On the range of the generator of a quantum Markov semigroup

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Received 31 March 2015
Accepted 2 September 2015

1. Introduction

Let h be a complex separable Hilbert space and $\mathcal{B}(\mathsf{h})$ the von Neumann algebra of all bounded operators on h. A bounded self-adjoint derivation δ on $\mathcal{B}(\mathsf{h})$ is a linear map with the property $\delta(xy) = x\delta(y) + \delta(x)y$ and $\delta(x^*) = \delta(x)^*$ for all $x, y \in \mathcal{B}(\mathsf{h})$. Such a derivation is inner, that is, there is a bounded self-adjoint operator H on h such that $\delta(x) = \mathrm{i}[H, x]$.

In general the range of a non-invertible linear transformation like δ is topologically small (first category) and has large codimension (see Ref. 14). However, it is well-known (Theorem 1 and Corollary 1 of Ref. 20) that the range of a bounded nonzero derivation on $\mathcal{B}(\mathsf{h})$ has trivial commutant and so, in a way, it is large.

A bounded self-adjoint derivation $\delta(x) = \mathrm{i}[H, x]$ generates a norm-continuous group of automorphisms $\alpha_t(x) = \mathrm{e}^{\mathrm{i}tH} x \mathrm{e}^{-\mathrm{i}tH}$.

A norm-continuous semigroup $(\mathcal{T}_t)_{t \geq 0}$ of normal, completely positive, identity preserving maps on $\mathcal{B}(\mathsf{h})$, namely a Quantum Markov Semigroup (QMS), is a natural generalisation of a group of automorphisms. Thus, it is natural to ask whether the range of its generator also has trivial commutant. Our motivation for studying this problem is the equivalence of the irreversible (H, β)-KMS condition and irreversible (H, β)-KMS condition in infinitesimal form proposed in Ref. 2.

In this note we prove that the commutant of the range $R(\mathcal{L})'$ of the generator $\mathcal{L}$ of a norm-continuous, non-identical ($\mathcal{T}_t$ are not identity maps) QMS with a faithful normal invariant state ρ, is trivial if the fixed point algebra $\mathcal{F}(\mathcal{T})$ of $\mathcal{T}$ (see (2.1)) is atomic, namely it is generated by its minimal projections. This is the case, for instance, when $\mathcal{F}(\mathcal{T})$ is finite-dimensional.

The plan of the paper is as follows. The main result is discussed in Sec. 2 and illustrated by some examples in Sec. 3. In Sec. 4 we prove, as an application, that the irreversible (H, β)-KMS condition, and the irreversible (H, β)-KMS condition in infinitesimal form are equivalent. Open problems and further extensions are briefly discussed in Sec. 5.

2. The Commutant of the Range of a Generator

Let $\mathcal{T}$ be a QMS on the von Neumann-algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on a complex separable Hilbert space h with a faithful normal invariant state ρ. Recall that the set $\mathcal{F}(\mathcal{T})$ of fixed points of $\mathcal{T}$

$$\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = x \ \forall\, t \geq 0\} \tag{2.1}$$

is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ (see Ref. 15). Indeed, if $\mathcal{T}_t(x) = x$, then by complete positivity $\mathcal{T}_t(x^*x) \geq \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$. Moreover, $\mathrm{tr}(\rho(\mathcal{T}_t(x^*x) - x^*x)) = 0$ by the invariance of ρ, thus $\mathcal{T}_t(x^*x) = x^*x$ because ρ is faithful. This proves that $\mathcal{F}(\mathcal{T})$ is a *-subalgebra of $\mathcal{B}(\mathsf{h})$. It is obviously σ-weakly closed and so it is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$.

A projection p in $\mathcal{B}(\mathsf{h})$ is *subharmonic* if $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$ (see Ref. 9). Since we consider QMS with a faithful invariant state, subharmonic projections are also fixed points because $\mathrm{tr}(\rho(\mathcal{T}_t(p) - p)) = 0$.

