Reflected BSDEs with nonpositive jumps, and controller-and-stopper games[☆]

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1. Introduction

Backward stochastic differential equations (BSDEs), introduced in the seminal paper by Pardoux and Peng [22], have emerged over the last years as a major topic in probability, especially through its deep connection with nonlinear PDEs and associated probabilistic numerical methods, and stochastic control in mathematical finance. A solution to a standard BSDE on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion W, is a pair of a progressively measurable process (Y, Z) satisfying:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{1.1}$$

where the generator F is a progressively measurable function, and the terminal data ξ is $\mathcal{F}_T$-measurable. In the Markovian case where $\xi(\omega) = g(W_T(\omega))$, $F(t, \omega, y, z) = f^0(W_t(\omega), y, z)$, for some continuous functions g and f^0 on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, it is well-known from [23] that BSDE (1.1) provides a Feynman–Kac formula to the semi-linear partial differential equation (PDE):

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}(D_x^2 v) + f^0(x, v, D_x v) = 0, \quad \text{on } [0, T) \times \mathbb{R}^d, \tag{1.2}$$

with terminal condition $v(T, \cdot) = g$, through the relation: $Y_t = v(t, W_t)$, $0 \leq t \leq T$. We also notice that when the function f^0 is in the form: $f^0(x, z) = \sup_{a\in A}[f(x, a) + a.z]$, for some function f on $\mathbb{R}^d \times A$, with A compact set of $\mathbb{R}^d$, then the semi-linear PDE (1.2) is the Hamilton–Jacobi–Bellman equation for a stochastic control problem, where the controller can affect only the drift of the Brownian motion: $W_t + \int_0^t \alpha_s ds$, by a progressively measurable process α valued in A, and with a running gain function f. The extension of a standard BSDE driven by a Brownian motion and an independent Poisson random measure was considered in [30,2], and is shown to be related in a Markovian framework to semi-linear integro-PDE.

The notion of reflected BSDE was introduced by El Karoui et al. [7], and consists in the addition (resp. subtraction) of a nondecreasing process to the standard BSDE (1.1) in order to keep the solution Y above (resp. below) a lower (resp. upper) obstacle, and chosen in a minimal way via the so-called Skorohod condition. Existence and uniqueness results for reflected BSDEs under general assumptions on the obstacle have been investigated in several papers, among others [9,18,26]. We also mention works by [12,8] for reflected BSDEs driven by Brownian motion and Poisson random measure. An important application of reflected BSDE is its connection to optimal stopping problems and its associated variational inequalities in the Markovian case.

The extension to fully nonlinear PDE, motivated in particular by uncertain volatility model and more generally by stochastic control problem where control can affect both drift and diffusion terms of the state process, generated important recent developments. Soner, Touzi and Zhang [29] introduced the notion of second order BSDEs (2BSDEs), whose basic idea is to require that the solution verifies the equation $\mathbb{P}^\alpha$ a.s. for every probability measure in a non dominated class of mutually singular measures. This theory is closely related to the notion of nonlinear and G-expectation of Peng [24]. Alternatively, Kharroubi and Pham [17], following [16], introduced the notion of BSDE with nonpositive jumps. The basic idea was to constrain the jumps-component solution to the BSDE driven by Brownian motion and Poisson random measure, to remain nonpositive, by adding a nondecreasing process in a minimal way. A key feature of this class of BSDEs is its formulation under a single probability measure in contrast with 2BSDEs, thus avoiding technical issues in quasi-sure analysis, and its connection with fully nonlinear HJB equation when considering a Markovian framework with a simulatable regime switching diffusion process, defined as a randomization of the controlled state process. This approach opens new perspectives for probabilistic scheme for fully nonlinear PDEs as currently investigated in [15].

In this paper, we define a class of reflected BSDEs with nonpositive jumps and upper obstacle. As in the case of doubly reflected BSDEs with lower and upper obstacles, related to Dynkin games, our BSDE formulation involves the introduction of two nondecreasing processes, one corresponding to the nonpositive jump constraint and added in a minimal way, and the other associated to the upper reflection, satisfying the Skorohod condition, and acting in the opposite direction. The first aim of this paper is to prove the existence and uniqueness of a minimal solution to reflected BSDEs with nonpositive jumps and upper obstacle. We use a double penalization approach, and the main issue is to obtain uniform estimates on both penalized nondecreasing processes associated on one hand to the nonpositive jumps constraint and on the other hand to the upper obstacle. This is achieved under some regularity assumptions on the upper obstacle. It is worth mentioning that the running order of the limits in the double penalization is crucial, in contrast with the case of upper and lower reflection. Indeed, we do not have comparison results on the jump-component solution of a BSDE, and so a priori rather few information on the sequence of nondecreasing processes associated to the jump constraint, whereas one can exploit comparison results on the Y-component of a BSDE in order to derive useful monotonicity property for the sequence of nondecreasing processes associated to the upper obstacle. Once, we get uniform estimates, we conclude by a monotonic convergence theorem for BSDEs. We also prove a dual game representation formula for the minimal solution to our BSDE, in terms of equivalent probability measures and discount processes.

The main motivation for considering such class of upper-reflected BSDEs with nonpositive jumps arises from a zero-sum stochastic differential game between a controller and a stopper: the controller can manipulate a state process X^α in $\mathbb{R}^d$ through the selection of the control α valued in A, while the stopper has the right to choose the duration of the game via a stopping time τ. The stopper would like to minimize his expected cost:

$$\mathbb{E}\Big[\int_0^\tau f(X_t^\alpha, \alpha_t)dt + g(X_\tau^\alpha)\Big], \tag{1.3}$$

over all choices of τ, while the controller plays against him by maximizing (1.3) over all choices of α. Controller-and-stopper game problem was studied in [13] when the state process X^α is a one-dimensional diffusion, in [14] by a martingale approach and in [10] by BSDE methods, but only when the drift is controlled. General existence results for optimal actions and saddle point were recently obtained in [21] in a non Markovian and non dominated framework by exploiting the theory of nonlinear expectations. We also mention the recent papers [20,19] where the authors considered 2BSDE with reflection, in connection with optimal stopping and Dynkin game under nonlinear expectation. In the Markovian case where both drift $b(X^\alpha, \alpha)$ and diffusion term $\sigma(X^\alpha, \alpha)$ of the state process X^α are controlled (hence in a non dominated framework), the recent paper [3] proved the existence of the game value, by a comparison principle for the associated Hamilton–Jacobi–Bellman Isaacs equation:

$$\max\Big[-\frac{\partial v}{\partial t} - \sup_{a\in A}\Big(b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\Big)\,;\, v - g\Big] = 0,$$
$$\text{on } [0,T)\times\mathbb{R}^d. \tag{1.4}$$

Our second main result is to connect the minimal solution to our reflected BSDE with nonpositive jumps to a general Markovian controller-and-stopper game problem through the HJB Isaacs equation (1.4). We follow the idea in [4,17] by a randomization of the state process X^α, and thus consider a regime switching forward diffusion process X with drift $b(X_t, I_t)$ and diffusion coefficient $\sigma(X_t, I_t)$, where I_t is a pure jump process associated to the Poisson random measure driving the BSDE. The minimal solution Y_t to the reflected BSDE with nonpositive jumps, with terminal data $\xi = g(X_T)$, upper obstacle $U_t = u(t, X_t)$, and generator $f(X_t, I_t, Y_t, Z_t)$, is written in this Markovian framework as: $Y_t = v(t, X_t, I_t)$ for some deterministic function v. It appears as in [17] that actually v does not depend on a in the interior of A as a consequence of the non positivity jumps constraint, and we show that v is a viscosity solution to the general HJB Isaacs equation (1.4) where the generator $f(x, a, v, \sigma^\intercal D_x v)$ may depend also on v and $D_x v$.

The rest of the paper is organized as follows. Section 2 gives a detailed formulation of reflected BSDE with nonpositive jumps and upper obstacle. Section 3 is devoted to the existence of a minimal solution to our BSDE by a double penalization approach. We derive in Section 4 a dual game representation formula for the BSDE minimal solution. Section 5 makes the connection of the minimal BSDE-solution to fully nonlinear variational inequalities of HJB Isaacs type. We conclude in Section 6 by indicating some possible extensions to our paper. Finally, in the appendix, we recall some useful comparison results for BSDE with jumps, and state a monotonic convergence theorem, which extends to the jump case the result in [26].

2. Reflected BSDE with nonpositive jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which are defined a d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$ and a Poisson random measure μ on $\mathbb{R}_+ \times A$, where A is a compact subset of $\mathbb{R}^q$, endowed with its Borel σ-field $\mathcal{B}(A)$. We assume that W and μ are independent, and μ has an intensity measure $\lambda(da)dt$ for some finite measure λ on $(A, \mathcal{B}(A))$. We set $\tilde{\mu}(dt, da) = \mu(dt, da) - \lambda(da)dt$ the compensated martingale measure associated to μ, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the completion of the natural filtration generated by W and μ.

We fix a finite time duration $T < \infty$ and we denote by $\mathcal{P}$ the σ-field of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$. Let us introduce some additional notations. We denote by:

- $\mathbf{L^p}(\mathcal{F}_t)$, $p \geq 1$, $0 \leq t \leq T$, the set of $\mathcal{F}_t$-measurable random variables X such that $\mathbb{E}|X|^p < \infty$.
- $\mathbf{S^2}$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{0\leq t\leq T}$ such that
$$\|Y\|^2_{\mathbf{S^2}} := \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big] < \infty.$$
- $\mathbf{L^p(0,T)}$, $p \geq 1$, the set of real-valued adapted processes $(\phi_t)_{0\leq t\leq T}$ such that
$$\|\phi\|^p_{\mathbf{L^p(0,T)}} := \mathbb{E}\Big[\int_0^T |\phi_t|^p dt\Big] < \infty.$$
- $\mathbf{L^p(W)}$, $p \geq 1$, the set of $\mathbb{R}^d$-valued $\mathcal{P}$-measurable processes $Z = (Z_t)_{0\leq t\leq T}$ such that
$$\|Z\|^p_{\mathbf{L^p(W)}} := \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty.$$
- $\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, the set of $\mathcal{P} \otimes \mathcal{B}(A)$-measurable maps $L: \Omega \times [0, T] \times A \to \mathbb{R}$ such that
$$\|L\|^p_{\mathbf{L^p}(\tilde{\mu})} := \mathbb{E}\Big[\Big(\int_0^T \int_A |L_t(a)|^2 \lambda(da)dt\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{L^2}(\lambda)$ the set of $\mathcal{B}(A)$-measurable maps $\ell: A \to \mathbb{R}$ such that

$$|\ell|^2_{\mathbf{L^2}(\lambda)} := \int_A |\ell(a)|^2 \lambda(da) < \infty.$$

- $\mathbf{K^2}$ the set of nondecreasing predictable processes $K = (K_t)_{0 \le t \le T} \in \mathbf{S^2}$ with $K_0 = 0$, so that

$$\|K\|^2_{\mathbf{S^2}} = \mathbb{E}|K_T|^2.$$

We are then given three objects:

1. A *terminal condition* $\xi \in \mathbf{L^2}(\mathcal{F}_T)$.
2. A *generator function* $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \to \mathbb{R}$, which is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{L^2}(\lambda))$-measurable map, satisfying:
   (i) The square integrability condition:
   $$\mathbb{E}\left[\int_0^T |F(t, 0, 0, 0)|^2 dt\right] < \infty.$$
   (ii) The uniform Lipschitz condition:
   $$|F(t, y, z, \ell) - F(t, y', z', \ell')| \le C_F\big(|y - y'| + |z - z'| + |\ell - \ell'|_{\mathbf{L^2}(\lambda)}\big),$$
   for all $t \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, and $\ell, \ell' \in \mathbf{L^2}(\lambda)$, where C_F is some positive constant.
   (iii) The monotonicity condition:
   $$F(t, y, z, \ell) - F(t, y, z, \ell') \le \int_A (\ell(a) - \ell'(a))\gamma(t, y, z, \ell, \ell', a)\lambda(da), \tag{2.1}$$
   for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, and $\ell, \ell' \in \mathbf{L^2}(\lambda)$, where $\gamma : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \times \mathbf{L^2}(\lambda) \times A \to \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{L^2}(\lambda)) \otimes \mathcal{B}(\mathbf{L^2}(\lambda)) \otimes \mathcal{B}(A)$-measurable map satisfying: $0 \le \gamma(t, y, z, \ell, \ell', a) \le C_\gamma$, for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell, \ell' \in \mathbf{L^2}(\lambda)$, and $a \in A$, for some positive constant C_γ.
3. An *upper barrier* $U \in \mathbf{S^2}$ satisfying $U_T \ge \xi$, almost surely.

Let us now consider our problem of reflected BSDE with nonpositive jumps. We say that a quintuple $(Y, Z, L, K^+, K^-) \in \mathbf{S^2} \times \mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ is a solution to the upper-reflected BSDE with nonpositive jumps with data (ξ, F, U) if the following relation holds:

$$\begin{aligned} Y_t = \xi &+ \int_t^T F(s, Y_s, Z_s, L_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \\ &- \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a)\mu(ds, da), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{2.2}$$

together with the jump constraint

$$L_t(a) \le 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \ \ a.e. \tag{2.3}$$

and the upper constraint

$$Y_t \le U_t, \quad 0 \le t \le T, \ a.s. \tag{2.4}$$

$$\int_0^T (U_{t^-} - Y_{t^-})dK_t^- = 0, \quad a.s. \tag{2.5}$$

We look for the *minimal solution* (Y, Z, L, K^+, K^-), in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ to the reflected BSDE with nonpositive jumps (2.2)–(2.5), it must hold that $Y \le \tilde{Y}$.

