)}{R}\,, \tag{11}$$

where the last equality comes from the fact that u is non-negative in Ω (cf. Remark 3). By (11), we have

$$u(x) \geq u(y)\Big(1 - \frac{|x-y|}{R}\Big) \qquad \forall x \in B_R(y)\,,$$

and hence

$$w(x) \leq w(y)\Big(1 - \frac{|x-y|}{R}\Big)^{3/4} \qquad \forall x \in B_R(y)\,. \tag{12}$$

Let us define the unit vector $\zeta := (x-x_1)/|x-x_1|$ and let $\nu = (y-x_1)/|y-x_1|$ denote the inner normal of $\partial\Omega$ at x_1. Since Ω is a convex set and $x \in \Omega$, we have that $\langle \zeta, \nu\rangle > 0$ and $x_1 + t\zeta \in B_R(y)$ for $t > 0$ small enough. Moreover, w_{**} is affine on $[x_1, x]$: indeed, since the epigraph of w_{**} is the convex envelope of the epigraph of w, it is readily seen that w_{**} is affine on the whole set of convex combinations of the points $\{x_1, \dots, x_k\}$. Taking into account that $w_{**}(x_1) = w(x_1) = 0$, we infer that there exists $\mu > 0$ such that

$$w(x_1 + t\zeta) \geq w_{**}(x_1 + t\zeta) = -\mu t \qquad \forall t \in [0, 1].$$

From (12) we obtain

$$-\mu t \leq w(y)\left(1 - \frac{|t\zeta - R\nu|}{R}\right)^{3/4} = w(y)\left(\langle \zeta, \nu\rangle\,\frac{t}{R} + o(t)\right)^{3/4}\,, \qquad t \to 0^+,$$

and, recalling that $w(y) < 0$,

$$\mu t^{1/4} \geq K + o(1), \qquad t \to 0^+$$

with $K > 0$, a contradiction.

Remark 5. As a consequence of (12), taking $x = x_1 + \lambda\nu$, and recalling that $w(x_1) = 0$, it is readily seen that

$$\lim_{\lambda\to 0^+} \frac{w(x_1 + \lambda\nu) - w(x_1)}{\lambda} \leq \lim_{\lambda\to 0^+} \frac{w(y)}{\lambda}\Big(\frac{\lambda}{R}\Big)^{3/4} = -\infty\,,$$

i.e. the normal derivative of w with respect to the external normal is $+\infty$ at every boundary point of Ω. This is the reason why the semiconcavity property of u is stated just *locally* in Ω and not up to the boundary, cf. the proof of Corollary 2.

On the basis of the lemma just proved, we can now establish that the convex envelope of a super-solution to (10) is still a super-solution.

Proposition 6. *Assume* $(h\Omega)$. *If* w *is a viscosity super-solution to* (10), *then also* w_{**} *is a viscosity super-solution to the same problem.*

Proof. Let $x \in \Omega$ and consider $(p, A) \in J^{2,-}_{\Omega} w_{**}(x)$. Recall that the second order subjet $J^{2,-}_{\Omega} v(x_0)$ of a function $v \in C(\overline{\Omega})$ at a point $x_0 \in \overline{\Omega}$ is by definition the set of pairs $(p, A) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\rm sym}$ such that

$$v(y) \geq v(x_0) + \langle p,\, y - x_0\rangle + \frac{1}{2}\langle A(y-x_0),\, y - x_0\rangle + o(|y-x_0|^2) \qquad \text{as } y \to x_0,\ y \in \overline{\Omega}\,,$$

whereas its "closure" $\overline{J}^{2,-}_{\Omega} v(x_0)$ is the set of $(p, A) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\rm sym}$ for which there is a sequence $(p_j, A_j) \in J^{2,-}_{\Omega} v(x_j)$ such that $(x_j, v(x_j), p_j, A_j) \to (x_0, v(x_0), p, A)$.
For every $\epsilon > 0$ small enough, applying Proposition 1 in [1] and Lemma 4, we obtain points $x_1, \dots, x_k \in \Omega$, positive numbers $\lambda_1, \dots, \lambda_k$ satisfying $\sum_{i=1}^k \lambda_i = 1$, and elements $(p, A_i) \in \overline{J}^{2,-}_{\Omega} w(x_i)$, with A_i positive semidefinite, such that

$$\sum_{i=1}^k \lambda_i x_i = x, \quad \sum_{i=1}^k \lambda_i w(x_i) = w_{**}(x), \quad A - \epsilon A^2 \leq \left(\sum_{i=1}^k \lambda_i A_i^{-1}\right)^{-1}.$$

We recall that, here and in the sequel, it is not restrictive to assume that the matrices A, $A_1, \dots, A_k$ are positive definite, since the case of degenerate matrices can be handled as in [1], p. 273.
Set for brevity $F(w, p, Q) := -\mathrm{tr}\big((p\otimes p)Q\big) - \frac{1}{w}\Big(\frac{1}{3}|p|^4 + c\Big)$, with $c := \Big(\frac{3}{4}\Big)^3$. Since w is a super-solution to (10), we have $F(w(x_i), p, A_i) \geq 0$, i.e.

$$-w(x_i) \leq \frac{1}{\langle A_i p,\, p\rangle}\left(\frac{1}{3}|p|^4 + c\right),$$

so that

$$-\frac{1}{\sum_{i=1}^k \lambda_i w(x_i)}\left(\frac{1}{3}|p|^4 + c\right) \geq \left(\sum_{i=1}^k \lambda_i \frac{1}{\langle A_i p,\, p\rangle}\right)^{-1}.$$

Then, using the degenerate ellipticity of F, we obtain

$$\begin{aligned} F(w_{**}(x), p, A - \epsilon A^2) &\geq -\left\langle \left(\sum_{i=1}^k \lambda_i A_i^{-1}\right)^{-1} p,\, p\right\rangle - \frac{1}{\sum_{i=1}^k \lambda_i w(x_i)}\left(\frac{1}{3}|p|^4 + c\right) \\ &\geq -\left\langle \left(\sum_{i=1}^k \lambda_i A_i^{-1}\right)^{-1} p,\, p\right\rangle + \left(\sum_{i=1}^k \lambda_i \frac{1}{\langle A_i p,\, p\rangle}\right)^{-1} \\ &\geq 0, \end{aligned}$$

where the last inequality follows from the concavity of the map $Q \mapsto 1/\mathrm{tr}\big((p\otimes p)Q^{-1}\big)$ proved in [1], p. 286.

Finally, Theorem 1 follows from Proposition 6 by invoking a comparison principle:

Proof of Theorem 1. Let u be the solution to problem (1), and let $w = -u^{3/4}$. Then w is a viscosity solution to (10) and, by Proposition 6, w_{**} is a viscosity super-solution to the same problem, which agrees with $w = 0$ on $\partial\Omega$. By the comparison principle holding for problem (10), we infer that $w_{**} \geq w$ in Ω. (Let us point out that the validity of the comparison principle for problem (10) can be readily deduced from the validity of the comparison principle for problem (1) established in [32, Thm. 3], combined with the observation that the map $u \mapsto w = -u^{3/4}$ is a bijection between viscosity sub- or super-solutions to problems (1) and sub- or super-solutions w to problem (10)). On the other hand, by definition, there holds $w_{**} \leq w$ in Ω. We conclude that $w_{**} = w$ in Ω, namely w is convex. □

Remark 7. An interesting question is whether is it possible to extend Theorem 1, and more generally the results of this paper, to the case of the so-called normalized infinity Laplace operator Δ_∞^N (for its definition, see for instance [33]). Actually, problem (1) with Δ_∞^N in place of Δ_∞ is known to have a unique solution u, and the natural conjecture is that u is (1/2)-concave, since one can readily check that the function $w := -u^{1/2}$ solves $-\Delta_\infty w = \frac{|\nabla w|^2}{2w}(2|\nabla w|^2 + 1)$. However, a careful inspection of the above proof of Theorem 1 reveals that the Alvarez-Lasry-Lions method does not extend straightforward to this situation, in particular because the bijection exploited in the last part of the proof fails. This is one of the reasons why we believe that the case of the normalized infinity Laplace operator cannot be handled merely as a parallel variant of the infinity Laplace operator. We consider it as a significant direction to be explored.

3. C^1 regularity of solutions

In this section we deal with the C^1 regularity of the unique solution to problem (1). Our strategy is as follows. As a first step, we prove a new estimate for locally semiconcave functions near singular points. Then we use such estimate as a crucial tool in order to construct suitable viscosity test functions, which prevent u from being a solution to the pde at singular points. Finally we exploit the local semiconcavity result obtained in the previous section to conclude that u is continuously differentiable.

