SELF-MAPPINGS OF THE QUATERNIONIC UNIT BALL: MULTIPLIER PROPERTIES, SCHWARZ-PICK INEQUALITY, AND NEVANLINNA–PICK INTERPOLATION PROBLEM

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Abstract. We study several aspects concerning slice regular functions mapping the quaternionic open unit ball $\mathbb{B}$ into itself. We characterize these functions in terms of their Taylor coefficients at the origin and identify them as contractive multipliers of the Hardy space $\mathrm{H}^2(\mathbb{B})$. In addition, we formulate and solve the Nevanlinna-Pick interpolation problem in the class of such functions presenting necessary and sufficient conditions for the existence and for the uniqueness of a solution. Finally, we describe all solutions to the problem in the indeterminate case.

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1. Introduction

Let $\mathbb{H}$ be the algebra of real quaternions $p = x_0 + ix_1 + jx_2 + kx_3$ where $x_\ell \in \mathbb{R}$ and i, j, k are imaginary units such that $ij = k$, $ki = j$, $jk = i$ and $i^2 = j^2 = k^2 = -1$. The conjugate, the absolute value, the real part and the imaginary part of a quaternion p are defined as $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, $|p| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, $\operatorname{Re} p = x_0$ and $\operatorname{Im} p = ix_1 + jx_2 + kx_3$, respectively By $\mathbb{S}$ we denote the unit sphere of purely imaginary quaternions. Any $I \in \mathbb{S}$ is such that $I^2 = -1$ so that the set $\mathbb{C}_I = \{x + Iy : x, y \in \mathbb{R}\}$ can be identified with the complex plane. We say that two quaternions p and q are *equivalent* if $p = h^{-1}qh$ for some nonzero $h \in \mathbb{H}$. Two quaternions p and q are equivalent if and only if $\operatorname{Re} p = \operatorname{Re} q$ and $|\operatorname{Im} p| = |\operatorname{Im} q|$ so the set of all quaternions equivalent to a given $p \in \mathbb{H}$ form a 2-sphere which will be denoted by $[p]$.

Since the algebra $\mathbb{H}$ is not commutative, function theory over $\mathbb{H}$ is quite different from that over the complex field. There are several notions of regularity for $\mathbb{H}$-valued functions. The most notable are due to Moisil [19], Fueter [12, 13], and Brackx, Delanghe, Sommen [8]. More recently, upon refining and developing Cullen's approach [11], Gentili and Struppa introduced in [15] the notion of slice regularity which comprises quaternionic polynomials and power series with quaternionic coefficients on one side. We recall it now.

Definition 1.1. Given an open set $\Omega \subset \mathbb{H}$, a real differentiable function $f : \Omega \to \mathbb{H}$ is called *left slice regular* (or just *slice regular*, in what follows) on Ω if for every $I \in \mathbb{S}$,

$$\frac{\partial}{\partial x} f_I(x + Iy) + I \frac{\partial}{\partial y} f_I(x + Iy) \equiv 0, \tag{1.1}$$

where f_I stands for the restriction of f to $\Omega \cap \mathbb{C}_I$.

We will denote by $\mathcal{R}(\Omega, \widetilde{\Omega})$ the set of all functions $f : \Omega \mapsto \widetilde{\Omega} \subset \mathbb{H}$ which are (left) slice regular on Ω and we will write $\mathcal{R}(\Omega)$ in case $\widetilde{\Omega} = \mathbb{H}$. It is clear that $\mathcal{R}(\Omega)$ is a right quaternionic vector space. As was shown in [15], *the identity* (3.1) *holds for* *a fixed $I \in \mathbb{S}$ if and only if for all $J \in \mathbb{S}$ orthogonal to I, there exist complex-valued holomorphic functions $F, G : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I(z) = F(z) + G(z)J$ for all $z = x + yI \in \Omega \cap \mathbb{C}_I$.*

The latter result called "the Splitting Lemma" clarifies the relation between the restriction of slice regular functions to a complex plane and complex holomorphy. It allows to get some of the analogs of basic principles of classical complex analysis (e.g. the uniqueness theorem, the maximum-minimum modulus principle), in the quaternionic setting. The theory of slice regular functions is a very active and fast developing area of analysis; we refer to recent books [10, 14] and references therein.

The parallels with the classical complex analysis become even stronger if one focuses on functions defined and slice regular on the unit ball $\mathbb{B} = \{p \in \mathbb{H} : |p| < 1\}$. Similarly to the complex case, the functions $f \in \mathcal{R}(\mathbb{B})$ admit power series expansion

$$f(p) = \sum_{k=0}^{\infty} p^k f_k \quad (f_k \in \mathbb{H}) \tag{1.2}$$

where the series on the right converges to f uniformly on compact subsets of $\mathbb{B}$; on the other hand, if $\limsup_k |f_k|^{\frac{1}{k}} \le 1$, the power series as in (1.2) converges absolutely on $\mathbb{B}$ and represents a slice regular function. We thus may identify the function from $\mathcal{R}(\mathbb{B})$ with power series of the form (1.2) with radius of convergence at least one.

The prominent role played in the classical complex analysis by analytic self-mappings of the open unit disk is well known. It is thus not surprising that the class $\mathcal{R}(\mathbb{B}, \mathbb{B})$ of slice regular self-mappings of the quaternionic unit ball $\mathbb{B}$ have already attracted much attention. A number of results in this direction (e.g., Möbius transformations, Schwarz Lemma, Bohr's inequality) are presented in [14, Chapter 9]. Among other results, we mention realizations for slice regular functions [1], Schwarz-Pick Lemma [6] and Blaschke products [2].

The present paper initiates the systematic study of interpolation theory in the class $\mathcal{R}(\mathbb{B}, \mathbb{B})$. In this context, it is convenient to extend the class $\mathcal{R}(\mathbb{B}, \mathbb{B})$ by unimodular constant functions. By the maximum modulus principle, this extended class equals $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, i.e., it consists of functions $f \in \mathcal{R}(\mathbb{B})$ such that $|f(p)| \le 1$ for all $p \in \mathbb{B}$. Although the unimodular constant case can be easily singled out, the results for the extended class $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ look more uniform as we will see below. Our first result (the analog of the celebrated result of I. Schur [24]) characterizes functions from $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ in terms of their Taylor coefficients at the origin.

Theorem 1.2. *Let S be slice regular on $\mathbb{B}$ and let $\mathbf{S}_n$ be the lower triangular Toeplitz matrix given by*

$$\mathbf{S}_n = \begin{bmatrix} S_0 & 0 & \dots & 0 \\ S_1 & S_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ S_n & \dots & S_1 & S_0 \end{bmatrix}, \quad \textit{where} \quad S(p) = \sum_{k=0}^{\infty} p^k S_k. \tag{1.3}$$

The function S belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ if and only if the matrix $\mathbf{I}_n - \mathbf{S}_n \mathbf{S}_n^*$ is positive semidefinite for all integers $n \ge 0$.

Here and in what follows, the symbol $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. The notions of adjoint matrices, of Hermitian matrices, of positive semidefinite and positive definite matrices over $\mathbb{H}$ are similar to those over $\mathbb{C}$.

The Hardy space $\mathrm{H}^2(\mathbb{B})$ of slice regular square summable power series has been recently introduced in [2]. Our second objective is to identify the class $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ with the unit ball of the multiplier algebra of $\mathrm{H}^2(\mathbb{B})$. This in turn will enable us to apply operator-theoretic tools to solve the quaternionic version of the Nevanlinna-Pick interpolation problem (we refer to [21, 20] for the classical origins) which is the main objective of the present paper and which is formulated as follows.

NP: *Given n distinct points $p_1, \dots, p_n \in \mathbb{B}$ and given n target values $s_1, \dots, s_n \in \mathbb{H}$, find a function $S \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ such that*

$$S(p_i) = s_i \qquad \text{for} \quad i = 1, \dots, n. \tag{1.4}$$

The interpolation problem is called *determinate* if it has a unique solution. By the convexity of the solution set, the indeterminate problem always has infinitely many solutions. The standard questions arising in any interpolation context are:

(1) Find necessary and sufficient conditions for the problem to have a solution (the solvability criterion).
(2) Find necessary and sufficient conditions for the problem to have a unique solution (the determinacy criterion).
(3) Describe all solutions in the indeterminate case.

As in the classical case the answers for these questions can be given in terms of the *Pick matrix* P of the problem which we define from the interpolation data as follows:

$$P = \left[\sum_{k=0}^{\infty} p_i^k (1 - s_i \bar{s}_j) \bar{p}_j^k \right]_{i,j=1}^{n}. \tag{1.5}$$

We remark that infinite series in (1.5) converge absolutely since $|p_i| < 1$ for all $i = 1, \dots, n$ and that the diagonal entries of P are equal to

$$P_{ii} = \frac{1 - |s_i|^2}{1 - |p_i|^2} \quad \text{for} \quad i = 1, \dots, n. \tag{1.6}$$

Our next result is the following analog of the classical Nevanlinna-Pick theorem.

Theorem 1.3. *The problem* **NP** *has a solution if and only if the Pick matrix P is positive semidefinite.*

Remark 1.4. It is known that the restriction of any slice regular function S to any 2-sphere is completely determined by the values of S at any two points of this sphere. Thus, if three interpolation nodes (say, p_1, p_2, p_3) are equivalent, then the value of s_3 must be uniquely specified by s_1 and s_2 in order the problem to have a solution. Theorem 1.3 asserts that condition $P \geq 0$ specifies s_3 in this unique way.

Therefore, once we know that the problem **NP** is solvable, there is no need to keep more than two interpolation conditions on the same 2-sphere. For each set of more than two conditions on the same 2-sphere, we keep any two of them and remove the others. In this way we reduce the original problem to the one for which

(**A**) : *None three of the interpolation nodes belong to the same 2-sphere.*

By Remark 1.4 the reduced problem will have the same solution set as the original one. Thus it is sufficient to get the uniqueness criterion and the description of the solution set for the reduced problem which is characterized by the property (**A**).

Theorem 1.5. *Under assumption* (**A**), *the problem* **NP** *is determinate if and only if the Pick matrix P of the problem is positive semidefinite and singular.*

A fairly explicit formula for this unique solution will be given in Lemma 4.6 below.