We begin by considering irreducible QMS (Definition II.2 of Refs. 9 and 10). Recall that a QMS is irreducible if the only if projections p satisfying $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$ are the trivial ones $0, \mathbb{1}$.

Proposition 2.1. *Suppose that the QMS $\mathcal{T}$ is irreducible, then*

(1) $\mathcal{F}(\mathcal{T}) = \mathbb{C}\mathbb{1}$,
(2) $\mathit{ker}(\mathcal{L}_*) = \mathbb{C}\rho$,

(3) *the closure of the range of $\mathcal{L}$ with respect to the σ-weak operator topology (i.e. the weak * topology) on $\mathcal{B}(\mathsf{h})$ is*

$$\overline{R(\mathcal{L})} = \{y \in \mathcal{B}(\mathsf{h}) \,|\, \mathrm{tr}(\rho y) = 0\}. \tag{2.2}$$

In particular, $R(\mathcal{L})' = \mathbb{C}\mathbb{1}$. Moreover, if the dimension of h is strictly larger than 1, then every nonzero $u \in \mathsf{h}$ is cyclic for the range of $\mathcal{L}$.

Proof. Suppose that x is a fixed point of $\mathcal{T}$ which is not a multiple of $\mathbb{1}$. By considering its self-adjoint and anti-self-adjoint parts $(x + x^*)/2$ and $(x - x^*)/(2\mathrm{i})$, we can find a self-adjoint $x \in \mathcal{F}(\mathcal{T})$ which is not a multiple of $\mathbb{1}$. Since $\mathcal{F}(\mathcal{T})$ is a *-subalgebra of $\mathcal{B}(\mathsf{h})$, every spectral projection of x belongs to $\mathcal{F}(\mathcal{T})$ contradicting irreducibility. This proves 1).

In order to prove 2) first recall that, since the invariant state ρ is faithful, by Theorem 1.1 of Ref. 16, the linear map

$$\mathcal{E}(x) = w^* - \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{T}_s(x)ds$$

defines a normal conditional expectation onto the von Neumann subalgebra $\mathcal{F}(\mathcal{T})$ which is trivial by 1. Consequently, for any trace class operator $\eta \in \ker(\mathcal{L}_*)$ and so $\mathcal{T}_{*s}$-invariant, we have $\mathrm{tr}(\eta\mathcal{T}_s(x)) = \mathrm{tr}(\eta x)$ for all $s \geq 0$ and all $x \in \mathcal{B}(\mathsf{h})$. Thus $\mathcal{E}(x) = c(x)\mathbb{1}$ for some constant $c(x)$ and, clearly,

$$\begin{aligned} c(x)\mathrm{tr}(\eta) &= \mathrm{tr}(\eta\mathcal{E}(x)) \\ &= \lim_{t\to\infty}\left(t^{-1}\int_0^t \mathrm{tr}(\eta\mathcal{T}_s(x))ds\right) = \mathrm{tr}(\eta x). \end{aligned}$$

Suppose that $\ker(\mathcal{L}_*)$ contains an element η (not necessarily positive). By considering its self-adjoint and anti-self-adjoint parts we can assume, without loss of generality, that η is self-adjoint. Moreover, we have

$$\mathrm{tr}(\eta x) = c(x)\mathrm{tr}(\eta) = \mathrm{tr}(\rho\mathcal{E}(x))\mathrm{tr}(\eta) = \mathrm{tr}(\rho x)\mathrm{tr}(\eta)$$

for all $x \in \mathcal{B}(\mathsf{h})$. If $\mathrm{tr}(\eta) \neq 0$ this implies $\eta = (\mathrm{tr}(\eta))^{-1}\rho$ and 2) follows. If $\mathrm{tr}(\eta) = 0$, replacing η by $\eta + \rho$ and repeating this argument we find again $\eta + \rho = (\mathrm{tr}(\eta + \rho))^{-1}\rho = \rho$. Thus $\eta = 0$ and 2) still follows.