Given a function $u \in C(\Omega)$, we denote by $\Sigma(u)$ the singular set of u, namely the set of points where u is not differentiable.

We recall that the *Fréchet super-differential* of u at a point $x_0 \in \Omega$ is defined by

$$D^+u(x_0) := \left\{ p \in \mathbb{R}^n \ : \ \limsup_{x \to x_0} \frac{u(x) - u(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \leq 0 \right\}.$$

We point out that, for every $x_0 \in \Sigma(u)$, $D^+u(x_0)$ is nonempty compact convex set which is not a singleton; in particular, $D^+u(x_0) \setminus \operatorname{extr} D^+u(x_0)$ is not empty and contains non-zero elements.

Then the result reads:

Theorem 8. *Let $u \colon \Omega \to \mathbb{R}$ be a locally semiconcave function, let $x_0 \in \Sigma(u)$, and let $p \in D^+u(x_0) \setminus \operatorname{extr} D^+u(x_0)$. Let $R > 0$ be such that $\overline{B}_R(x_0) \subset \Omega$, and let C denote the semiconcavity constant of u on $\overline{B}_R(x_0)$. Then there exist a constant $K > 0$ and a unit vector $\zeta \in \mathbb{R}^n$ satisfying the following property:*

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle - K \, | \langle \zeta, x - x_0 \rangle | + \frac{C}{2}|x - x_0|^2 \quad \forall x \in \overline{B}_R(x_0). \tag{13}$$

In particular, for every $c > 0$, setting $\delta := \min\{K/c, R\}$, it holds

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle - c \, \langle \zeta, x - x_0 \rangle^2 + \frac{C}{2}|x - x_0|^2 \quad \forall x \in \overline{B}_\delta(x_0). \tag{14}$$

Furthermore, if $p \neq 0$ then the vector ζ can be chosen so that $\langle \zeta, p \rangle \neq 0$.

Remark 9. By inspection of the proof below, one can derive some additional information on the constant K and the vector ζ appearing in (13). In fact, let $p_0, \ldots, p_k$ be points in $\operatorname{extr} D^+u(x_0)$ such that $p \in \operatorname{conv}\{p_0, \dots, p_k\}$ $(1 \le k \le n)$. Then the constant K can be chosen as as the distance between the origin and the boundary of the set $\operatorname{conv}\{p_0 - p, \dots, p_k - p\}$, whereas the vector ζ can be chosen in the set

$$Z := \left\{ \frac{z}{|z|} \ : \ z \in \operatorname{conv}\{p_0 - p, \dots, p_k - p\}, \ z \neq 0 \right\} . \tag{15}$$

Proof of Theorem 8. Since $p \in D^+u(x_0) \setminus \operatorname{extr} D^+u(x_0)$ there exist $p_0, \dots, p_k \in \operatorname{extr} D^+u(x_0)$ (with $1 \le k \le n$) and numbers $\lambda_0, \dots, \lambda_k \in (0,1)$ with $\sum_{i=0}^k \lambda_i = 1$ such that $p = \sum_{i=0}^k \lambda_i p_i$.
We divide the remaining of the proof in three steps.

Step 1. The following inequality holds:

$$u(x) \le u(x_0) + \min_{i=0,\dots,k} \langle p_i, \, x - x_0 \rangle + \frac{C}{2} |x - x_0|^2 \, , \qquad \forall x \in \overline{B}_R(x_0). \tag{16}$$

For every $i = 0, \dots, k$, since $p_i \in D^+u(x_0)$ by [11, Prop. 3.3.1] we have that

$$u(x) \le u(x_0) + \langle p_i, \, x - x_0 \rangle + \frac{C}{2} \, |x - x_0|^2 \qquad \forall x \in \overline{B}_R(x_0),$$

so that (16) easily follows.

Step 2. Let K denote the distance between the origin and the boundary of $\operatorname{conv}\{p_0 - p, \dots, p_k - p\}$. *Then for every unit vector ζ in the set Z defined in* (15), *one has*

$$\min_{i=0,\dots,k} \langle p_i - p, \, x \rangle \le -K \, | \langle \zeta, \, x \rangle \, | \, , \qquad \forall x \in \mathbb{R}^n. \tag{17}$$

Since $\sum_i \lambda_i (p_i - p) = 0$ we have that the set

$$F := \operatorname{span}\{p_0 - p, p_1 - p, \dots, p_k - p\}$$

is a subspace of $\mathbb{R}^n$ of dimension k. Let

$$Q := \operatorname{conv}\{p - p_0, p - p_1, \dots, p - p_k\} \, ;$$

since 0 belongs to the relative interior of the polytope Q, and since K is the distance between 0 and the boundary of Q, we clearly have $K > 0$ and $B := \overline{B}_K(0) \cap F \subseteq Q$. Hence

$$h_Q(x) := \max\{\langle q, \, x \rangle : \ q \in Q\} \ge \max\{\langle b, \, x \rangle : \ b \in B\} =: h_B(x), \qquad \forall x \in \mathbb{R}^n.$$

On the other hand, we have that

$$h_Q(x) = \max_{i=0,\dots k} \langle p - p_i, \, x \rangle = - \min_{i=0,\dots k} \langle p_i - p, \, x \rangle$$

whereas, if ζ is any unit vector in the set Z defined in (15), then $\pm K\zeta \in B$, so that

$$h_B(x) = \max\{\langle b, \, x \rangle : \ b \in B\} \ge K | \, \langle \zeta, \, x \rangle \, | \, .$$

Now (17) easily follows.

Step 3. Completion of the proof.

The estimate (13) is a direct consequence of (16) and (17). In order to prove (14) it is enough to observe that, given $c > 0$, the inequality $K \, |t| \ge c \, t^2$ holds for every $|t| < K/c$.

By exploiting the above geometric result for semiconcave functions, we obtain:

Theorem 10. *Let $u \in C(\Omega)$ be a viscosity solution to $-\Delta_\infty u = f(x,u)$ in Ω. If u is locally semiconcave in Ω, then u is everywhere differentiable (hence of class C^1) in Ω.*

Proof. Assume by contradiction that $\Sigma(u) \neq \emptyset$. Without loss of generality we can assume that $0 \in \Sigma(u)$. Let $p \in D^+u(0) \setminus \operatorname{extr} D^+u(0)$, $p \neq 0$. By Theorem 8, there exists a unit vector $\zeta \in \mathbb{R}^n$ such that $\langle \zeta, p \rangle \neq 0$ and, for every $c > 0$,

$$u(x) \leq \varphi(x) := u(0) + \langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{C}{2} |x|^2, \qquad \forall x \in B_\delta(0),$$

with δ depending on c. Since

$$\Delta_\infty \varphi(0) = -2c \langle \zeta, p \rangle^2 + C|p|^2,$$

choosing $c > 0$ large enough we get $-\Delta_\infty \varphi(0) > f(0, u(0))$, a contradiction.
Since u is differentiable everywhere in Ω, then by [11, Prop. 3.3.4] we conclude that $u \in C^1(\Omega)$.

Finally, by combining Theorem 1 and Corollary 2 with Theorem 10, we obtain:

Corollary 11. *Assume* $(h\Omega)$*, and let u be the solution to problem* (1)*. Then u is continuously differentiable in Ω.*

Proof. By Theorem 1 we know that u is locally semiconcave in Ω, hence from Theorem 10 we obtain that $u \in C^1(\Omega)$. □

4. The P-function along the gradient flow

In this section we investigate the behavior of the P-function associated with problem (1) according to the following

Definition 12. Let u be the solution to problem (1). We set

$$P(x) := \frac{|\nabla u(x)|^4}{4} + u(x), \qquad x \in \Omega. \tag{18}$$

The relevance of this P-function in connection with problems (1) and (2) is enlightened by the next two lemmas. In fact, such relevance is two-fold. On one hand, if it happens that P is constant on the whole Ω, this gives geometric information on Ω (see Proposition 13 and Corollary 23); we shall exploit this fact in Section 5 to study the overdedetemined boundary value problem (2). On the other hand, if it happens that P is constant along a steepest ascent line of u, i.e. along a trajectory of the Cauchy problem

$$\begin{cases} \dot\gamma(t) = \nabla u(\gamma(t)) \\ \gamma(0) = x \in \overline{\Omega}, \end{cases} \tag{19}$$

then it is possible to compute explicitly u along the trajectory (see Proposition 14); we shall exploit this fact in Section 6 in order to obtain regularity thresholds for the solution to the Dirichlet problem (1).
We recall that the cut locus $\overline{\Sigma}(\Omega)$ and the high ridge $\mathrm{M}(\Omega)$ of Ω are the set defined as in (7)-(8); moreover, ρ_Ω and ϕ_Ω denote respectively the inradius of Ω and the web function introduced in (9).