The outline of the paper is as follows. In Section 2 we recall some needed background on slice regular functions and positive kernels. In Section 3 we characterize functions $f \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ as contractive multipliers of the Hardy space $\mathrm{H}^2(\mathbb{B})$ of the unit ball and prove Theorem 1.2. In Section 4 we characterize solutions to the problem **NP** in terms of positive kernels of certain structure. Using this characterization, in Section 5 we give a linear fractional parametrization of all solutions to the problem **NP** in the indeterminate case and recover the Schwarz-Pick lemma as a consequence of this description. Finally the determinate case of the problem **NP** is handled in Section 6.

2. Slice hyperholomorphic functions and kernels

In this section we collect a number of basic facts needed in the sequel. Interpreting the set $\mathcal{R}(\mathbb{B})$ as the right quaternionic vector space of power series (3.4) converging in $\mathbb{B}$, one can introduce the ring structure on $\mathcal{R}(\mathbb{B})$ using the convolution multiplication

$$g \star f(p) = \sum_{k=0}^{\infty} p^k \cdot \left(\sum_{r=0}^{k} g_r f_{k-r} \right) \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k, \;\; g(p) = \sum_{k=0}^{\infty} p^k g_k \tag{2.1}$$

which is called (left) *slice regular multiplication* in the present context. As a convolution multiplication of the power series over the noncommutative ring, the $\star$-multiplication is associative and noncommutative. Since the series in (2.1) converge absolutely in $\mathbb{B}$, we can rearrange the terms getting

$$g \star f(p) = \sum_{k=0}^{\infty} p^k \left(\sum_{r=0}^{\infty} p^r g_r \right) f_k = \sum_{k=0}^{\infty} p^k g(p) f_k \tag{2.2}$$

which can also be written as

$$g \star f(p) = g(p) \sum_{k=0}^{\infty} (g(p)^{-1} p g(p))^k f_k = g(p) f(g(p)^{-1} p g(p)) \qquad (g(p) \neq 0), \tag{2.3}$$

with the understanding that $g \star f(p) = 0$ whenever $g(p) = 0$. Observe that the pointwise formula (2.3) makes sense even if the functions f and g are not in $\mathcal{R}(\mathbb{B})$ whereas formula (2.2) makes sense only for $f \in \mathcal{R}(\mathbb{B})$. We also observe that $g \star f(x) = g(x) f(x)$ for every $x \in \mathbb{R}$.

If the function $f \in \mathcal{R}(\mathbb{B})$ is as in (2.1), then we can construct its slice regular inverse $f^{-\star}$ as $f^{-\star}(p) = (f^c \star f)^{-1} f^c(p)$ where the *slice regular conjugate* f^c of f is defined by

$$f^c(p) = \sum_{k=0}^{\infty} p^k \overline{f}_k \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k \tag{2.4}$$

and $f^{-\star}$ is defined in $\mathbb{B}$ outside the zeros of $f^c \star f$. If f satisfies $f(0)=f_0\neq 0$, one can define its $\star$-inverse $f^{-\star}$ using the power series

$$f^{-\star}(p)=\sum_{k=0} p^k a_k, \quad \text{where} \quad a_0=f_0^{-1} \quad \text{and} \quad a_k=-f_0^{-1}\sum_{j=1}^{k} f_j a_{k-j} \ \ (k\geq 1)$$

with the coefficients a_k defined recursively. If $f(p)\neq 0$ for all $p\in\mathbb{B}$, the latter power series converges on $\mathbb{B}$. Equalities $f^{-\star}\star f=f\star f^{-\star}\equiv 1$ and $(g\star f)^{-\star}=f^{-\star}\star g^{-\star}$ are immediate. An application of (2.3) shows that

$$f^{-\star}(p)=f(\widetilde{p})^{-1}, \quad \text{where} \quad \widetilde{p}=f^c(p)^{-1}pf^c(p), \quad f(p)=\sum_{k=0}^{\infty} p^k f_k. \tag{2.5}$$

2.1. **Right slice regular functions.** A real differentiable function $f:\Omega\to\mathbb{H}$ is called *right slice regular* on Ω (in notation, $f\in\mathcal{R}^r(\Omega)$) if for every $I\in\mathbb{S}$ its restriction f_I to $\Omega\cap\mathbb{C}_I$ is subject to

$$\frac{\partial}{\partial x}f_I(x+Iy)+\frac{\partial}{\partial y}f_I(x+Iy)I\equiv 0.$$

The results for right slice regular functions are completely parallel to those for (left) regular ones. The functions in $f\in\mathcal{R}^r(\mathbb{B})$ can be identified with power series $f(p)=\sum_{k=0}^{\infty} f_k p^k$ converging on $\mathbb{B}$. The set $\mathcal{R}^r(\mathbb{B})$ itself is a left quaternionic vector space and it becomes a ring once we introduce the *right slice multiplication*

$$g\star_r f(p)=\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k} g_r f_{k-r}\right)p^k \quad \text{if} \quad f(p)=\sum_{k=0}^{\infty} f_k p^k, \ \ g(p)=\sum_{k=0}^{\infty} g_k p^k$$

which can be written alternatively (analogously to formulas (2.2) and (2.3)) as

$$g\star_r f(p)=\sum_{k=0}^{\infty} g_k f(p)p^k=\begin{cases} g(f(p)pg(p)^{-1})f(p) & \text{if} \quad f(p)=0,\\ 0 & \text{if} \quad f(p)\neq 0.\end{cases} \tag{2.6}$$

2.2. **Positive kernels.** A matrix-valued function $K(p,q):\Omega\times\Omega\to\mathbb{H}^{m\times m}$ is called a positive kernel (in notation, $K\succeq 0$) if the block matrix $[K(q_i,q_j)]_{i,j=1}^r$ is positive semidefinite for any choice of finitely many points $q_1,\ldots,q_r$. Equivalently,

$$\sum_{i,j=1}^{r} c_i^* K(q_i,q_j)c_j\geq 0 \quad \text{for all} \quad r\in\mathbb{N}, \ c_1,\ldots,c_r\in\mathbb{H}^m, \ q_1,\ldots,q_r\in\Omega.$$

Definition 2.1. *We say that the kernel $K(p,q):\Omega\times\Omega\to\mathbb{H}^{m\times m}$ is* slice sesquiregular *on an open set $\Omega\subset\mathbb{H}$ if it is (left) slice regular in p and right slice regular in $\bar{q}$.*

Several simple statements on positive kernels are collected in the next proposition.

Proposition 2.2. *Let $\Omega\subset\mathbb{H}$ and let $K:\Omega\times\Omega\to\mathbb{H}^{m\times m}$ be a positive kernel. Then*

(1) *For every $A:\Omega\to\mathbb{H}^{k\times m}$, the kernel $A(p)K(p,q)A(q)^*$ is positive on $\Omega\times\Omega$.*
(2) *For every positive definite matrix $P\in\mathbb{H}^{k\times k}$ and any function $B:\Omega\to\mathbb{H}^{m\times k}$, the kernel $\begin{bmatrix} P & B(q)^* \\ B(p) & K(p,q)\end{bmatrix}$ is positive if and only if the Schur complement of P defined below is positive semidefinite:*

$$K(p,q)-B(p)P^{-1}B(q)^*\succeq 0.$$

(3) *If in addition, $m = 1$, Ω is open and contains the origin, and $K : \Omega \times \Omega \to \mathbb{H}$ is slice sesquiregular, then for every (left) slice regular function $A : \Omega \to \mathbb{H}$, the kernel $(A \star K \star_r \overline{A})(p, q)$ is positive and slice sesquiregular.*

Proof. Statement (1) follows by the definition of the positive kernel and the corresponding property of positive semidefinite matrices. Due to factorization

$$\begin{bmatrix} P & B(q)^* \\ B(p) & K(p,q) \end{bmatrix} = A(p) \begin{bmatrix} P & 0 \\ 0 & K(p,q) - B(p)P^{-1}B(q)^* \end{bmatrix} A(q)^*,$$

where $A(p) = \begin{bmatrix} \mathbf{I}_k & 0 \\ B(p)P^{-1} & \mathbf{I}_m \end{bmatrix}$, part (2) follows from part (1) and the fact that the matrix $A(p)$ is invertible for every $p \in \Omega$. For part (3), see [3, Proposition 5.3].

3. The space $\mathrm{H}^2(\mathbb{B})$ and its contractive multipliers

In this section we show that the class $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ can be identified with the class of contractive multipliers of the quaternionic Hardy space H^2 of the unit ball $\mathbb{B}$. This space is defined as the space of square summable (left) slice regular power series:

$$\mathrm{H}^2 = \left\{ f(p) = \sum_{k=0}^{\infty} p^k f_k : \|f\|^2_{\mathrm{H}^2} := \sum_{k=0}^{\infty} |f_k|^2 < \infty \right\}. \tag{3.1}$$

The space H^2 is a right quaternionic Hilbert space with inner product

$$\langle f, g \rangle = \sum_{k=0}^{\infty} \bar{g}_k f_k \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k, \;\; g(p) = \sum_{k=0}^{\infty} p^k g_k. \tag{3.2}$$

A power-series computation followed by integration of (uniformly converging on compact sets) power series shows that for f as in (3.2) and for a fixed $I \in \mathbb{S}$,

$$\begin{aligned} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta &= \int_0^{2\pi} \left(\sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k e^{I(j-k)\theta} f_j \right) d\theta \\ &= \sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k \left(\int_0^{2\pi} e^{I(j-k)\theta} \theta \right) f_j = 2\pi \cdot \sum_{n=0}^{\infty} r^{2n} |f_n|^2. \end{aligned}$$

The latter formula implies that the norm in H^2 can be equivalently defined as

$$\|f\|^2_{\mathrm{H}^2} = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta \tag{3.3}$$

where the value of the integral on the right is the same for each $I \in \mathbb{S}$. Observe that the supremum in the last formula can be replaced by the limit as r tends to one.

