A SYMMETRY PROBLEM FOR THE INFINITY LAPLACIAN

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ABSTRACT. Aim of this paper is to prove necessary and sufficient conditions on the geometry of a domain $\Omega \subset \mathbb{R}^n$ in order that the homogeneous Dirichlet problem for the infinity-Laplace equation in Ω with constant source term admits a viscosity solution depending only on the distance from $\partial\Omega$. This problem was previously addressed and studied by Buttazzo and Kawohl in [7]. In the light of some geometrical achievements reached in our recent paper [14], we revisit the results obtained in [7] and we prove strengthened versions of them, where any regularity assumption on the domain and on the solution is removed. Our results require a delicate analysis based on viscosity methods. In particular, we need to build suitable viscosity test functions, whose construction involves a new estimate of the distance function $d_{\partial\Omega}$ near singular points.

1. INTRODUCTION

In [7], Buttazzo and Kawohl considered the following overdetermined boundary value problem for the infinity Laplacian:

$$
(1) \qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega \,. \end{cases}
$$

Here Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary, ν denotes the unit inner normal to $\partial\Omega$, and c is a positive constant. We recall that the infinity Laplacian operator Δ_∞ is defined on smooth functions by

$$
\Delta_\infty u = \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \qquad \text{for all } u \in C^2(\Omega).
$$

In the last decade, pde's involving this operator, first discovered by Aronsson in the pioneering work [1], have attracted an increasing amount of interest; without any attempt of completeness, we refer to the monograph [3] and to the representative works [2, 6, 9, 10, 22, 27, 28, 31, 36].

In view of the identity

$$
\Delta_p u = \nabla \cdot (|Du|^{p-2} Du) = (p-2)|Du|^{p-4} \left(\Delta_\infty u + \frac{|Du|^2 \, \Delta u}{p-2} \right)
$$

and of standard convergence results for viscosity solutions (see for instance [11]), if a sequence of p-harmonic functions converges locally uniformly as $p \to +\infty$, the limit function u is an infinity-harmonic function, i.e. a solution to $\Delta_\infty u = 0$. This is the reason why, with a mathematical abuse, (1) can be regarded as the limit as $p \to +\infty$ of the overdetermined

boundary value problems

$$(2) \qquad \begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega \,. \end{cases}$$

The latter have been extensively studied in the literature. The first result was proved by Serrin in the seminal paper [34] and states that, in the linear case of the Laplacian (namely when $p = 2$), problem (2) admits a solution if and only if Ω is a ball. Since then, several generalizations and related results have been proved, see for instance [4, 19, 25, 30, 23, 24]. The methods adopted in the literature to treat problem (2) are no longer exploitable when dealing with problem (1), because the infinity Laplacian operator Δ_∞ is highly degenerate. In particular, solutions can be no longer intended either in classical or in weak sense, respectively because they are not expected to be of class C^2 (cf. [20, 21]), and because Δ_∞ is not in divergence form. Thus, the notion of solution has to be understood in the sense of viscosity (the definition is recalled for convenience at the end of the Introduction). Moreover, as long as one wants to understand both the boundary conditions in (1) pointwise, one has to restrict attention to solutions which are C^1 up to the boundary.

These difficulties led to consider a simplified version of problem (1), which consists in investigating the existence of viscosity solutions to the Dirichlet problem

$$(3) \qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

within the class of functions depending only on the distance to the boundary of Ω, namely functions having the same level sets as the distance function

$$(4) \qquad d_{\partial\Omega}(x) := \min_{y \in \partial\Omega} |x - y|, \qquad x \in \overline{\Omega} \,.$$

The existence and uniqueness of a viscosity solution to problem (3) (actually of a more general version of it, allowing a constant-sign source term) has been established by Lu and Wang in [31]. The problem is then to establish for which geometries of Ω such solution turns out to depend only on $d_{\partial\Omega}$, and in particular whether this occurs only if Ω is a ball. Following [26], by *web functions* in the sequel we mean continuous functions depending only on $d_{\partial\Omega}$ (the name comes from the fact that, in case of planar polygons, level lines of the distance functions recall the pattern of a spider web). To the best of our knowledge, these functions firstly appeared in the monograph by Pólya and Szegö [32, Section 1.29]; more recently, they have found application in different variational problems, see [12, 13, 15, 16, 17, 18].

Clearly, asking that the solution of problem (3) is a web function is a more severe restriction than imposing just the constancy of its normal derivative along the boundary as in (1). However, this restriction is somehow natural, for instance because it is known that $d_{\partial\Omega}$ is the uniform limit as $p \to +\infty$ of the solution u_p to the first two eqs. in problem (2) (see [5, 29]), as well as the unique infinity ground state on Ω up to constant factors (see [36]). This latter result holds under the restriction that the cut locus and high ridge of Ω coincide. Such geometric property is precisely the same found by Buttazzo and Kawohl as a necessary and sufficient condition for the existence of a web solution to problem (3). Let us recall that the singular set $\Sigma(\Omega)$ of the distance function $d_{\partial\Omega}$ is defined as the set of points in Ω where $d_{\partial\Omega}$ is not differentiable, or equivalently the set of points $x \in \Omega$ whose projection $\pi_{\partial\Omega}(x)$ onto $\partial\Omega$ is not a singleton (being $\{\pi_{\partial\Omega}(x) := \{y \in \partial\Omega \ : \ d_{\partial\Omega}(x) =$

$|x - y|\}$). Then *cut locus* and the *high ridge* of Ω are defined respectively as

(5) $\qquad\qquad \overline{\Sigma}(\Omega) :=$ the closure of the singular set $\Sigma(\Omega)$ of $d_{\partial\Omega}$

(6) $\qquad\qquad \mathrm{M}(\Omega) :=$ the set where $d_{\partial\Omega}(x) = \rho_\Omega := \max_{\overline{\Omega}} d_{\partial\Omega}$.

Moreover, let us introduce the function ϕ_Ω which is the natural candidate to be a web solution to (3), as it can be easily seen via a one-dimensional ansatz (cf. [7, Section 2]):

(7) $\qquad\qquad \phi_\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_{\partial\Omega}(x))^{4/3} \right] , \qquad$ where $c_0 := 3^{4/3}/4$.

With this notation, the result by Buttazzo and Kawohl reads:

Theorem 1. [7, Theorem 1]
Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain, with $\partial\Omega$ of class C^2.

 (a) *Assume that $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$. Then ϕ_Ω is the unique web viscosity solution of class $C^1(\overline{\Omega})$ to problem (3).*

 (b) *Conversely, assume that problem (3) admits a web viscosity solution of class $C^1(\overline{\Omega})$. Then $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$.*

To some extent surprisingly, this result seems to indicate in particular that symmetry does not hold for the problem under study, namely that there exists some regular domain, different from a disk, where (3) admits a web viscosity solution. Actually the examples given in [7] of non-spherical domains Ω with $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ are of the form

$$\Omega_\gamma := \left\{ x \in \mathbb{R}^2 \ : \ d_\gamma(x) < r \right\},$$

where d_γ is the distance from a $C^{1,1}$-curve $\gamma : [0, L] \to \mathbb{R}^2$, with $\gamma(0) \neq \gamma(L)$.
The starting point of our investigation is the observation that, in fact, none of the domains Ω_γ can have a C^2 boundary, unless γ is a singleton and Ω_γ is a disk. More generally, the simultaneous validity of the two conditions $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ and $\partial\Omega \in C^2$, implies that Ω is a ball as soon as $\Omega \subset \mathbb{R}^2$ is simply connected, or $\Omega \subset \mathbb{R}^n$ is convex. This follows from some geometrical results we proved in a recent paper (see [14, Thm. 6 and Thm. 12]).
In this perspective, it is natural to inquire about the validity of Theorem 1 when no regularity assumptions on the domain Ω and on the solution u are made. Our results provide a complete answer to this question and can be summarized as follows:

 - Assume that Ω is an open bounded domain, satisfying $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ (and no further regularity requirement). Then the function ϕ_Ω is still the unique solution to problem (3) (see Theorem 2).