Now, the weak* closure of $R(\mathcal{L})$ (i.e. the closure with respect to the σ-weak, σ-strong and σ-strong* operator topology by Theorem 2.4.7 of Ref. 6, p. 71) is immediately characterised by (2.2) because it is the orthogonal of $\ker(\mathcal{L}_*)$ with respect to the duality $(\eta, x) = \mathrm{tr}(\eta x)$.

The commutant of $R(\mathcal{L})$ is trivial because any operator in $R(\mathcal{L})'$ commutes with all operators of the form $x - \mathrm{tr}(\rho x)\mathbb{1}$thus it commutes with all operators.

Finally, let u be a nonzero vector in h and let v be another vector orthogonal to $\{\mathcal{L}(x)u \mid x \in \mathcal{B}(\mathsf{h})\}$. The identity

$$0 = \langle v, \mathcal{L}(x)u\rangle = \mathrm{tr}(|v\rangle\langle u|\mathcal{L}(x)) = \mathrm{tr}(\mathcal{L}_*(|u\rangle\langle v|)x)$$

for all $x \in \mathcal{B}(\mathsf{h})$, implies that $\mathcal{L}_*(|u\rangle\langle v|) = 0$, and by 2), $|u\rangle\langle v|$ is a multiple of ρ. This is clearly a contradiction if $\dim \mathsf{h} > 1$ because $\ker(\mathcal{L}_*)$ consists of scalar multiples of ρ, which is faithful. □

Remark. Notice that, if $\dim \mathsf{h} = 1$, then by $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$, the range of $\mathcal{L}$ is the trivial subspace $\{0\}$. Indeed, in this case, the QMS is trivial even if irreducible according to our definition.

In order to deal with situations where fixed points are one-dimensional projections, we prove the following lemma.

Lemma 2.1. *Let $\mathcal{T}$ be a QMS and let $p = |u\rangle\langle u|$ be a one-dimensional projection in $\mathcal{F}(\mathcal{T})$. If $\mathcal{L}(|v\rangle\langle u|) = 0$ for all $v \in \mathsf{h}$, then $\mathcal{L} = 0$.*

13 **Proof.** Since $\mathcal{F}(\mathcal{T})$ is a *-subalgebra of $\mathcal{B}(\mathsf{h})$, if $\mathcal{L}(|v\rangle\langle u|) = 0$ for all $v \in \mathsf{h}$ then $|v\rangle\langle v| = |v\rangle\langle u| \cdot |v\rangle\langle u|^*$ also belongs to $\mathcal{F}(\mathcal{T})$. It follows that $\mathcal{L}$ vanishes on all one-dimensional projections. Thus, it vanishes on all self-adjoint operators and, as a consequence on the whole algebra $\mathcal{B}(\mathsf{h})$. □

A von Neumann algebra $\mathcal{A}$ is atomic if there exists a maximal family of nonzero mutually orthogonal projections $(p_j)_{j\ge 1}$ such that $\sum_j p_j = \mathbb{1}$.

Proposition 2.2. *Let $\mathcal{T}$ be a non-identical QMS on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state ρ and atomic fixed point algebra. Every self-adjoint element of $R(\mathcal{L})'$ can be written as*

$$\sum_{j\ge 1} c_j p_j, \quad c_j \in \mathbb{R}, \tag{2.3}$$

where $(p_j)_{j\ge 1}$ is a family of minimal projections in $\mathcal{F}(\mathcal{T})$. In particular, $R(\mathcal{L})'$ is contained in the fixed point algebra $\mathcal{F}(\mathcal{T})$.

Proof. Let $(p_j)_{j\ge 1}$ be a maximal family of minimal (nonzero) projections in $\mathcal{F}(\mathcal{T})$ that are mutually orthogonal, namely $p_j p_k = 0$ for $j \neq k$. Note that $\sum_{j\ge 1} p_j = \mathbb{1}$. Indeed, if this is not the case, the projection $\mathbb{1} - \sum_{j\ge 1} p_j$ is a nonzero fixed point of $\mathcal{T}$.