Proposition 13. *Assume that the unique solution u to the Dirichlet problem* (1) *is of class $C^1(\Omega)$, and that the P-function introduced in Definition 12 satisfies*

$$P(x) = \lambda \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{20}$$

for some $\lambda \leq c_0\rho_\Omega^{4/3}$. Then $\lambda = c_0\rho_\Omega^{4/3}$ and $u = \phi_\Omega$, where ϕ_Ω is the function defined in (9)*, and it holds $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$.*

Proof. It is clear that $\lambda = \max u$. On the other hand, $\max u \geq \max v = c_0\rho_\Omega^{4/3}$, where v is the radial solution of the Dirichlet problem in a ball $B_{\rho_\Omega} \subseteq \Omega$. Hence $\lambda = c_0\rho_\Omega^{4/3}$.
Let $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the Hamiltonian defined by

$$H(u,p) := \frac{1}{4}|p|^4 + u - \lambda.$$

Then the equality (20) can be rewritten as

$$H(u(x), \nabla u(x)) = 0, \qquad \mathcal{L}^n\text{-a.e. on } \Omega\ . \tag{21}$$

Since u is of class $C^1(\Omega)$, then it follows that it is a classical (hence also a viscosity) solution of the Dirichlet problem

$$\begin{cases} H(u, \nabla u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Since the solution to this Dirichlet problem is unique (see *e.g.* [5, Theorem III.1]), to prove that $u = \phi_\Omega$ it is enough to show that also ϕ_Ω is a viscosity solution to (22).
It is readily seen that ϕ_Ω is differentiable at every point $x \in \Omega\backslash S$, where $S := \Sigma(\Omega)\backslash M(\Omega)$, and $H(\phi_\Omega(x), \nabla\phi_\Omega(x)) = 0$.
Hence it is enough to show that, for every $x \in S$, one has

$$H(\phi_\Omega(x), p) \leq 0, \quad \forall p \in D^+\phi_\Omega(x), \qquad H(\phi_\Omega(x), q) \geq 0, \quad \forall q \in D^-\phi_\Omega(x), \tag{23}$$

where the symbols $D^+\psi$ and $D^-\psi$ denote respectively the super and sub-differential of a function ψ. Since $x \in S$, then $x \notin M(\Omega)$, so that $\delta := d_{\partial\Omega}(x) < \rho_\Omega$. As a consequence

$$D^\pm\phi_\Omega(x) = g'(\delta)\, D^\pm d_{\partial\Omega}(x),$$

where $g(t) := c_0[\rho_\Omega^{4/3} - (\rho_\Omega - t)^{4/3}]$ and $g'(\delta) > 0$. In particular $D^-\phi_\Omega(x) = \emptyset$, since the sub-differential of the distance function is empty at singular points (see [11, Corollary 3.4.5]), so that the second condition in (23) is trivially satisfied.
Let now consider $p \in D^+\phi_\Omega(x)$. Since $D^+d_{\partial\Omega}(x)$ is contained in the closed unit ball, there exists $\xi \in \mathbb{R}^n$, $|\xi| \leq 1$, such that $p = g'(\delta)\,\xi$, hence

$$H(\phi_\Omega(x), p) = \frac{1}{4}\, g'(\delta)^4\, |\xi|^4 + \phi_\Omega(x) - \lambda \leq \frac{1}{4}\, g'(\delta)^4 + g(\delta) - \lambda. \tag{24}$$

On the other hand, if $(x_k) \subset \Omega \setminus \Sigma(\Omega)$ is a sequence converging to x, we get

$$0 = H(\phi_\Omega(x_k), \nabla\phi_\Omega(x_k)) = \frac{1}{4} g'(d_{\partial\Omega}(x_k))^4 + g(d_{\partial\Omega}(x_k)) - \lambda \longrightarrow \frac{1}{4}\, g'(\delta)^4 + g(\delta) - \lambda,$$

that, together with (24), proves the first condition in (23).

Proposition 14. *Let u be the solution to problem* (1)*, and let $\gamma : [0, \delta) \to \Omega$ be a local solution to problem* (19)*, starting at a point x with $\nabla u(x) \neq 0$. Assume that u is differentiable at $\gamma(t)$ for $\mathcal{L}^1$-a.e. $t \in [0, \delta)$, and that the P-function introduced in Definition 12 satisfies*

$$P(\gamma(t)) = \lambda \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, \delta)\ .$$

Then, setting $m := u(x)$, it holds $\lambda > m$ and, for all $t \in [0,\delta)$, the function $\varphi(t) := u(\gamma(t))$ agrees with the function

$$\overline{\varphi}(t) := \begin{cases} \lambda - (\sqrt{\lambda - m} - t)^2, & \text{if } t \in [0, \sqrt{\lambda - m}) \\ \lambda, & \text{if } t \geq \sqrt{\lambda - m}\,. \end{cases}$$

Proof. The assumed equality $P(\gamma(t)) = \lambda$ $\mathcal{L}^1$-a.e. on $[0,\delta)$ implies that $\lambda > m$ (since γ is a steepest ascent line of u and $\nabla u(x) \neq 0$). Moreover, the function $\varphi(t) := u(\gamma(t))$ is in $AC([0,\delta))$, because $u \in \mathrm{Lip}(\Omega)$ (see e.g.[4, Lemma 2.9]) and $\gamma \in AC([0,\delta))$, and it is a solution to the Cauchy problem

$$\begin{cases} \dot{\varphi}(t) = 2\sqrt{\lambda - \varphi(t)} & \mathcal{L}^1\text{-a.e. on } [0,\delta) \\ \varphi(0) = m. \end{cases}$$

It is readily seen that this Cauchy problem admits a unique global solution, given precisely by $\overline{\varphi}$. Therefore, we conclude that φ agrees with $\overline{\varphi}$ on $[0,\delta)$.

We start now investigating what can be said about the behaviour P along trajectories of problem (19) and globally over Ω. A first elementary observation comes from the pde interpreted pointwise at points of two-differentiability of u:

Lemma 15. *Let u be the solution to problem* (1)*, and let $\gamma : [0,\delta) \to \Omega$ be a local solution to problem* (19)*. Assume that u is twice differentiable at $\gamma(t)$ for $\mathcal{L}^1$-a.e. $t \in [0,\delta)$. Then it holds*

$$\frac{d}{dt}\big(P(\gamma(t))\big) = 0 \qquad \mathcal{L}^1\text{-a.e. in } [0,\delta)\,. \tag{25}$$

Proof. At every point x where u is twice differentiable, it holds

$$\nabla P(x) = |\nabla u(x)|^2 D^2 u(x)\, \nabla u(x) + \nabla u(x)\,;$$

we infer that

$$\begin{aligned} \langle \nabla P(x), \nabla u(x) \rangle &= |\nabla u(x)|^2 \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle + |\nabla u(x)|^2 \\ &= |\nabla u(x)|^2\, (\Delta_\infty u(x) + 1) = 0\,. \end{aligned}$$

Thus, since by assumption u is twice differentiable at $\gamma(t)$ for $\mathcal{L}^1$-a.e. $t \in [0,\delta)$, it holds

$$\frac{d}{dt}\big(P(\gamma(t))\big) = \Big\langle \nabla P(\gamma(t)), \frac{d}{dt}\gamma(t) \Big\rangle = \Big\langle \nabla P(\gamma(t)), \nabla u(\gamma(t)) \Big\rangle = 0$$

for $\mathcal{L}^1$-a.e. $t \in [0,\delta)$.