 - Assume that problem (3) admits a web viscosity solution u (which a priori has no further regularity besides continuity). Then $u = \phi_\Omega$ and there holds $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ (see Theorem 13). Consequently, if the space dimension is $n = 2$, Ω has the special form of a tubular neighborhood around a $C^{1,1}$ manifold, so that it fails in general to have radial symmetry (see Corollary 14). In spite, if one assumes that $\partial\Omega \in C^2$, then Ω must be necessarily a ball under the following additional restrictions: either Ω is convex, or $n = 2$ and Ω is simply connected (see Corollary 15).

We advertise that these results cannot be obtained by minor modifications of the arguments used in [7] to prove Theorem 1, but require a delicate analysis based on viscosity methods. In particular, the proof of Theorem 13 relies on the construction of suitable viscosity test functions and involves a result which may have an autonomous interest, that is a new estimate of the distance function $d_{\partial\Omega}$ near singular points (see Theorem 5).

It remains by now an open problem, which seems to be quite challenging, to establish whether the conclusion $\overline{\Sigma}(\Omega) = M(\Omega)$ of Theorem 13 remains valid under the weaker assumption that the overdetermined problem (1) admits a solution. A major difficulty to deal with this problem is the lackness of any information about the regularity properties of the solution u to (3) beyond the local Lipschitz regularity. In particular, the expected regularity is not higher than that expected for infinity-harmonic functions, namely $C^{1,\alpha}$ regularity. (At present, such regularity has been settled only in two space dimensions by Evans and Savin in [20]; let us also recall that infinity-harmonic functions turn out to be everywhere differentiable in arbitrary space dimensions, see [21].) The study of such regularity issues for the solution to problem (3), as well as the investigation of its possible concavity-like properties, are in our opinion interesting topics for further research.

OUTLINE OF THE PAPER. In Section 2 we deal with the sufficiency of the condition $\overline{\Sigma}(\Omega) = M(\Omega)$ for the existence of web solution to problem (3). Necessity is more delicate and it is proved in Section 4, relying on a geometric result given in Section 3. Finally in the Appendix we show how the proof of necessity can be considerably simplified if the solution is assumed a priori to be differentiable (which might happen to be not restrictive in the light of the results in [21, 20]).

NOTATION. Throughout the paper, Ω will always denote a non-empty open bounded domain of $\mathbb{R}^n$. A point $x \in \Omega$ will be called *regular* if the distance function from the boundary $d_{\partial\Omega}$ is differentiable at x, and *singular* otherwise. The *singular set* of Ω (or of $d_{\partial\Omega}$), i.e. the set of all singular points of Ω, will be denoted by $\Sigma(\Omega)$. We shall denote by $\overline{\Sigma}(\Omega)$ and $M(\Omega)$ the sets introduced respectively in (5) and (6), by ρ_Ω the maximum of $d_{\partial\Omega}$ on $\overline{\Omega}$, and by ϕ_Ω the function defined in (7). Moreover, we set

$$(8) \qquad g(t) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - t)^{4/3} \right], \qquad t \in [0, \rho_\Omega],$$

so that the function ϕ_Ω can also be rewritten as

$$\phi_\Omega(x) = g(d_{\partial\Omega}(x)), \qquad x \in \overline{\Omega}.$$

Following [11], a viscosity solution to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C^0(\Omega)$ which is both a viscosity subsolution, i.e.

$$(9) \quad -\Delta_\infty \varphi(x_0) - 1 \le 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } \varphi - u \text{ has a local minimum at } x_0,$$

and a viscosity super-solution, i.e.

$$(10) \quad -\Delta_\infty \varphi(x_0) - 1 \ge 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } \varphi - u \text{ has a local maximum at } x_0.$$

By a viscosity solution to the Dirichlet problem (3) we mean a function $u \in C^0(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and u is a viscosity solution to $-\Delta_\infty u = 1$ in Ω.

2. SUFFICIENCY OF THE CONDITION $\overline{\Sigma}(\Omega) = M(\Omega)$.

In this section we prove:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain, satisfying $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$. Then the function ϕ_Ω is the unique viscosity solution to the Dirichlet boundary value problem* (3).
Moreover, if $\partial\Omega$ is of class C^1, then the function ϕ_Ω is also the unique viscosity solution to the overdetermined boundary value problem (1), *with $c = (3\rho_\Omega)^{1/3}$.*

Remark 3. Notice that the validity of the condition $\partial\Omega$ of class C^1 is not guaranteed by the coincidence of cut locus and high ridge. A simple example of domain Ω with $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ but $\partial\Omega \notin C^1$ can be found in [14, Remark 7]. However, under the sole assumption $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$, the existence of a solution to the overdetermined boundary problem (1) can be inferred up to replacing Ω by a parallel set. More precisely, set $S := \overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ and, for $r > 0$, denote by S_r the set of points with distance from S less than r. It turns out that, for r sufficiently small, $\overline{\Sigma}(S_r) = \mathrm{M}(S_r)$ and $\partial S_r \in C^1$ (see [14, Proposition 13 and Lemma 16]). Hence, for such values of r, there exists a unique viscosity solution to the overdetermined boundary value problem (1) on S_r, given by the corresponding function ϕ_{S_r}.

For the proof of Theorem 2 we need the following simple lemma.

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain and let g be defined by* (8). *For every fixed point $x_0 \in \mathrm{M}(\Omega)$, the radial-profile function*

$$(11) \qquad\qquad v(x) := g(\rho_\Omega - |x - x_0|), \qquad x \in \Omega.$$

is a viscosity solution the equation to $-\Delta_\infty v = 1$ in Ω.

Proof. We observe that v is a classical solution to the equation $-\Delta_\infty v = 1$ on the set $\Omega \setminus \{x_0\}$, because it is therein of class C^2, with

$$-\Delta_\infty v(x) = -g''(\rho_\Omega - |x - x_0|) \left[g'(\rho_\Omega - |x - x_0|) \right]^2 = 1 \qquad \forall x \in \Omega \setminus \{x_0\}.$$

On the other hand, by the assumption $d(x_0) = \rho_\Omega$, the conditions $g'(\rho_\Omega) = 0$ and $g''(\rho_\Omega) = -\infty$ ensure respectively that any C^2 function φ touching v from above at x_0 satisfies $-\Delta_\infty \varphi(x_0) = 0 \leq 1$, and that the class of C^2 functions touching v at x_0 from below is empty. Thus v satisfies the definition of viscosity solution also at x_0. $\qquad\square$

Proof of Theorem 2.
Let us prove that the function ϕ_Ω is a viscosity solution to problem (3).
Clearly, ϕ_Ω satisfies the null Dirichlet condition on the boundary.
We have to show that ϕ_Ω satisfies the definition of viscosity solution to the pde in (3) at every point $x_0 \in \Omega$. To that aim, we distinguish the two cases $x_0 \in \overline{\Sigma}(\Omega)$ and $x_0 \notin \overline{\Sigma}(\Omega)$. For simplicity of notation, since no ambiguity may arise, in the remaining the proof we denote by $d := d_{\partial\Omega}$ the distance function from $\partial\Omega$.