Let

$$\mathcal{L}(x) = G^* x + \sum_{\ell\ge 1} L_\ell^* x L_\ell + xG$$

be a Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) representation of $\mathcal{L}$ (see
30 Theorem 30.16 of Ref. 18) with $G = -\mathrm{i}H - \frac{1}{2}\sum_{\ell\ge 1} L_\ell^* L_\ell$ and $H = H^*$. Here, in

case there are no Kraus operators L_ℓ, we find $\mathcal{L}(x) = \mathrm{i}[H, x]$ namely $G = -\mathrm{i}H$ with H self-adjoint. Since the algebra $\mathcal{F}(\mathcal{T})$ is the commutant of $\{G, G^*, L_\ell, L_\ell^* \mid \ell \geq 1\}$ (see Ref. 15 and also Proposition 6.2 of Ref. 12, p. 191), each p_j commutes with all the operators L_ℓ, L_ℓ^*, G, G^*, hence

$$\mathcal{L}(p_j x) = p_j \mathcal{L}(p_j x), \quad \mathcal{L}(x p_j) = \mathcal{L}(x p_j) p_j$$

for all j and $x \in \mathcal{B}(\mathsf{h})$.

Let $\mathcal{T}^j$ be the reduced QMS on $\mathcal{B}(p_j\mathsf{h})$ defined by $\mathcal{T}_t^j(p_j x p_j) = \mathcal{T}_t(p_j x p_j)$ for all $t \geq 0$ and $x \in \mathcal{B}(\mathsf{h})$. Clearly, $p_j \rho p_j / \mathrm{tr}(p_j \rho p_j)$ is a faithful $\mathcal{T}^j$- invariant state. The QMS $\mathcal{T}^j$ on $\mathcal{B}(p_j\mathsf{h})$ is irreducible because a subharmonic projection p of $\mathcal{T}^j$, by the existence of a faithful invariant state, is a fixed point and hence it must be either 0 or p_j by minimality.

Let $y \in R(\mathcal{L})'$. Assume, without loss of generality, that y is self-adjoint and prove (2.3). For all j we have

$$y p_j \mathcal{L}(p_j x p_j) = y \mathcal{L}(p_j x p_j) = \mathcal{L}(p_j x p_j) y = \mathcal{L}(p_j x p_j) p_j y \tag{2.4}$$

and thus, left and right multiplying by p_j we find

$$(p_j y p_j) \mathcal{L}(p_j x p_j) = \mathcal{L}(p_j x p_j)(p_j y p_j).$$

It follows then, from 3) of Proposition 2.1, that $p_j y p_j$ is a scalar multiple of p_j. Moreover, right multiplying (2.4) by p_j and left multiplying by p_k for a $k \neq j$ we find

$$(p_k y p_j) \mathcal{L}(p_j x p_j) = p_k \mathcal{L}(p_j x p_j)(p_j y p_j) = 0,$$

namely $(p_k y p_j) \mathcal{L}(p_j x p_j) u = 0$ for all u. It follows that, if $\dim p_j > 1$ then $p_k y p_j = 0$ and $p_j y p_k = (p_k y p_j)^* = 0$ by Proposition 2.1. If $\dim p_j = 1$ but $\dim p_k > 1$, it suffices to exchange j and k to conclude again $p_j y p_k = (p_k y p_j)^* = 0$.

Consider now the case where both p_j and p_k are one-dimensional projections, $p_j = |u_j\rangle\langle u_j|$ and $p_k = |u_k\rangle\langle u_k|$, say, with unit vectors u_j and u_k. Recalling Lemma 2.1, there exists a $v \in \mathsf{h}$ such that $\mathcal{L}(|v\rangle\langle u_j|) \neq 0$ and, by $\mathcal{L}(|u_j\rangle\langle u_j|) = 0$, we can also choose v orthogonal to u_j. Since u_j is an eigenvector for G^* and all the operators L_ℓ^*, we have $\mathcal{L}(|v\rangle\langle u_j|) = |\tilde{v}\rangle\langle u_j|$ for some nonzero $\tilde{v}$.