Remark 2.1. We have chosen to formulate the BSDE (2.2) directly in terms of the random measure μ instead of the compensated random measure $\tilde{\mu}$ since we dealt with finite intensity measure $\lambda(A) < \infty$. Of course, one can formulate equivalently the BSDE (2.2) in terms of $\tilde{\mu}$ by changing the generator F to:

$$\tilde{F}(t, y, z, \ell) = F(t, y, z, \ell) - \int_A \ell(a)\lambda(da).$$

In this case, the monotonicity condition (2.1) for $\tilde{F}$ holds with a measurable map $\tilde{\gamma}$ satisfying: $-1 \leq \tilde{\gamma}(t, y, z, \ell, \ell', a) \leq C_{\tilde{\gamma}}$, for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell, \ell' \in \mathbf{L^2}(\lambda)$, and $a \in A$, for some positive constant $C_{\tilde{\gamma}}$. This condition is consistent with the assumption required in comparison Theorem 4.2 in [27]. □

Remark 2.2 (*Uniqueness of the Minimal Solution*)**.** . Uniqueness of a minimal solution holds in the following sense: if (Y, Z, L, K^+, K^-) and $(Y, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ are minimal solutions to (2.2)–(2.5), then $Y = Y'$, $Z = Z'$, $L = L'$, and $K^+ - K^- = \tilde{K}^+ - \tilde{K}^-$. As a matter of fact, the uniqueness of the Y component is clear by definition. Then, denoting by $K := K^+ - K^-$, and $\tilde{K} := \tilde{K}^+ - \tilde{K}^-$, which are predictable finite variation processes, we have

$$\int_0^t \left[F(s, Y_s, Z_s, L_s) - F(s, Y_s, \tilde{Z}_s, \tilde{L}_s)\right]ds + K_t - \tilde{K}_t$$
$$+ \int_0^t (\tilde{Z}_s - Z_s)dW_s + \int_0^t \int_A (\tilde{L}_s(a) - L_s(a))\mu(ds, da) = 0,$$

for all $t \in [0, T]$, almost surely. The uniqueness of $Z = \tilde{Z}$ follows by identifying the Brownian part and the finite variation part, while the uniqueness of $(L, K) = (\tilde{L}, \tilde{K})$ is obtained by identifying the predictable part, and by recalling that the jumps of μ are totally inaccessible. □

The main feature in this class of BSDEs is to consider a reflection constraint on Y in addition to the nonpositive jump constraint as already studied in [16,17]. Moreover, we deal with an upper barrier U associated to a nondecreasing process K^-, which is subtracted in (2.2) from the nondecreasing process K^+ associated to the nonpositive constrained jumps. In order to ensure that the problem of getting a minimal solution to (2.2)–(2.5) is well-posed, and similarly as in [17], we make the assumption that there exists a supersolution to the BSDE with nonpositive jumps, namely:

(H0) There exists $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ satisfying the BSDE with nonpositive jumps:

$$\bar{Y}_t = \xi + \int_t^T F(s, \bar{Y}_s, \bar{Z}_s, \bar{L}_s)ds + \bar{K}_T^+ - \bar{K}_t^+$$
$$- \int_t^T \bar{Z}_s dW_s - \int_t^T \int_A \bar{L}_s(a)\mu(ds, da), \quad 0 \leq t \leq T, \ a.s. \tag{2.6}$$

and

$$\bar{L}_t(a) \leq 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \quad a.e. \tag{2.7}$$

We shall see later in the Markovian case (see Remark 5.2) how this condition **(H0)** is directly satisfied.

3. Existence and approximation by double penalization

This section is devoted to the existence of the minimal solution to (2.2)–(2.5). We use a penalization approach and introduce the doubly indexed sequence of BSDEs with jumps:

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m})ds + K_T^{n,m,+} - K_t^{n,m,+} - (K_T^{n,m,-} - K_t^{n,m,-}) \\ &\quad - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da), \end{aligned} \tag{3.1}$$

for $n, m \in \mathbb{N}$, where $K^{n,m,+}$ and $K^{n,m,-}$ are the nondecreasing continuous processes in $\mathbf{K^2}$ defined by

$$K_t^{n,m,+} = m \int_0^t \int_A (L_s^{n,m}(a))_+ \lambda(da)ds, \qquad K_t^{n,m,-} = n \int_0^t (U_s - Y_s^{n,m})_- ds.$$

Here we use the notation $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ to denote the positive and negative parts of f. Notice that this penalized BSDE can be written as

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T F_{n,m}(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m})ds - \int_t^T Z_s^{n,m} dW_s \\ &\quad - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da), \end{aligned}$$

with a generator $F_{n,m}$ given by

$$F_{n,m}(t, y, z, \ell) = F(t, y, z, \ell) + m \int_A (\ell(a))_+ \lambda(da) - n(U_t - y)_-, \quad a.s.$$

for $(t, y, z, \ell) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda)$. Observe that the generator $F_{n,m}$ satisfies the assumptions of square integrability and uniform Lipschitzianity, which ensure by Lemma 2.4 in [30] the existence and uniqueness of a solution $(Y^{n,m}, Z^{n,m}, L^{n,m}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$ to the BSDE with jumps (3.1). Notice also that $F_{n,m}$ satisfies the monotonicity condition (2.1), is increasing in m for any fixed n, and decreasing in n for any fixed m. Thus, by the comparison Theorem A.1, we deduce that $(Y^{n,m})_{n,m}$ inherits the same property:

$$Y^{n+1,m} \le Y^{n,m} \le Y^{n,m+1}, \quad \forall n, m \in \mathbb{N}. \tag{3.2}$$

We shall first fix m, and let n to infinity, and then let m to infinity (the order of the limits is important here, see Remark 3.2). The key point, as in the case of doubly reflected BSDEs related to Dynkin games, is to deal with the difference of the nondecreasing processes $K^{n,m+}$ and $K^{n,m,-}$, and the main difficulty is to prove their convergence towards respectively the nondecreasing processes K^+ and K^-, which appear in the minimal solution to the reflected BSDE with nonpositive jumps we are looking for. We have to impose some regularity conditions on the upper barrier process that will be precised later.

For fixed m, let us now consider the reflected BSDE with jumps:

$$\begin{aligned} Y_t^m &= \xi + \int_t^T F_m(s, Y_s^m, Z_s^m, L_s^m)ds - (K_T^{m,-} - K_t^{m,-}) \\ &\quad - \int_t^T Z_s^m dW_s - \int_t^T \int_A L_s^m(a)\mu(ds, da), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{3.3}$$

and

$$Y_t^m \le U_t, \quad 0 \le t \le T, \ a.s. \tag{3.4}$$

$$\int_0^T (U_{t^-} - Y_{t^-}^m) dK_t^{m,-} = 0, \quad a.s. \tag{3.5}$$

where

$$F_m(t, y, z, \ell) = F(t, y, z, \ell) + m \int_A (\ell(a))_+ \lambda(da), \quad a.s. \tag{3.6}$$

for $(t, y, z, \ell) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda)$. We know from Theorem 4.2 in [12] that there exists a unique solution $(Y^m, Z^m, L^m, K^{m,-}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ to the reflected BSDE with jumps (3.3)–(3.5).

Remark 3.1. Note that in [12] the existence of $(Y^m, Z^m, L^m, K^{m,-})$ is proved using a fixed point argument and not through the penalized sequence $(Y^{n,m}, Z^{n,m}, L^{n,m})$, except for the particular case where the generator $F_{n,m}(t, \omega)$ does not depend on y, z, ℓ, see Theorem 4.1 and Remark 4.1(i) in [12]. The reason is that in [12] the authors do not impose any monotonicity condition on the generator F and therefore they do not have at disposal a comparison theorem for BSDEs with jumps. Nevertheless, under our monotonicity condition (2.1) and by means of the comparison Theorem A.1, the existence of $(Y^m, Z^m, L^m, K^{m,-})$ can be proved via the penalized sequence $(Y^{n,m}, Z^{n,m}, L^{n,m})$. This program is carried out in [8], Theorem 5.1, even though under the additional hypothesis that the barrier U is a $\mathcal{P}$-measurable process. More precisely, it can be shown that Y^m is obtained as the decreasing limit of $Y^{n,m}$ when n goes to infinity:

$$Y_t^m = \lim_{n\to\infty} \downarrow Y_t^{n,m}, \quad 0 \le t \le T, a.s.$$

and this convergence also holds in $\mathbf{L^2(0, T)}$. Furthermore, $(Z^{n,m}, L^{n,m})$ converges weakly to (Z^m, L^m) in $\mathbf{L^2}(W) \times \mathbf{L^2}(\tilde{\mu})$, and we have the strong convergence

$$(Z^{n,m}, L^{n,m}) \to (Z^m, L^m) \quad \text{in } \mathbf{L^p(W)} \times \mathbf{L^p}(\tilde{\mu}), \text{ as } n \to \infty,$$

for any $p \in [1, 2)$, while

$$K_t^{n,m,-} \rightharpoonup K_t^{m,-} \quad \text{weakly in } \mathbf{L^2}(\mathcal{F}_t), \text{ as } n \to \infty$$

for all $0 \le t \le T$. □

We first derive the following important property on the sequence of nondecreasing processes $(K^{m,-})$.

Lemma 3.1. *The sequence of processes* $(K^{m,-})_m$ *satisfies:*

$$K_t^{m,-} - K_s^{m,-} \le K_t^{m+1,-} - K_s^{m+1,-}, \quad 0 \le s \le t \le T, \ a.s., \forall m \in \mathbb{N}. \tag{3.7}$$

Proof. By definition of $K^{n,m,-}$, and from (3.2), we clearly have for all $n, m \in \mathbb{N}$:

$$K_t^{n,m,-} - K_s^{n,m,-} \le K_t^{n,m+1,-} - K_s^{n,m+1,-}, \quad 0 \le s \le t \le T, \ a.s.$$

Thus, by passing to the (weak) limit as n goes to infinity, we get the required result. □

By (3.2), we see that $(Y^m)_m$ is a nondecreasing sequence: $Y^m \leq Y^{m+1}$, and we denote:

$$\underline{Y}_t := Y_t^0, \quad 0 \leq t \leq T,$$

which thus provides a lower bound for the sequences (Y^m) and $(Y^{n,m})$:

$$\underline{Y}_t \leq Y_t^m \leq Y_t^{n,m}, \quad 0 \leq t \leq T, \ \forall n, m \in \mathbb{N}. \tag{3.8}$$

Moreover, under condition **(H0)**, we observe that the quintuple $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+, \bar{K}^-)$ satisfies $\int_A (\bar{L}_t(a))_+ \lambda(da) = 0 \; dt \otimes d\mathbb{P}$ a.e. so that

$$F_{n,m}(t, \bar{Y}_t, \bar{Z}_t, \bar{L}_t) \leq F(\bar{Y}_t, \bar{Z}_t, \bar{L}_t), \qquad dt \otimes d\mathbb{P} \quad a.e.$$

By the comparison Theorem A.1, we then get an upper bound for the sequences (Y^m) and $(Y^{n,m})$:

$$Y_t^m \leq Y_t^{n,m} \leq \bar{Y}_t, \quad 0 \leq t \leq T, \forall n, m \in \mathbb{N}. \tag{3.9}$$

By standard arguments, we now state some estimates on the doubly indexed sequence $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+})$ expressed in terms of $(K^{n,m,-})$.

Lemma 3.2. *Let assumption* **(H0)** *hold. Then there exists a positive constant C, such that for all $n, m \in \mathbb{N}$,*

$$\begin{aligned} &\|Y^{n,m}\|^2_{\mathbf{S^2}} + \|Z^{n,m}\|^2_{\mathbf{L^2(W)}} + \|L^{n,m}\|^2_{\mathbf{L^2}(\tilde{\mu})} + \|K^{n,m,+}\|^2_{\mathbf{S^2}} \\ &\quad \leq C\left(\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}} + \|K^{n,m,-}\|^2_{\mathbf{S^2}} \right). \end{aligned} \tag{3.10}$$

Proof. In what follows we shall denote by $C > 0$ a generic positive constant depending only on T, $\lambda(A)$, and the Lipschitz constant of F, which may vary from line to line. Proceeding as in the proof of Lemma 3.3 in [17], we apply Itô's formula to $|Y_s^{n,m}|^2$ between t and T, and get after some rearrangement:

$$\begin{aligned} &\mathbb{E}|Y_t^{n,m}|^2 + \|Z^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \|L^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2}(\tilde{\mu})} \\ &\quad = \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T Y_s^{n,m} F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds - 2\mathbb{E}\int_t^T \int_A Y_{s^-}^{n,m} L_s^{n,m}(a)\lambda(da)ds \\ &\qquad + 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,+} - 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,-}. \end{aligned} \tag{3.11}$$

By the linear growth condition on F, the inequality $ab \leq a^2/2 + b^2/2$, and recalling that $\lambda(A) < \infty$, we get

$$\begin{aligned} &2\mathbb{E}\int_t^T Y_s^{n,m} F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds - 2\mathbb{E}\int_t^T \int_A Y_{s^-}^{n,m} L_s^{n,m}(a)\lambda(da)ds \\ &\quad \leq C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \frac{1}{2}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds \\ &\qquad + \frac{1}{2}\|Z^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \frac{1}{2}\|L^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2}(\tilde{\mu})}. \end{aligned} \tag{3.12}$$

From the bounds (3.8)–(3.9) on $Y^{n,m}$: $\underline{Y} \leq Y^{n,m} \leq \bar{Y}$, and thanks to the inequality $2ab \leq a^2/\alpha + \alpha b^2$ for any constant $\alpha > 0$, we have

$$
\begin{aligned}
&2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,+} - 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,-} \\
&\quad \leq \frac{1}{\alpha}\Big(\|\underline{Y}\|_{\mathbf{S^2}}^2 + \|\bar{Y}\|_{\mathbf{S^2}}^2\Big) + \alpha\mathbb{E}|K_T^{n,m,+} - K_t^{n,m,+}|^2 + \alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \\
&\quad \leq \frac{1}{\alpha}\Big(\|\underline{Y}\|_{\mathbf{S^2}}^2 + \|\bar{Y}\|_{\mathbf{S^2}}^2\Big) + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 + 2\alpha\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2,
\end{aligned}
$$

where we set $K_t^{n,m} := K_t^{n,m,+} - K_t^{n,m,-}$, so that $\mathbb{E}|K_T^{n,m,+} - K_t^{n,m,+}|^2 \leq 2\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2 + 2\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2$. Together with (3.12) and (3.11), this yields:

$$
\begin{aligned}
&\mathbb{E}|Y_t^{n,m}|^2 + \frac{1}{2}\|Z^{n,m}1_{[t,T]}\|_{\mathbf{L^2(W)}}^2 + \frac{1}{2}\|L^{n,m}1_{[t,T]}\|_{\mathbf{L^2}(\tilde{\mu})}^2 \\
&\quad \leq C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \mathbb{E}|\xi|^2 + \frac{1}{2}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \frac{1}{\alpha}\Big(\|\underline{Y}\|_{\mathbf{S^2}}^2 + \|\bar{Y}\|_{\mathbf{S^2}}^2\Big) \\
&\qquad + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 + 2\alpha\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2. \qquad (3.13)
\end{aligned}
$$

Now, from the relation (3.1), we have

$$
\begin{aligned}
K_T^{n,m} - K_t^{n,m} &= Y_t^{n,m} - \xi - \int_t^T F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds \\
&\qquad + \int_t^T Z_s^{n,m} dW_s + \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da),
\end{aligned}
$$

so that by the linear growth condition on F:

$$
\begin{aligned}
&\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2 \leq C\bigg(\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \mathbb{E}|Y_t^{n,m}|^2 \\
&\quad + \mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \|Z^{n,m}1_{[t,T]}\|_{\mathbf{L}}^{\mathbf{2}}(\mathbf{W})^2 + \|L^{n,m}1_{[t,T]}\|_{\mathbf{L^2}(\tilde{\mu})}^2\bigg). \qquad (3.14)
\end{aligned}
$$