Case $x_0 \in \overline{\Sigma}(\Omega)$.
We check first that ϕ_Ω satisfies (9) at x_0. We notice that, if v is defined by (11), the functions ϕ_Ω and v satisfy

$$(12) \qquad \phi_\Omega(x_0) = v(x_0) \qquad \text{and} \qquad \phi_\Omega(x) \geq v(x) \quad \forall x \in \overline{\Omega} \setminus \{x_0\}.$$

Namely, the first condition is due to the equality $d(x_0) = \rho_\Omega$ (recall that by hypothesis $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$), and the second one follows by using the monotonicity of g and the triangular inequality $d(x) \geq \rho_\Omega - |x - x_0| = d(x_0) - |x - x_0|$.

In view of (12), if φ is a C^2 function touching ϕ_Ω at x_0 from above, then φ touches also v from above. Since from Lemma 4 we know that $-\Delta_\infty v = 1$ in Ω, we infer that $-\Delta_\infty \varphi(x_0) \leq 1$, and hence that ϕ_Ω satisfies (9) at x_0.

The proof that ϕ_Ω satisfies also (10) at x_0 is straightforward. Indeed, since $\overline{\Sigma}(\Omega) = M(\Omega)$ and $g''(\rho_\Omega) = -\infty$, the class of C^2 functions touching ϕ_Ω at x_0 from below turns out to be empty.

Case $x_0 \notin \overline{\Sigma}(\Omega)$.

Let $p_0 \in \overline{\Sigma}(\Omega)$ and $q_0 \in \pi_{\partial\Omega}(x_0)$ be such that $x_0 \in]p_0, q_0[$ (the open segment joining p_0 and q_0), with $|q_0 - x_0| = d(x_0)$ and $|p_0 - q_0| = \rho_\Omega$ (where the latter equality holds true by the assumption $\overline{\Sigma}(\Omega) = M(\Omega)$). Set

$$\nu := \frac{p_0 - q_0}{|p_0 - q_0|} = \nabla d(x_0),$$

and let φ be a C^2 function touching ϕ_Ω at x_0 from above. Let us compare the two functions of one real variable defined for $t \in [0, \rho_\Omega]$ by

$$(13) \qquad h(t) := \varphi(y_0 + t\nu) \qquad \text{and} \qquad g(t) = \phi_\Omega(y_0 + t\nu).$$

Taking into account that φ touches ϕ_Ω at x_0 from above, and that ϕ_Ω is differentiable at x_0, the functions h and g satisfy:

$$h(d(x_0)) = \varphi(x_0) = \phi_\Omega(x_0) = g(d(x_0))$$
$$h'(d(x_0))\nabla d(x_0) = \nabla\varphi(x_0) = \nabla\phi_\Omega(x_0) = g'(d(x_0))\nabla d(x_0)$$
$$h(t) \geq g(t) \qquad \text{near } t = d(x_0).$$

We infer that

$$h''(d(x_0)) \geq g''(d(x_0)),$$

and in turn, recalling also that $g'(d(x_0)) = h'(d(x_0)) > 0$, that

$$(14) \qquad (h'(d(x_0))^2 h''(d(x_0)) \geq (g'(d(x_0))^2 g''(d(x_0)).$$

Now, by definition the infinity-Laplacian of φ is given by

$$\Delta_\infty \varphi(x_0) = \langle D^2\varphi(x_0)\nabla\varphi(x_0), \nabla\varphi(x_0)\rangle = h'(d(x_0))^2 h''(d(x_0)).$$

Then, by using (14) we obtain

$$-\Delta_\infty \varphi(x_0) \leq -(g'(d(x_0))^2 g''(d(x_0)) = 1,$$

where the last equality readily follows from the definition of g. Hence ϕ_Ω satisfies (9) at x_0. The proof that ϕ_Ω satisfies also (10) at x_0 is completely analogous: if φ is a C^2 function touching ϕ_Ω at x_0 from below, it is enough to compare the two functions of one real variable defined by (13), and argue in a similar way as above.

We conclude that ϕ_Ω is a viscosity solution to the PDE in (3). Since, by [31, Thm. 5], the viscosity solution to the Dirichlet boundary value problem (3) is unique, the assertion that ϕ_Ω is the unique viscosity solution follows.

Finally we observe that, if Ω is of class C^1, then the distance function d is differentiable also on $\partial\Omega$, and

$$\nabla\phi_\Omega(y) = g'(0)\nu(y) = (3\rho_\Omega)^{1/3}\nu(y) \qquad \forall\, y \in \partial\Omega.$$

Therefore, ϕ_Ω solves also the overdetermined boundary value problem (1), with the value of the constant c equal to $(3\rho_\Omega)^{1/3}$. $\qquad\square$

3. A GEOMETRIC RESULT ON THE DISTANCE FUNCTION

In this section we prove a new estimate on the distance function near singular points, which will be used as a crucial tool to construct suitable viscosity test functions for problem (3). We recall that the *Fréchet super-differential* of a function $u \in C^0(\Omega)$ at $x \in \Omega$ is defined by

$$D^+u(x) := \left\{ p \in \mathbb{R}^n \ : \ \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

If u is a locally Lipschitz function in Ω, the set $D^*u(x)$ of *reachable gradients* of u at $x \in \Omega$ is the set of vectors $p \in \mathbb{R}^n$ for which there exists a sequence $\{x_h\} \subset \Omega \setminus \{x\}$, with u differentiable at x_h, such that

$$\lim_h x_h = x \qquad \text{and} \qquad \lim_n \nabla u(x_h) = p.$$

Then the result reads:

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $x_0 \in \Sigma(\Omega)$. Then for every $p \in D^+d_{\partial\Omega}(x_0) \cap B_1(0)$ there exist a constant $K > 0$ and a unit vector $\zeta \in \mathbb{R}^n$ satisfying the following property:*

$$(15) \quad d_{\partial\Omega}(x) \leq d_{\partial\Omega}(x_0) + \langle p, x - x_0 \rangle - K \, |\langle \zeta, x - x_0 \rangle| + \frac{1}{2d_{\partial\Omega}(x_0)} |x - x_0|^2, \quad \forall x \in \Omega.$$

In particular for every $c > 0$ the inequality

$$(16) \qquad d_{\partial\Omega}(x) \leq d_{\partial\Omega}(x_0) + \langle p, x - x_0 \rangle - c \langle \zeta, x - x_0 \rangle^2 + \frac{1}{2d_{\partial\Omega}(x_0)} |x - x_0|^2$$

holds for every $x \in B_\delta(x_0) \cap \Omega$ with $\delta = K/c$. Furthermore, if $p \neq 0$ then the vector ζ can be chosen so that $\langle \zeta, p \rangle \neq 0$.