The space H^2 can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel

$$k_{\mathrm{H}^2}(p, q) = \sum_{n=0}^{\infty} p^n \overline{q}^n. \tag{3.4}$$

The latter means that the function $k_{\mathrm{H}^2}(\cdot, q)$ belongs to H^2 for every $q \in \mathbb{B}$ and for any function $f \in \mathrm{H}^2$ as in (3.2),

$$\langle f,\, k_{\mathrm{H}^2}(\cdot, q)\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} q^k f_k = f(q). \tag{3.5}$$

Proposition 3.1. *The finite collection of functions $\{k_{\mathrm{H}^2}(\cdot, q_i)\}$ based on distinct points $q_1, \ldots, q_k \in \mathbb{B}$ is (right) linearly independent in H^2 if and only if none three of these points belong to the same 2-sphere.*

Proof. Let us assume that $\sum_{i=1}^k k_{\mathrm{H}^2}(p, q_i)\alpha_i \equiv 0$. Substituting the power series expansions (3.4) for $k^2_{\mathrm{H}}(\cdot, q_i)$ into this identity and equating the corresponding coefficients we conclude that the columns of the Vandermonde matrix $V = \left[q_i^{j-1}\right]_{i,j=1}^k$ are linearly dependent which is the case if and only if there are three equivalent points in $\{q_1, \ldots, q_k\}$; we refer to [18] for more details. $\square$

Remark 3.2. The linear dependence of three functions $k_{\mathrm{H}^2}(\cdot, p_i)$ based on equivalent points implies that the restriction of any function $f \in \mathrm{H}^2$ to any 2-sphere is completely determined by the values of f at any two points of this sphere. Indeed, if

$$p_i = x + yI_i, \quad (x, y \in \mathbb{R},\ I_i \in \mathbb{S},\ i = 1, 2, 3),$$

then it is readily checked that

$$\overline{p}_3^n = \overline{p}_1^n (I_1 - I_2)^{-1}(I_3 - I_2) + \overline{p}_2^n (I_1 - I_2)^{-1}(I_1 - I_3) \quad \text{for all} \quad n \geq 0$$

which implies the identity

$$k_{\mathrm{H}^2}(p, p_3) \equiv k_{\mathrm{H}^2}(p, p_1)(I_1 - I_2)^{-1}(I_3 - I_2) + k_{\mathrm{H}^2}(p, p_2)(I_1 - I_2)^{-1}(I_1 - I_3). \tag{3.6}$$

Combining the latter identity with (3.5) leads us to

$$\begin{aligned} f(p_3) = \langle f,\, k_{\mathrm{H}^2}(\cdot, p_3)\rangle_{\mathrm{H}^2} &= \left\langle f,\, k_{\mathrm{H}^2}(\cdot, p_1)(I_1 - I_2)^{-1}(I_3 - I_2)\right\rangle_{\mathrm{H}^2} \\ &\quad + \left\langle f,\, k_{\mathrm{H}^2}(\cdot, p_2)(I_1 - I_2)^{-1}(I_1 - I_3)\right\rangle_{\mathrm{H}^2} \\ &= (\overline{I}_1 - \overline{I}_2)^{-1}(\overline{I}_3 - \overline{I}_2) f(p_1) + (\overline{I}_1 - \overline{I}_2)^{-1}(\overline{I}_1 - \overline{I}_3) f(p_2) \\ &= (I_2 - I_1)^{-1}\left\{(I_2 - I_3) f(p_1) + (I_3 - I_1) f(p_2)\right\}. \end{aligned} \tag{3.7}$$

The latter representation was established in [9] for general slice regular functions on axially symmetric s-domains. We now pass to the main result of this section.

Theorem 3.3. *Let $S : \mathbb{B} \to \mathbb{H}$. The following are equivalent:*

(1) *S is slice regular on $\mathbb{B}$ and $|S(p)| \leq 1$ for all $p \in \mathbb{B}$.*

(2) *The operator M_S of left $\star$–multiplication by S*

$$M_S :\ f \mapsto S \star f \tag{3.8}$$

is a contraction on H^2, that is, $\|S \star f\|_{\mathrm{H}^2} \leq \|f\|_{\mathrm{H}^2}$ for all $f \in \mathrm{H}^2$.

(3) *The kernel*

$$K_S(p, q) = \sum_{k=0}^{\infty} p^k (1 - S(p)\overline{S(q)})\bar{q}^k \tag{3.9}$$

is positive on $\mathbb{B} \times \mathbb{B}$.

(4) $S \in \mathcal{R}(\mathbb{B})$ *and* $\mathbf{I}_n - \mathbf{S}_n\mathbf{S}_n^* \geq 0$ *for all* $n \geq 0$ *where* $\mathbf{S}_n$ *is the matrix given in* (1.3).

Proof. We first remark that the operator M_S can be defined via formula (2.2) which does not assume any regularity of S. However, if M_S maps H^2 into itself, then the function $S = M_S 1$ belongs to H^2 and hence is slice regular.

Proof of (2) $\Longrightarrow$ (3): Let us assume that $M_S : \mathrm{H}^2 \to \mathrm{H}^2$ is a contraction. Combining formulas (2.2) and (3.4) gives

$$M_S k_{\mathrm{H}^2}(\cdot, q) = \sum_{j=0}^{\infty} p^k S(p)\bar{q}^k$$

which together with reproducing kernel property (3.5) implies

$$\begin{aligned}(M_S^* k_{\mathrm{H}^2}(\cdot, q))(p) &= \langle M_S^* k_{\mathrm{H}^2}(\cdot, q),\, k_{\mathrm{H}^2}(\cdot, p)\rangle_{\mathrm{H}^2} \\ &= \langle k_{\mathrm{H}^2}(\cdot, q),\, S \star k_{\mathrm{H}^2}(\cdot, p)\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} p^k \overline{S(q)}\bar{q}^k, \qquad (3.10)\end{aligned}$$

and subsequently,

$$\langle (I - M_S M_S^*) k_{\mathrm{H}^2}(\cdot, q),\, k_{\mathrm{H}^2}(\cdot, p)\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} p^k (1 - S(p)\overline{S(q)})\bar{q}^k.$$

Therefore, for any function $f \in \mathrm{H}^2$ of the form

$$f = \sum_{i=1}^{r} k_{\mathrm{H}^2}(\cdot, p_i)\alpha_i, \qquad r \in \mathbb{N},\ p_i \in \mathbb{B},\ \alpha_i \in \mathbb{H}, \qquad (3.11)$$

we have

$$\begin{aligned}\langle (I - M_S M_S^*) f,\, f\rangle_{\mathrm{H}^2} &= \langle f,\, f\rangle_{\mathrm{H}^2} - \langle M_S^* f,\, M_S^* f\rangle_{\mathrm{H}^2} \\ &= \sum_{i,j=1}^{r} \overline{\alpha}_i k_{\mathrm{H}^2}(p_i, p_j)\alpha_j - \sum_{i,j=1}^{r}\sum_{k=0}^{\infty} \overline{\alpha}_i p_i^k S(p_i)\overline{S(p_j)}\overline{p}_i^k \alpha_j \\ &= \sum_{i,j=1}^{r} \overline{\alpha}_i K_S(p_i, p_j)\alpha_j. \qquad (3.12)\end{aligned}$$

Since $M_S : \mathrm{H}^2 \to \mathrm{H}^2$ is a contraction, the inner product on the left hand side of (3.12) is nonnegative. Consequently, the quadratic form on the right hand side of (3.12) is nonnegative so that K_S is a positive kernel.

Proof of (3) $\Longrightarrow$ (2): Let us assume that the kernel (3.9) is positive on $\mathbb{B} \times \mathbb{B}$. Observing that the function on the right side of (3.10) belongs to H^2 (for each fixed $q \in \mathbb{B}$) with H^2-norm equal $\frac{|S(q)|^2}{1-|q|^2}$, we define the operator $T : \mathrm{H}^2 \to \mathrm{H}^2$ by letting $T : k_{\mathrm{H}^2}(\cdot, q) \mapsto \sum_{k=0}^{\infty} p^k \overline{S(q)}\bar{q}^k$ (that is, using the formula (3.8) obtained earlier for M_S^*) with subsequent extention by linearity to functions f of the form (3.11) and then, since such functions are dense in H^2, extending by continuity to all of H^2. Due to this density, the calculation (3.12) (with T instead of M_S^*) shows that T is a contraction on H^2. We then calculate its adjoint getting $T^*f = S \star f = M_S f$. Since T is a contraction on H^2, its adjoint M_S is a contraction as well.

Proof of (3) $\Longrightarrow$ (1): If the kernel K_S is positive on $\mathbb{B} \times \mathbb{B}$, we have, in particular,

$$0 \le K_S(q,q) = \sum_{k=0}^{\infty} q^k (1 - |S(q)|^2) \bar{q}^k = \frac{1 - |S(q)|^2}{1 - |q|^2}$$

and therefore, $|S(q)| \le 1$ for every $q \in \mathbb{B}$. On the other hand, by implication (3) $\Longrightarrow$ (2), the operator M_S maps H^2 into itself and thus $S = M_S 1$ belongs to $\mathrm{H}^2 \subset \mathcal{R}(\mathbb{B})$.

Proof of (1) $\Longrightarrow$ (2): We now assume that S is in $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, i.e., S is slice regular and with $|S(p)| \le 1$ for all $p \in \mathbb{B}$. By formulas (3.3) and (2.3), we have for every $f \in \mathrm{H}^2$ and every $I \in \mathbb{S}$,

$$\begin{aligned} \|f \star S\|^2_{\mathrm{H}^2} &= \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f \star S(re^{I\theta})|^2 d\theta \\ &= \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta}) S(f(re^{I\theta})^{-1} re^{I\theta} f(re^{I\theta})|^2 d\theta \\ &\le \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta = \|f\|^2_{\mathrm{H}^2}. \end{aligned} \tag{3.13}$$

Let S^c and f^c be the slice conjugates of S and f defined via formula (2.4). Due to obvious relations $(f^c)^c = f$, $\|f\|_{\mathrm{H}^2} = \|f^c\|_{\mathrm{H}^2}$, $(f \star g)^c = g^c \star f^c$ holding for all $f, g \in \mathrm{H}^2$, we have from (3.13)

$$\|S^c \star f^c\|_{\mathrm{H}^2} = \|(f \star S)^c\|_{\mathrm{H}^2} = \|f \star S\|_{\mathrm{H}^2} \le \|f\|_{\mathrm{H}^2} = \|f^c\|_{\mathrm{H}^2}. \tag{3.14}$$

Therefore the operator $M_{S^c} : f \mapsto S^c \star f$ is a contraction on H^2. By implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) which have been already proved, we conclude that $S^c \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$. We then apply (3.14) to S^c rather than to S concluding that the operator $M_{(S^c)^c} = M_S$ is a contraction on H^2.