The main and crucial difficulty in exploiting the information given by Lemma 15 is the possible lackness of regularity of the function $P \circ \gamma$. In particular, we are not able to ensure that this map is in $AC([0,\delta))$, so to infer from (25) that P is constant along γ.
Moreover, in order to obtain some information on the *global* behavior of P on Ω, and not merely on a single trajectory, we need to study the solutions to problem (19) under the aspects of local uniqueness and continuation property.
This is the main reason why hereafter we require both the assumptions $(h\Omega)$-(hu). Several comments on these assumptions are postponed after the main result of this section, which reads:

Theorem 16. *Assume* $(h\Omega)$–(hu). *Then the P-function introduced in Definition 12 satisfies*

$$\min_{\partial\Omega}\frac{|\nabla u|^4}{4}\le P(x)\le\max_{\overline{\Omega}}u\qquad\forall x\in\overline{\Omega}\,. \tag{26}$$

Remark 17. (i) It is a natural question to ask whether $(h\Omega)$ implies (hu). Indeed, under the assumption $(h\Omega)$, the set Ω is of class C^1 and, by Corollary 11, the unique viscosity solution u to problem (1) belongs to $C^1(\Omega)$. Hence, if we assume $(h\Omega)$, requiring the additional condition (hu) amounts to ask just that u is of class C^1 *up to the boundary.* In view of the results proved in [39, 27, 28], it seems reasonable to guess that the latter condition is not always fulfilled when $(h\Omega)$ holds, but becomes true under the additional regularity requirement that $\partial\Omega$ is C^2. However, this boundary regularity property seems to be a highly technical point, which goes beyond the purposes of this paper, and we limit ourselves to address it as an interesting open problem.
(ii) Under the assumptions $(h\Omega)$–(hu), if u is the unique viscosity solution to problem (1), the set of critical points of u agrees with the set $\operatorname{argmax}_{\overline{\Omega}} u$ where u attains its maximum over $\overline{\Omega}$. Indeed, by Theorem 1, the function $u^{3/4}$ is concave (and strictly positive) in Ω; hence its gradient vanishes only on points of maximum of u.
(iii) Since the unique solution u to (1) is strictly positive in Ω (cf. Remark 3), it can be extended to a continuously differentiable function, still denoted by u, in an open set $\widetilde{\Omega}\supset\overline{\Omega}$, such that $u<0$ in $\widetilde{\Omega}\setminus\overline{\Omega}$. Consequently, also the P-function in (18) can be extended to a continuous function in $\widetilde{\Omega}$, still denoted by P.

The remaining of this section is devoted to the proof of Theorem 16.
In the sequel, we set for brevity

$$K:=\operatorname{argmax}_{\overline{\Omega}}u\,,\qquad \mu:=\max_{\overline{\Omega}}u\,. \tag{27}$$

A first key step is the construction of the *gradient flow* $\mathbf{X}$ associated with u, and the location of its terminal points:

Lemma 18. *Assume* $(h\Omega)$–(hu), *and let* $x\in\overline{\Omega}\setminus K$. *Then there exists a unique solution* $\mathbf{X}(\cdot,x)$ *to* (19) *defined in a interval* $[0,T(x))$, *where* $T(x)\in(0,+\infty]$ *is given by*

$$T(x):=\sup\{t\ge 0:\ \nabla u(\mathbf{X}(t,x))\neq 0\}\,. \tag{28}$$

Moreover,

$$\lim_{t\to T(x)^-}\mathbf{X}(t,x)\in K,\qquad \lim_{t\to T(x)^-}\nabla u(\mathbf{X}(t,x))=0\,. \tag{29}$$

Finally, there exist $x_0\in\partial\Omega$ *and* $t_0\in[0,T(x_0))$ *such that* $x=\mathbf{X}(t_0,x_0)$.

Proof. The local uniqueness of forward solutions to (19) follows from Corollary 2 and [12, Theorem 3.2 and Example 3.6]. (Note that we think of u as extended to $\widetilde{\Omega}\supset\overline{\Omega}$ according to Remark 17 (iii).) Moreover,

$$\frac{d}{dt}u(\gamma(t))=\nabla u(\gamma(t))\cdot\dot{\gamma}(t)=|\nabla u(\gamma(t))|^2\ge 0. \tag{30}$$

Hence local forward solutions to (19) remain into $\overline{\Omega}$ for every t at which they are defined. As a consequence, they are actually global forward solutions, i.e., they are defined in $[0,+\infty)$ (and remain into $\overline{\Omega}$ for all $t\in[0,+\infty)$).

For a fixed $x \in \overline{\Omega} \setminus K$, by local uniqueness, all forward solutions to (19) must coincide in the interval $[0, T(x))$, with $T(x)$ defined as in (28): from now on, we denote by $\mathbf{X}(\cdot, x)$ *the unique solution* to (19) in $[0, T(x))$.
Now, in order to prove (29), we distinguish the two cases $T(x) < +\infty$ and $T(x) = +\infty$.
If $T(x) < +\infty$, we have

$$\lim_{t \to T(x)^-} \mathbf{X}(t, x) \in K \,,$$

so that

$$\lim_{t \to T(x)^-} \nabla u(\mathbf{X}(t, x)) = 0 \,.$$

If $T(x) = +\infty$, assume by contradiction that

$$m := \lim_{t \to +\infty} u(\mathbf{X}(t, x)) < \mu \tag{31}$$

(note that the limit which defines m exists by monotonicity in view of (30)). Since ∇u is continuous and strictly positive in the compact set $\{x \in \overline{\Omega} \ : \ u(x) \leq m\}$, from (30) there exists $\alpha > 0$ such that

$$\frac{d}{dt} u(\mathbf{X}(t, x)) \geq \alpha > 0 \qquad \forall t \in [0, +\infty) \,,$$

which clearly contradicts (31). Therefore, it must be $m = \mu$, and, taking into account that $\frac{d}{dt}\mathbf{X}(t, x)$ is bounded, (29) is proved also in case $T(x) = +\infty$.
Now let γ: $(T^-, +\infty)$, $T^- \in [-\infty, 0)$, be a maximal solution to (19) in Ω. (From the discussion above we have $\gamma(t) = \mathbf{X}(t, x)$ for every $t \in [0, T(x))$.) By (30), the map $t \mapsto u(\gamma(t))$ is strictly monotone increasing for $t \in (T^-, T(x))$, so that

$$\gamma(t) \in K' := \{y \in \overline{\Omega} : \ u(y) \leq u(x)\} \,, \qquad \forall t \in (T^-, 0].$$

Since ∇u is continuous and $\nabla u \neq 0$ in the compact set K', we have that

$$\frac{d}{dt} u(\gamma(t)) \geq \alpha' > 0 \qquad \forall t \in (T^-, 0].$$

Hence T^- must be finite and, being $\dot{\gamma}$ bounded, $\lim_{t \to T^-} \gamma(t) =: x_0 \in \partial\Omega$. By construction, it holds

$$\mathbf{X}(t, x_0) = \gamma(T^- + t) \qquad \forall t \in (0, T(x_0)),$$

hence setting $t_0 := -T^- > 0$ we have that $\mathbf{X}(t_0, x_0) = \gamma(0) = x$.

As we have already mentioned after Lemma 15, such result cannot be directly exploited to infer the constancy of the P-function along the flow $\mathbf{X}$, because of the possible lack of absolute continuity of P. In order to overcome this difficulty, we approximate u via its supremal convolutions, defined for $\varepsilon > 0$ by

$$u^\varepsilon(x) := \sup_{y \in \mathbb{R}^n} \left\{ \tilde{u}(y) - \frac{|x - y|^2}{2\varepsilon} \right\} \qquad \forall x \in \mathbb{R}^n \,, \tag{32}$$

where $\tilde{u}$ is a Lipschitz extension of u to $\mathbb{R}^n$ with $\mathrm{Lip}_{\mathbb{R}^n}(\tilde{u}) = \mathrm{Lip}_{\overline{\Omega}}(u)$.
In the next lemma we state the basic properties of the functions u^ε which we are going to use in the sequel.
Let us recall that, according to [11, Lemma 3.5.7], there exists $R > 0$, depending only on $\mathrm{Lip}_{\mathbb{R}^n}(\tilde{u})$, such that any point y at which the supremum in (32) is attained satisfies $|y - x| < \varepsilon R$. Thus, setting

$$U_\varepsilon := \left\{ x \in \Omega \ : \ u(x) > \varepsilon \right\} , \qquad A_\varepsilon := \left\{ x \in U_\varepsilon \ : \ d_{\partial U_\varepsilon}(x) > \varepsilon R \right\} , \tag{33}$$

there holds

$$u^\varepsilon(x) = \sup_{y\in U_\varepsilon} \Big\{ u(y) - \frac{|x-y|^2}{2\varepsilon} \Big\} \qquad \forall x \in A_\varepsilon\,. \tag{34}$$

Moreover, let us define

$$m_\varepsilon := \max_{\partial A_\varepsilon} u^\varepsilon, \qquad \Omega_\varepsilon := \{x \in A_\varepsilon : \ u^\varepsilon(x) > m_\varepsilon\}\,. \tag{35}$$

Lemma 19. *Assume* (hΩ)–(hu)*. Let* u^ε *and* Ω_ε *be defined respectively as in* (32) *and* (35)*. Then:*

(i) u^ε *is of class* $C^{1,1}$ *on* Ω_ε*;*
(ii) u^ε *is a sub-solution to* $-\Delta_\infty u - 1 = 0$ *in* Ω_ε*;*
(iii) *as* $\varepsilon \to 0^+$*, it holds*

$$u^\varepsilon \to u \qquad \text{uniformly in } \overline{\Omega},$$
$$\nabla u^\varepsilon \to \nabla u \qquad \text{uniformly in } \overline{\Omega}$$

(so that $m_\varepsilon \to 0$ *and* Ω_ε *converges to* Ω *in Hausdorff distance).*