Remark 6. We shall see in a moment (cf. Proposition 9) that, since $x_0 \in \Sigma(\Omega)$, the set $D^+d_{\partial\Omega}(x_0) \cap B_1(0)$ is not empty and contains non-zero elements. Moreover, every $p \in D^+d_{\partial\Omega}(x_0) \cap B_1(0)$ can be written as a convex combination of points $p_0, \ldots p_k \in D^+d_{\partial\Omega}(x_0) \cap \partial B_1(0)$, with $k \leq n$. We shall show, during the proof of Theorem 5, that the constant K appearing in (15) can be chosen as the distance between the origin and the boundary of the set $\operatorname{conv}\{p_0 - p, \ldots, p_k - p\}$, whereas the vector ζ can be chosen in the set

$$(17) \qquad Z := \left\{ \frac{z}{|z|} \ : \ z \in \operatorname{conv}\{p_0 - p, \ldots, p_k - p\}, \ z \neq 0 \right\}.$$

Remark 7. If Ω is a set of positive reach, i.e., if there exists $R > 0$ such that every point of $\{z \in \mathbb{R}^n \setminus \overline{\Omega} : \ d_{\overline{\Omega}}(z) < R\}$ has a unique projection on $\overline{\Omega}$, then estimate (15) can be improved in the following way:

$$(18) \qquad d_{\partial\Omega}(x) \leq d_{\partial\Omega}(x_0) + \langle p, x - x_0 \rangle - K \, |\langle \zeta, x - x_0 \rangle| + \frac{1}{2(d_{\partial\Omega}(x_0) + R)} |x - x_0|^2$$

(see Step 2 in the proof of Theorem 5). In particular, if Ω is a convex set, then we get

$$(19) \qquad d_{\partial\Omega}(x) \leq d_{\partial\Omega}(x_0) + \langle p, x - x_0 \rangle - K \, |\langle \zeta, x - x_0 \rangle|, \qquad \forall x \in \Omega.$$

Analogous improvements hold also for inequality (16).

Remark 8. By writing (16) at $x = x_0 \pm h$ and summing up, one arrives at

$$d_{\partial\Omega}(x_0 + h) + d_{\partial\Omega}(x_0 - h) \leq 2d_{\partial\Omega}(x_0) - 2c \langle \zeta, h \rangle^2 + \frac{1}{d(x_0)} |h|^2 \, .$$

This shows that Theorem 5 can be seen as a refinement at singular points of the concavity estimate for the distance function given in [8, Prop. 2.2.2].

The remaining of this section is devoted to prove Theorem 5, and then to exemplify it in the cases of two simple geometries, such as a square and a triangle.

Let us recall some definitions and known results on the non-smooth analysis of the distance function. Let $d_S(x) := \min_{y \in S} |x - y|$ denote the distance from a nonempty closed set S. By regular (resp. singular) points of d_S we mean points where d_S is differentiable (resp. not differentiable). Moreover, we denote by $\pi_S(x) = \{y \in S : d_S(x) = |x - y|\}$ the projection of a point x onto S.

Proposition 9. *Let S be a nonempty closed set in $\mathbb{R}^n$ and let $x \notin S$. Then the following hold:*

    (i) *x is a regular point of d_S if and only if $\pi_S(x)$ is a singleton, and in this case there holds*

$$\nabla d_S(x) = \frac{x - \pi_S(x)}{d_S(x)} \, .$$

    (ii) *If x is a singular point of d_S, there holds*

$$(20) \qquad D^* d_S(x) = \left\{ \frac{x - y}{|x - y|} \; : \; y \in \pi_S(x) \right\}$$

$$(21) \qquad D^+ d_S(x) = \mathrm{conv}\, D^* d_S(x) = \frac{x - \mathrm{conv}(\pi_S(x))}{d_S(x)} \, .$$

    (iii) *The set $\mathrm{extr}\, D^+ d_S(x)$ of extremal points of $D^+ d_S(x)$ is given by*

$$\mathrm{extr}\, D^+ d_S(x) = D^* d_S(x) = D^+ d_S(x) \cap \partial B_1(0) \, .$$

Proof. The properties (i) and (ii) follow from Corollary 3.4.5 in [8]. To prove (iii) notice firstly that, by (20)–(21), it holds $\mathrm{extr}\, D^+ d_S(x) \subseteq D^* d_S(x) \subset \partial B_1(0)$. On the other hand, since $D^+ d_S(x) \subseteq \overline{B}_1(0)$, if $p \in D^+ d_S(x) \cap \partial B_1(0)$ then necessarily $p \in \mathrm{extr}\, D^+ d_S(x)$. $\qquad\square$

Remark 10. As a consequence of Proposition 9 (ii)-(iii), we have that, if Ω is an open subset of $\mathbb{R}^n$ and $x \in \mathrm{M}(\Omega)$, then

$$x \in \mathrm{conv} \left(\partial\Omega \cap \partial B_{\rho_\Omega}(x) \right) \, .$$

Namely, since $x \in \mathrm{M}(\Omega)$, then $0 \in D^+ d_{\partial\Omega}(x)$, so that there exist $p_0, \ldots, p_k \in D^* d_{\partial\Omega}(x)$ and numbers $\lambda_0, \ldots, \lambda_k \in [0, 1]$ with $\sum_{i=0}^{k} \lambda_i = 1$ such that $\sum_{i=0}^{k} \lambda_i p_i = 0$. On the other hand, for every i there exists $y_i \in \pi_{\partial\Omega}(x)$ such that $p_i = (x - y_i)/\rho_\Omega$, so that

$$0 = \rho_\Omega \sum_{i=0}^{k} \lambda_i p_i = \sum_{i=0}^{k} \lambda_i (x - y_i)$$

from which we conclude that $x = \sum_{i=0}^{k} \lambda_i y_i$.

We are now in a position to give:

Proof of Theorem 5. Set for brevity $d := d_{\partial\Omega}$. Since $p \in D^+d(x_0) \setminus D^*d(x_0)$ and $D^+d(x_0) = \text{conv } D^*d(x_0)$, there exist $p_0, \ldots, p_k \in D^*d(x_0)$ (with $1 \le k \le n$) and numbers $\lambda_0, \ldots, \lambda_k \in (0,1)$ with $\sum_{i=0}^k \lambda_i = 1$ such that $p = \sum_{i=0}^k \lambda_i p_i$.
We divide the remaining of the proof in three steps.

Step 1. The following inequality holds:

$$(22) \qquad d(x) \le d(x_0) + \min_{i=0,\ldots,k} \langle p_i, \, x - x_0 \rangle + \frac{1}{2d(x_0)} |x - x_0|^2 \,, \qquad \forall x \in \Omega.$$

Since the points $y_i := x_0 - d(x_0)p_i$, $i = 0, \ldots, k$, belong to $\pi_{\partial\Omega}(x_0)$, and recalling that $\sqrt{1+t} \le 1 + t/2$ for every $t \ge -1$, we have the estimate

$$d(x) \le |x - y_i| = |x - x_0 + d(x_0)p_i| = \left[|x - x_0|^2 + 2d(x_0) \langle p_i, \, x - x_0 \rangle + d(x_0)^2 \right]^{1/2}$$

$$= d(x_0) \left[1 + \frac{2}{d(x_0)} \langle p_i, \, x - x_0 \rangle + \frac{1}{d(x_0)^2} |x - x_0|^2 \right]^{1/2}$$

$$\le d(x_0) + \langle p_i, \, x - x_0 \rangle + \frac{1}{2d(x_0)} |x - x_0|^2 \,.$$

Since this inequality holds for every i, (22) follows. We observe that, if Ω is a set of positive reach (see Remark 7), then we can obtain the improved estimate

$$(23) \qquad d(x) \le d(x_0) + \min_{i=0,\ldots,k} \langle p_i, \, x - x_0 \rangle + \frac{1}{2(d(x_0) + R)} |x - x_0|^2 \,,$$

by using, in place of y_i, the points $\widetilde{y}_i := x_0 - (d(x_0) + R)p_i$ and the fact that $B_R(\widetilde{y}_i) \cap \Omega = \emptyset$. If in Steps 2 and 3 below we use the improved estimate (23) in place of (22), we can then obtain (18) instead of (15).