Proof of (2) $\Longrightarrow$ (4): The proof is similar to that of (2) $\Longleftrightarrow$ (4) with the only difference that instead of functions of the form (3.11) we will use slice regular polynomials, namely the polynomials with coefficients written on the right. We first assume (2). The calculation analogous to that in (3.10) shows that for S with the Taylor series as in (1.3),

$$M_S^* : \, p^k \mapsto \sum_{j=0}^{k} p^j \overline{S}_{k-j} \quad \text{for all} \quad k \ge 0$$

which extends by linearity to

$$M_S^* : \, f(p) = \sum_{k=0}^{n} p^k f_k \mapsto \sum_{k=0}^{n} p^k \left(\sum_{j=k}^{n} \overline{S}_{j-k} f_j \right).$$

Letting $\mathbf{f} := \begin{bmatrix} f_0 & f_1 & \dots & f_n \end{bmatrix}^\top$ we get the following analog of (3.12) in terms of the matrix $\mathbf{S}_n$ from (1.3):

$$\|f\|^2_{\mathrm{H}^2} - \|M_S^* f\|^2_{\mathrm{H}^2} = \sum_{k=0}^{n} |f_k|^2 - \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \overline{S}_{j-k} f_j \right|^2 = \mathbf{f}^* \left(\mathbf{I}_n - \mathbf{S}_n \mathbf{S}_n^* \right) \mathbf{f}. \tag{3.15}$$

If M_S is a contraction on H^2, the latter expression is nonnegative for every vector $\mathbf{f} \in \mathbb{H}^{n+1}$ and therefore the matrix $\mathbf{I}_n - \mathbf{S}_n\mathbf{S}_n^*$ is positive semidefinite. Conversely, if this matrix is positive semidefinite for each $n \geq 1$, then equality (3.15) shows that M_S^* acts contractively (in H^2-metric) on any polynomial. Since the polynomials are dense in H^2, the operators M_S^* and M_S are contractions on the whole H^2. □

We point out several consequences of the last theorem.

Corollary 3.4. *Let $S : V \to \mathbb{H}$ be such that the kernel (3.9) is positive on $V \times V$, where V is an open subset of $\mathbb{B}$. Then S extends to a function from $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$.*

Proof. The proof is the same as that of implication (3) $\Longrightarrow$ (2) in Theorem 3.3 once we observe that the functions of the form (3.11) with $p_i \in V$ (rather than in $\mathbb{B}$) are still dense in H^2. □

In analogy to the classical case we may introduce the Hardy space $\mathrm{H}^\infty(\mathbb{B})$ of bounded slice regular functions on $\mathbb{B}$ with norm $\|S\|_\infty = \sup_{p\in\mathbb{B}} |S(p)| < \infty$ and the space $\mathcal{M}(\mathrm{H}^2)$ of bounded multipliers, that is, the functions $S : \mathbb{B} \to \mathbb{H}$ such that the operator M_S of left $\star$–multiplication (3.8) is bounded on H^2. By the very definition, $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ is the closed unit ball of $\mathrm{H}^\infty(\mathbb{B})$. The following conclusion is a consequence of Theorem 3.3.

Corollary 3.5. $\mathrm{H}^\infty(\mathbb{B}) = \mathcal{M}(\mathrm{H}^2)$ *and* $\|S\|_\infty = \|M_S\|$ *for every* $S \in \mathrm{H}^\infty(\mathbb{B})$.

Proof. If $S \in \mathrm{H}^\infty(\mathbb{B})$ with $\|S\|_\infty = r > 0$, then the scaled function $\frac{1}{r}S$ belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ and by Theorem 3.3, the operator $M_{\frac{1}{r}S} : \mathrm{H}^2 \to \mathrm{H}^2$ is a contraction. Since $M_{\frac{1}{r}S} = \frac{1}{r}M_S$, we conclude that $\|M_S\| = \|rM_{\frac{1}{r}S}\| \leq r$ and thus, $S \in \mathcal{M}(\mathrm{H}^2)$ with $\|M_S\| \leq \|S\|_\infty$. In particular, $\mathrm{H}^\infty(\mathbb{B}) \subset \mathcal{M}(\mathrm{H}^2)$. The reverse inclusion and the reverse norm inequality is established in much the same way. □

As another consequence of Theorem 3.3, we get the necessity part in Theorem 1.3.

Corollary 3.6. *If the problem* **NP** *has a solution, then the Pick matrix P is positive semidefinite.*

Proof. Let S be a solution to the problem **NP**. Since $S \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, the kernel K_S is positive on $\mathbb{B} \times \mathbb{B}$. Then the matrix $[K_S(p_i, p_j)]_{i,j=1}^n$ is positive semidefinite. Since S satisfies the interpolation conditions (1.4),

$$K_S(p_i, p_j) = \sum_{k=0}^{\infty} p_i^k(1 - S(p_i)\overline{S(p_j)})\overline{p}_j^k = \sum_{k=0}^{\infty} p_i^k(1 - s_i\overline{s}_j)\overline{p}_j^k. \tag{3.16}$$

Comparing (3.16) with (1.5) we see that the matrix $[K_S(p_i, p_j)]_{i,j=1}^n$ is equal to the Pick matrix P which is therefore positive semidefinite.

4. Characterization of solutions to the problem NP in terms of positive kernels

The classical complex-valued Nevanlinna-Pick problem has been studied using different approaches, including in particular, the iterative Schur algorithm [20], the Commutatnt Lifting approach [23], the Grassmanian approach [5], Potapov's method of fundamental matrix inequalities [22] and its far-reaching extension the Abstract Interpolation Problem approach [16, 17]. Each method has its strengths and weaknesses; so it would be interesting to clarify how each of them extends to the quaternionic setting. The method we chose for the present paper has its origins in [22]. The first step is carried out in the next theorem which characterizes solutions of the problem **NP** in terms of positive kernels of special structure.

Theorem 4.1. *A function $S:\mathbb{B}\to\mathbb{H}$ is a solution to the problem* **NP** *if and only if the following kernel is positive on $\mathbb{B}\times\mathbb{B}$:*

$$\widehat{K}_S(p,q) := \begin{bmatrix} P & B^S(q)^* \\ B^S(p) & K_S(p,q)\end{bmatrix} \succeq 0, \tag{4.1}$$

where P is given in (1.5) *and where*

$$\begin{aligned} B^S(p) &= \begin{bmatrix} B_1^S(p) & B_2^S(p) & \dots & B_n^S(p)\end{bmatrix} \\ &= \sum_{k=0}^{\infty} p^k \begin{bmatrix} (1-S(p)\bar{s}_1)\bar{p}_1^k & (1-S(p)\bar{s}_2)\bar{p}_2^k & \dots & (1-S(p)\bar{s}_n)\bar{p}_n^k\end{bmatrix}. \end{aligned} \tag{4.2}$$

Proof. For the necessity part we modify the argument used in the previous section. If S is a solution to the problem **NP**, it belongs to $\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$ and therefore, the kernel K_S is positive on $\mathbb{B}\times\mathbb{B}$. Let us pick finitely many points $q_1,\dots,q_r\in\mathbb{B}$ and let us consider the (positive semidefinite) matrix $R=[K_S(\zeta_i,\zeta_j)]_{i,j=1}^{(n+1)r}$ based on the $(n+1)r$ points ζ_j chosen as follows:

$$\zeta_j = \begin{cases} p_i & \text{if} \quad j = i \bmod (n+1), \\ q_\ell & \text{if} \quad j=(n+1)\ell. \end{cases}$$

Since S satisfies interpolation conditions (1.4), the entries $K_S(p_i,p_j)$ in R are given as in (3.16). On the other hand, in view of (4.2),

$$K_S(q_i,p_j) = \sum_{k=0}^{\infty} q_i^k(1-S(q_i)\overline{S(p_j)})\overline{p}_j^k = \sum_{k=0}^{\infty} q_i^k(1-S_i\overline{s}_j)\overline{p}_j^k = B_j^S(q_i). \tag{4.3}$$

A careful but straightforward verification based on (3.16) and (4.3) confirms that the matrix $R=[K_S(\zeta_i,\zeta_j)]_{i,j=1}^{(n+1)r}$ can be written in the block-matrix form as

$$R=\left[\widehat{K}_S(q_i,q_j)\right]_{i,j=1}^{r}$$

where $\widehat{K}$ is the kernel defined in (4.1). Since R is positive semidefinite and the points $q_1,\dots,r$ were chosen arbitrarily in $\mathbb{B}$, the kernel (4.1) is positive on $\mathbb{B}\times\mathbb{B}$.

The proof of the sufficiency part is based on the operator-theoretic argument involving Schur complements and multiplication operators which is adapted from [4, 7]. Let us assume that the kernel (4.1) is positive on $\mathbb{B}\times\mathbb{B}$. Then in particular, the kernel K_S is positive on $\mathbb{B}\times\mathbb{B}$ and therefore, $S\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$. Furthermore, it follows from (4.1) that the following 2×2 matrix valued kernel is positive

$$K_i(p,q) = \begin{bmatrix} P_{ii} & B_i^S(q)^* \\ B_i^S(p) & K_S(p,q)\end{bmatrix} \succeq 0 \tag{4.4}$$

for each $i = 1, \dots, n$. The positivity condition (4.4) is equivalent to the positivity of the operator

$$\mathbf{P}_i = \begin{bmatrix} P_{ii} & M^*_{B^S_i} \\ M_{B^S_i} & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathbb{H} \\ \mathrm{H}^2 \end{bmatrix} \to \begin{bmatrix} \mathbb{H} \\ \mathrm{H}^2 \end{bmatrix} \tag{4.5}$$

due to the identity

$$\left\langle \mathbf{P}_i \begin{bmatrix} \alpha \\ k_{\mathrm{H}^2}(\cdot, q)\beta \end{bmatrix}, \begin{bmatrix} \alpha' \\ k_{\mathrm{H}^2}(\cdot, p)\beta' \end{bmatrix} \right\rangle_{\mathbb{H}\oplus\mathrm{H}^2} = \left\langle K_i(p,q) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} \right\rangle_{\mathbb{H}^2}$$

holding for all $\alpha, \alpha', \beta, \beta' \in \mathbb{H}$ and all $p, q \in \mathbb{B}$, and since linear combinations of vectors of the form $\alpha \oplus k_{\mathrm{H}^2}(\cdot, q)\beta$ ($\alpha, \beta \in \mathbb{H}$, $q \in \mathbb{B}$) are dense in $\mathbb{H} \oplus \mathrm{H}^2$.