Now, if y commutes with $\mathcal{L}(|v\rangle\langle u_j|)$, we have $|y\tilde{v}\rangle\langle u_j| = |\tilde{v}\rangle\langle y u_j|$. Thus u_j is an eigenvector of y, thus $p_k y p_j = 0$ for all $k \neq j$ and y has the form (2.3).

This completes the proof. $\square$

We now show that the commutant of the range on a nonzero generator is trivial.

Theorem 2.1. *Let $\mathcal{T}$ be a non-identical QMS on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state ρ, atomic fixed point algebra and let $\mathcal{L}$ be its nonzero generator. The commutant $R(\mathcal{L})'$ of the range of $\mathcal{L}$ consists of multiples of $\mathbb{1}$.*

Proof. Consider a self-adjoint $y \in R(\mathcal{L})'$ written in the form (2.3) for some minimal projections $(p_j)_{j\geq 1}$ in $\mathcal{F}(\mathcal{T})$. Clearly, each p_j belongs to $R(\mathcal{L})'$ because $R(\mathcal{L})'$ is a

*-algebra; moreover each p_j commutes with G, G^*, L_ℓ, L_ℓ^*, because it is a fixed point for $\mathcal{T}$.

We seek a proof by contradiction. Suppose that y is not a multiple of $\mathbb{1}$ so that none of the p_j is $\mathbb{1}$ or 0. For all $x \in \mathcal{B}(\mathsf{h})$ we have

$$\mathcal{L}(p_j x) = p_j \mathcal{L}(x) = \mathcal{L}(x) p_j = \mathcal{L}(x p_j),$$

namely $\mathcal{L}(p_j x - x p_j) = 0$ and $p_j x - x p_j$ belongs to $\mathcal{F}(\mathcal{T})$.

Let u_j, u_k, for $j \neq k$, be unit vectors with $p_j u_j = u_j$ and $p_k u_k = u_k$ and let $x = |u_j\rangle\langle u_k|$. The operator $|u_j\rangle\langle u_k| = p_j x - x p_j$ is a fixed point for $\mathcal{T}$ and, since $\mathcal{F}(\mathcal{T})$ is a *-algebra, so is $|u_j\rangle\langle u_j| = |u_j\rangle\langle u_k|(|u_k\rangle\langle u_j|)^*$. Thus, by minimality, $p_j = |u_j\rangle\langle u_j|$ is one-dimensional.

By Lemma 2.1, there exists a $v_j \in \mathsf{h}$ such that $\mathcal{L}(|v_j\rangle\langle u_j|) \neq 0$ and, using the same notation as in Lemma 2.1, $\mathcal{L}(|v_j\rangle\langle u_j|) = |\widetilde{v}_j\rangle\langle u_j|$ for some nonzero vector $\widetilde{v}_j \in \mathsf{h}$. This (as in the proof of Proposition 2.2) commutes with p_j, therefore

$$|p_j \widetilde{v}_j\rangle\langle u_j| = |\widetilde{v}_j\rangle\langle u_j| = |\widetilde{v}_j\rangle\langle p u_j|.$$

The above identity implies that $\widetilde{v}_j = u_j$, hence $\mathcal{L}(|u_j\rangle\langle u_j|) = |u_j\rangle\langle u_j|$ contradicting $\mathcal{L}(p_j) = \mathcal{L}(|u_j\rangle\langle u_j|) = 0$.

3. Examples

In this section we give three examples illustrating the structure of $\mathcal{F}(\mathcal{T})$, $R(\mathcal{L})$, and its commutant. The Hilbert space h is always finite-dimensional $\mathsf{h} = \mathbb{C}^d$.