Step 2. Let K denote the distance between the origin and the boundary of $\text{conv}\{p_0 - p, \ldots, p_k - p\}$. Then for every unit vector ζ in the set Z defined in (17), one has

$$(24) \qquad \min_{i=0,\ldots,k} \langle p_i - p, \, x \rangle \le -K \, |\langle \zeta, \, x \rangle| \,, \qquad \forall x \in \mathbb{R}^n.$$

Since $\sum_i \lambda_i(p_i - p) = 0$ we have that the set

$$F := \text{span}\{p_0 - p, p_1 - p, \ldots, p_k - p\}$$

is a subspace of $\mathbb{R}^n$ of dimension k. Let

$$Q := \text{conv}\{p - p_0, p - p_1, \ldots, p - p_k\} \,;$$

since 0 belongs to the relative interior of the polytope Q, and since K is the distance between 0 and the boundary of Q, we clearly have $K > 0$ and $B := \overline{B}_K(0) \cap F \subseteq Q$. Hence

$$h_Q(x) := \max\{\langle q, \, x \rangle : \ q \in Q\} \ge \max\{\langle b, \, x \rangle : \ b \in B\} =: h_B(x), \qquad \forall x \in \mathbb{R}^n.$$

On the other hand, we have that

$$h_Q(x) = \max_{i=0,\ldots k} \langle p - p_i, \, x \rangle = - \min_{i=0,\ldots k} \langle p_i - p, \, x \rangle$$

whereas, if ζ is any unit vector in the set Z defined in (17), then $\pm K\zeta \in B$, so that

$$h_B(x) = \max\{\langle b, \, x \rangle : \ b \in B\} \ge K |\langle \zeta, \, x \rangle| \,.$$

Now (24) easily follows.

Step 3. Completion of the proof.

The estimate (15) is a direct consequence of (22) and (24). In order to prove (16) it is enough to observe that, given $c > 0$, the inequality $K\,|t| \geq c\,t^2$ holds for every $|t| < K/c$. $\qquad\square$
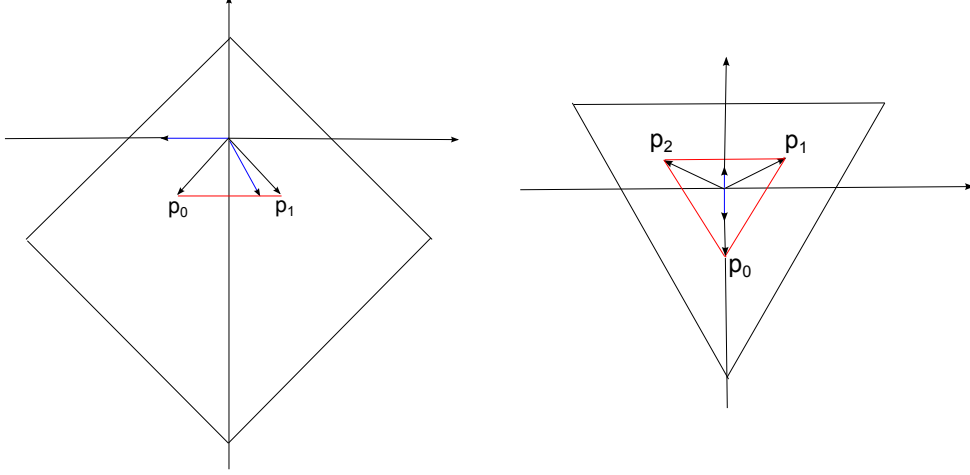

FIGURE 1. Application of Theorem 5 to a square and a triangle

Example 11. Let Q be the square with sides of length 4, having two vertices at $(0, \sqrt{2})$ and $(0, -3\sqrt{2})$. Let us show how the construction of Theorem 5 applies at the point $x_0 = (0,0) \in \Sigma(Q)$, see Figure 1 left. Adopting the same notation as in the above proof, there holds

$$D^* d_{\partial Q}(x_0) = \{p_0, p_1\}, \ \text{ with } p_0 = \Big(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big) \text{ and } p_1 = \Big(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big).$$

Choosing

$$p = \frac{1}{2}(p_0 + p_1) = \Big(0, -\frac{1}{\sqrt{2}}\Big) \in D^+ d_{\partial Q}(x_0)$$

and

$$\zeta = \frac{p_0 - p}{|p_0 - p|} = (-1, 0),$$

we have that $K = \min\{|p - p_0|, |p - p_1|\} = 1/\sqrt{2}$, and the estimate (19) (for convex Ω) takes the form

$$d_{\partial Q}(x) \leq 1 - \frac{1}{\sqrt{2}}\,x_2 - \frac{1}{\sqrt{2}}\,|x_1|.$$

A direct computation of $d_{\partial Q}$ shows that, indeed, the inequality above turns out to be an equality on the upper half–square.

Example 12. Let T be triangle with sides of length 2, having vertices at $\big(1, \frac{\sqrt{3}}{3}\big)$, $\big(-1, \frac{\sqrt{3}}{3}\big)$ and $\big(0, -2\frac{\sqrt{3}}{3}\big)$. Let us show how the construction of Theorem 5 applies at the point

$x_0 = (0,0)$; notice that in this case there holds $x_0 \in \mathrm{M}(T)$, as $d_{\partial T}(x_0) = \frac{\sqrt{3}}{3}$, see Figure 1 right. There holds

$$D^* d_{\partial T}(x_0) = \{p_0, p_1, p_2\}, \text{ with } p_0 = (0,-1), p_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ and } p_2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Choosing

$$p = (0,0) = \frac{1}{3}\big(p_0 + p_1 + p_2\big) \in D^+ d_{\partial T}(x_0), \qquad \zeta = \frac{p_0 - p}{|p_0 - p|} = (0,-1),$$

the resulting estimate (for convex sets) reads

$$d_{\partial T}(x) \leq \frac{\sqrt{3}}{3} - \frac{1}{2}\,|x_2|.$$

This estimate can be compared with the exact value of $d_{\partial T}$:

$$d_{\partial T}(x) = \frac{\sqrt{3}}{3} + \min\left\{-\frac{\sqrt{3}}{2}\,|x_1| + \frac{x_2}{2}, -x_2\right\}.$$

4. Necessity of the condition $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$

In this section we establish the converse statement of Theorem 2, which reads:

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain. Assume there exists a web viscosity solution u to the Dirichlet boundary value problem (3). Then it holds $u = \phi_\Omega$ and $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$.*

By combining Theorem 13 with the geometrical results we proved in [14, Thm. 6 and Thm. 12], one readily gets the following two corollaries.
The first one allows to view the shape of domains where problem (3) admits a web viscosity solution:

Corollary 14. *Under the same hypotheses of Theorem 13, assume in addition that $n = 2$. Then the set $S := \overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ is either a singleton or a 1-dimensional manifold of class $C^{1,1}$, and Ω is the tubular neighborhood*

$$\Omega = S_{\rho_\Omega} := \{x \in \mathbb{R}^2 \ : \ d_S(x) < \rho_\Omega\}.$$

The second one allow to establish symmetry under suitable assumptions on the space dimension and on the topology of Ω:

Corollary 15. *Under the same hypotheses of Theorem 13, assume in addition that Ω is of class C^2. If either $n = 2$ and Ω is simply connected, or n is arbitrary and Ω is convex, then Ω is a ball.*

Remark 16. If the conclusion $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$ of Theorem 13 should remain valid when one assumes merely the existence of a solution to problem (1), in view of Corollaries 14 and 15 a crucial difference would emerge between the overdetermined boundary value problems (2) for the classical Laplacian and (1) for the ∞-Laplacian: while for the former symmetry holds under very mild conditions on $\partial\Omega$ [33, 35], for the latter symmetry would be still true if Ω is simply connected and $\partial\Omega \in C^2$, but false below such threshold of regularity.