We next fix $i \in \{1, \dots, n\}$ and introduce two operators $T_1, T_2 : \mathbb{H} \to \mathrm{H}^2$ as follows:

$$T_1\alpha = k_{\mathrm{H}^2}(\cdot, p_i)\alpha \quad \text{and} \quad T_2\alpha =: \begin{cases} k_{\mathrm{H}^2}(\cdot, s_i^{-1}p_i s_i)\bar{s}_i\alpha, & \text{if } s_i \neq 0, \\ 0, & \text{if } s_i = 0. \end{cases} \tag{4.6}$$

Since k_{H^2} is the reproducing kernel for H^2 we have

$$T_1^*T_1 - T_2^*T_2 = \begin{cases} k_{\mathrm{H}^2}(p_i, p_i) - s_i k_{\mathrm{H}^2}(s_i^{-1}p_i s_i, s_i^{-1}p_i s_i)\bar{s}_i = \frac{1-|s_i|^2}{1-|p_i|^2} & \text{if} \quad s_i \neq 0, \\ k_{\mathrm{H}^2}(p_i, p_i) = \frac{1}{1-|p_i|^2} & \text{if} \quad s_i = 0, \end{cases}$$

which being compared with (1.6) gives

$$T_1^*T_1 - T_2^*T_2 = P_{ii}.$$

We also observe from (4.2) that the function B_i^S can be written as

$$B_i^S = k_{\mathrm{H}^2}(\cdot, p_i) - S \star k_{\mathrm{H}^2}(\cdot, s_i^{-1}p_i s_i)\bar{s}_i,$$

so that $M_{B_i^S} = T_1 - M_S T_2$. Therefore, we can rewrite (4.5) as

$$\mathbf{P}_i = \begin{bmatrix} T_1^*T_1 - T_2^*T_2 & T_1^* - T_2^*M_S \\ T_1 - M_S T_2 & I - M_S M_S^* \end{bmatrix}.$$

The operator $\mathbf{P}_i$ equals the Schur complement of the left top block in the extended operator

$$\widehat{\mathbf{P}}_i = \begin{bmatrix} I & T_2 & M_S^* \\ T_2^* & T_1^*T_1 & T_1^* \\ M_S & T_1 & I \end{bmatrix} : \begin{bmatrix} \mathrm{H}^2 \\ \mathbb{H} \\ \mathrm{H}^2 \end{bmatrix} \to \begin{bmatrix} \mathrm{H}^2 \\ \mathbb{H} \\ \mathrm{H}^2 \end{bmatrix},$$

and therefore, $\mathbf{P}_i \geq 0$ if and only if $\widehat{\mathbf{P}}_i \geq 0$. But then the Schur complement of the right bottom block in $\widehat{\mathbf{P}}_i$ is also positive semidefinite:

$$\begin{bmatrix} I - M_S^*M_S & T_2 - M_S^*T_1 \\ T_2^* - T_1^*M_S & T_1^*T_1 - T_1^*T_1 \end{bmatrix} = \begin{bmatrix} I - M_S^*M_S & T_2 - M_S^*T_1 \\ T_2^* - T_1^*M_S & 0 \end{bmatrix} \geq 0$$

from which we conclude $T_2 - M_S^*T_1 = 0$. We next use (3.10) and definitions (4.6) to rewrite the last equality as

$$0 \equiv T_2 - M_S^*T_1 = k_{\mathrm{H}^2}(p, s_i^{-1}p_i s_i)\bar{s}_i - \sum_{k=0}^{\infty} p^k \overline{S(p_i)}\bar{p}_i^k$$

and finally, letting $p = 0$ we get $\bar{s}_i = \overline{S(p_i)}$ which is equivalent to (1.4). Thus, S solves the problem **NP**. □

Remark 4.2. The positivity condition (4.1) implies $P \geq 0$; thus Theorem 4.1 contains the necessity part of Theorem 1.3.

Remark 4.3. The Pick matrix P of the problem **NP** satisfies the Stein equality

$$P - TPT^* = EE^* - NN^* \tag{4.7}$$

where

$$T = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad N = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}. \tag{4.8}$$

The entry-wise verification of (4.7) is immediate. In fact, if T is any square matrix with right spectrum contained in $\mathbb{B}$, then the Stein equation $P - TPT^* = D$ has a unique solution given by converging series $P = \sum_{k\geq 0} T^k D T^{*k}$. In particular, if $D = EE^* - NN^*$, this series produces P as in (1.5).

Remark 4.4. Let us note that the function (4.2) can be written in terms of (4.8) as

$$B^S(p) = \sum_{k=0}^{\infty} p^k \left(E^* - S(p)N^*\right) T^{*k} = \begin{bmatrix} 1 & -S(p) \end{bmatrix} \star \left(\sum_{k=0}^{\infty} p^k \begin{bmatrix} E^* \\ N^* \end{bmatrix} T^{*k} \right) \tag{4.9}$$

Therefore, all the entries in the kernel inequality (4.1) are defined in terms of given E, N, T and an unknown function S. The description of all functions S satisfying the latter inequality does not rely on the specific formulas (1.5), (4.8); it will be established under the assumptions that (1) the right spectrum of T is contained in $\mathbb{B}$ and (2) the unique solution P of the Stein equation (4.7) is positive semidefinite.

We conclude this section with two results which substantially simplify the subsequent analysis. The first one is about the "consistency" of interpolation data set.

Lemma 4.5. *Let us assume that the Pick matrix P (1.5) is positive semidefinite and that three interpolation nodes, say p_1, p_2 and p_3 belong to the same 2-sphere:*

$$p_i = x + yI_i, \quad (x, y \in \mathbb{R}, \ I_i \in \mathbb{S}, \ i = 1, 2, 3). \tag{4.10}$$

Then the three top rows in P are left linearly dependent and the target values s_1, s_2 and s_3 are related by

$$s_3 = (I_2 - I_1)^{-1} \left\{ (I_2 - I_3)s_1 + (I_3 - I_1)s_2 \right\}. \tag{4.11}$$

Proof. Let us define two positive semidefinite matrices

$$P_1 = \sum_{k=0}^{\infty} T^k EE^* T^{*k} \quad \text{and} \quad P_2 = \sum_{k=0}^{\infty} T^k NN^* T^{*k}$$

and observe that

$$P_1 = TP_1T^* + EE^*, \quad P_2 = TP_2T^* + NN^* \quad \text{and} \quad P = P_1 - P_2. \tag{4.12}$$

The matrix P_1 can be written more explicitly as

$$P_1 = \left[k_{\mathrm{H}^2}(p_i, p_j)\right]_{i,j=1}^n = \left[\langle k_{\mathrm{H}^2}(\cdot, p_j), \, k_{\mathrm{H}^2}(\cdot, p_i)\rangle\right]_{i,j=1}^n$$

and is, therefore, the gram matrix of the set $\{k_{\mathrm{H}^2}(\cdot, p_i)\}_{i=1}^n$. Due to identity (3.6), the three top rows in P are left linearly dependent and moreover,

$$\mathbf{x}P_1 = 0, \quad \text{where} \quad \mathbf{x} = \begin{bmatrix}(I_1 - I_2)^{-1}(I_2 - I_3) & (I_1 - I_2)^{-1}(I_3 - I_1) & 1 & 0 & \dots & 0\end{bmatrix}.$$

Since $P = P_1 - P_2 \geq 0$, it also follows that $\mathbf{x}P_2 = 0$ and therefore, by the second relation in (4.12), $\mathbf{x}N = 0$. Substituting explicit formulas for $\mathbf{x}$ and N into the latter equality gives (4.11). □

Comparing (4.11) and (3.7) shows that condition $P \geq 0$ indeed guarantees that the target value s_3 at p_2 for the unknown slice regular interpolant is consistent with its values at p_1 and p_2. This advances us toward establishing the "if" part in Theorem 1.3: now it suffices to prove Theorem 1.3 under the assumption (**A**).

Lemma 4.6. *Let us assume that* (**A**) *holds and the Pick matrix* $P \geq 0$ *is singular. Then the problem* **NP** *has at most one solution which, if exists, is given by the formula*

$$S(p) = R \star Q(p)^{-\star}, \tag{4.13}$$

where

$$R = \sum_{i=1}^n k_{\mathrm{H}^2}(\cdot, p_i)\alpha_i, \quad Q = \sum_{i: s_i \neq 0} k_{\mathrm{H}^2}(\cdot, s_i^{-1}p_i s_i)\overline{s}_i\alpha_i \tag{4.14}$$

and where $\mathbf{y} = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix} \in \mathbb{H}^n$ *is any nonzero vector such that* $P\mathbf{y} = 0$.

Proof. Let us assume that S is a solution to the problem **NP**. Then the matrix

$$\begin{bmatrix} P & B^S(p)^* \\ B^S(p) & \frac{1-|S(p)|^2}{1-|p|^2}\end{bmatrix} \geq 0 \quad \text{for all} \quad p \in \mathbb{B},$$

is positive semidefinite for every $p \in \mathbb{B}$. From this positivity and from the equality $P\mathbf{y} = 0$ we conclude that $B^S(p)\mathbf{y} \equiv 0$. Making use of the formula (4.2) for B^S, we write the latter identity more explicitly as

$$\begin{aligned} 0 \equiv \sum_{i=1}^n \sum_{k=0}^\infty p^k \left(1 - S(p)\overline{s}_i\right) \overline{p}_i^k \alpha_i &= \sum_{i=0}^n k_{\mathrm{H}^2}(p, p_i)\alpha_i - \sum_{s_i \neq 0}\sum_{k=0}^\infty p^k S(p) \left(\overline{s_i^{-1}p_i s_i}\right)^k \overline{s}_i\alpha_i \\ &= R(p) - S \star Q(p) \end{aligned}$$

where the last step follows by formulas (4.14) and the definition of the $\star$-product. Thus any solution S to the problem **NP** must satisfy

$$S \star Q(p) = R(p) \quad \text{for all} \quad p \in \mathbb{B}. \tag{4.15}$$

By Proposition 3.1 and due to assumption (**A**), the function R is not vanishing identically. Then it follows from (4.15) that Q is not vanishing identically as well. Therefore, the formula (4.13) holds (first on an open subset of $\mathbb{B}$ and then by continuity on the whole $\mathbb{B}$, since S is assumed to be in $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$). So the solution (if exists) is unique, and this uniqueness implies in particular, that the representation (4.13) does not depend on the particular choice of $\mathbf{y} \in \operatorname{Ker} P$. □

It seems tempting to verify directly that the function S defined in (4.13) belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ and satisfies interpolation conditions (1.4). The first part can be achieved easily using the extension arguments (similar to those used in the proof of Theorem 5.2

below). Verification of interpolation equalities is much harder, so the existence part will be proven in Section 6 using the reduction method.