Proof. In order to prove (i), by [11, Corollary 3.3.8], it is enough to show that u^ε is both semiconcave and semiconvex on Ω_ε. We have $u^\varepsilon = -(-u)_\varepsilon$, where $(-u)_\varepsilon$ is the infimal convolution defined by

$$(-u)_\varepsilon(x) := \inf_{y\in U_\varepsilon} \Big\{ -u(y) + \frac{|x-y|^2}{2\varepsilon} \Big\} \qquad \forall x \in \Omega_\varepsilon\,.$$

From [11, Proposition 2.1.5], it readily follows that $(-u)_\varepsilon$ is semiconcave on Ω_ε, and hence that u^ε is semiconvex on Ω_ε. In order to show that $(-u)_\varepsilon$ is semiconvex on Ω_ε (and hence that u^ε is semiconcave on Ω_ε), let $x_i \in \Omega_\varepsilon$, $i = 1, 2$, be fixed, and let $y_i \in U_\varepsilon$ be points where the infima which define $(-u)_\varepsilon(x_i)$ are attained. Denoting by C_ε the semiconcavity constant of u on U_ε, we have

$$\begin{aligned}
&(-u)_\varepsilon(x_1) + (-u)_\varepsilon(x_2) - 2(-u)_\varepsilon\Big(\tfrac{x_1+x_2}{2}\Big) \geq \\
&-u(y_1) - u(y_2) + 2u\Big(\tfrac{y_1+y_2}{2}\Big) + \tfrac{|x_1-y_1|^2}{2\varepsilon} + \tfrac{|x_2-y_2|^2}{2\varepsilon} - \tfrac{2}{2\varepsilon}\Big|\tfrac{x_1+x_2}{2} - \tfrac{y_1+y_2}{2}\Big|^2 \\
&\geq -C_\varepsilon\Big|\tfrac{y_1-y_2}{2}\Big|^2 + \tfrac{1}{2\varepsilon}\Big(|x_1-y_1|^2 + |x_2-y_2|^2 - 2\Big|\tfrac{x_1+x_2}{2} - \tfrac{y_1+y_2}{2}\Big|^2\Big) \\
&\geq -\tfrac{2C_\varepsilon}{2-\varepsilon C_\varepsilon}|x_1-x_2|^2\,.
\end{aligned}$$

Thus $(-u)_\varepsilon$ is semiconvex with constant $\frac{2C_\varepsilon}{2-\varepsilon C_\varepsilon}$ on Ω_ε.

Let us now prove (ii). Let $x \in \Omega_\varepsilon$, and let $(p, X) \in J^{2,+}_{\Omega_\varepsilon} u^\varepsilon(x)$. It follows from magical properties of supremal convolution (*cf.* [14, Lemma A.5]) that $(p, X) \in J^{2,+}_{\Omega_\varepsilon} u(y)$, where y is a point at which the supremum which defines $u^\varepsilon(x)$ is attained. Since $y \in U_\varepsilon \subset \Omega_\varepsilon$, it holds $J^{2,+}_{\Omega} u(y) = J^{2,+}_{\Omega_\varepsilon} u^\varepsilon(x)$; therefore, we have $(p, X) \in J^{2,+}_{\Omega} u(y)$, which implies $-\langle Xp, p\rangle - 1 \leq 0$.

Finally, let us turn to the proof of the convergence properties (iii). For the uniform convergence of u^ε to u in $\overline{\Omega}$, see for instance [11, Thm. 3.5.8]. In order to show the uniform convergence of the gradients, recall first that there holds

$$\nabla u^\varepsilon(x) = \nabla u(y_\varepsilon(x)) \qquad \forall x \in \Omega_\varepsilon\,,$$

where $y_\varepsilon(x)$ is a point where the supremum which defines $u^\varepsilon(x)$ is attained [25, Thm. 3.1 (a)]. Moreover, as already mentioned above, there exists $R > 0$, depending only on $\mathrm{Lip}_{\overline{\Omega}}(u)$, such that $|y_\varepsilon(x) - x| < \varepsilon R$ for every $x \in \overline{\Omega}$; in particular, for $x \in \Omega_\varepsilon$, there holds $y_\varepsilon(x) \in U_\varepsilon \subset \Omega$. Then we can write:

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \sup_{x\in\overline{\Omega}} |\nabla u^\varepsilon(x) - \nabla u(x)| &= \lim_{\varepsilon\to 0^+} \sup_{x\in\Omega_\varepsilon} |\nabla u^\varepsilon(x) - \nabla u(x)| \\ &= \lim_{\varepsilon\to 0^+} \sup_{x\in\Omega_\varepsilon} |\nabla u(y_\varepsilon(x)) - \nabla u(x)| \\ &\le \mathrm{Lip}_{\overline{\Omega}}(u) \lim_{\varepsilon\to 0^+} |y_\varepsilon(x) - x| \\ &\le \mathrm{Lip}_{\overline{\Omega}}(u) \lim_{\varepsilon\to 0^+} (\varepsilon R) = 0\,.\end{aligned}$$

Next we observe that, for every $\varepsilon > 0$, one can consider the gradient flow $\mathbf{X}_\epsilon$ associated with u^ε. Namely, for every $x_\varepsilon \in \overline{\Omega_\varepsilon}$, the Cauchy problem

$$\begin{cases}\dot\gamma_\varepsilon(t) = \nabla u^\varepsilon(\gamma_\varepsilon(t))\,, \\ \gamma_\varepsilon(0) = x_\varepsilon \in \overline{\Omega_\varepsilon}\,,\end{cases} \tag{36}$$

admits a unique solution $\mathbf{X}_\epsilon(\cdot, x_\varepsilon)\colon [0,+\infty) \to \overline{\Omega_\varepsilon}$. Indeed, the fact that $\mathbf{X}_\epsilon(\cdot, x_\varepsilon)$ is defined in $[0,+\infty)$ follows from the estimate

$$\frac{d}{dt} u^\varepsilon(\gamma_\varepsilon(t)) = |\nabla u^\varepsilon(\gamma_\varepsilon(t))|^2 \ge 0,$$

so that $\gamma_\varepsilon(t) \in \overline{\Omega_\varepsilon}$ for every $t \ge 0$, while uniqueness follows from the $C^{1,1}$ regularity of u^ε stated in Lemma 19(i).

In the following key lemma, we establish the behavior, along the flow $\mathbf{X}_\varepsilon$, of the approximate P-function defined by

$$P_\varepsilon(x) := \frac{|\nabla u^\varepsilon(x)|^4}{4} + u^\varepsilon(x)\,, \qquad x \in \overline{\Omega_\varepsilon}\,. \tag{37}$$

In fact we show that P_ε increases along $\mathbf{X}_\varepsilon$:

Lemma 20. *Assume* $(h\Omega)$*–*(hu)*. Let* u^ε*,* Ω_ε*, and* P_ε *be defined respectively as in* (32)*,* (35)*, and* (37)*. Then, for* $\mathcal{H}^{n-1}$*-a.e.* $x_\varepsilon \in \partial\Omega_\varepsilon$*, it holds*

$$P_\varepsilon(\mathbf{X}_\epsilon(t_1, x_\varepsilon)) \le P_\varepsilon(\mathbf{X}_\epsilon(t_2, x_\varepsilon)) \qquad \forall\, t_1, t_2 \ \text{ with } 0 \le t_1 \le t_2\,.$$

Proof. At every point x where u^ε is twice differentiable, it holds

$$\nabla P_\varepsilon(x) = |\nabla u^\varepsilon(x)|^2 D^2 u^\varepsilon(x)\, \nabla u^\varepsilon(x) + \nabla^\varepsilon u(x)\,;$$

we infer that

$$\langle \nabla P_\varepsilon(x), \nabla u^\varepsilon(x)\rangle = |\nabla u^\varepsilon(x)|^2\, (\Delta_\infty u^\varepsilon(x) + 1) \ge 0\,,$$

where the last inequality follows from Lemma 19(ii).