Let us give some preliminary results needed for the proof of Theorem 13.
Below, $D^\pm f(\rho - |z_0|)$ denote the Fréchet super and sub-differentials of f at $\rho - |z_0|$.

Lemma 17. *Let $f \colon [0, \rho] \to \mathbb{R}$ be a continuous function and, for $z \in B_\rho(0)$, set $v(z) := f(\rho - |z|)$. Let $z_0 \in B_\rho(0) \setminus \{0\}$ be fixed and assume that ψ is a C^1 function.*

(a) *If ψ touches v from above at z_0, then $D^+ f(\rho - |z_0|) \neq \emptyset$ and*

$$\nabla \psi(z_0) = \alpha \frac{z_0}{|z_0|}, \quad \text{with} \ \ \alpha \in -D^+ f(\rho - |z_0|).$$

(b) *If ψ touches v from below at z_0, then $D^- f(\rho - |z_0|) \neq \emptyset$ and*

$$\nabla \psi(z_0) = \alpha \frac{z_0}{|z_0|}, \quad \text{with} \ \ \alpha \in -D^- f(\rho - |z_0|).$$

Proof. We prove only statement (a), as the proof of (b) is completely analogous.
Let $\zeta_0 := z_0 / |z_0|$ and define the map $z \colon [0, \rho] \to B_\rho(0)$ by $z(s) := (\rho - s)\zeta_0$.
Setting $s_0 := \rho - |z_0|$, we have

$$\begin{aligned}
f(s) = f(\rho - |z(s)|) &= v(z(s)) \\
&\leq \psi(z(s)) = \psi(z_0) + \langle \nabla\psi(z_0), z(s) - z_0 \rangle + o(|z(s) - z_0|) \\
&= f(s_0) - (s - s_0) \langle \nabla\psi(z_0), \zeta_0 \rangle + o(|s - s_0|).
\end{aligned}$$

This shows that, setting $\alpha := \langle \nabla\psi(z_0), \zeta_0 \rangle$, it holds $-\alpha \in D^+ f(s_0)$.
Let now n_0 be a unit vector orthogonal to ζ_0, and consider an arc of circumference $\gamma(s)$, $s \in (-\epsilon, \epsilon)$, such that $\gamma(0) = z_0$, $\gamma'(0) = n_0$ and $|\gamma(s)| = |z_0|$. We have

$$\begin{aligned}
\psi(z_0) = f(\rho - |z_0|) = f(\rho - |\gamma(s)|) &= v(\gamma(s)) \\
&\leq \psi(\gamma(s)) = \psi(z_0) + \langle \nabla\psi(z_0), n_0 \rangle \, s + o(s).
\end{aligned}$$

Then it must be $\langle \nabla\psi(z_0), n_0 \rangle = 0$, completing the proof. $\qquad \square$

Proposition 18. *Let $\Omega \subset \mathbb{R}^n$ be a non-empty bounded open set, let $f \colon [0, \rho_\Omega] \to \mathbb{R}$ be a continuous function, and assume that $u(x) := f(d_{\partial\Omega}(x))$ is a web viscosity solution of*

$$(25) \qquad\qquad -\Delta_\infty u = 1 \qquad \text{in } \Omega.$$

Then:

(i) *the map $t \mapsto f(t)$ is monotone increasing on $[0, \rho_\Omega]$;*

(ii) *the function $v(z) := f(\rho_\Omega - |z|)$ is a viscosity solution of*

$$(26) \qquad\qquad -\Delta_\infty v = 1 \qquad \text{in } B_{\rho_\Omega}(0) \setminus \{0\}.$$

Proof. (i) Assume by contradiction that $t \mapsto f(t)$ is not monotone increasing on $[0, \rho_\Omega]$: let $t_1, t_2 \in [0, \rho_\Omega]$ be such that $t_1 < t_2$ but $f(t_1) > f(t_2)$. Then the absolute minimum of the continuous function f on the interval $[t_1, \rho_\Omega]$ is attained at some point $t_0 > t_1$; in particular, there exists a point $t_0 \in (0, \rho_\Omega]$ which is of local minimum for the map f. Let us show that this fact is not compatible with the assumption that $u(x) = f(d_{\partial\Omega}(x))$ is a web viscosity solution to (25). Since $t_0 > 0$, there exists a point x_0 lying in Ω such that $d_{\partial\Omega}(x_0) = t_0$. Since t_0 is a local minimum for the map f, the point x_0 is a local minimum for the function u. Then, we can construct a C^2 function φ which touches u from below at x_0, and is locally constant in a neighborhood of x_0, namely $\varphi(x) = u(x_0)$ for every $x \in B_r(x_0)$ (for some $r > 0$). Clearly it holds $-\Delta_\infty \varphi = 0 < 1$, against the fact that u is a viscosity super-solution.

(ii) Let $z_0 \in B_{\rho_\Omega}(0) \setminus \{0\}$ be fixed. Let us prove that v is a viscosity sub-solution to (26) at z_0. If ψ is a C^2 function touching v from above at z_0, we have to show that

$$(27) \qquad -\Delta_\infty \psi(z_0) = -\left\langle D^2\psi(z_0)\nabla\psi(z_0), \nabla\psi(z_0)\right\rangle \le 1.$$

We choose a maximal ray $[p_0, q_0]$, with $p_0 \in M(\Omega)$ and $q_0 \in \partial\Omega$, that is, p_0 is the center of a ball of radius $\rho_\Omega = |p_0 - q_0|$ contained into Ω. We pick a point $x_0 \in \Omega$ such that

$$x_0 \in]p_0, q_0[\qquad \text{and} \qquad d_{\partial\Omega}(x_0) = \rho_\Omega - |z_0|$$

and, for x belonging to a neighborhood of x_0, we set

$$z(x) := \left[\rho_\Omega - |x - q_0|\right]\zeta_0 , \qquad \text{with } \zeta_0 := \frac{z_0}{|z_0|} .$$

In particular, notice that by construction there holds $z(x_0) = z_0$.
We now consider the composite map

$$\varphi(x) := \psi(z(x)) ,$$

which is clearly of class C^2 in a neighborhood of x_0. We claim that φ touches u from above at x_0. Indeed, by the definitions of u, v, and z, and since ψ touches v from above at $z_0 = z(x_0)$, there holds

$$u(x_0) = f(d_{\partial\Omega}(x_0)) = f(\rho_\Omega - |z_0|) = v(z_0) = \psi(z_0) = \varphi(x_0).$$

Moreover there exists $r > 0$ such that

$$u(x) = f(d_{\partial\Omega}(x)) \le f(\rho_\Omega - |z(x)|) = v(z(x)) \le \psi(z(x)) = \varphi(x) \qquad \forall x \in B_r(x_0).$$

Notice that the first inequality in the line above follows from statement (i) already proved (taking into account that $|z(x)| \le \rho_\Omega - d_{\partial\Omega}(x)$), while the second one holds for r sufficiently small by the assumption that ψ touches v from above at z_0 and the continuity the map z at x_0.
Then, since φ touches u from above at x_0 and by assumption u is a viscosity solution to (25), we deduce that

$$(28) \qquad -\Delta_\infty \varphi(x_0) = -\left\langle D^2\varphi(x_0)\nabla\varphi(x_0), \nabla\varphi(x_0)\right\rangle \le 1.$$