5. The indeterminate case

In this section we handle the case where the Pick matrix P of the problem **NP** is positive definite. By Lemma 4.5 this may occur only if none three of the interpolation nodes belong to the same 2-sphere. On the other hand, Lemma 4.6 tells us that this is the only option for the indeterminacy. We will show that in this case the problem **NP** indeed has infinitely many solutions and we will describe all solutions in terms of a linear fractional formula. Thus, assuming that P is positive definite and making use of notation (4.8), we introduce the 2×2 matrix-valued function

$$\Theta(p)=\mathbf{I}_2+(p-1)\sum_{k=0}^{\infty}p^k\begin{bmatrix}E^*\\ N^*\end{bmatrix}T^{*k}P^{-1}(\mathbf{I}_n-T)^{-1}\begin{bmatrix}E & -N\end{bmatrix} \tag{5.1}$$

which is clearly slice regular in $\mathbb{B}$.

Proposition 5.1. *Under assumptions (4.7) and (4.8) in Remark 4.3, let Θ be defined by formula* (5.1) *and let*

$$\Theta(z)=\begin{bmatrix}\Theta_{11}(p) & \Theta_{12}(p)\\ \Theta_{21}(p) & \Theta_{22}(p)\end{bmatrix}\quad \textit{and}\quad J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{5.2}$$

Then the kernel

$$K_{\Theta,J}(p,q)=\sum_{k=0}^{\infty}p^k\left(J-\Theta(p)J\Theta(q)^*\right)\bar{q}^k \tag{5.3}$$

is positive on $\mathbb{B}\times\mathbb{B}$. Furthermore, $|\Theta_{22}(p)|>1$ for every $p\in\mathbb{B}$ and the functions $\Theta_{22}^{-\star}$ and $\Theta_{22}^{-\star}\star\Theta_{21}$ are both in $\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$.

Proof. A straightforward computation relying solely on the identity (4.8) shows that

$$K_{\Theta,J}(p,q)=\left(\sum_{k=0}^{\infty}p^k\begin{bmatrix}E^*\\ N^*\end{bmatrix}T^{*k}\right)P^{-1}\left(\sum_{k=0}^{\infty}T^k\begin{bmatrix}E & N\end{bmatrix}\bar{q}^k\right) \tag{5.4}$$

from which the positivity of $K_{\Theta,J}$ follows. The bottom diagonal entry $K^{22}_{\Theta,J}$ of this kernel equals (as is easily seen from (5.2) and (5.3))

$$K^{22}_{\Theta,J}(p,q)=\sum_{k=0}^{\infty}p^k\left(-1-\Theta_{21}(p)\overline{\Theta_{21}(q)}+\Theta_{22}(p)\overline{\Theta_{22}(q)}\right)\bar{q}^k \tag{5.5}$$

and is also positive. Therefore,

$$\begin{aligned}K^{22}_{\Theta,J}(p,p)&=\sum_{k=0}^{\infty}p^k\left(-1-|\Theta_{21}(p)|^2+|\Theta_{22}(p)|^2\right)\bar{p}^k\\ &=\frac{-1-|\Theta_{21}(p)|^2+|\Theta_{22}(p)|^2}{1-|p|^2}\geq 0\end{aligned}$$

and in particular, $|\Theta_{22}(p)|>1$ for all $p\in\mathbb{B}$. Therefore, its slice regular inverse $f=\Theta_{22}^{-\star}$ is defined on $\mathbb{B}$ as well as the function $g=f\star\Theta_{21}=\Theta_{22}^{-\star}\star\Theta_{21}$. Using for now this compact notation, observe that the kernel $f\star K^{22}_{\Theta,J}\star_r \overline{f}$ is positive on $\mathbb{B}\times\mathbb{B}$ by Proposition 2.2 (part (3)). According to (5.5), this kernel equals

$$\begin{aligned}
&f(p)\star\left(\sum_{k=0}^{\infty}p^k\left(-1-\Theta_{21}(p)\overline{\Theta_{21}(q)}+\Theta_{22}(p)\overline{\Theta_{22}(q)}\right)\bar{q}^k\right)\star_r\overline{f(q)}\\
&=\sum_{k=0}^{\infty}p^kf(p)\left(-1-\Theta_{21}(p)\overline{\Theta_{21}(q)}+\Theta_{22}(p)\overline{\Theta_{22}(q)}\right)\overline{f(q)}\bar{q}^k\\
&=\sum_{k=0}^{\infty}p^k\left(1-f(p)\overline{f(q)}-g(p)\overline{g(q)}\right)\bar{q}^k\succeq 0,
\end{aligned}$$

and thus, both f and g are in $\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$.

Theorem 5.2. *Let us assume that $P>0$ and let $\Theta=\left[\begin{smallmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{smallmatrix}\right]$ be defined as in* (5.1). *Then all solutions S to the problem* **NP** *are given by the formula*

$$S=(\Theta_{11}\star\mathcal{E}+\Theta_{12})\star(\Theta_{21}\star\mathcal{E}+\Theta_{22})^{-\star}\tag{5.6}$$

with the free parameter $\mathcal{E}$ running through the class $\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$.

Proof. By Proposition 5.1, the function Θ_{22} is left $\star$–invertible and $\Theta_{22}^{-\star}\star\Theta_{21}\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$. It is seen from formula (5.1) that Θ is continuous on the closed unit ball $\overline{\mathbb{B}}$ and that $\Theta(1)=\mathbf{I}_2$. Therefore $\Theta_{21}(1)=0$, $\Theta_{22}(1)=1$ and therefore $\Theta_{22}^{-\star}\star\Theta_{21}$ is not a unimodular constant. Hence, $|\Theta_{22}^{-\star}\star\Theta_{21}(p)|<1$ by the maximum modulus principle. Therefore, $|\Theta_{22}^{-\star}\star\Theta_{21}\star\mathcal{E}(p)|<1$ for all $p\in\mathbb{B}$ and for any $\mathcal{E}\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$. Consequently, the function

$$\Theta_{21}\star\mathcal{E}+\Theta_{22}=\Theta_{22}\star\left(\Theta_{22}^{-\star}\star\Theta_{21}\star\mathcal{E}+1\right)$$

is $\star$–invertible and the formula (5.6) makes sense for every $\mathcal{E}\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$.

By Theorem 4.1, a function $S:\mathbb{B}\to\mathbb{H}$ solves the problem **NP** if and only if the kernel (4.1) is positive, which in turn is equivalent (by part (3) in Proposition 2.2) to

$$\widetilde{K}_S(p,q):=K_S(p,q)-B^S(p)P^{-1}B^S(q)^*\succeq 0\quad(p,q\in\mathbb{B}).\tag{5.7}$$

Multiplying both parts in (5.4) by $\begin{bmatrix}1 & -S\end{bmatrix}$ on the left and by its adjoint on the right and taking into account (4.9) we get

$$\begin{bmatrix}1 & -S(p)\end{bmatrix}\star K_{\Theta,J}(p,q)\star_r\begin{bmatrix}1\\ -\overline{S(q)}\end{bmatrix}=B^S(p)P^{-1}B^S(q)^*.$$

On the other hand, the kernel K_S in (3.9) can be written as

$$K_S(p,q)=\begin{bmatrix}1 & -S(p)\end{bmatrix}\star\left(\sum_{k=0}^{\infty}p^kJ\bar{q}^k\right)\star_r\begin{bmatrix}1\\ -\overline{S(q)}\end{bmatrix}.$$

Substituting the two latter representations into the right side of (5.7) and taking into account the formula (5.4) for $K_{\Theta,J}$ gives

$$\begin{aligned}\widetilde{K}_S(p,q) &= \begin{bmatrix} 1 & -S(p)\end{bmatrix} \star \left(\sum_{k=0}^{\infty} p^k \Theta(p) J \Theta(q)^* \bar{q}^k\right) \star_r \begin{bmatrix} 1 \\ -S(q)\end{bmatrix} \\ &= \sum_{k=0}^{\infty} p^k \begin{bmatrix} 1 & -S(p)\end{bmatrix} \star \Theta(p) J \Theta(q)^* \star_r \begin{bmatrix} 1 \\ -S(q)\end{bmatrix} \bar{q}^k \succeq 0. \end{aligned} \tag{5.8}$$

It remains to show that S satisfies inequality (5.8) if and only if it is of the form (5.4) for some $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$. For the "only if" direction, let us assume that (5.8) holds and let us introduce the functions

$$u = \Theta_{11} - S \star \Theta_{21} \qquad \text{and} \quad v = \Theta_{12} - S \star \Theta_{22} \tag{5.9}$$

so that $\begin{bmatrix} u & v\end{bmatrix} = \begin{bmatrix} 1 & -S\end{bmatrix} \star \Theta$. Substituting the latter equality into (5.8) and making use of the formula for J in (5.2) we get

$$\widetilde{K}_S(p,q) := \sum_{k=0}^{\infty} p^k (u(p)\overline{u(q)} - v(p)\overline{v(q)}) \bar{q}^k \succeq 0 \quad (p, q \in \mathbb{B}). \tag{5.10}$$

Since $\Theta_{11}(1) = 1$, $\Theta_{21}(1) = 0$ (by formula (5.1)) and since $|S(p)| \le 1$ for all $p \in \mathbb{B}$, it follows that

$$\limsup_{r \to 1^-} |u(r)| \ge \limsup_{r \to 1^-} (|\Theta_{11}(r)| - |S(r)| \cdot |\Theta_{21}(r)|) = 1.$$

By continuity, u is not vanishing in a real interval $[r_1, r_2]$ near 1 and therefore, by compactness, on an open set $V \subset \mathbb{B}$ containing this interval. Therefore we may introduce the function $\mathcal{E} := u^{-\star} \star v$ and rewrite (5.10) (at least for $p, q \in V$) in terms of this function as

$$u(p) \star \left(\sum_{k=0}^{\infty} p^k \left(1 - \mathcal{E}(p)\overline{\mathcal{E}(q)}\right) \bar{q}^k\right) \star_r \overline{u(q)} \succeq 0 \quad (p, q \in V).$$

The inverses $u^{-\star}$ and $u^{-\star_r}$ exist on V and we conclude by part (3) in Proposition 2.2 that

$$\sum_{k=0}^{\infty} p^k (1 - \mathcal{E}(p)\overline{\mathcal{E}(q)}) \bar{q}^k \succeq 0 \quad (p, q \in V).$$

By Remark 3.4 $\mathcal{E}$ can be extended to a function from $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, which we still denote by $\mathcal{E}$. By the uniqueness theorem, the equality

$$v = u \star \mathcal{E} \tag{5.11}$$

holds on the whole $\mathbb{B}$. Substituting equalities (5.9) into (5.11) gives

$$\Theta_{12} - S \star \Theta_{22} = (\Theta_{11} - S \star \Theta_{21}) \star \mathcal{E}$$

which can be written as

$$S \star (\Theta_{21} \star \mathcal{E} + \Theta_{22}) = \Theta_{11} \star \mathcal{E} + \Theta_{12}. \tag{5.12}$$

Since the function $\Theta_{21} \star \mathcal{E} + \Theta_{22}$ is slice invertible for any $\mathcal{E} \in \mathcal{S}$, the latter equality implies (5.6).