By choosing $\alpha > 0$ such that $2\alpha C \leq 1/4$, and plugging this estimate of $\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2$ into (3.13), we get for all $0 \leq t \leq T$:

$$
\begin{aligned}
&\frac{3}{4}\mathbb{E}|Y_t^{n,m}|^2 + \frac{1}{4}\|Z^{n,m}1_{[t,T]}\|_{\mathbf{L^2(W)}}^2 + \frac{1}{4}\|L^{n,m}1_{[t,T]}\|_{\mathbf{L^2}(\tilde{\mu})}^2 \\
&\quad \leq C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \frac{5}{4}\mathbb{E}|\xi|^2 + \frac{3}{4}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds \\
&\qquad + \frac{1}{\alpha}\Big(\|\underline{Y}\|_{\mathbf{S^2}}^2 + \|\bar{Y}\|_{\mathbf{S^2}}^2\Big) + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \\
&\quad \leq C\bigg(\|\underline{Y}\|_{\mathbf{S^2}}^2 + \|\bar{Y}\|_{\mathbf{S^2}}^2 + \mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds\bigg) + 12\alpha\|K^{n,m,-}\|_{\mathbf{S^2}}^2, \qquad (3.15)
\end{aligned}
$$

where we used again the bounds $\underline{Y} \leq Y^{n,m} \leq \bar{Y}$ and the inequality $\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \leq 4\mathbb{E}|K_T^{n,m,-}|^2$. This proves, taking $t = 0$ in (3.15), the required estimate (3.10) for

$(Z^{n,m}, L^{n,m})$, and also for $K^{n,m,+}$ by (3.14), and recalling that $\mathbb{E}|K_T^{n,m,+}|^2 \leq 2\mathbb{E}|K_T^{n,m}|^2 + 2\mathbb{E}|K_T^{n,m,-}|^2$. Finally, the estimate for $\|Y^{n,m}\|_{\mathbf{S^2}}$ in (3.10) follows as usual from the relation (3.1), Burkholder–Davis–Gundy inequality, and the estimates for $(Z^{n,m}, L^{n,m}, K^{n,m,+})$. □

The key point is now to obtain a uniform estimate on $K^{n,m,-}$, and consequently uniform estimates on $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+})$ in view of Lemma 3.2. Let us introduce the following set of probability measures. For $m \in \mathbb{N}$, let $\mathcal{V}_m$ be the set of $\mathcal{P} \otimes \mathcal{B}(A)$-measurable processes valued in $(0, m]$, $\mathcal{V} = \cup_m \mathcal{V}_m$, and given $\nu \in \mathcal{V}$, consider the probability measure $\mathbb{P}^\nu$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with Radon–Nikodym density:

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \zeta_t^\nu := \mathcal{E}_t\bigg(\int_0^{\cdot}\int_A (\nu_s(a) - 1)\tilde{\mu}(ds, da)\bigg),$$

where $\mathcal{E}_t(\cdot)$ is the Doléans–Dade exponential. Indeed, since $\nu \in \mathcal{V}$ is essentially bounded, and $\lambda(A) < \infty$, it is known that ζ^ν is a uniformly integrable martingale (see e.g. Lemma 4.1 in [17]), and so defines a probability measure $\mathbb{P}^\nu$. Moreover, $\zeta_T^\nu \in \mathbf{L^p}(\mathcal{F}_T)$ for any $p \geq 1$. Notice that the Brownian motion W remains a Brownian motion W under $\mathbb{P}^\nu$, while the effect of the probability measure $\mathbb{P}^\nu$, by Girsanov's theorem, is to change the compensator $\lambda(da)dt$ of μ under $\mathbb{P}$ to $\nu_t(a)\lambda(da)dt$ under $\mathbb{P}^\nu$. We then denote by $\tilde{\mu}^\nu(dt, da) := \mu(dt, da) - \nu_t(a)\lambda(da)dt$ the compensated martingale measure of μ under $\mathbb{P}^\nu$.

Inspired by [11] (see also [5]), we make the following regularity assumption on the upper barrier:

(H1) There exists a nonincreasing sequence of processes $(U^k)_k$ such that:

(i) $\lim_{k\to\infty} U_t^k = U_t$, for all $0 \leq t \leq T$, a.s.

(ii) For any $k \in \mathbb{N}$, U^k is in the form:

$$U_t^k = U_0^k + \int_0^t \upsilon_s^k ds + \int_0^t \vartheta_s^k dW_s, \quad 0 \leq t \leq T, \ a.s.$$

where $(\upsilon^k)_k \subset \mathbf{L^2(0, T)}$ and $(\vartheta^k)_k \subset \mathbf{L^2(W)}$.

(iii) There exists some $p > 2$ such that:

$$\sup_{k\in\mathbb{N}} \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{t\leq s\leq T}\big(|U_s^k|^p + |\upsilon_s^k|^p + |\vartheta_s^k|^p\big)|\mathcal{F}_t\big]\Big]dt$$
$$+ \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{t\leq s\leq T}\big|F(s, 0, 0, 0)\big|^p|\mathcal{F}_t\big]\Big]dt < \infty.$$

We shall see later in the Markovian framework how Assumption **(H1)** is automatically satisfied, see Remark 5.3. The following key lemma states a uniform estimate for $K^{n,m,-}$ under condition **(H1)**.

Lemma 3.3. *Under condition* **(H1)***, we have*

$$\sup_{n,m\in\mathbb{N}} \big\|K^{n,m,-}\big\|_{\mathbf{S^2}} < \infty.$$

Proof. Let $(U^k)_k$ be in the form as in assumption **(H1)**(ii) and consider for positive integers n, m, k, the difference $\bar{Y}^{n,m,k} := Y^{n,m} - U^k$, which is then expressed in backward form as:

$$\begin{aligned}\bar{Y}_t^{n,m,k} &= \xi - U_T^k + \int_t^T \big(F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) + \upsilon_s^k\big)ds \\ &\quad + m\int_t^T \int_A (L_s^{n,m}(a))_+ \lambda(da)ds - n\int_t^T (U_s - U_s^k - \bar{Y}_s^{n,m,k})_- ds \\ &\quad - \int_t^T \big(Z_s^{n,m} - \vartheta_s^k\big)dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da). \end{aligned} \tag{3.16}$$

Now, by the Lipschitz condition of F in (y, z), and the monotonicity condition (2.1) of F in ℓ, we have for all $n, m \in \mathbb{N}$:

$$\begin{aligned} F(t, Y_t^{n,m}, Z_t^{n,m}, L_t^{n,m}) &= F(t, 0, 0, 0) + \alpha_t^{n,m} Y_t^{n,m} + \beta_t^{n,m}.Z_t^{n,m} \\ &\quad + \int_A \gamma_t^{n,m}(a) L_t^{n,m}(a)\lambda(da) - \delta_t^{n,m}, \end{aligned}$$

for some sequence of bounded predictable processes $(\alpha^{n,m})$ valued in $\mathbb{R}$, $(\beta^{n,m})$ valued in $\mathbb{R}^d$, uniformly bounded in n, m, a nonnegative sequence of predictable process $(\delta^{n,m})$, and a nonnegative sequence of bounded $\mathcal{P} \otimes \mathcal{B}(A)$-measurable maps $(\gamma^{n,m})$, uniformly bounded in n, m. Plug this decomposition of F into (3.16), and let us consider the process $\{\Gamma_{ts}^{n,m}, t \leq s \leq T\}$ of dynamics:

$$d\Gamma_{ts}^{n,m} = \Gamma_{ts}^{n,m}[(\alpha_s^{n,m} - n)ds + \beta_s^{n,m} dW_s], \quad t \leq s \leq T, \qquad \Gamma_{tt}^{n,m} = 1,$$

and given explicitly by:

$$\Gamma_{ts}^{n,m} = e^{-n(s-t)} e^{\int_t^s \alpha_u^{n,m} du} M_{ts}^{n,m}, \qquad M_{ts}^{n,m} = \frac{\mathcal{E}_s\big(\int_0^\cdot \beta_u^{n,m} dW_u\big)}{\mathcal{E}_t\big(\int_0^\cdot \beta_u^{n,m} dW_u\big)}, \quad t \leq s \leq T,$$

where $\mathcal{E}_t(\cdot)$ is the Doléans–Dade exponential. Since $\beta^{n,m}$ is a bounded process, we see that $\{M_{ts}^{n,m},\ t \leq s \leq T\}$ is a uniformly integrable martingale, with $M_{tT}^{n,m} \in \mathbf{L^p}(\mathcal{F}_T)$ for any $p \geq 1$. By applying Itô's formula to the product $\{\Gamma_{ts}^{n,m} \bar{Y}_s^{n,m,k}, t \leq s \leq T\}$, we then obtain:

$$\begin{aligned} \bar{Y}_t^{n,m,k} &= \Gamma_{tT}^{n,m}\big(\xi - U_T^k\big) + \int_t^T \Gamma_{ts}^{n,m}\big(F(s, 0, 0, 0) + \alpha_s^{n,m} U_s^k + \beta_s^{n,m}\vartheta_s^k + \upsilon_s^k\big)ds \\ &\quad + \int_t^T \Gamma_{ts}^{n,m}\big[n\bar{Y}_s^{n,m,k} - n(U_s - U_s^k - \bar{Y}_s^{n,m,k})_- - \delta_s^{n,m}\big]ds \\ &\quad + \int_t^T \int_A \Gamma_{ts}^{n,m}\big[\gamma_s^{n,m}(a)L_s^{n,m}(a) + m(L_s^{n,m}(a))_+ - \nu_s(a)L_s^{n,m}(a)\big]\lambda(da)ds \\ &\quad - \int_t^T \Gamma_{ts}^{n,m}\big(Z_s^{n,m} - \vartheta_s^k + \bar{Y}_s^{n,m,k}\beta_s^{n,m}\big)dW_s \\ &\quad - \int_t^T \int_A \Gamma_{ts}^{n,m} L_s^{n,m}(a)\tilde{\mu}^\nu(ds, da), \end{aligned}$$

for any $\nu \in \mathcal{V}$, where we introduced the compensated measure $\tilde{\mu}^\nu$ of μ under $\mathbb{P}^\nu$. By choosing $\nu = \nu^{n,m,\varepsilon} \in \mathcal{V}$ defined by: $\nu_t^{n,m,\varepsilon}(a) = (\gamma_t^{n,m}(a) + m)1_{\{L_t^{n,m}(a)\geq 0\}} + (\gamma_t^{n,m}(a) + \varepsilon)1_{\{L_t^{n,m}(a)<0\}}$, for some arbitrary $\varepsilon > 0$, we see that:

$$\gamma_t^{n,m}(a)L_t^{n,m}(a) + m(L_t^{n,m}(a))_+ - \nu_t^{n,m}(a)L_t^{n,m}(a) = -\varepsilon L_t^{n,m}(a)1_{\{L_t^{n,m}(a)<0\}}.$$

Observe also that

$$n\bar{Y}_t^{n,m,k} - n(U_t - U_t^k - \bar{Y}_t^{n,m,k})_- - \delta_s^{n,m} \le 0, \quad 0 \le t \le T, \ a.s.$$

since $U \le U^k$, and $\delta^{n,m} \ge 0$. Recalling that $\xi \le U_T \le U_T^k$, the explicit expression of $\Gamma^{n,m}$, and the fact that $(\alpha^{n,m})$, $(\beta^{n,m})$ are uniformly bounded in (t, ω, n, m), we then get the existence of some positive constant C such that:

$$\begin{aligned}
\bar{Y}_t^{n,m,k} \le C & \int_t^T e^{-n(s-t)} M_{ts}^{n,m} \big(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|\big) ds \\
& - \varepsilon \int_t^T \int_A \Gamma_{ts}^{n,m} L_s^{n,m}(a) 1_{\{L_s^{n,m}(a)<0\}} \lambda(da) ds \\
& - \int_t^T \Gamma_{ts}^{n,m} \big(Z_s^{n,m} - \vartheta_s^k + \bar{Y}_s^{n,m,k} \beta_s^{n,m}\big) dW_s \\
& - \int_t^T \int_A \Gamma_{ts}^{n,m} L_s^{n,m}(a) \tilde{\mu}^{\nu^{n,m,\varepsilon}}(ds, da),
\end{aligned} \tag{3.17}$$

for any $n, m, k \in \mathbb{N} \setminus \{0\}$, $\varepsilon > 0$. Denote by $S_t^{n,m,k} = \int_0^t \Gamma_{0s}^{n,m} \big(Z_s^{n,m} - \vartheta_s^k + \bar{Y}_s^{n,m,k} \beta_s^{n,m}\big) dW_s$, $0 \le t \le T$, which is a $\mathbb{P}^\nu$-local martingale, for any $\nu \in \mathcal{V}$, by recalling that W remains a Brownian motion under $\mathbb{P}^\nu$. From Burkholder–Davis–Gundy, Bayes formula, Cauchy–Schwarz, and Doob inequalities, we have

$$\begin{aligned}
& \mathbb{E}^\nu\big[\sup_{0\le t\le T} |S_t^{n,m,k}|\big] \\
& \le C\mathbb{E}^\nu\Big[\sqrt{\langle S^{n,m,k}\rangle_T}\Big] = C\mathbb{E}^\nu\Big[\sqrt{\int_0^T |\Gamma_{0t}^{n,m}|^2 |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k} \beta_t^{n,m}|^2 dt}\Big] \\
& \le C\mathbb{E}\Big[\zeta_T^\nu \sup_{0\le t\le T} \Gamma_{0t}^{n,m} \sqrt{\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k} \beta_t^{n,m}|^2 dt}\Big] \\
& \le C\Big(\mathbb{E}\big[|\zeta_T^\nu|^4\big]\mathbb{E}\big[\sup_{0\le t\le T} |\Gamma_{0t}^{n,m}|^4\big]\Big)^{\frac{1}{4}} \sqrt{\mathbb{E}\Big[\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k} \beta_t^{n,m}|^2 dt\Big]} \\
& \le C\Big(\mathbb{E}\big[|\zeta_T^\nu|^4\big]\mathbb{E}\big[|M_{0T}^{n,m}|^4\big]\Big)^{\frac{1}{4}} \sqrt{\mathbb{E}\Big[\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k} \beta_t^{n,m}|^2 dt\Big]} \\
& < \infty,
\end{aligned} \tag{3.18}$$

where we used the fact that $\alpha^{n,m}$, $\beta^{n,m}$ are bounded processes, $Z^{n,m}$, ϑ^k lie in $\mathbf{L^2(W)}$, and $\bar{Y}^{n,m,k}$ in $\mathbf{L^2(0,T)}$. Therefore, $S^{n,m,k}$ is a uniformly $\mathbb{P}^\nu$-integrable martingale for any $\nu \in \mathcal{V}$, and similarly we show that $\int_0^t \int_A \Gamma_{ts}^{n,m} L_s^{n,m}(a) \tilde{\mu}^\nu(ds, da)$ is a $\mathbb{P}^\nu$-martingale. Hence, by taking conditional expectation with respect to $\mathbb{P}^{\nu^{n,m,\varepsilon}}$ into (3.17), we have for all $n, m, k \in \mathbb{N} \setminus \{0\}$, $\varepsilon > 0$:

$$\begin{aligned}
\bar{Y}_t^{n,m,k} \le & \frac{C}{n} \mathbb{E}^{\nu^{n,m,\varepsilon}}\Big[\sup_{t\le s\le T} M_{ts}^{n,m} \big(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|\big) |\mathcal{F}_t\Big] \\
& - \varepsilon \mathbb{E}^{\nu^{n,m,\varepsilon}}\Big[\int_t^T \int_A \Gamma_{ts}^{n,m} L_s^{n,m}(a) 1_{\{L_s^{n,m}(a)<0\}} \lambda(da) ds |\mathcal{F}_t\Big]
\end{aligned}$$

$$\le \frac{C}{n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^{\nu}\Big[\sup_{t\le s\le T} M_{ts}^{n,m}\big(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|\big)|\mathcal{F}_t\Big]$$

$$+\varepsilon\mathbb{E}\Big[\frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}}\int_t^T\int_A \Gamma_{ts}^{n,m}|L_s^{n,m}(a)|\lambda(da)ds|\mathcal{F}_t\Big], \quad 0\le t\le T, \tag{3.19}$$

from Bayes formula. Now, for $\varepsilon \le m$, we see that $\nu^{n,m,\varepsilon} \le \bar{\nu}^{n,m} := \gamma^{n,m} + m$, and so:

$$0 \le \frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}} \le \frac{\zeta_T^{\bar{\nu}^{n,m}}}{\zeta_t^{\bar{\nu}^{n,m}}}\exp\bigg(\int_t^T\int_A \bar{\nu}_s^{n,m}(a)\lambda(da)ds\bigg). \tag{3.20}$$