Since the sum of the dimensions of $R(\mathcal{L})$ and $\ker(\mathcal{L}) = \mathcal{F}(\mathcal{T})$ is d^2; the bigger is $\mathcal{F}(\mathcal{T})$, the smaller is $R(\mathcal{L})$. Now, since $\mathcal{X} \subseteq \mathcal{Y}$ implies $\mathcal{X}' \supseteq \mathcal{Y}'$ for $\mathcal{X}, \mathcal{Y}$ sets of operators, as a consequence, the bigger is $\mathcal{F}(\mathcal{T})$, the bigger is $R(\mathcal{L})'$.

3.1. *Derivations*

Let H be a self-adjoint operator and let $\mathcal{T}$ be the QMS defined by $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH}$. The normalised trace is obviously a faithful invariant state. The generator is the derivation $\mathcal{L}(x) = \mathrm{i}[H, x]$ and the fixed point algebra $\mathcal{F}(\mathcal{T})$ is clearly the commutant of H.

Let $(e_i)_{1 \leq i \leq d}$ be an orthonormal basis of h of eigenvalues of H giving a spectral decomposition

$$H = \sum_{i=1}^{d} \epsilon_i |e_i\rangle\langle e_i|.$$

Clearly, $[H, |e_i\rangle\langle e_j|] = (\epsilon_i - \epsilon_j)|e_i\rangle\langle e_j|$, thus $|e_i\rangle\langle e_j|$ belongs to the range of $\mathcal{L}$ if $\epsilon_i \neq \epsilon_j$. Moreover, for any operator $x = \sum_{i,j} x_{ij} |e_i\rangle\langle e_j|$ such that $x_{ij} = 0$ whenever

$\epsilon_i = \epsilon_j$, we have

$$[H, \widetilde{x}\,] = x \quad \text{with } \widetilde{x} = \sum_{i,j} (\epsilon_i - \epsilon_j)^{-1} x_{ij} |e_i\rangle\langle e_j|.$$

It follows that $R(\mathcal{L})$ is the vector space of matrices with zero entries on diagonal blocks determined by eigenspaces of H.

By well-known results (see e.g., Corollary 1 of Ref. 20 also for H bounded and h infinite dimensional) the commutant $R(\mathcal{L})'$ is trivial if H is not a multiple of the identity operator. This is also easily verified because any y in this algebra commutes with all $|e_i\rangle\langle e_j|$ such that $\epsilon_i \neq \epsilon_j$ therefore both e_i and e_j are eigenvectors of y with the same eigenvalue.

3.2. *Squares of derivations*

Let p be a nontrivial projection on a finite-dimensional h and let $\mathcal{T}$ be generated by $\mathcal{L}(x) = -\frac{1}{2}[p, [p, x]]$. As in the previous example, the normalised trace is obviously a faithful invariant state and fixed point algebra $\mathcal{F}(\mathcal{T})$ is clearly the commutant of p.

Let $(e_i)_{1\leq j\leq d}$ be an orthonormal basis of eigenvectors of p. It is easy to see as in the previous example that the range of $\mathcal{L}$ contains the vector space generated by rank-one operators $|e_i\rangle\langle e_j|$ with $pe_i = e_i$ and $pe_j = 0$ or resp. $pe_i = 0$ and resp. $pe_j = e_j$. Here again its commutant is trivial.

3.3. *A circulant QMS*

Let $(e_i)_{1\leq i\leq d}$ be the canonical orthonormal basis of $\mathbb{C}^d$ and let S be the unitary operator $Se_i = e_{i+1}$, where the sum is understood modulo d. Fix $\alpha \in]0, 1[$ and consider the QMS $\mathcal{T}$ on $d \times d$ matrices generated by

$$\mathcal{L}(x) = \alpha S^* x S + (1 - \alpha) S x S^* - x. \tag{3.1}$$

This is a special case of circulant QMS studied in Ref. 5 (see also Example 7.1 of Ref. 13). The normalised trace is a faithful invariant state and the fixed point algebra $\mathcal{F}(\mathcal{T})$ is the commutant of S which is a normal operator.