Conversely, if S is of the form (5.6) for some parameter $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, then equivalently, S and $\mathcal{E}$ are related as in (5.12). This means that u and v defined as in (5.9) satisfy equality (5.11). Then the formula (5.10) for $\widetilde{K}_S$ takes the form

$$\widetilde{K}_S(p,q) = \sum_{k=0}^{\infty} p^k u(p)\,(1-\mathcal{E}(p)\mathcal{E}(q)^*)\,\overline{u(q)}\bar{q}^k = u(p)\star K_{\mathcal{E}}(p,q)\star_r \overline{u(q)}$$

and is positive by Proposition 2.2 (part (3)), since $\mathcal{E}$ belongs to $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ so that $K_{\mathcal{E}}(p,q) \succeq 0$. Thus, inequality (5.8) holds which completes the proof. □

5.1. **Schwarz-Pick inequalities.** The goal of this subsection is to demonstrate that even the single-point version of Theorem 2.6 provides some non-trivial information. Let us observe that in case $n = 1$, the formulas (1.5) and (4.8) amount to $P = \frac{1-|s_1|^2}{1-|p_1|^2}$, $T = p_1$, $E = 1$, $N = s_1$ so that the formula (5.1) simplifies to

$$\Theta(p) = \mathbf{I}_2 + (p-1)\sum_{k=0}^{\infty} p^k \begin{bmatrix} 1 \\ \overline{s}_1 \end{bmatrix} \overline{p}_1^k \frac{1-|p_1|^2}{1-|s_1|^2}(1-p_1)^{-1}\begin{bmatrix} 1 & -s_1 \end{bmatrix}. \tag{5.13}$$

Upon specifying Theorem 2.6 to the single-point case and we conclude: *Given p_1, $s_1 \in \mathbb{B}$, all functions $S \in \mathcal{R}(\mathbb{B}, \mathbb{B})$ mapping p_1 to s_1 are given by the formula* (5.6) *where $\mathcal{E}$ is the parameter from $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ and where Θ is given as in* (5.13).

We next observe that for any S of the form (5.6) with Θ given by (5.13),

$$\begin{aligned}(S(p)-s_1)\star(1-\overline{s}_1\star S(p))^{-\star} =&(p-p_1)\star(1-p\overline{p}_1)^{-\star}\gamma \\ &\star(\mathcal{E}(p)-s_1)\star(1-\overline{s}_1\star\mathcal{E}(p))^{-\star}\end{aligned} \tag{5.14}$$

where we have set for short $\gamma = (1-\overline{p}_1)(1-p_1)^{-1}$. Indeed, upon substituting the linear fractional formula (5.6) for S into the left hand side of (5.14) and canceling out the factors $(\Theta_{21}\star\mathcal{E}+\Theta_{22})^{-\star}$ we get

$$\begin{aligned}(S-s_1)\star(1-\overline{s}_1\star S)^{-\star} =&(\Theta_{11}\star\mathcal{E}+\Theta_{12}-s_1\star(\Theta_{21}\star\mathcal{E}+\Theta_{22})) \\ &\star(\Theta_{21}\star\mathcal{E}+\Theta_{22}-\overline{s}_1\star(\Theta_{11}\star\mathcal{E}+\Theta_{12}))^{-\star} \\ =&\begin{bmatrix} 1 & -s_1\end{bmatrix}\star\Theta\star\begin{bmatrix}\mathcal{E}\\ 1\end{bmatrix}\star\left(\begin{bmatrix}-\overline{s}_1 & 1\end{bmatrix}\star\Theta\star\begin{bmatrix}\mathcal{E}\\ 1\end{bmatrix}\right)^{-\star}.\end{aligned} \tag{5.15}$$

Furthermore, it follows from (5.13) by direct verifications that

$$\begin{bmatrix} 1 & -s_1\end{bmatrix}\star\Theta\star\begin{bmatrix}\mathcal{E}\\ 1\end{bmatrix}(p) = (p-p_1)\star(1-p\overline{p}_1)^{-\star}\gamma\star(\mathcal{E}(p)-s_1),$$

$$\begin{bmatrix}-\overline{s}_1 & 1\end{bmatrix}\star\Theta\star\begin{bmatrix}\mathcal{E}\\ 1\end{bmatrix}(p) = 1-\overline{s}_1\star\mathcal{E}(p),$$

and substituting the two last equalities into (5.15) gives (5.14). The Schwarz-Pick lemma for slice regular functions established recently in [6] is an immediate consequence of (5.14).

Lemma 5.3. *For any $S \in \mathcal{R}(\mathbb{B}, \mathbb{B})$ and $p_1 \in \mathbb{B}$,*

$$|(S(p)-S(p_1))\star(1-\overline{S(p_1)}\star S(p))^{-\star}| \le |(p-p_1)\star(1-p\overline{p}_1)^{-\star}| \tag{5.16}$$

with equality holding if and only if S is an automorphism of $\mathbb{B}$.

Proof. The function S solves the interpolation problem with the single interpolation node p_1 and the target value $s_1 := S(p_1)$. Therefore, S is of the form (5.6) for some $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, and identity (5.14) holds with $S(p_1)$ instead of s_1. Since $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$, we have

$$|(\mathcal{E}(p) - s_1) \star (1 - \overline{s}_1 \star \mathcal{E}(p))^{-\star}| \leq 1$$

with equality if and only if $|\mathcal{E}(p)| = 1$; see [2]. Since $|\gamma| = |(1 - \overline{p}_1)(1 - p)^{-1})| = 1$, we conclude from (5.14) that inequality (5.16) holds with equality if and only if (by the maximum modulus principle) $\mathcal{E}$ is a unimodular constant function. The latter is equivalent (as it is easily seen again from (5.14)) to S be an automorphism of the unit ball. ◻

Remark 5.4. *Letting $s_1 = 0$ in formula (5.14) we get Schwarz lemma: If $S \in \mathcal{R}(\mathbb{B}, \mathbb{B})$ vanishes at $p_0 \in \mathbb{B}$, then S is equal to the Blaschke factor multiplied by some function $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$.*

6. The determinate case

Still assuming that none three of interpolation nodes belong to the same 2-sphere, we will assume in addition that the Pick matrix P of the problem has rank $d < n$. In fact, it can be shown that any $d \times d$ principal submatrix of P is positive definite. Instead of proving this result which is beyond the scope of this paper, we will permute indices (if necessary) and assume without loss of generality that the *leading* principal $d \times d$ submatrix of P is invertible. In order to keep notation from the previous section we proceed slightly differently. We extend the problem **NP** to the problem $\widetilde{\mathbf{NP}}$ by k additional conditions

$$S(p_{n+i}) = s_{n+i} \quad (i = 1, \dots, d) \tag{6.1}$$

still assuming that no three interpolation nodes from the extended set $\{p_1, \dots, p_{n+d}\}$ belong to the same 2-sphere, that the Pick matrix of the extended problem $\widetilde{\mathbf{NP}}$ (with interpolation conditions (1.4) *and* (6.1)) is positive semidefinite

$$\mathbb{P} = \begin{bmatrix} P & P_1^* \\ P_1 & P_2 \end{bmatrix} = \left[\sum_{k=0}^{\infty} p_i^k (1 - s_i \bar{s}_j) \bar{p}_j^k \right]_{i,j=1}^{n+d} \geq 0, \tag{6.2}$$

and that

$$\operatorname{rank} \mathbb{P} = \operatorname{rank} P = n. \tag{6.3}$$

Theorem 6.1. *Under assumptions* (6.2) *and* (6.3)*, the problem* $\widetilde{\mathbf{NP}}$ *has a unique solution.*

Proof. Let Θ be defined as in (5.1); the formula makes sense since P is invertible. Any solution S to the extended problem $\widetilde{\mathbf{NP}}$ (if exists) is also a solution to the problem **NP**, so that it is necessarily of the form (5.6) for some parameter $\mathcal{E} \in \mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$. The functions S and $\mathcal{E}$ are related as in (5.12) or (which is the same) as in (5.11), where u and v are defined as in (5.9). Evaluating (5.11) at $p = p_{n+i}$ implies that S of the form (5.6) satisfies the additional interpolation conditions (6.1) if and only if the corresponding parameter $\mathcal{E}$ satisfies conditions

$$v(p_{n+i}) = u \star \mathcal{E}(p_{n+i}) = u(p_{n+i}) \mathcal{E}(u(p_{n+i})^{-1} p_{n+i} u(p_{n+i})) \tag{6.4}$$

for $i=1,\dots,d$, where according to (5.9) and (6.1),

$$u_{n+i}:=u(p_{n+i})=\Theta_{11}(p_{n+i})-s_{n+i}\Theta_{21}(s_{n+i}^{-1}p_{n+i}s_{n+i}),\tag{6.5}$$
$$v_{n+i}:=v(p_{n+i})=\Theta_{12}(p_{n+i})-s_{n+i}\Theta_{22}(s_{n+i}^{-1}p_{n+i}s_{n+i}).\tag{6.6}$$

Let us assume for a moment that the numbers defined in (6.5), (6.6) are subject to relations

$$|u_{n+i}|=|u_{n+i}|\neq 0,\quad u_{n+i}^{-1}v_{n+i}=u_{n+j}^{-1}v_{n+j}=\gamma\in\partial\mathbb{B}\tag{6.7}$$

for all $i,j=1,\dots,k$. We then conclude from (6.4) that in order for S to be a solution to the extended problem $\widetilde{\mathbf{NP}}$, it is necessary and sufficient that S is of the form (5.6) for some $\mathcal{E}\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$ such that

$$\mathcal{E}(u_{n+i}^{-1}p_{n+i}u_{n+i})=\gamma\quad\text{for}\quad i=1,\dots,d.$$

Since $|\gamma|=1$, it then follows by the maximum modulus principle that a unique $\mathcal{E}\in\mathcal{R}(\mathbb{B},\overline{\mathbb{B}})$ satisfying the latter conditions is the constant function $\mathcal{E}\equiv\gamma$.