Thus, if u^ε is twice differentiable at $\mathbf{X}_\epsilon(t, x_\varepsilon)$, we have

$$\frac{d}{dt}\big(P_\varepsilon(\mathbf{X}_\epsilon(t, x_\varepsilon))\big) = \Big\langle \nabla P_\varepsilon(\mathbf{X}_\epsilon(t, x_\varepsilon)), \nabla u^\varepsilon(\mathbf{X}_\epsilon(t, x_\varepsilon))\Big\rangle \ge 0\,. \tag{38}$$

Let us now show that, for $\mathcal{H}^{n-1}$-a.e. $x_\varepsilon \in \partial\Omega_\varepsilon$, the inequality (38) is satisfied $\mathcal{L}^1$-a.e. on $[0,+\infty)$. To that aim we have to prove that, for $\mathcal{H}^{n-1}$-a.e. $x_\varepsilon \in \partial\Omega_\varepsilon$, u^ε is twice differentiable at $\mathbf{X}_\epsilon(t, x_\varepsilon)$ for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$. Namely we have to show that, setting

$$N(x_\varepsilon) := \Big\{ t \ge 0 \ :\ u^\varepsilon \text{ is not twice differentiable at } \mathbf{X}_\epsilon(t, x_\varepsilon) \Big\}\,, \qquad x_\varepsilon \in \partial\Omega_\varepsilon\,,$$

it holds

$$\mathcal{L}^1(N(x_\varepsilon)) = 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_\varepsilon \in \partial\Omega_\varepsilon\,. \tag{39}$$

By construction the set

$$E_\varepsilon := \Big\{ \mathbf{X}_\epsilon(t, x_\varepsilon) \ : \ x_\varepsilon \in \partial\Omega_\varepsilon\,,\ t \in N(x_\varepsilon) \Big\}$$

is contained into the set of points where u^ε is not twice differentiable.
By Lemma 19(i), we know that u^ε is twice differentiable $\mathcal{L}^n$-a.e. on Ω_ε. This implies that the set E_ε is Lebesgue negligible. By the area formula, we have

$$0 = \mathcal{L}^n(E_\varepsilon) = \int_{\partial\Omega_\varepsilon} d\mathcal{H}^{n-1}(x_\varepsilon) \int_{N(x_\varepsilon)} J\mathbf{X}_\epsilon(t, x_\varepsilon)\, dt\,,$$

where $J\mathbf{X}_\epsilon$ is the Jacobian of the function $\mathbf{X}_\epsilon$ with respect to the second variable. Since this Jacobian is strictly positive (*cf.* [2, eq. (5)]), we infer that (39) holds true. Hence, for $\mathcal{H}^{n-1}$-a.e. $x_\varepsilon \in \partial\Omega_\varepsilon$, the inequality (38) is satisfied $\mathcal{L}^1$-a.e. on $[0, +\infty)$. Then, for $\mathcal{H}^{n-1}$-a.e. $x_\varepsilon \in \partial\Omega_\varepsilon$, integrating (38) over $[t_1, t_2]$ and taking into account that, by Lemma 19(i), the map $t \mapsto P_\varepsilon \circ \mathbf{X}_\epsilon(t, x_\varepsilon)$ is locally Lipschitz continuous on $[0, +\infty)$, the lemma is proved. □

Remark 21. We point out that an analogous procedure as the one adopted above does not work with the infimal convolutions u_ε in place of the supremal convolutions u^ε. The reason is simply the fact that such infimal convolutions are not necessarily of class $C^{1,1}$.

We are finally in a position to give the

Proof of Theorem 16. Let us show firstly that

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \leq \mu\,. \tag{40}$$

Let $B = B_\rho(z_0)$ be a ball internally tangent to Ω at y_0, and let ϕ_B denote the unique solution to (1) on B. By the comparison principle proved in [32, Thm. 3], it holds $u \geq \phi_B$ on B. Hence,

$$u(z_0) \geq \phi_B(z_0) = c_0 \rho^{4/3}$$

and

$$|\nabla u(y_0)| \leq |\nabla \phi_B(y_0)| = (3\rho)^{1/3}\,.$$

Therefore, we have

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \leq \frac{|\nabla u|^4(y_0)}{4} \leq c_0 \rho^{4/3} \leq u(z_0) \leq \mu\,.$$

Let us now prove (26). It is enough to show that (26) holds for all $x \in \Omega \setminus K$ (otherwise $P(x) = u(x)$ and we are done by the inequality (40)). By Lemma 18, given $x \in \Omega \setminus K$, there exist $x_0 \in \partial\Omega$ and $t_0 \in [0, T(x_0))$ such that $x = \mathbf{X}(t_0, x_0)$. By Lemma 20, we may find a sequence of points $x_\varepsilon \in \partial\Omega_\varepsilon$ converging to x_0 such that, for every $t \geq t_0$, we have

$$P_\varepsilon(x_\varepsilon) \leq P_\varepsilon(\mathbf{X}_\epsilon(t_0, x_\varepsilon)) \leq P_\varepsilon(\mathbf{X}_\epsilon(t, x_\varepsilon))\,.$$

We now pass to the limit as $\varepsilon \to 0^+$ in the above inequalities: by using the continuous dependence for ordinary differential equations (see e.g. [26, Lemma 3.1]), and the uniform convergences stated in Lemma 19(iii), we get

$$P(x_0) \leq P(x) \leq P(\mathbf{X}(t, x_0))\,. \tag{41}$$

We have
$$P(x_0)=\frac{|\nabla u(x_0)|^4}{4}\geq \min_{\partial\Omega}\frac{|\nabla u|^4}{4};$$
on the other hand, from (29), it holds
$$\lim_{t\to T(x_0)^-} P(\mathbf{X}(t,x_0))=\lim_{t\to T(x_0)^-} u(\mathbf{X}(t,x_0))\leq \mu\,.$$
Then (26) follows from (41).

5. Geometric results for Serrin's problem

Theorem 22. *Assume* $(h\Omega)$*–*(hu)*. Further, assume that there exists an inner ball* B *of radius* ρ_Ω *which meets* $\partial\Omega$ *at two points lying on the same diameter of* B*. If the overdetermined boundary value problem* (2) *admits a solution* u*, then it holds* $u=\phi_\Omega$ *(with* $\rho_\Omega=a^3/3$*), and* $\overline{\Sigma}(\Omega)=\mathrm{M}(\Omega)$*.*

Proof. Let $B=B_{\rho_\Omega}(x_0)$ be an inner ball of radius ρ_Ω which meets $\partial\Omega$ at two diametral points $y_\pm$. We claim that there exists a domain D with $\overline{\Sigma}(D)=\mathrm{M}(D)$ (and $\rho_D=\rho_\Omega$), such that

$$B\subset\Omega\subset D\,,\qquad \partial B\cap\partial\Omega=\partial B\cap\partial D=\{y_+,y_-\}\,. \tag{42}$$

Namely, assume without loss of generality that the center x_0 of B is the origin, and that $y_\pm=\pm\rho_\Omega e_n$. Let $S\subset\{x_n=0\}$ be a $(n-1)$-dimensional disk centered at the origin, with radius sufficiently large so that Ω is contained into the cylinder $S\times[-\rho_\Omega,\rho_\Omega]$. Then the conditions (42) are satisfied by taking
$$D:=\{x\in\mathbb{R}^n\ :\ d_S(x)<\rho_\Omega\}\,.$$
Now, denote by ϕ_B and ϕ_D the web functions defined according to (9) (respectively on B and on D), and by γ the diameter of B containing y_+ and y_-. Since $-\Delta_\infty u=-\Delta_\infty\phi_B=1$ on B, and $u\geq 0=\phi_B$ on ∂B, by the comparison principle proved in [32, Thm. 3], it holds $u\geq\phi_B$ on B. In the same way, we get the inequality $u\leq\phi_D$ on D. We thus have

$$\phi_B(x)\leq u(x)\leq\phi_D(x)\qquad\forall x\in B\,. \tag{43}$$

We can deduce several consequences from these inequalities. Firstly we observe that, since both the functions ϕ_B and ϕ_D have a relative maximum at x_0, by (43) the same property holds true for u. Hence x_0 is a critical point of u. In turn, by Remark 17, we know that

$$x_0\in K\,. \tag{44}$$

Moreover we notice that, since the distance functions $d_{\partial B}$ and $d_{\partial D}$ agree on γ (as both coincide with $d_{\partial\Omega}$), there holds

$$\phi_B(x)=\phi_D(x)\qquad\forall x\in\gamma\,. \tag{45}$$

As a consequence of (43) and (45), we deduce that
$$u(x)=\phi_B(x)=\phi_D(x)\qquad\forall x\in\gamma\,.$$
Namely, there holds

$$u(x)=c_0\left[\rho_\Omega^{4/3}-(\rho_\Omega-d_{\partial\Omega}(x))^{4/3}\right]\qquad\forall x\in\gamma\,. \tag{46}$$

It follows from (46) that the following relationship holds between the value of $|\nabla u|$ at the boundary points $y_\pm$ and the inradius:
$$|\nabla u(y_\pm)|=\frac{4}{3}c_0\rho_\Omega^{1/3}=(3\rho_\Omega)^{1/3}\qquad i=1,2\,.$$

Recalling that by assumption u satisfies the Neumann condition $|\nabla u|(y) = a$ for all $y \in \partial\Omega$, we deduce that the value of the parameter a is related to the inradius by the equality

$$a = (3\rho_\Omega)^{1/3} . \tag{47}$$

Thus, using (44) and (47), we get

$$\mu = u(x_0) = c_0 \rho_\Omega^{4/3} = \frac{a^4}{4} .$$

By Theorem 16, this implies that the P-function associated with u according to (18) is constant on $\overline{\Omega}$:

$$P(x) \equiv \frac{a^4}{4} \qquad \forall x \in \overline{\Omega} .$$

By Proposition 13, this implies that $u = \phi_\Omega$ (with $\rho_\Omega = a^3/3$), and $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$.