This shows that

$$\lim_{\varepsilon\to 0}\varepsilon\mathbb{E}\Big[\frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}}\int_t^T\int_A \Gamma_{ts}^{n,m}|L_s^{n,m}(a)|\lambda(da)ds|\mathcal{F}_t\Big] = 0, \quad 0\le t\le T, \tag{3.21}$$

and so by sending ε to zero in (3.19):

$$\begin{aligned}
(U_t^k - Y_t^{n,m})_- &= (\bar{Y}_t^{n,m,k})_+ \\
&\le \frac{C}{n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^{\nu}\Big[\sup_{t\le s\le T} M_{ts}^{n,m}\big(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|\big)|\mathcal{F}_t\Big] \\
&\le \frac{C}{n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^{\nu}\Big[\sup_{t\le s\le T} |M_{ts}^{n,m}|^{\frac{p}{p-2}} + \sup_{t\le s\le T}\big(|F(s,0,0,0)|^{\frac{p}{2}} \\
&\quad + |U_s^k|^{\frac{p}{2}} + |\vartheta_s^k|^{\frac{p}{2}} + |\upsilon_s^k|^{\frac{p}{2}}\big)|\mathcal{F}_t\Big]
\end{aligned}$$

for all $0\le t\le T$, and $p>2$, by Young inequality. Recall that W is a Brownian motion under $\mathbb{P}^\nu$, and so $\{M_{ts}^{n,m}, t\le s\le T\}$ is a martingale under $\mathbb{P}^\nu$, for any $\nu\in\mathcal{V}$. By Doob's inequality, we then have with $q = p/(p-2) > 1$:

$$\begin{aligned}
\mathbb{E}^{\nu}\Big[\sup_{t\le s\le T}|M_{ts}^{n,m}|^q|\mathcal{F}_t\Big] &\le \Big(\frac{q}{q-1}\Big)^q \mathbb{E}^{\nu}\big[|M_{tT}^{n,m}|^q|\mathcal{F}_t\big] \\
&\le \Big(\frac{q}{q-1}\Big)^q \exp\big(q(q-1)\|\beta\|_\infty^2(T-t)\big),
\end{aligned}$$

where $\|\beta\|_\infty$ is a uniform bound of $(\beta^{n,m})$, hence independent of n, m and $\nu\in\mathcal{V}$. We then deduce that

$$\begin{aligned}
&(U_t^k - Y_t^{n,m})_- \\
&\quad\le \frac{C}{n}\Big(1 + \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^{\nu}\Big[\sup_{t\le s\le T}\big(|F(s,0,0,0)|^{\frac{p}{2}} + |U_s^k|^{\frac{p}{2}} + |\vartheta_s^k|^{\frac{p}{2}} + |\upsilon_s^k|^{\frac{p}{2}}\big)|\mathcal{F}_t\Big]\Big)
\end{aligned}$$

for all $0\le t\le T$, $n, m, k\in\mathbb{N}\setminus\{0\}$. By Cauchy–Schwarz inequality, we then obtain:

$$\begin{aligned}
&\mathbb{E}\Big[n\int_0^T (U_t^k - Y_t^{n,m})_- dt\Big]^2 \\
&\quad\le C\bigg(1 + \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^{\nu}\Big[\sup_{t\le s\le T}\big(|F(s,0,0,0)|^p + |U_s^k|^p + |\vartheta_s^k|^p + |\upsilon_s^k|^p\big)|\mathcal{F}_t\Big]\Big]dt\bigg).
\end{aligned}$$

By taking $p>2$ as in Assumption **(H1)**(iii), and then sending k to infinity in the l.h.s. of the above inequality, we get the required uniform estimate on $K^{n,m,-}$. □

Corollary 3.1. *Let assumptions* **(H0)** *and* **(H1)** *hold. Then, we have*

$$\sup_{m\in\mathbb{N}}\Big(\|Y^m\|_{\mathbf{S^2}}+\|Z^m\|_{\mathbf{L^2(W)}}+\|L^m\|_{\mathbf{L^2}(\tilde{\mu})}+\|K^{m,+}\|^2_{\mathbf{S}}+\|K^{m,-}\|_{\mathbf{S^2}}\Big)<\infty,$$

where $K_t^{m,+}:=m\int_0^t\int_A\big(L_s^m(a)\big)_+\lambda(da)ds$.

Proof. From the bounds (3.8) and (3.9), we already have the uniform estimate for $\|Y^m\|_{\mathbf{S^2}}$. Moreover, by Lemmata 3.2 and 3.3, we have the uniform estimates:

$$\sup_{n,m\in\mathbb{N}}\Big(\|Z^{n,m}\|_{\mathbf{L^2(W)}}+\|L^{n,m}\|_{\mathbf{L^2}(\tilde{\mu})}+\|K^{n,m,+}\|_{\mathbf{S^2}}+\|K^{n,m,-}\|_{\mathbf{S^2}}\Big)<\infty.$$

We deduce that the weak limits $(Z^m, L^m, K^{m,-})$ of $(Z^{m,n}, L^{m,n}, K^{n,m,-})$ when n goes to infinity, are also uniformly bounded in $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{S^2}$. From the strong convergence of $L^{n,m}$ to L^m in $\mathbf{L^p}(\tilde{\mu})$, $1\le p<2$, we see by definition of $K^{n,m,+}$ and $K^{m,+}$ that $K_T^{n,m,+}$ converges strongly to $K_T^{m,+}$ in $\mathbf{L^p}(\mathcal{F}_T)$, when n goes to infinity. Moreover, since $(K_T^{n,m,+})_n$ is uniformly bounded in $\mathbf{L^2}(\mathcal{F}_T)$, it also converges weakly to $K_T^{m,+}$ in $\mathbf{L^2}(\mathcal{F}_T)$. It follows that $(K^{m,+})_m$ inherits from $(K^{n,m,+})_{n,m}$ the uniform estimate in $\mathbf{S^2}$. □

We can now state the main result of this section as a consequence of the monotonic convergence theorem stated in Appendix B, which extends to the Brownian–Poisson filtration framework the result of Peng and Xu [26].

Theorem 3.1. *Let assumptions* **(H0)** *and* **(H1)** *hold. Then there exists a minimal solution* $(Y, Z, L, K^+, K^-)\in\mathbf{S^2}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{K^2}\times\mathbf{K^2}$ *to the reflected BSDE with nonpositive jumps (2.2)–(2.5), where:*

- (i) *Y is the increasing limit of $(Y^m)_m$.*
- (ii) *(Z, L) is the strong (resp. weak) limit of $(Z^m, L^m)_m$ in $\mathbf{L^p(W)}\times\mathbf{L^p}(\tilde{\mu})$, with $p\in[1,2)$, (resp. in $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})$).*
- (iii) *K_t^+ is the weak limit of $(K_t^{m,+})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, and K_t^- is the strong limit of $(K_t^{m,-})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, for any $0\le t\le T$.*

Proof. We already know that $(Y^m)_m$ is a nondecreasing sequence in $\mathbf{S^2}$, which converges to some Y, which satisfies $\underline{Y}\le Y\le\bar{Y}$ from (3.8) and (3.9), and so lies in $\mathbf{S^2}$. By Lemma 3.1 and Corollary 3.1, we then see that the sequence $(Y^m, Z^m, L^m, K^{m,+}, K^{m,-})_m$ solution to the BSDE (3.3) satisfies all the conditions of the monotonic limit Theorem B.1. This provides the existence of $(Z, L, K^+, K^-)\in\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{K^2}\times\mathbf{K^2}$ as in the assertions (ii) and (iii) of Theorem 3.1 such that the quintuple (Y, Z, L, K^+, K^-) solves (2.2).

From the strong convergence in $\mathbf{L^1}(\tilde{\mu})$ of $(L^m)_m$ to L, and since $\lambda(A)<\infty$, we have

$$\mathbb{E}\Big[\int_0^T\int_A\big(L_t^m(a)\big)_+\lambda(da)dt\Big]\longrightarrow\mathbb{E}\Big[\int_0^T\int_A\big(L_t(a)\big)_+\lambda(da)dt\Big],$$

as m goes to infinity. Moreover, since $K_T^{m,+}=m\int_0^T(L_t(a))_+\lambda(da)dt$ is bounded in m in $\mathbf{L^2}(\mathcal{F}_T)$, this implies that

$$\mathbb{E}\Big[\int_0^T\int_A\big(L_t(a)\big)_+\lambda(da)dt\Big]=0,$$

which means that the constraint (2.3) is satisfied. The upper reflection (2.4) is obviously satisfied from (3.4) and by sending m to infinity. Let us now check the Skorohod reflecting condition (2.5). We recall from (3.5) that $\int_0^T(U_{t^-}-Y_{t^-}^m)dK_t^{m,-}=0$. Together with the fact that

$U_{t^-} - Y^m_{t^-} \geq U_{t^-} - Y_{t^-} \geq 0$, this yields $\int_0^T (U_{t^-} - Y_{t^-}) dK^{m,-}_t = 0$. Since $(K^{m,-}_t)_m$ converges strongly to K^-_t in $\mathbf{L}^2(\mathcal{F}_t)$ for all t, and by Lemma 3.1, this implies that the measure $dK^{m,-}$ converges weakly to dK^-, and so $\int_0^T (U_{t^-} - Y_{t^-}) dK^-_t = 0$ a.s.

It remains to prove the minimality condition. Let $(\tilde{Y}, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ be another solution to the reflected BSDE with nonpositive jumps (2.2)–(2.5). We then see that $\int_0^t \int_A (\tilde{L}_s(a))_+ \lambda(da) ds = 0$, and thus $F(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{L}_t) = F_m(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{L}_t)$, for $0 \leq t \leq T$. From the comparison Theorem A.2, we deduce that $Y^m_t \leq \tilde{Y}_t$, $0 \leq t \leq T$. Taking the limit with respect to m, this proves the minimality condition: $Y_t \leq \tilde{Y}_t$, $0 \leq t \leq T$. □

Remark 3.2. The order of the limits: first let n to infinity, and then let m to infinity, is crucial in our approach. Indeed, by sending first n to infinity, we get a nondecreasing sequence of processes $(K^{m,-})_m$ (see Lemma 3.1), which is a required property for applying the monotonic convergence theorem in Theorem 3.1. On the other hand, if we would first let m to infinity in the double sequence $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+}, K^{n,m,-})$, then we would obtain a minimal solution $(\hat{Y}^n, \hat{Z}^n, \hat{K}^{n,+})$ to the BSDE with nonpositive jumps:

$$\begin{aligned} \hat{Y}^n_t &= \xi + \int_t^T F(s, \hat{Y}^n_s, \hat{Z}^n_s, \hat{L}^n_s) ds - n \int_t^T (U_s - \hat{Y}^n_s)_- ds + \hat{K}^{n,+}_T - \hat{K}^{n,+}_t \\ &\quad - \int_t^T \hat{Z}^n_s dW_s - \int_t^T \int_A \hat{L}^n_s(a) \mu(ds, da), \quad 0 \leq t \leq T, \\ \hat{L}^n_t(a) &\leq 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \quad a.e. \end{aligned} \tag{3.22}$$

and $(\hat{Y}^n)_n$ is a nonincreasing sequence, converging to some $\hat{Y} \geq Y$ by (3.2). But neither $K^{n,+}$, which is the weak limit of $K^{n,m,+}$, as m goes to infinity, nor $K^{n,-}_t := n \int_0^t (U_s - \hat{Y}^n_s)_- ds$, satisfy monotonicity properties in n, which prevents to apply the monotonic convergence theorem to the sequence $(\hat{Y}^n, \hat{Z}^n, \hat{K}^{n,+}, \hat{K}^{n,-})_n$, and thus to identify $\hat{Y} = Y$ as the minimal solution to the reflected BSDE with nonpositive jumps. This differs from the case of doubly reflected BSDEs where one can send indifferently first m or n to infinity. □

4. Dual game representation

In this section, we consider the case where the generator $F(t, \omega)$ does not depend on y, z, ℓ, and we provide a dual game representation of the minimal solution to the reflected BSDE with nonpositive jumps in terms of a family of equivalent probability measures and discount factors. In addition to the set of probability measures $\mathbb{P}^\nu$, $\nu \in \mathcal{V} = \cup_m \mathcal{V}_m$ defined in the previous section, let us introduce for any $n \in \mathbb{N}$, the set Θ_n of $\mathbb{F}$-progressively measurable processes valued in $[0, n]$, and set $\Theta = \cup_n \Theta_n$, which shall represent the set of discount processes. Inspired by Proposition 6.2 in [5] and the dual representation in Section 4 of [17], we prove an explicit representation formula for the minimal solution to the reflected BSDE with nonpositive jumps.