Setting $\delta(x) := |x - q_0|$, a direct computation yields

$$\nabla\varphi(x) = -\left\langle \nabla\psi(z(x)), \zeta_0\right\rangle \nabla\delta(x),$$
$$D^2\varphi(x) = \left\langle D^2\psi(z(x))\,\zeta_0, \zeta_0\right\rangle \nabla\delta(x) \otimes \nabla\delta(x) - \left\langle \nabla\psi(z(x)), \zeta_0\right\rangle D^2\delta(x) .$$

Taking into account the identities

$$[\nabla\delta(x) \otimes \nabla\delta(x)]\nabla\delta(x) = \nabla\delta(x),$$
$$D^2\delta(x)\,\nabla\delta(x) = 0,$$

we obtain

$$(29) \qquad \left\langle D^2\varphi(x_0)\nabla\varphi(x_0), \nabla\varphi(x_0)\right\rangle = \left\langle D^2\psi(z_0)\,\zeta_0, \zeta_0\right\rangle \left(\left\langle\nabla\psi(z_0), \zeta_0\right\rangle\right)^2 .$$

Now, from Lemma 17 (a) we have

$$\nabla\psi(z_0) = \alpha\zeta_0, \qquad \text{with } \alpha \in -D^+ f(\rho_\Omega - |z_0|) .$$

Therefore,

$$(30) \qquad \left\langle D^2\psi(z_0)\,\zeta_0, \zeta_0\right\rangle \left(\left\langle\nabla\psi(z_0), \zeta_0\right\rangle\right)^2 = \left\langle D^2\psi(z_0)\,\nabla\psi(z_0), \nabla\psi(z_0)\right\rangle .$$

In view of (29) and (30), we conclude that (27) follows from (28).

In order to prove that v is a viscosity super-solution to (26) at z_0, one can argue in a completely analogous way. More precisely, keeping the same definitions of ζ_0, p_0, q_0, and x_0 as above, one has just to modify the auxiliary function $z(x)$ into $\tilde{z}(x) := |x - p_0|\zeta_0$, then replace the distance function $\delta(x)$ by $\tilde{\delta}(x) := |x - p_0|$, and finally apply part (b) in place of part (a) of Lemma 17. $\qquad\square$

Proof of Theorem 13. Throughout the proof, we set for brevity $d := d_{\partial\Omega}$.

Let us first prove the equality $u = \phi_\Omega$. Since by assumption u is a web function and belongs to $C^0(\overline{\Omega})$ (because a viscosity solution to (3) is by definition continuous up to the boundary), there exists a continuous function $f : [0, \rho_\Omega] \to \mathbb{R}$ such that

$$u(x) = f(d(x)) \,.$$

We have to show that f agrees with the function g defined by (8).

Since u is assumed to be a viscosity solution to the Dirichlet problem (3), by Proposition 18 the function $v(z) := f(\rho_\Omega - |z|)$ is a viscosity solution to

$$(31) \qquad \begin{cases} -\Delta_\infty v = 1 & \text{in } B_{\rho_\Omega}(0) \setminus \{0\}, \\ v = 0 & \text{on } \partial B_{\rho_\Omega}(0) \\ v(0) = f(\rho_\Omega) \,. \end{cases}$$

Let us define, for every $r > 0$, the function

$$(32) \qquad g_r(t) := c_0 \left[r^{4/3} - (r - t)^{4/3} \right], \qquad t \in [0, r].$$

We claim that there exists $r \in [\rho_\Omega, +\infty)$ such that

$$(33) \qquad g_r(\rho_\Omega) = f(\rho_\Omega) \,.$$

To prove this claim, we observe that the function

$$r \mapsto g_r(\rho_\Omega) = c_0 \left[r^{4/3} - (r - \rho_\Omega)^{4/3} \right]$$

maps the interval $[\rho_\Omega, +\infty)$ onto $[c_0 \rho_\Omega^{4/3}, +\infty)$. Thus in order to show the existence of some r such that (33) holds, it is enough to prove the inequality

$$(34) \qquad f(\rho_\Omega) \geq c_0 \rho_\Omega^{4/3} \,.$$

In turn, this inequality readily follows by a comparison principle holding for the Dirichlet problem (3). Namely, let $x_0 \in M(\Omega)$. By Lemma 4, the function $w(x) := g(\rho_\Omega - |x - x_0|)$ solves $-\Delta_\infty w = 1$ in $B_{\rho_\Omega}(x_0)$ and $w = 0$ on $\partial B_{\rho_\Omega}(x_0)$. On the other hand, the function u solves $-\Delta_\infty u = 1$ in $B_{\rho_\Omega}(x_0)$ and $u \geq 0$ on $\partial B_{\rho_\Omega}(x_0)$. The latter inequality can be deduced by applying the following result proved in [31, Thm. 3]: if $w_1, w_2 \in C(\overline{A})$ are respectively a viscosity sub- and super-solution to $-\Delta_\infty w = 1$ in A, and $w_1 \leq w_2$ on ∂A, then $w_1 \leq w_2$ in A.

Again by applying the same result, we deduce that $u(x) \geq g(\rho_\Omega - |x - x_0|)$ in $B_{\rho_\Omega}(x_0)$. This implies in particular

$$f(\rho_\Omega) = u(x_0) \geq g(\rho_\Omega) = c_0 \rho_\Omega^{4/3}$$

and concludes the proof of the claim.

Now, we have that the function

$$g_r(\rho_\Omega - |z|), \qquad z \in B_{\rho_\Omega}(0),$$

is a classical solution (and hence a viscosity solution) to problem (31). (Notice that in particular the third equation in (31) is satisfied thanks to (33)).

From [31, Theorems 1 and 5], we know that there exists a unique viscosity solution to (31). We conclude that, for some $r \geq \rho_\Omega$, it holds $v(z) = g_r(\rho_\Omega - |z|)$, that is

$$(35) \qquad f(\rho_\Omega - |z|) = g_r(\rho_\Omega - |z|),$$

or equivalently $u(x) = g_r(d(x))$.

To conclude, we have to show the following equalities:

$$u = \phi_\Omega \qquad \text{and} \qquad \overline{\Sigma}(\Omega) = M(\Omega).$$

Proof of the equality $u = \phi_\Omega$.

Since we know that $u(x) = g_r(d(x))$ for some $r \geq \rho_\Omega$, all we have to prove is that $r = \rho_\Omega$. We recall that, since $r \geq \rho_\Omega$, then $g'_r(\rho_\Omega) \geq 0$, and that $g'_r(\rho_\Omega) = 0$ if and only if $r = \rho_\Omega$. Assume by contradiction that $g'_r(\rho_\Omega) > 0$. Let $x_0 \in M(\Omega)$. Without loss of generality, assume that $x_0 = 0$. Thanks to the concavity of g_r, we have

$$(36) \qquad u(x) = g_r(d(x)) \leq u(0) + g'_r(\rho_\Omega)(d(x) - \rho_\Omega).$$

Let $p, \zeta \in \mathbb{R}^n$ be associated with the point $x_0 = 0$ according to Theorem 5, with $\langle \zeta, p \rangle \neq 0$, and set

$$\psi(x) := \langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{1}{2\rho_\Omega}|x|^2.$$

Then, by (36) and Theorem 5, it holds

$$u(x) \leq \varphi(x) := u(0) + g'_r(\rho_\Omega)\psi(x)$$

so that the function φ touches u from above. Some straightforward computations give

$$\Delta_\infty \varphi(0) = g'_r(\rho_\Omega)^3 \, \Delta_\infty \psi(0) = g'_r(\rho_\Omega)^3 \, \left(-2c \langle \zeta, p \rangle^2 + \frac{1}{\rho_\Omega}|p|^2\right)$$

Since $g'_r(\rho_\Omega) > 0$ and $\langle \zeta, p \rangle \neq 0$, it is enough to choose $c > 0$ large enough in order to have $\Delta_\infty \varphi(0) < -1$, contradiction.