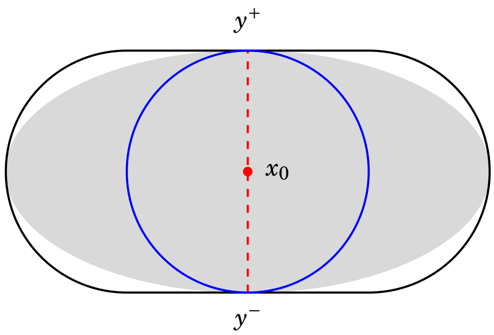

FIGURE 1. A domain Ω (in gray) as in Theorem 22, with $B \subset \Omega \subset D$.

By combining Theorem 22 with the geometric results we obtained in [17], we can provide some geometric information on the shape of domains where the Serrin-type problem (2) admits a solution, according to Corollary 23 below. We emphasize that symmetry may hold or may fail according in particular to the boundary regularity of Ω.

Corollary 23. *Under the same hypotheses of Theorem 22, we have:*

- (i) *if Ω is of class C^2, then Ω is a ball;*
- (ii) *if $n = 2$, the set $S := \overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ is a line segment (possibly degenerated into a point), and Ω is the stadium-like domain*

$$\Omega = \{x \in \mathbb{R}^2 \ : \ \mathrm{dist}(x, S) < \rho_\Omega\} .$$

Proof. Statement (i) follows directly from [17, Thm. 12]. Concerning statement (ii), from [17, Thm. 6] we obtain then the set $S := \overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ is either a singleton or a 1-dimensional manifold of class $C^{1,1}$, and Ω is the tubular neighborhood $\Omega = \{x \in \mathbb{R}^2 \ : \mathrm{dist}(x, S) < \rho_\Omega\}$. Since we assumed that Ω is convex, the manifold S must be necessarily a line segment, possibly degenerated into a point.

6. Regularity thresholds for the Dirichlet problem

As mentioned in the Introduction, it is well-known that for infinity–harmonic functions one cannot expect $C^{1,1}$ regularity; the expected regularity is in fact $C^{1,\alpha}$ with $\alpha \le 1/3$, which has been by now proved only in two space dimensions.
In this section we show that a similar situation occurs also for the solution u to problem (1), specifically in view of its behavior near the set K defined in (27).
We start with the following lemma, which allows to define the gradient flow associated with u under the assumption that it is $C^{1,1}$ outside K.

Lemma 24. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open bounded set, and let u be the solution to problem* (1)*. Assume that $u \in C^{1,1}(\Omega \setminus K)$. Then, for a.e. $x \in \Omega \setminus K$, there exists a unique solution $\mathbf{X}(\cdot, x)$ to* (19)*, defined in a interval $[0, T(x))$, where $T(x)$ is defined by*

$$T(x) := \sup\{t \ge 0 : \ \mathbf{X}(s, x) \in \Omega \setminus K \quad \forall s \in [0, t]\} .$$

Moreover, it holds $T(x) < +\infty$ and

$$\lim_{t \to T(x)^-} \mathbf{X}(t, x) \in K. \tag{48}$$

Proof. For every $x \in \Omega \setminus K$, any solution of (19) cannot exit from $\{u \ge u(x)\}$ by the same argument given in the proof of Lemma 18, and hence they are actually defined on $[0, +\infty)$. The uniqueness of the gradient flow associated with u in $\Omega \setminus K$ follows from the local Lipschitz regularity of ∇u assumed therein.
Let us now prove (48). Recall that μ is defined according to (27). We first prove the following

Claim: There exists a set $L \subseteq (0, \mu)$ with $|L| = \mu$ such that, for all $m \in L$, the condition (48) *is satisfied for $\mathcal{H}^{n-1}$-a.e. $x \in \{u = m\}$.*

Let us define L as the set of values $m \in (0, \mu)$ such that u is twice differentiable $\mathcal{H}^{n-1}$a.e. on $\{u = m\}$. By the coarea formula, if Z is the set of points in $\Omega \setminus K$ where u is not twice differentiable, we have

$$0 = \int_Z |\nabla u| \, dx = \int_0^\mu dm \int_{\{u=m\} \cap Z} d\mathcal{H}^{n-1}(y) \, . \tag{49}$$

We observe that $|\nabla u|$ remains strictly positive $\mathcal{L}^n$-a.e. in $\Omega \setminus K$; otherwise, since u is twice differentiable $\mathcal{L}^n$-a.e. in $\Omega \setminus K$, the pde $\Delta_\infty u = -1$ would not be satisfied. Hence we infer from (49) that, for $\mathcal{L}^1$-a.e. $m \in (0, \mu)$, the set $\{u = m\} \cap Z$ is $\mathcal{H}^{n-1}$-negligible, so that L is of full measure in $(0, \mu)$.
From now on, let m denote a fixed value in L. For $x \in \{u = m\}$, set

$$N(x) := \Big\{ t \in [0, T(x)] \ : \ u \text{ is not twice differentiable at } \mathbf{X}(t, x) \Big\} . \tag{50}$$

By repeating the arguments given in the proof of Lemma 20, we obtain that $\mathcal{L}^1(N(x)) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{u = m\}$.
Let us prove that (48) holds for every $x_0 \in \{u = m\}$ such that both the conditions $\mathcal{L}^1(N(x_0)) = 0$ and u twice differentiable at x_0 hold.
Let x_0 be such a point, and let $p(x_0) := \lim_{t \to T(x_0)^-} \mathbf{X}(t, x_0)$ (observe that such limit exists since $\frac{d}{dt}\mathbf{X}(t, x_0)$ is bounded).
Since u is twice differentiable at x_0, it cannot be $\nabla u(x_0) = 0$; otherwise, as already noticed above, the pde $-\Delta_\infty u = 1$ would not be satisfied.

Then, by the very definition of $T(x_0)$, in order to prove (48) is is enough to show that $T(x_0) < +\infty$. Indeed in this case we have that $p(x_0) \in \partial\Omega \cup K$, but the possibility that $p(x_0) \in \partial\Omega$ is excluded by the fact that u increases along the flow.
Let us show that $T(x_0) < +\infty$. Let

$$\gamma(t) := \mathbf{X}(t, x_0), \quad \varphi(t) := u(\gamma(t)) , \qquad t \in [0, T),\ T := T(x_0).$$

Since $\mathcal{L}^1(N(x_0)) = 0$, the P-function is constant along γ. Then by Proposition 14 for some $\lambda > m$ we have $\varphi(t) = \overline{\varphi}(t)$ for every $t \in [0, T)$, with

$$\overline{\varphi}(t) := \begin{cases} \lambda - (\sqrt{\lambda - m} - t)^2, & \text{if } t \in [0, \sqrt{\lambda - m}), \\ \lambda, & \text{if } t \geq \sqrt{\lambda - m}\,. \end{cases}$$

Let us show that $T = \sqrt{\lambda - m}$. It is clear that $T \geq \sqrt{\lambda - m}$, since $\dot\varphi(t) \neq 0$ for $t \in [0, \sqrt{\lambda - m})$, so that $\gamma(t) \notin K$ for $t \in [0, \sqrt{\lambda - m})$ because $\nabla u = 0$ on K (recall that, by [31], u is differentiable everywhere in Ω). On the other hand, the trajectory γ enters in finite time $\sqrt{\lambda - m}$ in a point p where $\nabla u(p) = 0$, which cannot happen if ∇u is locally Lipschitz continuous in a neighborhood of p (since otherwise uniqueness would be contradicted). Hence $p \in K$, $\lambda = \mu$ and $T(x_0) = \sqrt{\lambda - m}$. This concludes the proof of the claim.
Finally, let us prove that the statement of the lemma follows from the claim. Let F denote the set of points $x \in \Omega \setminus K$ such that (48) is false. Note that the complement of F, namely the set where (48) holds, is closed by continuous dependence on initial data. In particular, this ensures that F is $\mathcal{L}^n$-measurable. Then, by the claim and the coarea formula, we have

$$0 = \int_0^\mu dm \int_{\{u=m\}\cap F} d\mathcal{H}^{n-1}(y) = \int_F |\nabla u|\, dx\,. \tag{51}$$

We recall that, since by assumption $u \in C^{1,1}(\Omega \setminus K)$, u is twice differentiable $\mathcal{L}^n$-a.e. on $\Omega \setminus K$, and hence $|\nabla u| > 0$ $\mathcal{L}^n$-a.e. on $\Omega \setminus K$ (because the pde $\Delta_\infty u = -1$ is not fulfilled at points where $\nabla u = 0$). Then by (51) we deduce that $|F| = 0$.