Proposition 4.1. (i) *For any $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$, the solution to the penalized BSDE (3.1) admits the following dual representation formula:*

$$Y^{n,m}_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}_m} \operatorname*{ess\,inf}_{\theta \in \Theta_n} G_t(\nu, \theta) = \operatorname*{ess\,inf}_{\theta \in \Theta_n} \operatorname*{ess\,sup}_{\nu \in \mathcal{V}_m} G_t(\nu, \theta),$$

for all $0 \leq t \leq T$, where

$$G_t(\nu, \theta) := \mathbb{E}^\nu \Big[e^{-\int_t^T \theta_s ds} \xi + \int_t^T e^{-\int_t^s \theta_r dr} \big(F(s) + \theta_s U_s \big) ds | \mathcal{F}_t \Big].$$

(ii) *Under assumptions* **(H0)** *and* **(H1)**, *the minimal solution to the reflected BSDE with nonpositive jumps (2.2)–(2.5) is explicitly represented as:*

$$Y_t = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu,\theta), \quad 0 \le t \le T. \tag{4.1}$$

Proof. (i) Fix $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$. For $\theta \in \Theta$, by applying Itô's rule to the product of the processes $e^{-\int_0^{\cdot} \theta_s ds}$ and $Y^{n,m}$ in (3.1), and by introducing the compensated measure $\tilde{\mu}^\nu(dt,da)$ under $\mathbb{P}^\nu$ for $\nu \in \mathcal{V}$, we obtain:

$$\begin{aligned} Y_t^{n,m} &= e^{-\int_t^T \theta_s ds}\xi + \int_t^T e^{-\int_t^s \theta_r dr}\big(F(s) + \theta_s U_s\big)ds \\ &\quad + \int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^{n,m}(a))_+ - \nu_s(a)L_s^{n,m}(a)\big)\lambda(da)ds \\ &\quad - \int_t^T e^{-\int_t^s \theta_r dr}\big(n(U_s - Y_s^{n,m})_- + \theta_s(U_s - Y_s^{n,m})\big)ds \\ &\quad - \int_t^T e^{-\int_t^s \theta_r dr} Z_s^{n,m} dW_s - \int_t^T \int_A e^{-\int_t^s \theta_r dr} L_s^{n,m}(a)\tilde{\mu}^\nu(ds,da). \end{aligned}$$

By same arguments as in (3.18) (see also Lemma 4.2 in [17]), we can check that the $\mathbb{P}^\nu$ local martingales $\{\int_t^s e^{-\int_t^u \theta_r dr} Z_u^{n,m} dW_u, t \le s \le T\}$ and $\{\int_t^s \int_A e^{-\int_t^u \theta_r dr} L_u^{n,m}(a)\tilde{\mu}^\nu(du,da), t \le s \le T\}$ are actually uniformly integrable $\mathbb{P}^\nu$-martingales, so that by taking conditional expectation under $\mathbb{P}^\nu$:

$$\begin{aligned} Y_t^{n,m} &= G_t(\nu,\theta) + \mathbb{E}^\nu\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^{n,m}(a))_+ - \nu_s(a)L_s^{n,m}(a)\big)\lambda(da)ds \\ &\quad - \int_t^T e^{-\int_t^s \theta_r dr}\big(n(U_s - Y_s^{n,m})_- + \theta_s(U_s - Y_s^{n,m})\big)ds \big| \mathcal{F}_t\Big], \end{aligned} \tag{4.2}$$

and this relation holds for any $\nu \in \mathcal{V}$, and $\theta \in \Theta$. Now, observe that for any $\nu \in \mathcal{V}_m$, hence valued in $(0,m]$, we have

$$m(L_t^{n,m}(a))_+ - \nu_t(a)L_t^{n,m}(a) \ge 0, \quad 0 \le t \le T,\ a \in A,\ a.s.$$

and for $\nu = \nu^\varepsilon \in \mathcal{V}_m$ defined by: $\nu_t^\varepsilon(a) = m1_{\{L_t^{n,m}(a)\ge 0\}} + \varepsilon 1_{\{L_t^{n,m}(a)<0\}}$, for arbitrary $\varepsilon \in (0,m]$, we have

$$m(L_t^{n,m}(a))_+ - \nu_t^\varepsilon(a)L_t^{n,m}(a) = -\varepsilon L_t^{n,m}(a)1_{\{L_t^{n,m}(a)<0\}}, \quad 0 \le t \le T,\ a \in A,\ a.s.$$

Similarly, for any $\theta \in \Theta_n$, hence valued in $[0,n]$, we have

$$n(U_t - Y_t^{n,m})_- + \theta_t(U_t - Y_t^{n,m}) \ge 0, \quad 0 \le t \le T,\ a.s.$$

and for $\theta^* \in \Theta_n$ defined by: $\theta_t^* = n1_{\{Y_t^{n,m}\ge U_t\}}$, we have

$$n(U_t - Y_t^{n,m})_- + \theta_t^*(U_t - Y_t^{n,m}) = 0, \quad 0 \le t \le T,\ a.s.$$

Therefore, by (4.2), we get

$$\begin{aligned} G_t(\nu,\theta^*) \le Y_t^{n,m} &= G_t(\nu^\varepsilon,\theta^*) + \varepsilon R_t^{n,m,\varepsilon}(\theta^*), \quad \forall \nu \in \mathcal{V}_m, && (4.3) \\ &\le G_t(\nu^\varepsilon,\theta) + \varepsilon R_t^{n,m,\varepsilon}(\theta), \\ &\le G_t(\nu^\varepsilon,\theta) + \varepsilon R_t^{n,m,\varepsilon}(0), \quad \forall \theta \in \Theta_n, && (4.4) \end{aligned}$$

for all $\varepsilon \in (0, m]$, where we set:

$$R_t^{n,m,\varepsilon}(\theta) := \mathbb{E}^{\nu^\varepsilon}\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr} |L_s^{n,m}(a)|\lambda(da)ds|\mathcal{F}_t\Big].$$

For fixed m, and by viewing the BSDE (3.1) as a penalized BSDE in n for the upper-reflected BSDE with generator F_m in (3.6), we have by standard arguments based on Itô's lemma, uniform estimates in n for $(Y^{n,m}, Z^{n,m}, L^{n,m})$ in $\mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$ (see Theorem 4.2 in [8]). Actually, these arguments show that for all $0 \le t \le T$, there exists some real-valued $\mathcal{F}_t$-measurable random variable C_t^m such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_t^T \int_A |L_s^{n,m}(a)|^2\lambda(da)ds|\mathcal{F}_t\Big] \le C_t^m. \tag{4.5}$$

Moreover, since $\nu^\varepsilon \le m$, we see as in (3.20) that $\zeta_T^{\nu^\varepsilon}/\zeta_t^{\nu^\varepsilon} \le e^{m(T-t)\lambda(A)}\zeta_T^m/\zeta_t^m$, where ζ^m is the Radon–Nikodym density of $d\mathbb{P}^\nu/d\mathbb{P}$ for $\nu = m$. Thus, by Cauchy–Schwarz inequality, there exists some real-valued $\mathcal{F}_t$-measurable random variable $\tilde{C}_t^m$ such that

$$\sup_{n\in\mathbb{N}} R_t^{n,m,\varepsilon}(0) \le \tilde{C}_t^m, \tag{4.6}$$

for all $\varepsilon \in (0, m]$. Now, by (4.3), we have: $\operatorname{ess\,inf}_{\theta\in\Theta_n} \operatorname{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu, \theta) \le Y_t^{n,m}$, and by (4.4), we get:

$$Y_t^{n,m} \le \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu, \theta) + \varepsilon R_t^{n,m,\varepsilon}(0).$$

By (4.6), we see in particular that $\varepsilon R_t^{n,m,\varepsilon}(0) \to 0$ a.s. as ε goes to zero. Since we always have $\operatorname{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname{ess\,inf}_{\theta\in\Theta_n} G_t(\nu, \theta) \le \operatorname{ess\,inf}_{\theta\in\Theta_n} \operatorname{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu, \theta)$, this shows that

$$\begin{aligned} Y_t^{n,m} = \lim_{\varepsilon\to 0} G_t(\nu^\varepsilon, \theta^*) &= \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu, \theta) \\ &= \operatorname*{ess\,inf}_{\theta\in\Theta_n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu, \theta), \end{aligned} \tag{4.7}$$

i.e. $(\nu^\varepsilon, \theta^*) \in \mathcal{V}_m \times \Theta_n$ is an ε-saddle point for $G_t(\nu, \theta)$.

(ii) By sending m to infinity into (4.7), and recalling that $Y^m = \lim_n Y^{n,m}$, we get:

$$Y_t^m = \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu, \theta) \ \ge\ \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu, \theta). \tag{4.8}$$

On the other hand, for arbitrary $n_0 \in \mathbb{N}$, we see that for any $\theta \in \Theta_{n_0}$ and any $n \ge n_0$:

$$n(U_t - Y_t^{n,m})_- + \theta_t(U_t - Y_t^{n,m}) \ge 0, \quad 0 \le t \le T, \ a.s.,$$

which implies, from (4.2),

$$\begin{aligned} Y_t^{n,m} &\le G_t(\nu, \theta) \\ &\quad + \mathbb{E}^\nu\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^{n,m}(a))_+ - \nu_s(a)L_s^{n,m}(a)\big)\lambda(da)ds|\mathcal{F}_t\Big], \end{aligned} \tag{4.9}$$

for any $\nu \in \mathcal{V}$, $\theta \in \Theta_{n_0}$, and $n \ge n_0$. Now note that, since $L^{n,m} \to L^m$ strongly in $\mathbf{L^p}(\tilde{\mu})$, $p \in [1, 2)$, then, up to a subsequence, $L^{n,m} \to L^m\ d\mathbb{P} \otimes dt \otimes \lambda(da)$ almost everywhere.

Moreover, as already recalled in step (i) of the proof, we have uniform estimates in n for $(L^{n,m}) \in \mathbf{L^2}(\tilde{\mu})$, namely, from (4.5) with $t = 0$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \int_A |L_s^{n,m}(a)|^2 \lambda(da)ds\Big] \leq C_0^m, \tag{4.10}$$

for some positive constant C_0^m. Then, sending n to infinity in (4.9) we obtain, from Lebesgue's dominated convergence theorem,

$$\begin{aligned} Y_t^m \leq\ & G_t(\nu,\theta) \\ & + \mathbb{E}^\nu\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^m(a))_+ - \nu_s(a)L_s^m(a)\big)\lambda(da)ds|\mathcal{F}_t\Big], \end{aligned} \tag{4.11}$$

for any $\nu \in \mathcal{V}$, $\theta \in \Theta_{n_0}$. Since $\Theta = \cup_n \Theta_n$, from the arbitrariness of n_0 we conclude that (4.11) remains true for all $\theta \in \Theta$. Take $\tilde{\nu}^\varepsilon \in \mathcal{V}_m$ defined by: $\tilde{\nu}_t^\varepsilon(a) = m1_{\{L_t^m(a)\geq 0\}} + \varepsilon 1_{\{L_t^m(a)<0\}}$, for arbitrary $\varepsilon \in (0, m]$, so that

$$m(L_t^m(a))_+ - \nu_t^\varepsilon(a)L_t^m(a) = -\varepsilon L_t^m(a)1_{\{L_t^m(a)<0\}}, \quad 0 \leq t \leq T,\ a \in A,\ a.s.,$$

and thus by (4.11):

$$Y_t^m \leq G_t(\tilde{\nu}^\varepsilon,\theta) + \varepsilon\tilde{R}_t^{m,\varepsilon}(\theta) \leq\ G_t(\tilde{\nu}^\varepsilon,\theta) + \varepsilon\tilde{R}_t^{m,\varepsilon}(0), \quad \forall\theta \in \Theta, \tag{4.12}$$

for all $\varepsilon \in (0, m]$, where we set:

$$\tilde{R}_t^{m,\varepsilon}(\theta) := \mathbb{E}^{\tilde{\nu}^\varepsilon}\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}|L_s^m(a)|\lambda(da)ds|\mathcal{F}_t\Big].$$

Using again the uniform estimate (4.10) and the fact that, up to a subsequence, $L^{n,m} \to L^m$ $d\mathbb{P} \otimes dt \otimes \lambda(da)$ a.e., we obtain, from (4.5) and Lebesgue's dominated convergence theorem,

$$\mathbb{E}\Big[\int_t^T \int_A |L_s^m(a)|^2 \lambda(da)ds|\mathcal{F}_t\Big] \leq C_t^m.$$

Moreover, as in step (i) of the proof, since $\tilde{\nu}^\varepsilon \leq m$ we see that $\zeta_T^{\tilde{\nu}^\varepsilon}/\zeta_t^{\tilde{\nu}^\varepsilon} \leq e^{m(T-t)\lambda(A)}\zeta_T^m/\zeta_t^m$. Thus, by Cauchy–Schwarz inequality, it follows that, for all $\varepsilon \in (0, m]$,

$$\tilde{R}_t^{m,\varepsilon}(0) \leq \tilde{C}_t^m,$$

with the same real-valued $\mathcal{F}_t$-measurable random variable $\tilde{C}_t^m$ as in (4.6). Then, from (4.12) we get

$$Y_t^m \leq \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu,\theta) + \varepsilon\tilde{C}_t^m,$$

for all $\varepsilon \in (0, m]$. By sending ε to zero, and combining with (4.8), we obtain:

$$\begin{aligned} Y_t^m &= \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta) \\ &= \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu,\theta). \end{aligned} \tag{4.13}$$

Finally, by sending m to infinity into (4.13), we obtain the dual relation (4.1) for $Y = \lim_m Y^m$. □

Remark 4.1. We do not know in general if one can switch in (4.1) the essential infimum and supremum. Actually, by considering $\hat{Y}^n = \lim_m Y^{n,m}$ the minimal solution to the BSDE with nonnegative jumps (3.22), one could show by similar arguments as in the second part (ii) of Proposition 4.1 that:

$$\hat{Y}_t^n = \underset{\theta \in \Theta_n}{\operatorname{ess\,inf}} \underset{\nu \in \mathcal{V}}{\operatorname{ess\,sup}} G_t(\nu, \theta) = \underset{\nu \in \mathcal{V}}{\operatorname{ess\,sup}} \underset{\theta \in \Theta_n}{\operatorname{ess\,inf}} G_t(\nu, \theta),$$

so that $\hat{Y} := \lim_n \hat{Y}^n$ satisfies:

$$\hat{Y}_t = \underset{\theta \in \Theta}{\operatorname{ess\,inf}} \underset{\nu \in \mathcal{V}}{\operatorname{ess\,sup}} G_t(\nu, \theta).$$

However, as pointed out in Remark 3.2, we cannot conclude whether $\hat{Y}_t$ is equal or strictly greater than Y_t. □

5. Connection with HJB Isaacs equation for controller-and-stopper games

In this section, we show how the minimal solution to our class of reflected BSDEs with nonpositive jumps provides a probabilistic representation (hence a Feynman–Kac formula) to fully nonlinear variational inequalities of Hamilton–Jacobi–Bellman (HJB) Isaacs type arising in a controller/stopper game, when considering a suitable Markovian framework.

5.1. The Markovian framework

We are given two measurable functions $b : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^{d\times d}$ and we introduce the forward Markov regime-switching process (X, I) in $\mathbb{R}^d \times \mathbb{R}^q$ governed by:

$$dX_t = b(X_t, I_t)dt + \sigma(X_t, I_t)dW_t \tag{5.1}$$

$$dI_t = \int_A (a - I_{t^-})\mu(dt, da). \tag{5.2}$$

Therefore, the coefficients b and σ, appearing in the dynamics of the diffusion process X, change according to the pure jump process I, which is associated to the Poisson random measure μ on $\mathbb{R}_+ \times A$. We make the following standard assumption on the forward coefficients b and σ:

(HFC) There exists a constant C such that

$$|b(x, a) - b(x', a')| + |\sigma(x, a) - \sigma(x', a')| \le C\big(|x - x'| + |a - a'|\big),$$

for all $x, x' \in \mathbb{R}^d$ and $a, a' \in \mathbb{R}^q$.