The spectral decomposition of S is well known, see Ref. 17. Let $\omega_j = \mathrm{e}^{2\pi \mathrm{i} j/d}$, with $0 \leq j \leq d-1$, be the dth roots of unit and let v_j be the vectors $(1, \omega_j, \omega_j^2, \ldots, \omega_j^{d-1})$. Each ω_j^{d-1} is an eigenvalue with eigenvector v_j and we have

$$S = \sum_{j=0}^{d-1} \omega_j^{d-1}\, |v_j\rangle\langle v_j|.$$

It follows that $\mathcal{F}(\mathcal{T})$ is the d-dimensional algebra of matrices which are diagonal in the basis $(v_j)_{0\leq j\leq d-1}$.

Clearly

$$\mathcal{L}(|e_j\rangle\langle e_k|) = \alpha |e_{j-1}\rangle\langle e_{k-1}| + (1 - \alpha)|e_{j+1}\rangle\langle e_{k+1}| - |e_j\rangle\langle e_k|.$$

Thus the d-dimensional vector space $\mathcal{O}_m$ $(0 \le m \le d-1)$ generated by rank-one operators $|e_j\rangle\langle e_k|$ with $j-k=m$ modulo d is $\mathcal{L}$ invariant for all m and the action of $\mathcal{L}$ on each $\mathcal{O}_m$ is given by the circulant matrix $\alpha S + (1-\alpha)S^* - \mathbb{1}$ whose eigenvalues are $\lambda_j = \alpha\omega_j^{d-1} + (1-\alpha)\omega_j - 1$. By the known spectral properties of the circulant generator $\mathcal{L}$ (see Theorem 5 of Ref. 4) $\lambda_0 = 0$ is an $\mathcal{L}$-eigenvalue and the matrix $S^m = \sum_{1\le k\le d} |e_{k+m}\rangle\langle e_k|$ is the only associated eigenvector in $\mathcal{O}_m$. Since if $0 < j \le d-1$ we have

$$|\lambda_j|^2 = |1-\cos(2\pi \mathrm{i} j/d)|^2 + (1-2\alpha)^2|\sin(2\pi \mathrm{i} j/d)|^2 \ge |1-\cos(2\pi \mathrm{i} j/d)|^2 > 0,$$

the kernel of $\mathcal{L}$ restricted to $\mathcal{O}_m$ is the one-dimensional space generated by S^m. The same conclusion holds for $\mathcal{L}_*$ which is obtained exchanging α and $1-\alpha$. It follows that the range of $\mathcal{L}$, the orthogonal space to $\ker(\mathcal{L}_*)$, is given by matrices x such that $\mathrm{tr}(xS^m) = 0$ for all $m = 0, \ldots, d-1$.

Now one can check by elementary manipulations that $R(\mathcal{L})'$ is trivial.

Remark. We could check directly that also the commutant of the range of the generator of a generic QMS (see Refs. 1 and 7 the references therein), certain QMSs with unbounded generators (see e.g., examples in Refs. 8 and 12, and the two-photon absorption and emission process[11]) which is trivial.

4. The Irreversible (H,β)-KMS Condition

Let $\mathcal{T} = (\mathcal{T}_t)_{t\ge 0}$ be a norm continuous quantum Markov semigroup (QMS) on $\mathcal{B}(\mathsf{h})$ with faithful normal invariant state ρ and let $\mathcal{L}$ denote its generator.

The following definition has been proposed in Ref. 2 (see also Ref. 3).

Definition 4.1. Given a self-adjoint H and a function $\beta : \mathbb{R} \to \mathbb{R}_+$, the QMS $\mathcal{T}$ satisfies the irreversible (H,β)-KMS condition if

$$\mathrm{tr}(\rho x\, \mathrm{e}^{-\beta(H)H}\mathcal{T}_t(y)\mathrm{e}^{\beta(H)H}) = \mathrm{tr}(\rho\mathcal{T}_t(y)x) \tag{4.1}$$

for all $x, y \in \mathcal{B}(\mathsf{h})$ and all $t \ge 0$. It satisfies the irreversible (H,β)-KMS condition in infinitesimal form if

$$\mathrm{tr}(\rho x\, \mathrm{e}^{-\beta(H)H}\mathcal{L}(y)\mathrm{e}^{\beta(H)H}) = \mathrm{tr}(\rho\mathcal{L}(y)x) \tag{4.2}$$

for all $x, y \in \mathcal{B}(\mathsf{h})$.