Corollary 25. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open bounded set, and let u be the solution to problem* (1)*. Assume that $u \in C^{1,1}(\Omega \setminus K)$. Then*

$$P(x) = \mu \qquad \forall x \in \Omega \setminus K\,.$$

In particular, if (hu) *holds, we have*

$$\frac{|\nabla u|^4(y)}{4} = \mu \qquad \forall y \in \partial\Omega\,. \tag{52}$$

Proof. Since by assumption P is continuous on $\Omega\setminus K$, it is enough to show that the equality $P(x) = \mu$ holds almost everywhere on $\Omega \setminus K$. Namely, let us show that it holds for every $x \in \Omega \setminus K$ such that (48) holds and $\mathcal{L}^1(N(x)) = 0$. (Actually, both these conditions are satisfied up to a $\mathcal{L}^n$-negligible set, by the same arguments used in the proof of Lemma 24). Let $x \in \Omega \setminus K$ be such that (48) holds and $\mathcal{L}^1(N(x)) = 0$. Set $\gamma(t) := \mathbf{X}(t, x)$, for $t \in [0, T(x))$. Since $\mathcal{L}^1(N(x)) = 0$, P is contant along γ and, since (48) holds, we have $P(\gamma(t)) = \mu$ on $[0, T(x))$. In particular, $P(x) = \mu$. Finally, under assumption (hu), the equality (52) follows immediately by combining the equality $P(x) \equiv \mu$ holding on $\Omega \setminus K$ with the Dirichlet condition $u = 0$ satisfied on $\partial\Omega$.

Proposition 26. *Assume that Ω satisfy the following conditions: $(h\Omega)$, $\overline{\Sigma}(\Omega) \neq \mathrm{M}(\Omega)$, and there exists an inner ball B of radius ρ_Ω which meets $\partial\Omega$ at two points lying on the same diameter of B. Further, assume that the unique solution to problem* (1) *satisfies* (hu). *Then $u \notin C^{1,1}(\Omega \setminus K)$.*

Proof. Assume by contradiction that $u \in C^{1,1}(\Omega \setminus K)$. Then, by assumption (hu) and Corollary 25, we have that (52) holds. Hence, u is a solution to the overdetermined boundary value problem (2). Since we have assumed also $(h\Omega)$ and the existence of an inner ball B which meets $\partial\Omega$ at two diametral points, by Theorem 22 we infer that $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$, contradiction. □

The assumptions made on Ω in the above proposition are satisfied for instance when Ω is an ellipse. For general domains, assuming that the solution u to problem (1) is $C^{1,1}$ near K, we obtain the following result which gives an indication that the optimal expected regularity of u (up to K) is $C^{1,1/3}$.

Proposition 27. *Let u be the solution to problem* (1), *and let A be a neighborhood of K. Assume that $u \in C^{1,1}(A \setminus K)$. Then for every $\alpha > 1/3$ it holds $u \notin C^{1,\alpha}(A)$.*

Proof. Assume that $u \in C^{1,\alpha}(A)$, i.e. $u \in C^1(A)$ and there exists $C > 0$ such that

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^\alpha \qquad \forall x, y \in A\,.$$

We are going to show that necessarily it must be $\alpha \leq 1/3$.
Let us choose m such that $E := \{x \in \Omega : \ u(x) \geq m\} \subset A$.
Since $u \in C^{1,1}(E \setminus K)$ and u is differentiable on K, we can associate with the restriction of u to E the gradient flow $\mathbf{X}$ according to Lemma 24.
Let $L \subseteq (0, \mu)$ be as in the Claim contained in the proof of Lemma 24. Let $m' \in L \cap (0, m)$ be such that (48) holds $\mathcal{H}^{n-1}$-a.e. on $\{u = m'\}$. By repeating the arguments given in the proof of Lemma 20, we obtain that $\mathcal{L}^1(N(x)) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{u = m'\}$, with $N(x)$ defined as in (50).
Then we can pick $x_0 \in \{u = m'\}$ be such that $\mathcal{L}^1(N(x_0)) = 0$ and (48) holds at x_0. Set

$$\gamma(t) := \mathbf{X}(t, x_0), \quad \varphi(t) := u(\gamma(t)), \qquad t \in [0, T),\ T := T(x_0).$$

Since $\mathcal{L}^1(N(x_0)) = 0$, the P-function is constant along γ. Moreover, since (48) holds at x_0, the value of the constant is equal to μ, namely it holds

$$P(\gamma(t)) = \frac{1}{4}|\nabla u(\gamma(t))|^4 + u(\gamma(t)) = \mu \qquad \forall t \in [0, T).$$

Then, by Proposition 14, we have $\varphi(t) = \overline{\varphi}(t)$ for every $t \in [0, T)$, with

$$\overline{\varphi}(t) := \begin{cases} \mu - (\sqrt{\mu - m} - t)^2, & \text{if } t \in [0, \sqrt{\mu - m}), \\ \mu, & \text{if } t \geq \sqrt{\mu - m}\,. \end{cases}$$

We recall that the trajectory γ cannot reach a maximum point of u in a time $t < T$, whereas it approaches K as $t \to T^-$, i.e.

$$\varphi(t) < \mu \quad \forall t \in [0, T)\,, \qquad \lim_{t \to T^-} \varphi(t) \in K\,. \tag{53}$$

Thus we deduce that T is finite and

$$T = \sqrt{\mu - m}, \qquad \varphi(t) = \mu - (T - t)^2 \quad \forall t \in [0, T]. \tag{54}$$

(For later convenience we have extended φ up to time T.)

Incidentally, notice that the finitenes of T shown in (54) already implies that $u \notin C^{1,1}(A)$ (otherwise by uniqueness it should be $T = +\infty$). For every $t \in [0, T]$ we have

$$\begin{aligned} \mu - \varphi(t) = \varphi(T) - \varphi(t) &= \int_t^T |\nabla u(\gamma(s))|^2 \, ds \\ &= \int_t^T |\nabla u(\gamma(s)) - \nabla u(\gamma(T))|^2 \, ds \leq C^2 \int_t^T |\gamma(s) - \gamma(T)|^{2\alpha} \, ds \, . \end{aligned} \tag{55}$$

In order to estimate the last integral in (55), let us consider the auxiliary function $z(t) := |\gamma(t) - \gamma(T)|$, $t \in [0, T]$. Since $z(t) > 0$ for every $t \in [0, T)$, for such values of t we have

$$\begin{aligned} \dot{z}(t) = \dot{\gamma}(t) \cdot \frac{\gamma(t) - \gamma(T)}{|\gamma(t) - \gamma(T)|} &\geq -|\dot{\gamma}(t)| = -|\nabla u(\gamma(t))| \\ = -|\nabla u(\gamma(t)) - \nabla u(\gamma(T))| &\geq -C|\gamma(t) - \gamma(T)|^\alpha = -C \, z(t)^\alpha \, . \end{aligned}$$

Since the maximal solution in $[0, T]$ of the Cauchy problem $\dot{z} = -C \, z^\alpha$, $z(T) = 0$, is

$$\overline{z}(t) := [C(1 - \alpha)(T - t)]^{1/(1-\alpha)} \, , \qquad t \in [0, T],$$

we conclude that

$$|\gamma(t) - \gamma(T)| = z(t) \leq \overline{z}(t) = C_1 \, (T - t)^{1/(1-\alpha)}, \qquad \forall t \in [0, T] \, , \tag{56}$$

where $C_1 := [C(1 - \alpha)]^{1/(1-\alpha)}$. By (55) and (56), we deduce that

$$\mu - \varphi(t) \leq C_2 \int_t^T (T - s)^{2\alpha/(1-\alpha)} \, ds = C_2 (T - t)^{(1+\alpha)/(1-\alpha)} \, , \qquad \forall t \in [0, T],$$

where $C_2 := C^2 \, C_1^{2\alpha}$. Taking into account the explicit form of φ given in (54) we obtain

$$(T - t)^2 \leq C_2 (T - t)^{(1+\alpha)/(1-\alpha)} \, , \qquad \forall t \in [0, T],$$

that clearly cannot be satisfied if $2 < (1 + \alpha)/(1 - \alpha)$, i.e. if $\alpha > 1/3$.

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