It is well-known that under hypothesis **(HFC)** there exists a unique solution $(X^{t,x,a}, I^{t,a}) = (X_s^{t,x,a}, I_s^{t,a})_{t\le s\le T}$ to (5.1)–(5.2) starting from $(x, a) \in \mathbb{R}^d \times \mathbb{R}^q$ at time $s = t \in [0, T]$. Furthermore, we have the standard estimates: for all $p \ge 2$, there exists some constant C_p such that

$$\mathbb{E}\Big[\sup_{t\le s\le T} \big(|X_s^{t,x,a}|^p + |I_s^{t,a}|^p\big)\Big] \le C_p\big(1 + |x|^p + |a|^p\big), \tag{5.3}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$.

Remark 5.1. Notice that the constant C_p in (5.3) depends only on p, T, and the growth linear condition of b, σ in **(HFC)**. Since the dynamics (5.1) of X is not changed by the change of probability measure $\mathbb{P}^\nu$, $\nu \in \mathcal{V}$ (recall that W remains a Brownian motion under $\mathbb{P}^\nu$), we then see that for all $p \geq 2$:

$$\mathbb{E}^\nu\Big[\sup_{s\leq r\leq T}\big(|X_r^{t,x,a}|^p + |I_r^{t,a}|^p\big)|\mathcal{F}_s\Big] \leq C_p\big(1 + |X_s^{t,x,a}|^p + |I_s^{t,a}|^p\big), \quad t \leq s \leq T,$$

for all $\nu \in \mathcal{V}$, and thus:

$$\int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\leq r\leq T}\big(|X_r^{t,x,a}|^p + |I_r^{t,a}|^p\big)|\mathcal{F}_s\big]\Big]ds \leq C_p(1 + |x|^p + |a|^p), \tag{5.4}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. □

Regarding the reflected BSDE with nonpositive jumps, the terminal condition, the generator function, and the barrier are given respectively by some continuous functions $g : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$. We make the following assumptions on the BSDE coefficients:

(HBC)

(i) The functions g, $f(\cdot, \cdot, 0, 0)$ and u satisfy a polynomial growth condition:

$$\sup_{x\in\mathbb{R}^d, a\in\mathbb{R}^q} \frac{|f(x, a, 0, 0)|}{1 + |x|^h + |a|^h} + \sup_{t\in[0,T], x\in\mathbb{R}^d} \frac{|g(x)| + |u(t, x)|}{1 + |x|^h} < \infty,$$

for some $h \geq 0$.

(ii) There exists some constant C such that:

$$|f(x, a, y, z) - f(x, a, y', z')| \leq C\big(|y - y'| + |z - z'|\big),$$

for all $x \in \mathbb{R}^d$, $a \in \mathbb{R}^q$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$.

(iii) $u(T, x) \geq g(x)$, for all $x \in \mathbb{R}^d$, and there exists a nonincreasing sequence of functions $(u^k)_k$ lying in $C^{1,2}([0, T] \times \mathbb{R}^d)$, and converging pointwisely to u such that the following polynomial growth condition holds

$$\sup_{k\in\mathbb{N}} \sup_{t\in[0,T], x\in\mathbb{R}^d} \frac{\left|\frac{\partial u^k}{\partial t}(t, x)\right| + |D_x u^k(t, x)| + |D_x^2 u^k(t, x)|}{1 + |x|^h} < \infty,$$

for some $h \geq 0$.

In this Markovian framework, the reflected BSDE with nonpositive jumps (2.2)–(2.5) takes the form:

$$\begin{aligned} Y_t = {} & g(X_T) + \int_t^T f(X_s, I_s, Y_s, Z_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \\ & - \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a)\mu(ds, da), \quad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{5.5}$$

with

$$L_t(a) \leq 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \quad a.e. \tag{5.6}$$

and

$$Y_t \leq u(t, X_t), \quad 0 \leq t \leq T, \ a.s. \tag{5.7}$$

$$\int_0^T (u(t, X_t) - Y_{t^-}) dK_t^- = 0, \quad a.s. \tag{5.8}$$

Notice that under **(HFC)** and **(HBC)** the terminal condition $\xi(\omega) = g(X_T(\omega))$, the generator $F(t, \omega, y, z, \ell) = f(X_t(\omega), I_{t^-}(\omega), y, z)$, and the barrier $U_t(\omega) = u(t, X_t(\omega))$ clearly satisfy the standing assumptions 1–4 in Section 2. Let us now discuss about conditions **(H0)** and **(H1)** in the two following remarks.

Remark 5.2. Condition **(H0)** is satisfied in our Markovian framework. Actually, it is shown in Lemma 5.1 in [17] that under **(HFC)** and **(HBC)**(i), (ii), there exists for any initial condition $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, a solution $\{(\bar{Y}_s^{t,x,a}, \bar{Z}_s^{t,x,a}, \bar{L}_s^{t,x,a}, \bar{K}_s^{t,x,a,+}), t \le s \le T\}$ to the BSDE with nonpositive jumps (2.6)–(2.7) when $(X, I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$, with $\bar{Y}_s^{t,x,a} = \bar{v}(s, X_s^{t,x,a})$ for some deterministic function $\bar{v}$ on $[0, T] \times \mathbb{R}^d$ satisfying the polynomial growth condition:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|\bar{v}(t, x)|}{1 + |x|^r} < \infty$$

for some $r \ge 2$. Such solution is constructed by Itô's lemma from a smooth supersolution to

$$-\frac{\partial \bar{v}}{\partial t} - \sup_{a\in A}[\mathcal{L}^a \bar{v} + f(\cdot, a, \bar{v}, \sigma^\intercal(\cdot, a) D_x \bar{v})] \ge 0, \quad \text{on } [0, T) \times \mathbb{R}^d$$
$$\bar{v}(T, x) \ge g(x), \quad x \in \mathbb{R}^d,$$

where

$$\mathcal{L}^a \varphi = b(x, a).D_x \varphi + \frac{1}{2} \mathrm{tr}(\sigma\sigma^\intercal(x, a) D_x^2 \varphi),$$

which can be chosen equal to $\bar{v}(t, x) = \bar{C} e^{\rho(T-t)}(1 + |x|^r)$, with $r = \max(2, h)$, for $\bar{C}$ and ρ positive large enough. $\square$

Remark 5.3. We also observe that assumption **(H1)** is satisfied in the present framework. More precisely, given an initial condition $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, let us consider the process U^k, $k \in \mathbb{N}$, defined by:

$$U_s^k := u^k(s, X_s^{t,x,a}), \quad t \le s \le T.$$

By Itô's formula, U^k is in the form of condition **(H1)**(ii), with

$$\begin{aligned} \upsilon_s^k &= \frac{\partial u^k}{\partial t}(s, X_s^{t,x,a}) + b(X_s^{t,x,a}, I_s^{t,a}).D_x u^k(s, X_s^{t,x,a}) \\ &\quad + \frac{1}{2}\mathrm{tr}\big(\sigma\sigma^\intercal(X_s^{t,x,a}, I_s^{t,a}) D_x^2 u^k(s, X_s^{t,x,a})\big), \\ \vartheta_s^k &= D_x u^k(s, X_s^{t,x,a})^\intercal \sigma(X_s^{t,x,a}, I_s^{t,a}), \end{aligned}$$

for all $t \le s \le T$, a.s., and we clearly see from **(HFC)**, **(HBC)**(iii), and (5.3) that

$$\mathbb{E}\Big[\int_t^T |\upsilon_s^k|^2 ds\Big] + \mathbb{E}\Big[\int_t^T |\vartheta_s^k|^2 ds\Big] < \infty.$$

Moreover, by using (5.4), and again from the polynomial growth conditions on b, σ, F and u^k in **(HFC)**, **(HBC)**, there exists some $p > 2$ such that

$$\sup_{k\in\mathbb{N}} \int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\le r\le T} \big(|U_r^k|^p + |\upsilon_r^k|^p + |\vartheta_r^k|^p\big)|\mathcal{F}_s\big]\Big]ds$$

$$+ \int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\le r\le T} \big|f(X_r^{t,x,a}, I_r^{t,a}, 0, 0)\big|^p|\mathcal{F}_s\big]\Big]ds$$

$$\le C_p(1 + |x|^p + |a|^p)$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. □

From Theorem 3.1, we get, for any initial condition $(t, x, a) \in [0, T]\times\mathbb{R}^d\times\mathbb{R}^q$, the existence of a minimal solution $\{(Y_s^{t,x,a}, Z_s^{t,x,a}, L_s^{t,x,a}, K_s^{t,x,a,+}, K_s^{t,x,a,-}), t \le s \le T\}$ to the Markovian reflected BSDE with nonpositive jumps (5.5)–(5.8) when $(X, I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$. Moreover, as we shall see in the next paragraph, this minimal solution is written in this Markovian context as: $Y_s^{t,x,a} = v(s, X_s^{t,x,a}, I_s^{t,a})$, where v is a real-valued deterministic function defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ by

$$v(t, x, a) := Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.9}$$

We aim at proving that this function v does not depend actually on the argument a in the interior of A, and is connected to the fully nonlinear variational inequality of HJB Isaacs type:

$$\max\Big[-\frac{\partial v}{\partial t} - \sup_{a\in A}\big(\mathcal{L}^a v + f(\cdot, a, v, \sigma^\intercal(\cdot, a)D_x v)\big); v - u\Big] = 0, \quad \text{on } [0, T) \times \mathbb{R}^d \tag{5.10}$$

$$v(T, x) = g(x), \quad x \in \mathbb{R}^d. \tag{5.11}$$

5.2. *Viscosity property of the penalized BSDE*

Let us consider the Markovian penalized BSDE associated to (5.5)–(5.8)

$$\begin{aligned} Y_t^{n,m} &= g(X_T) + \int_t^T f(X_s, I_s, Y_s^{n,m}, Z_s^{n,m})ds \\ &\quad + m\int_t^T \int_A \big(L_s^{n,m}(a)\big)_+ \lambda(da)ds - n\int_t^T \big(u(s, X_s) - Y_s^{n,m}\big)_- ds \\ &\quad - \int_t^T Z_s^{n,m}dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da), \quad 0 \le t \le T, \end{aligned} \tag{5.12}$$

and denote by $\{(Y_s^{n,m,t,x,a}, Z_s^{n,m,t,x,a}, L_s^{n,m,t,x,a}), t \le s \le T\}$ the unique solution to (5.12) when $(X, I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$ for any initial condition $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. From the Markov property of the jump–diffusion process (X, I), we recall from [2] that $Y_s^{n,m,t,x,a} = v^{n,m}(s, X_s^{t,x,a}, I_s^{t,a})$, $t \le s \le T$, for some deterministic function $v^{n,m}$ defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ by

$$v^{n,m}(t, x, a) := Y_t^{n,m,t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.13}$$

Next, for fixed m, let us consider the limiting BSDE of (5.12) as n goes to infinity, that is the reflected BSDE:

$$\begin{aligned} Y_t^m &= g(X_T) + \int_t^T f(X_s, I_s, Y_s^m, Z_s^m)ds + m\int_t^T \int_A \big(L_s^m(a)\big)_+ \lambda(da)ds \\ &- (K_T^{m,-} - K_t^{m,-}) - \int_t^T Z_s^m dW_s - \int_t^T \int_A L_s^m(a)\mu(ds, da), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{5.14}$$

and

$$Y_t^m \le u(t, X_t), \quad 0 \le t \le T, \ a.s. \tag{5.15}$$

$$\int_0^T (u(t, X_t) - Y_{t^-}^m) dK_t^{m,-} = 0, \quad a.s. \tag{5.16}$$

and denote by $\{(Y_s^{m,t,x,a}, Z_s^{m,t,x,a}, L_s^{m,t,x,a}, K_s^{m,t,x,a,+}), t \le s \le T\}$ the unique solution to (5.14)–(5.16) when $(X, I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$ for any initial condition $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. Since $Y^{n,m,t,x,a}$ converges to $Y^{m,t,x,a}$ as n goes to infinity, we see from (5.13) that $Y^{m,t,x,a}$ may be written as $Y_s^{m,t,x,a} = v^m(s, X_s^{t,x,a}, I_s^{t,a})$, $t \le s \le T$, where v^m is the deterministic function defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ by:

$$v^m(t, x, a) := \lim_{n\to\infty} v^{n,m}(t, x, a) = Y_t^{m,t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.17}$$

From the convergence of $Y^{m,t,x,a}$ to the minimal solution $Y^{t,x,a}$, when m goes to infinity, as stated in Theorem 3.1, we deduce that $Y^{t,x,a}$ has indeed the form $Y_s^{t,x,a} = v(s, X_s^{t,x,a}, I_s^{t,a})$, with a deterministic function v defined as the pointwise (nondecreasing) limit of $(v^m)_m$:

$$v(t, x, a) := \lim_{m\to\infty} v^m(t, x, a) = Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.18}$$

From the bounds (3.8)–(3.9), we have for all $m \in \mathbb{N}$: $\underline{v}(t, x, a) \le v^m(t, x, a) \le \bar{v}(t, x)$, $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, where $\underline{v} := v^0$ is associated to the reflected BSDE Y^m for $m = 0$, and $\bar{v}$ is the supersolution as defined in Remark 5.2. By the polynomial growth condition on $\bar{v}$, and also on $\underline{v}$ (see e.g. Lemma 3.2 in [6]), we deduce that v^m, and thus also v by passing to the limit, satisfy a polynomial growth condition: there exist some positive constant C and some $p \ge 2$, such that, for all $m \in \mathbb{N}$:

$$|v^m(t, x, a)| + |v(t, x, a)| \le C(1 + |x|^p + |a|^p), \tag{5.19}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. As expected, for fixed m, the function $v^m = v^m(t, x, a)$ associated to the reflected BSDE with jumps (5.14)–(5.16) is connected to the integro-differential variational inequality:

$$\begin{aligned} \max\Big[&-\frac{\partial v^m}{\partial t} - b(x, a).D_x v^m - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x, a)D_x^2 v^m) - f(x, a, v^m, \sigma^\intercal(x, a)D_x v^m) \\ &- \int_A \big(v^m(t, x, a') - v^m(t, x, a)\big)\lambda(da') \\ &- m\int_A \big(v^m(t, x, a') - v^m(t, x, a)\big)_+ \lambda(da'); v^m(t, x, a) - u(t, x)\Big] = 0, \end{aligned} \tag{5.20}$$

for $(t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$, together with the terminal condition:

$$v^m(T,x,a) = g(x), \quad (x,a) \in \mathbb{R}^d \times \mathbb{R}^q. \tag{5.21}$$

More precisely, we have the following result, which may be proved by extending to the multidimensional case Lemma 3.1 and Theorem 3.4 of [6], and by using Theorem A.1 as comparison theorem for BSDEs with jumps.