Here we assume, for simplicity, that H is bounded and the function β is locally bounded.

The irreversible (H,β)-KMS condition (4.1) clearly implies the irreversible (H,β)-KMS condition in infinitesimal form (4.2) by differentiation of (4.1) at $t=0$. Moreover, as noted in Ref. 2, by a simple argument (see p. 82 of Ref. 19), if (4.2) holds, then, by the cyclic property of the trace, we have

$$\mathrm{tr}(x\mathrm{e}^{-\beta(H)H}\mathcal{L}(y)\mathrm{e}^{\beta(H)H}\rho) = \mathrm{tr}(x\rho\mathcal{L}(y)).$$

By the arbitrarity of x, we find then

$$\mathrm{e}^{-\beta(H)H}\mathcal{L}(y)\mathrm{e}^{\beta(H)H}\rho = \rho\mathcal{L}(y)$$

for all y, namely, left multiplying by $\mathrm{e}^{\beta(H)H}$,

$$\mathcal{L}(y)\mathrm{e}^{\beta(H)H}\rho = \mathrm{e}^{\beta(H)H}\rho\mathcal{L}(y). \tag{4.3}$$

If the commutant $R(\mathcal{L})'$ of the range of $\mathcal{L}$ consists of multiples of the identity operator, it follows that $\mathrm{e}^{\beta(H)H}\rho$ is a multiple of the identity operator, namely $\rho = \mathrm{e}^{-\beta(H)H}/\mathrm{tr}(\mathrm{e}^{-\beta(H)H})$.

As a consequence of Theorem 2.1 we find the following.

Theorem 4.1. *Let $\mathcal{T}$ be a non-identical QMS on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state ρ and atomic fixed point algebra and let $\mathcal{L}$ be its nonzero generator. An invariant state ρ satisfies the irreversible (H,β)-KMS condition in infinitesimal form* (4.2) *if and only if $\rho = \mathrm{e}^{-\beta(H)H}/\mathrm{tr}(\mathrm{e}^{-\beta(H)H})$, namely, it satisfies the (H,β)-KMS condition* (4.1).

Thus, the irreversible (H,β)-KMS condition in infinitesimal form (4.2) turns out to be equivalent to the irreversible (H,β)-KMS condition (4.1) for all QMS $\mathcal{T}$, except in the case where there $\mathcal{T}$ acts identically on $\mathcal{B}(\mathsf{h})$.

Now, since (4.1) holds for all t if and only if it is true for $t=0$ (replacing y by $\mathcal{T}_t(y)$), it follows that it is a property of the state ρ independent of the QMS $\mathcal{T}$. Therefore the irreversible (H,β)-KMS condition of Definition 4.1, in the form (4.1), does not single out any relevant class of QMS.

5. Open Problems

The commutant of the range of a derivation $\mathrm{i}[H,\cdot]$ (with a self-adjoint H which is not a multiple of the identity operator) is trivial by Theorem 1 and Corollary 1 of Ref. 20. In the proof of this result the existence of a faithful normal invariant state, implying that H has pure point spectrum, is not assumed. It would be desirable and it might be possible to extend our result in some directions.

(1) Drop the existence of a faithful normal invariant state. In this case $\mathcal{F}(\mathcal{T})$ might not be an algebra.
(2) Suppose that there exists faithful normal invariant state, the assumption on the fixed point algebra might be relaxed keeping into account that it is the range of a conditional expectation.
(3) Consider uniformly continuous QMSs on an arbitrary von Neumann algebra with Christensen–Evans generators.

These problems will be investigated in the future.

Acknowledgment

The financial support No. 236072 from CONACYT (México) through the program "Estancias postdoctorales 2015" is gratefully acknowledged.

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