We now verify (6.7). Let, in analogy to (4.8),

$$\widetilde{T}=\begin{bmatrix}p_{n+1} & & 0\\ & \ddots & \\ 0 & & p_{n+d}\end{bmatrix},\quad \widetilde{E}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix},\quad \widetilde{N}=\begin{bmatrix}s_{n+1}\\ \vdots\\ s_{n+d}\end{bmatrix}\tag{6.8}$$

so that the block entries P_1 and P_2 in (6.2) can be alternatively defined as unique solutions to the Stein equations

$$P_1-\widetilde{T}P_1T^*=\widetilde{E}E^*-\widetilde{N}N^*,\qquad P_2-\widetilde{T}P_2\widetilde{T}^*=\widetilde{E}\widetilde{E}^*-\widetilde{N}\widetilde{N}^*.\tag{6.9}$$

Equating the i-th rows in the first of the two last equalities we conclude that the i-th row P_{1i} of P_1 satisfies

$$P_{1i}-p_{n+i}P_{1i}T^*=E^*-s_{n+i}N^*\tag{6.10}$$

and is recovered from this equality by the formula

$$P_{1i}=\sum_{k=0}^{\infty}p_{n+i}^k(E^*-s_{n+i}N^*)T^{*k}\tag{6.11}$$

Similarly, one can see from the second equality in (6.9) that the ij-entry of P_2 satisfies the linear equation

$$P_{2,ij}-p_{n+i}P_{2,ij}\overline{p}_{n+j}=1-s_{n+i}\overline{s}_{n+j}.\tag{6.12}$$

We now plug in formula (5.1) into (6.5) and (6.6) and then make use of (6.11) to get

$$\begin{aligned}u_{n+i}&=1+(p_{n+i}-1)\cdot\sum_{k=0}^{\infty}p_{n+i}^k\left(E^*-s_{n+i}N^*\right)T^{*k}P^{-1}(I-T)^{-1}E\\ &=1+(p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}E,\end{aligned}\tag{6.13}$$

$$\begin{aligned}v_{n+i}&=-s_{n+i}-(p_{n+i}-1)\cdot\sum_{k=0}^{\infty}p_{n+i}^k\left(E^*-s_{n+i}N^*\right)T^{*k}P^{-1}(I-T)^{-1}N\\ &=-s_{n+i}-(p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}N,\end{aligned}\tag{6.14}$$

so that

$$\begin{bmatrix}u_{n+i} & v_{n+i}\end{bmatrix}=\begin{bmatrix}1 & -s_{n+i}\end{bmatrix}+(p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}\begin{bmatrix}E & N\end{bmatrix}$$

We next use the two latter formulas and the formula (5.2) for J to compute

$$\begin{aligned} u_{n+i}\overline{u}_{n+j} - v_{n+i}\overline{v}_{n+j} &= \begin{bmatrix} u_{n+i} & v_{n+i} \end{bmatrix} J \begin{bmatrix} \overline{u}_{n+j} \\ \overline{v}_{n+j} \end{bmatrix} \\ &= 1 - s_{n+i}\overline{s}_{n+j} + (p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}(E - N\overline{s}_{n+j}) \\ &\quad + (E^* - s_{n+i}N^*)(I-T^*)^{-1}P^{-1}P^*_{1,j}(\overline{p}_{n+j}-1) \\ &\quad + (p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}(EE^* - NN^*) \\ &\qquad \times (I-T^*)^{-1}P^{-1}P^*_{1j}(\overline{p}_{n+j}-1). \end{aligned}$$

The latter expression can be further simplified due to (6.10) and the equality

$$P^{-1}(I-T)^{-1}(EE^* - NN^*)(I-T^*)^{-1}P^{-1} = (I-T^*)^{-1}(P^{-1} - T^*P^{-1}T)(I-T)^{-1}$$

which is a fairly straightforward consequence of the Stein identity (4.7), as follows:

$$\begin{aligned} &u_{n+i}\overline{u}_{n+j} - v_{n+i}\overline{v}_{n+j} \\ &= 1 - s_{n+i}\overline{s}_{n+j} + (p_{n+i}-1)\cdot P_{1i}P^{-1}(I-T)^{-1}(P^*_{ij} - TP^*_{1j}\overline{p}_{n+j}) \\ &\quad + (P_{1i} - p_{n+i}P_{1i}T^*)(I-T^*)^{-1}P^{-1}P^*_{1,j}(\overline{p}_{n+j}-1) \\ &\quad + (p_{n+i}-1)\cdot P_{1i}(I-T^*)^{-1}(P^{-1} - T^*P^{-1}T)(I-T)^{-1}P^*_{1j}(\overline{p}_{n+j}-1). \end{aligned}$$

Further simplification follows thanks to (6.12) and the equality

$$(P_{1i} - p_{n+i}P_{1i}T^*)(I-T^*)^{-1} = p_{n+i}P_{1i} - (p_{n+i}-1)P_{1i}(I-T^*)^{-1}.$$

We get

$$\begin{aligned} u_{n+i}\overline{u}_{n+j} - v_{n+i}\overline{v}_{n+j} =& 1 - s_{n+i}\overline{s}_{n+j} + p_{n+i}P_{1i}P^{-1}P^*_{1j}(\overline{p}_{n+j}-1) \\ &+ (p_{n+i}-1)P_{1i}P^{-1}P^*_{1j}\overline{p}_{n+j} \\ &- (p_{n+i}-1)P_{1i}P^{-1}P^*_{1j}(\overline{p}_{n+j}-1) \\ =& P_{2,ij} - p_{n+i}P_{2,ij}\overline{p}_{n+j} + p_{n+i}P_{1i}P^{-1}P^*_{1j}\overline{p}_{n+j} - P_{1i}P^{-1}P^*_{1j} \\ =& P_{2,ij} - P_{1i}P^{-1}P^*_{1j} - p_{n+i}(P_{2,ij} - P_{1i}P^{-1}P^*_{1j})\overline{p}_{n+j}. \end{aligned} \tag{6.15}$$

Due to factorization

$$\mathbb{P} = \begin{bmatrix} I_n & 0 \\ P_1P^{-1} & I_d \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P_2 - P_1P^{-1}P_1^* \end{bmatrix} \begin{bmatrix} I_n & P^{-1}P_1 \\ 0 & I_d \end{bmatrix},$$

the rank condition (6.3) implies $P_2 = P_1P^{-1}P_1^*$ or entry-wise,

$$P_{2,ij} = P_{1i}P^{-1}P_{1j} \quad \text{for} \quad i,j = 1,\ldots,d$$

which together with (6.15) implies

$$u_{n+i}\overline{u}_{n+j} = v_{n+i}\overline{v}_{n+j} \quad \text{for} \quad i,j = 1,\ldots,d. \tag{6.16}$$

Letting $i = j$ in the latter equalities gives $|u_{n+i}| = |v_{n+i}|$ for $i = 1,\ldots,d$. To show that u_{n+i} and v_{n+i} are nonzero we will argue via contradiction. Assuming that

$$u_{n+i} = v_{n+i} = 0. \tag{6.17}$$

for some $i \in \{1, \ldots, d\}$ we then get

$$\begin{aligned}
0 =& u_{n+i}E^* + v_{n+i}N^* \\
=& E^* - s_{n+i}N^* + (p_{n+i} - 1) \cdot P_{1i}P^{-1}(I - T)^{-1}(EE^* - NN^*) \\
=& P_{1i} - p_{n+i}P_{1i}T^* + (p_{n+i} - 1) \cdot P_{1i}P^{-1}(I - T)^{-1}(P - TPT^*) \\
=& \left(p_{n+i}P_{1i}P^{-1} - P_{1i}P^{-1}T\right)(I - T)^{-1}P(I - T^*).
\end{aligned} \tag{6.18}$$

We remark that the second equality in the latter computation follows from formulas (6.13), (6.14), the third equality is a consequence of relations (4.7) and (6.10) while the last equality is easily verified directly. Since the matrices P and $I - T$ are invertible (recall that P is Hermitian and T is diagonal), it follows from (6.18) that

$$p_{n+i}P_{1i}P^{-1} = P_{1i}P^{-1}T. \tag{6.19}$$

Substituting the latter equality into (6.13), (6.14) results in

$$u_{n+i}1 = 1 + P_{1i}P^{-1}T(I - T)^{-1}E - P_{1i}P^{-1}(I - T)^{-1}E = 1 - P_{1i}P^{-1}E$$

which being combined with the assumption (6.17), leads us to

$$P_{1i}P^{-1}E = 1. \tag{6.20}$$

Let $\mathbf{e}_j$ denote the i-th column in the identity matrix $\mathbf{I}_n$. Multiplying both sides in (6.18) by $\mathbf{e}_j$ on the right and making use of the diagonal structure (4.8) of T we get

$$p_{n+i}P_{1i}P^{-1}\mathbf{e}_j = P_{1i}P^{-1}\mathbf{e}_j p_j \quad \text{for} \quad j = 1, \ldots, n. \tag{6.21}$$

Therefore,

$$P_{1i}P^{-1}\mathbf{e}_j = 0, \quad \text{whenever} \quad p_j \notin [p_{n+i}]. \tag{6.22}$$

Due to the assumption that no three points from the set $\{p_1, \ldots, p_{n+d}\}$ belong to the same 2-sphere, the intersection of the set $\{p_1, \ldots, p_n\}$ with the 2-sphere $[p_{n+i}]$ is either empty or a singleton. We will show that either case leads to a contradiction.

Case 1. If $p_j \notin [p_{n+i}]$ for all $j = 1, \ldots, n$, then it follows from (6.22) that $P_{1i}P^{-1} = 0$ which contradicts to (6.20).

Case 2. Without loss of generality we assume that $p_1 \in [p_{n+i}]$ and $p_j \notin [p_{n+i}]$ for $j = 2, \ldots, n$. Therefore, equalities (6.22) hold for all $j = 2, \ldots, n$ and then we conclude from (6.20)

$$1 = P_{1i}P^{-1}E = P_{1i}P^{-1}(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n) = P_{1i}P^{-1}\mathbf{e}_1.$$

Due to this latter relation, the equality (6.21) for $j = 1$ simplifies to $p_{n+i} = p_1$ which contradicts to the assumption that all interpolation nodes are distinct.

The derived contradictions show that equalities (6.17) cannot be in force which completes the proof of the first part in (6.7). Once we know that $u_{n+1} \neq 0$, the second part in (6.7) follows from (6.16). $\square$

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