Proof of the equality $\overline{\Sigma}(\Omega) = M(\Omega)$.

Since we have just proved that $u = \phi_\Omega$, we know that $u(x) = g(d(x))$, with g as in (8). Assume by contradiction that there exists $x_0 \in \Sigma(\Omega) \setminus M(\Omega)$. Without loss of generality, assume that $x_0 = 0$, and set $d_0 = d(0)$. Since we are assuming $x_0 \notin M(\Omega)$, it holds $d_0 < \rho_\Omega$, which implies $g'(d_0) > 0$. Then, we can reach a contradiction by arguing similarly as above. Namely, thanks to the concavity of g, we have

$$(37) \qquad u(x) \leq u(0) + g'(d_0)(d(x) - d_0).$$

Let $p, \zeta \in \mathbb{R}^n$ be associated with the point $x_0 = 0$ according to Theorem 5, with $\langle \zeta, p \rangle \neq 0$, and set

$$\psi(x) := \langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{1}{2d_0}|x|^2.$$

By (37) and Theorem 5, we have that

$$u(x) \leq \varphi(x) := u(0) + g'(d_0)\psi(x),$$

so that the function φ touches u from above. Moreover,

$$\Delta_\infty \varphi(0) = g'(d_0)^3 \, \Delta_\infty \psi(0) = g'(d_0)^3 \, \left(-2c \langle \zeta, p \rangle^2 + \frac{1}{d_0}|p|^2\right).$$

Since $g'(d_0) > 0$ and $\langle \zeta, p \rangle \neq 0$, it is enough to choose $c > 0$ large enough in order to have $\Delta_\infty \varphi(0) < -1$, contradiction. We have thus shown that $\Sigma(\Omega) \subseteq M(\Omega)$. Since the converse inclusion holds true for all Ω, and since $M(\Omega)$ is a closed set, we conclude that the required equality $\overline{\Sigma}(\Omega) = M(\Omega)$ holds. $\square$

5. Appendix

In this Appendix we show how the proof of Theorem 13 can be simplified under the additional assumption that the solution u is differentiable.

One can follow the proof given in Section 4 up to arriving at the equality (35). Then the conclusions $u = \phi_\Omega$ and $\overline{\Sigma}(\Omega) = M(\Omega)$ can be achieved as follows, with no need to apply Theorem 5 or construct test functions. As usual, we set for brevity $d := d_{\partial\Omega}$.

Proof of the equality $u = \phi_\Omega$.

We claim that the function f is differentiable in $(0, \rho_\Omega)$, and

$$(38) \qquad f'_-(\rho_\Omega) := \lim_{t \to \rho_\Omega^-} \frac{f(t) - f(r)}{t - r} = 0.$$

Namely, let us first show that f is differentiable at an arbitrary fixed point $t_0 \in (0, \rho_\Omega)$. Let $p_0 \in M(\Omega)$, that is, p_0 is the center of a ball of radius $\rho_\Omega = |p_0 - q_0|$ contained into Ω, with $q_0 \in \pi_{\partial\Omega}(p_0)$. We take $x_0 \in]p_0, q_0[$ such that $d(x_0) = t_0$. Setting $\nu_0 := \frac{p_0 - q_0}{\rho_\Omega}$, as $h \to 0$ there holds

$$u(x_0 + h\nu_0) - u(x_0) - h \langle \nabla u(x_0), \nu_0 \rangle = o(h),$$

which implies

$$f(t_0 + h) - f(t_0) - h \langle \nabla u(x_0), \nu_0 \rangle = o(h).$$

It remains to prove (38). Let p_0 be the center of a maximal ball as above. By Remark 10 and Carathéodory Theorem there exist points $q_0, q_1, \ldots, q_k \in (\partial\Omega \cap \partial B_{\rho_\Omega}(p_0))$, with k equal at most n, such that $p_0 \in \mathrm{conv}(\{q_0, q_1, \ldots q_k\})$, i.e.

$$p_0 = \sum_{i=0}^k \lambda_i q_i, \qquad \sum_{i=0}^k \lambda_i = 1, \quad \lambda_i > 0 \ \ \forall i = 0, \ldots, k.$$

Let us define the unit vectors

$$\nu_i := \frac{p_0 - q_i}{|p_0 - q_i|} = \frac{p_0 - q_i}{\rho_\Omega}, \qquad i = 0, \ldots, k.$$

Since u is differentiable at p_0, we have that, for every $i = 0, \ldots, k$ and $h \in (0, \rho_\Omega)$

$$u(p_0 - h\nu_i) - u(p_0) + h \langle \nabla u(p_0), \nu_i \rangle = o(h),$$

that is

$$f(\rho_\Omega - h) - f(\rho_\Omega) + h \langle \nabla u(p_0), \nu_i \rangle = o(h).$$

In turn, this equality yields

$$f'_-(\rho_\Omega) = \langle \nabla u(p_0), \nu_i \rangle, \qquad i = 0, \ldots, k.$$

Since $\sum_{i=0}^k \lambda_i = 1$ and $\sum_{i=0}^k \lambda_i \nu_i = 0$, we have

$$f'_-(\rho_\Omega) = \sum_{i=0}^k \lambda_i f'_-(\rho_\Omega) = \sum_{i=0}^k \lambda_i \langle \nabla u(p_0), \nu_i \rangle = 0,$$

which proves the claim.

In view of the equality (35) already proved, condition (38) implies that r is uniquely determined as $r = \rho_\Omega$, and the proof of the equality $u = \phi_\Omega$ is achieved.

Proof of the equality $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$.

Let $x_0 \notin \mathrm{M}(\Omega)$ be fixed. Then $d(x_0) < \rho_\Omega$, which taking into account the explicit expression (8) of the function g implies $g'(d(x_0)) > 0$. Using the equality $u(x) = g(d(x))$ already proved, the differentiability of u at x_0, and the inequality $|d(x) - d(x_0)| \leq |x - x_0|$, we see that

$$
\begin{aligned}
o(|x - x_0|) &= u(x) - u(x_0) - \langle \nabla u(x_0), x - x_0 \rangle = g(d(x)) - g(d(x_0)) - \langle \nabla u(x_0), x - x_0 \rangle \\
&= g'(d(x_0))(d(x) - d(x_0)) + o(d(x) - d(x_0)) - \langle \nabla u(x_0), x - x_0 \rangle \\
&= g'(d(x_0))(d(x) - d(x_0)) - \langle \nabla u(x_0), x - x_0 \rangle + o(|x - x_0|).
\end{aligned}
$$

This, combined with the inequality $g'(d(x_0)) > 0$ noticed above, implies that d is differentiable at x_0. We conclude that $\Sigma(\Omega) \subseteq \mathrm{M}(\Omega)$ and in turn, since $\mathrm{M}(\Omega)$ is a closed subset of $\overline{\Sigma}(\Omega)$, that $\overline{\Sigma}(\Omega) = \mathrm{M}(\Omega)$. $\qquad\square$

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