Proposition 5.1. *Let assumptions* (**HFC**) *and* (**HBC**) *hold. The function* v^m *in* (5.17) *is a continuous viscosity solution to* (5.20)–(5.21), *i.e., it is continuous on* $[0,T]\times\mathbb{R}^d\times\mathbb{R}^q$, *a viscosity supersolution (resp. subsolution) to* (5.21), *i.e.*

$$v^m(T,x,a) \geq (resp. \ \leq) g(x)$$

for any $(x,a) \in \mathbb{R}^d \times \mathbb{R}^q$, *and a viscosity supersolution (resp. subsolution) to* (5.20), *i.e.*

$$\begin{aligned} \max\Big[&-\frac{\partial \varphi}{\partial t}(t,x,a) - b(x,a).D_x\varphi(t,x,a) - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x,a)D_x^2\varphi(t,x,a)) \\ &- f(x,a,v^m(t,x,a),\sigma^{\intercal}(x,a)D_x\varphi(t,x,a)) - \int_A \big(\varphi(t,x,a') - \varphi(t,x,a)\big)\lambda(da') \\ &- m\int_A \big(\varphi(t,x,a') - \varphi(t,x,a)\big)_+\lambda(da') \ ; \ v^m(t,x,a) - u(t,x)\Big] \geq (resp. \ \leq) \ 0 \end{aligned} \tag{5.22}$$

for any $(t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$ *and any* $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ *such that*

$$(v^m - \varphi)(t,x,a) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi) \quad (resp. \max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi)). \tag{5.23}$$

Remark 5.4. Notice that

$$v^m(t,x,a) \leq u(t,x), \quad \text{for all } (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.24}$$

Indeed, for any $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, since $Y_s^{m,t,x,a} = v^m(s, X_s^{t,x,a}, I_s^{t,a})$, $t \leq s \leq T$, we deduce, from (5.15) that

$$\mathbb{E}\Big[\frac{1}{s-t}\int_t^s \big(v^m(r, X_r^{t,x,a}, I_r^{t,a}) - u(r, X_r^{t,x,a})\big)dr\Big] \leq 0$$

for all $t < s \leq T$. Since $(X^{t,x,a}, I^{t,a})$ is càdlàg, in particular it is right-continuous at time t. Therefore, (5.24) follows from the continuity of v^m and u. □

5.3. HJB Isaacs equation

This paragraph is devoted to the derivation of the equation satisfied in the viscosity sense by the function v in (5.18), by passing to the limit, as m goes to infinity, in the equation satisfied by v^m. The first step is to prove that v does not depend on a, which is basically a consequence of the nonpositive jump constraint:

$$L_s^{t,x,a}(a') = v(s, X_s^{t,x,a}, a') - v(s, X_s^{t,x,a}, I_{s^-}^{t,a}) \leq 0, \qquad d\mathbb{P} \otimes ds \otimes \lambda(da') \quad a.e.$$

providing that the function v is continuous. However, as we do not know a priori that the function v is continuous, we shall rely on (discontinuous) viscosity solution arguments as in [17], and make the following conditions on the set A and the intensity measure λ:

($\mathbf{H}A$) The interior set $\mathring{A}$ of A is connected, and $A = \mathrm{Adh}(\mathring{A})$, the closure of its interior.

($\mathbf{H}\lambda$)

(i) The measure λ supports the whole set $\mathring{A}$: for any $a \in \mathring{A}$ and any open neighborhood $\mathcal{O}$ of a in $\mathbb{R}^q$ we have $\lambda(\mathcal{O} \cap \mathring{A}) > 0$.

(ii) The boundary of A: $\partial A = A \backslash \mathring{A}$, is negligible with respect to λ, i.e., $\lambda(\partial A) = 0$.

Proposition 5.2. *Let assumptions* (**HFC**), (**HBC**), ($\mathbf{H}A$), *and* ($\mathbf{H}\lambda$) *hold. Then the function* v *does not depend on the variable* a *on* $[0, T) \times \mathbb{R}^d \times \mathring{A}$*:*

$$v(t, x, a) = v(t, x, a'), \quad a, a' \in \mathring{A}, \tag{5.25}$$

for all $(t, x) \in [0, T) \times \mathbb{R}^d$.

Proof. The proof borrows most arguments from Section 5.3 in [17], and we only report here the main steps and the points to be modified. First, we see from (5.24), and sending m to infinity that:

$$v \le u \quad \text{on } [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.26}$$

We next show that the function v is a viscosity supersolution to:

$$-|D_a v(t, x, a)| = 0, \quad (t, x, a) \in [0, T) \times \mathbb{R}^d \times \mathring{A}, \tag{5.27}$$

i.e., for any $(t, x, a) \in [0, T) \times \mathbb{R}^d \times \mathring{A}$ and any function $\varphi \in C^{1,2}([0, T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that $(v - \varphi)(t, x, a) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v - \varphi)$, we have

$$-\big|D_a\varphi(t, x, a)\big| \ge 0, \quad \text{i.e. } D_a\varphi(t, x, a) = 0.$$

Indeed, let $(t, x, a) \in [0, T) \times \mathbb{R}^d \times \mathring{A}$ and $\varphi \in C^{1,2}([0, T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that $0 = (v - \varphi)(t, x, a) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v - \varphi)$. We distinguish two cases.

(i) $v(t, x, a) = u(t, x)$. From (5.26), we have

$$\varphi(t, x, a') \le v(t, x, a') \le u(t, x), \quad \forall\, a' \in \mathbb{R}^q$$

and $\varphi(t, x, a) = v(t, x, a) = u(t, x)$. It follows that $\varphi(t, x, a) = \max_{a'\in\mathbb{R}^q} \varphi(t, x, a')$, which yields: $D_a\varphi(t, x, a) = 0$, since $a \in \mathring{A}$.

(ii) $v(t, x, a) < u(t, x)$. We may assume, without loss of generality, that φ satisfies the polynomial growth condition $\sup_{(t,x,a)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^q} \frac{|\varphi(t,x,a)|}{1+|x|^p+|a|^p} < \infty$, with p as in (5.19). Then, for any $\varepsilon > 0$, consider the test function

$$\varphi^\varepsilon(t', x', a') = \varphi(t', x', a') - \varepsilon\big(|t' - t|^2 + |x' - x|^{2p} + |a' - a|^{2p}\big),$$

for all $(t', x', a') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. Since $\varphi^\varepsilon(t, x, a) = \varphi(t, x, a)$ and $\varphi^\varepsilon \le \varphi$, with equality if and only if $(t', x', a') = (t, x, a)$, we see that

$$(v - \varphi^\varepsilon)(t, x, a) = \text{strict} \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v - \varphi^\varepsilon).$$

From the continuity and the growth conditions of v^m and φ, we see that there exists a bounded sequence $(t_m, x_m, a_m)_m$ (we omit the dependence on ε) in $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ such that

$$(v^m - \varphi^\varepsilon)(t_m, x_m, a_m) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi^\varepsilon).$$

By standard arguments, we obtain, up to a subsequence,

$$(t_m, x_m, a_m, v^m(t_m, x_m, a_m)) \overset{m\to\infty}{\longrightarrow} (t, x, a, v(t, x, a)).$$

From the viscosity supersolution property of v^m to (5.22) at (t_m, x_m, a_m), we find

$$\begin{aligned}
&-\frac{\partial \varphi^\varepsilon}{\partial t}(t_m, x_m, a_m) - \mathcal{L}^{a_m}\varphi^\varepsilon(t_m, x_m, a_m) \\
&\quad - f(x_m, a_m, v^m(t_m, x_m, a_m), \sigma^\intercal(x_m, a_m) D_x\varphi^\varepsilon(t_m, x_m, a_m)) \\
&\quad - \int_A \big(\varphi^\varepsilon(t_m, x_m, a') - \varphi^\varepsilon(t_m, x_m, a_m)\big)\lambda(da') \\
&\quad - m\int_A \big(\varphi^\varepsilon(t_m, x_m, a') - \varphi^\varepsilon(t_m, x_m, a_m)\big)_+ \lambda(da') \geq 0.
\end{aligned}$$

By sending m to infinity, and then ε to zero, we conclude as in the proof of Lemma 5.3 in [17] that: $\int_A \big(\varphi(t, x, a') - \varphi(t, x, a)\big)_+ \lambda(da') = 0$, which means under (**H**λ) that $\varphi(t, x, a) = \max_{a' \in \mathbb{R}^q} \varphi(t, x, a')$, i.e., $D_a\varphi(t, x, a) = 0$.

Finally, by arguing exactly as in Lemma 5.4 and Proposition 5.2 of [17], we obtain under the additional condition (**H**A) the non dependence of v on $a \in \mathring{A}$ from the viscosity supersolution property to (5.27). □

From Proposition 5.2, we can define by misuse of notation the function v on $[0, T) \times \mathbb{R}^d$ by:

$$v(t, x) = v(t, x, a), \quad (t, x) \in [0, T) \times \mathbb{R}^d,$$

for any $a \in \mathring{A}$, and we see that v satisfies a polynomial growth condition when x goes to infinity by (5.19). We finally state the viscosity property of v to the HJB Isaacs type equation (5.10)–(5.11). Recall the definition of lower semicontinuous envelope v_*, and upper semicontinuous envelope v^*:

$$v_*(t, x) = \liminf_{\substack{(t', x') \to (t, x) \\ t' < T}} v(t', x') \quad \text{and} \quad v^*(t, x) = \limsup_{\substack{(t', x') \to (t, x) \\ t' < T}} v(t', x'),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Theorem 5.1. *Let assumptions* (**HFC**), (**HBC**), (**H**A), *and* (**H**λ) *hold. Then v is a viscosity solution to* (5.10)–(5.11) *in the sense that it verifies:*

(i) *Viscosity supersolution property:*

$$v_*(T, x) \geq g(x), \tag{5.28}$$

for any $x \in \mathbb{R}^d$, and

$$\begin{aligned}
&\max\Big[-\frac{\partial \varphi}{\partial t}(t, x) - \sup_{a \in A}\Big(\mathcal{L}^a\varphi(t, x) + f\big(x, a, v_*(t, x), \sigma^\intercal(x, a) D_x\varphi(t, x)\big)\Big); \\
&\quad v_*(t, x) - u(t, x)\Big] \geq 0
\end{aligned} \tag{5.29}$$

for any $(t, x) \in [0, T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that $(v_* - \varphi)(t, x) =$ $\min_{[0,T] \times \mathbb{R}^d}(v_* - \varphi)$.

(ii) *Viscosity subsolution property:*

$$v^*(T, x) \leq g(x), \tag{5.30}$$

for any $x \in \mathbb{R}^d$, *and*

$$\max\Big[-\frac{\partial \varphi}{\partial t}(t,x) - \sup_{a\in A}\Big(\mathcal{L}^a\varphi(t,x) + f\big(x, a, v^*(t,x), \sigma^\intercal(x,a)D_x\varphi(t,x)\big)\Big);$$

$$v^*(t,x) - u(t,x)\Big] \leq 0 \tag{5.31}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$ *and any* $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ *such that* $(v^* - \varphi)(t,x) = \max_{[0,T]\times\mathbb{R}^d}(v^* - \varphi)$.

Proof. The proof is quite similar to the proof detailed in Section 5.4 of [17], and we report only the main arguments and the points to be modified with respect to the proof in [17].

• *Viscosity supersolution property* (5.29): Since v is the pointwise limit of the nondecreasing sequence of continuous functions (v^m), and recalling (5.25), we know (see e.g. [1]) that v is lower semicontinuous and so:

$$v(t,x) = v_*(t,x) = \lim_{m\to\infty} v^m(t,x,a), \quad \forall (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathring{A}.$$

Fix now $(t,x) \in [0,T) \times \mathbb{R}^d$, and let $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $(v_* - \varphi)(t,x) = \min_{[0,T]\times\mathbb{R}^d}(v_* - \varphi)$. We already know from (5.26) that $v_* \leq u$, and so distinguish two cases:

(1) $v_*(t,x) = u(t,x)$, then the viscosity supersolution property of v at (t,x) is obviously satisfied.

(2) We have $v(t,x) = v_*(t,x) < u(t,x)$. We may assume, without loss of generality, that φ satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|\varphi(t,x)|}{1+|x|^p} < \infty$, with p as in (5.19). Then, take $a \in \mathring{A}$ and consider, for any $\varepsilon > 0$, the test function

$$\varphi^\varepsilon(t',x',a') = \varphi(t',x') - \varepsilon\big(|t'-t|^2 + |x'-x|^{2p} + |a'-a|^{2p}\big),$$

for all $(t',x',a') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$. Proceeding as in the proof of Proposition 5.2, step (ii), we can find a bounded sequence $(t_m, x_m, a_m)_m$ (we omit the dependence on ε) in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ such that

$$(v^m - \varphi^\varepsilon)(t_m, x_m, a_m) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi^\varepsilon)$$

and, up to a subsequence,

$$(t_m, x_m, a_m, v^m(t_m, x_m, a_m)) \overset{m\to\infty}{\longrightarrow} (t, x, a, v(t,x)).$$

Therefore, recalling that $v(t,x) < u(t,x)$ and using the continuity of u, we see that $v^m(t_m, x_m, a_m) < u(t_m, x_m)$ for m large enough. As a consequence, from the viscosity supersolution property (5.22) of v^m at (t_m, x_m, a_m) with the test function φ^ε, we then get:

$$\begin{aligned}
-&\frac{\partial \varphi^\varepsilon}{\partial t}(t_m, x_m, a_m) - \mathcal{L}^{a_m}\varphi^\varepsilon(t_m, x_m, a_m) \\
&- f(x_m, a_m, v^m(t_m, x_m, a_m), \sigma^\intercal(x_m, a_m)D_x\varphi^\varepsilon(t_m, x_m, a_m)) \\
&- \int_A \big(\varphi^\varepsilon(t_m, x_m, a') - \varphi^\varepsilon(t_m, x_m, a_m)\big)\lambda(da') \\
&- m\int_A \big(\varphi^\varepsilon(t_m, x_m, a') - \varphi^\varepsilon(t_m, x_m, a_m)\big)_+ \lambda(da') \geq 0.
\end{aligned}$$

By sending firstly m to infinity, and afterwards ε to zero, then using that a is arbitrary in $\mathring{A}$, together with the continuity of the coefficients b, σ, and f in the variable a, we obtain the required viscosity supersolution inequality:

$$-\frac{\partial\varphi}{\partial t}(t,x) - \sup_{a\in A}\Big(\mathcal{L}^a\varphi(t,x) + f(x,a,v_*(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x))\Big) \geq 0.$$

• *Viscosity subsolution property* (5.31): By (5.26), we have: $v^* \leq u$ on $[0,T)\times\mathbb{R}^d$, and so it remains to show the viscosity subsolution property of v to:

$$-\frac{\partial v}{\partial t} - \sup_{a\in A}\Big(\mathcal{L}^a v(t,x) + f\big(x,a,v(t,x),\sigma^\intercal(x,a)D_x v(t,x)\big)\Big) \leq 0.$$

This follows by same arguments as in [17] from the viscosity subsolution property of v^m to:

$$\begin{aligned}
&-\frac{\partial v^m}{\partial t} - b(x,a).D_x v^m - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v^m) - f(x,a,v^m,\sigma^\intercal(x,a)D_x v^m)\\
&-\int_A \big(v^m(t,x,a') - v^m(t,x,a)\big)\lambda(da')\\
&-m\int_A \big(v^m(t,x,a') - v^m(t,x,a)\big)_+\lambda(da') \leq 0,
\end{aligned}$$

and by sending m to infinity under $(\mathbf{H}\lambda)$(ii).

• Finally, the viscosity supersolution and subsolution inequalities (5.28), (5.30) are proved by same arguments as in [17]. □

Remark 5.5 (*Zero-Sum Controller/Stopper Game*)**.** Let us consider the particular and important case where the generator $f(x,a)$ does not depend on (y,z), and $u(t,x) = g(x)$. In this case, the nonlinear variational inequality (5.10)–(5.11) is the HJB Isaacs equation associated to the following zero-sum controller-and-stopper game: let us introduce the controlled diffusion process in $\mathbb{R}^d$

$$dX_s^\alpha = b(X_s^\alpha,\alpha_s)ds + \sigma(X_s^\alpha,\alpha_s)dW_s, \tag{5.32}$$

where the control $\alpha \in \mathcal{A}$ is an $\mathbb{F}^W$-progressively measurable process, valued in A, affecting both drift and diffusion coefficient, possibly degenerate. Here $\mathbb{F}^W$ denotes the natural filtration generated by the Brownian motion W. Notice that the laws $\mathbb{P}^\alpha$ of X^α under $\mathbb{P}$, for α varying in $\mathcal{A}$, belong to a non dominated set of probability measures. Given $(t,x)\in[0,T]\times\mathbb{R}^d$, and $\alpha\in\mathcal{A}$, we denote by $\{X_s^{t,x,\alpha}, t\leq s\leq T\}$ the solution to (5.32) starting from x at $s=t$. Let us also define $\mathcal{T}_{t,T}$ as the set of all $\mathbb{F}^W$-stopping times valued in $[t,T]$ for $0\leq t\leq T$, and consider $\Pi_{t,T}$ the set of stopping strategies $\pi : \mathcal{A}\mapsto\mathcal{T}_{t,T}$ satisfying a non-anticipative condition as defined in [3]. The upper and lower value functions of the controller/stopper game are given by:

$$\begin{aligned}
\overline{V}(t,x) &:= \inf_{\pi\in\Pi_{t,T}}\sup_{\alpha\in\mathcal{A}}\mathbb{E}\Big[\int_t^{\pi[\alpha]} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_{\pi[\alpha]}^{t,x,\alpha})\Big],\\
\underline{V}(t,x) &:= \sup_{\alpha\in\mathcal{A}}\inf_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\Big[\int_t^{\tau} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_\tau^{t,x,\alpha})\Big], \quad (t,x)\in[0,T]\times\mathbb{R}^d.
\end{aligned}$$

It is shown in [3] that this game has a value, i.e., $\overline{V} = \underline{V} = V$, and that V is the unique viscosity solution to (5.10)–(5.11) satisfying a polynomial growth condition. By combining this result with Theorem 5.1, this shows that $v = V$. In other words, we have provided a representation of HJB Isaacs equation, arising in zero-sum controller/stopper game, including control on possibly degenerate diffusion coefficient, in terms of minimal solution to reflected BSDE with nonpositive jumps. Furthermore, by combining with the dual game representation in Proposition 4.1, we obtain an original representation for the value function of the controller-and-stopper game:

$$\begin{aligned}
&\inf_{\pi\in\Pi_{0,T}}\sup_{\alpha\in\mathcal{A}}\mathbb{E}\Big[\int_0^{\pi[\alpha]} f(X_t^\alpha,\alpha_t)dt+g(X^\alpha_{\pi[\alpha]})\Big]\\
&\quad=\sup_{\alpha\in\mathcal{A}}\inf_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}\Big[\int_0^{\tau} f(X_t^\alpha,\alpha_t)dt+g(X^\alpha_{\tau})\Big]\\
&\quad=\sup_{\nu\in\mathcal{V}}\inf_{\theta\in\Theta}\mathbb{E}^\nu\Big[\int_0^{T} e^{-\int_0^t\theta_s ds}\big(f(X_t,I_t)+\theta_t g(X_t)\big)dt+e^{-\int_0^T\theta_t dt}g(X_T)\Big]. \quad\square
\end{aligned}$$

6. Conclusion

We introduced in this paper a class of reflected BSDEs with nonpositive jumps and upper obstacle, and showed in the Markov case its connection with fully nonlinear variational inequalities arising typically in controller-and-stopper games with control both on drift and diffusion term. Such representation suggests an original approach for probabilistic numerical schemes of HJB Isaacs equations by discretization and simulation of this reflected BSDE with nonpositive jumps. From a theoretical point of view, an open problem is to relate this class of BSDEs to general controller-and-stopper games in the non Markovian case. A variation of our class of BSDEs would be to consider reflected BSDEs with nonpositive jumps and lower obstacle, which is related to sup sup problem over control and stopping time, and in other words to optimal stopping under nonlinear expectation. Actually, the proof of existence of a minimal solution by a double penalization approach is simpler since it would involve the sum (instead of the difference) of two nondecreasing processes. Another possible extension is the class of doubly reflected BSDEs with nonpositive jumps motivated by Dynkin games under nonlinear expectation (see [19]).

Appendix A. Comparison theorems for sub and supersolutions to BSDEs with jumps

We provide in this section two comparison theorems for BSDEs with jumps. We first recall a comparison theorem for sub and supersolutions to BSDEs driven by the Brownian motion W and the Poisson random measure μ, for which we refer to Theorem 4.2 in [27] (see also Section 4.3 in [27] and Theorem 2.5 in [28]).

Theorem A.1. *Let* $\xi^1,\xi^2\in\mathbf{L^2}(\mathcal{F}_T)$ *be two terminal conditions and let* $F^1,F^2:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbf{L^2}(\lambda)\to\mathbb{R}$ *be two generators satisfying the assumptions* 2(i)–(iii) *of Section* 2*. Let* $(Y^1,Z^1,L^1,K^{1,-})\in\mathbf{S^2}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde\mu)\times\mathbf{K^2}$ *satisfying*

$$\begin{aligned}
Y_t^1&=\xi^1+\int_t^T F^1(s,Y_s^1,Z_s^1,L_s^1)ds-(K_T^{1,-}-K_t^{1,-})\\
&\quad-\int_t^T Z_s^1 dW_s-\int_t^T\int_A L_s^1(a)\mu(ds,da),\quad 0\le t\le T,\ a.s. \qquad (A.1)
\end{aligned}$$

and $(Y^2,Z^2,L^2,K^{2,+})\in\mathbf{S^2}\times\mathbf{L^2}(W)\times\mathbf{L^2}(\tilde\mu)\times\mathbf{K^2}$ *satisfying*

$$\begin{aligned}
Y_t^2&=\xi^2+\int_t^T F^2(s,Y_s^2,Z_s^2,L_s^2)ds+K_T^{2,+}-K_t^{2,+}\\
&\quad-\int_t^T Z_s^2 dW_s-\int_t^T\int_A L_s^2(a)\mu(ds,da),\quad 0\le t\le T,\ a.s. \qquad (A.2)
\end{aligned}$$

If $F^1(t,Y^1_t,Z^1_t,L^1_t) \le F^2(t,Y^1_t,Z^1_t,L^1_t)$ *(resp.* $F^1(t,Y^2_t,Z^2_t,L^2_t) \le F^2(t,Y^2_t,Z^2_t,L^2_t)$*),* $d\mathbb{P}\otimes dt$ *a.e., and* $\xi^1 \le \xi^2$ *a.s., then*

$$Y^1_t \le Y^2_t, \quad 0 \le t \le T, \ a.s.$$

We now state a comparison theorem between a Skorohod solution and a Skorohod supersolution, both driven by the Brownian motion W and the Poisson random measure μ. This slightly extends Theorem 5.2 in [8].

Theorem A.2. *Let* $\xi^1, \xi^2 \in \mathbf{L^2}(\mathcal{F}_T)$ *be two terminal conditions and let* $F^1, F^2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \to \mathbb{R}$ *be two generators satisfying assumptions* 2(i)–(iii) *of Section* 2*. Let* $(Y^1, Z^1, L^1, K^{1,-}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ *satisfying*

$$\begin{aligned} Y^1_t &= \xi^1 + \int_t^T F^1(s,Y^1_s,Z^1_s,L^1_s)ds - (K^{1,-}_T - K^{1,-}_t) \\ &\quad - \int_t^T Z^1_s dW_s - \int_t^T \int_A L^1_s(a)\mu(ds,da), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} &Y^1_t \le U_t, \quad 0 \le t \le T, \ a.s. \\ &\int_0^T (U_{t^-} - Y^1_{t^-})dK^{1,-}_t = 0, \quad a.s. \end{aligned}$$

Furthermore, let $(Y^2, Z^2, L^2, K^{2,+}, K^{2,-}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ *satisfying*

$$\begin{aligned} Y^2_t &= \xi^2 + \int_t^T F^2(s,Y^2_s,Z^2_s,L^2_s)ds + K^{2,+}_T - K^{2,+}_t - (K^{2,-}_T - K^{2,-}_t) \\ &\quad - \int_t^T Z^2_s dW_s - \int_t^T \int_A L^2_s(a)\mu(ds,da), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} &Y^2_t \le U_t, \quad 0 \le t \le T, \ a.s. \\ &\int_0^T (U_{t^-} - Y^2_{t^-})dK^{2,-}_t = 0, \quad a.s. \end{aligned}$$

If $\xi^1 \le \xi^2$ *a.s. and* $F^1(t,Y^1_t,Z^1_t,L^1_t) \le F^2(t,Y^1_t,Z^1_t,L^1_t)$*,* $d\mathbb{P}\otimes dt$ *a.e., then*

$$Y^1_t \le Y^2_t, \quad 0 \le t \le T, \ a.s.$$

Proof. Consider the following penalized BSDEs:

$$\begin{aligned} Y^{n,1}_t &= \xi^1 + \int_t^T F^1(s,Y^{n,1}_s,Z^{n,1}_s,L^{n,1}_s)ds - n\int_t^T (U_s - Y^{n,1}_s)^- ds \\ &\quad - \int_t^T Z^{n,1}_s dW_s - \int_t^T \int_A L^{n,1}_s(a)\mu(ds,da) \end{aligned}$$

and

$$\begin{aligned} Y^{n,2}_t &= \xi^2 + \int_t^T F^2(s,Y^{n,2}_s,Z^{n,2}_s,L^{n,2}_s)ds + K^{2,+}_T - K^{2,+}_t - n\int_t^T (U_s - Y^{n,2}_s)^- ds \\ &\quad - \int_t^T Z^{n,2}_s dW_s - \int_t^T \int_A L^{n,2}_s(a)\mu(ds,da), \end{aligned}$$

for all $0 \le t \le T$, almost surely. By comparison Theorem A.1 we get $Y_t^{n,1} \le Y_t^{n,2}$, for all $n \in \mathbb{N}$. Recalling Remark 3.1, we have that $Y_t^{n,1}$ converges to Y_t^1. It remains to prove the convergence of $Y_t^{n,2}$ towards Y_t^2.

Set $\tilde{Y}^{n,2} := Y^{n,2} + K^{2,+}$, $\tilde{U} := U + K^{2,+}$, $\tilde{\xi}^2 := \xi^2 + K_T^{2,+}$, and $\tilde{F}^2(t, y, z, \ell) := F^2(t, y - K_t^{2,+}, z, \ell)$, for all $0 \le t \le T$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell \in \mathbf{L^2}(\lambda)$, almost surely. Then

$$\tilde{Y}_t^{n,2} = \tilde{\xi}^2 + \int_t^T \tilde{F}^2(s, \tilde{Y}_s^{n,2}, Z_s^{n,2}, L_s^{n,2})ds - n\int_t^T (\tilde{U}_s - \tilde{Y}_s^{n,2})^- ds$$
$$- \int_t^T Z_s^{n,2} dW_s - \int_t^T \int_A L_s^{n,2}(a)\mu(ds, da),$$

for all $0 \le t \le T$, almost surely. Note that $\tilde{\xi}^2$ verifies the square integrability condition and $\tilde{F}^2$ satisfies assumptions 2(i)–(iii) of Section 2. Moreover, $\tilde{U}_T \in \mathbf{S^2}$ and $\tilde{U}_T \ge \tilde{\xi}^2$, almost surely. Now, again from Remark 3.1, we have that $\tilde{Y}^{n,2}$ converges to $\tilde{Y}^2 = Y^2 + K^{2,+}$, and hence $Y^{n,2}$ converges to Y^2. □

Appendix B. Monotonic limit theorem for BSDEs with jumps

We state a monotonic limit theorem for BSDEs driven by the Brownian motion W and the Poisson random measure μ. This extends the monotonic limit Theorem 3.1 in [26] to the jump case.

Theorem B.1. *Let $(Y^m, Z^m, L^m, K^{m,+}, K^{m,-})_m$ be a sequence in $\mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$, with $K^{m,+}$ continuous, solution to:*

$$Y_t^m = \xi + \int_t^T F(s, Y_s^m, Z_s^m, L_s^m)ds + K_T^{m,+} - K_t^{m,+} - (K_T^{m,-} - K_t^{m,-})$$
$$- \int_t^T Z_s^m dW_s - \int_t^T \int_A L_s^m(a)\mu(ds, da), \quad 0 \le t \le T, \ a.s. \tag{B.1}$$

such that

$$\sup_{m\in\mathbb{N}} \Big(\|Y^m\|_{\mathbf{S^2}} + \|Z^m\|_{\mathbf{L^2(W)}} + \|L^m\|_{\mathbf{L^2}(\tilde{\mu})} + \|K^{m,+}\|_{\mathbf{S^2}} + \|K^{m,-}\|_{\mathbf{S^2}} \Big) < \infty, \tag{B.2}$$

and $(Y^m)_m$ converges increasingly to $Y \in \mathbf{S^2}$. Suppose also that the sequence $(K^{m,-})_m$ satisfies:

$$K_t^{m,-} - K_s^{m,-} \le K_t^{m+1,-} - K_s^{m+1,-}, \quad 0 \le s \le t \le T, \ a.s., \tag{B.3}$$

for all $m \in \mathbb{N}$. Then there exists $(Z, L, K^+, K^-) \in \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ such that

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, L_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-)$$
$$- \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a)\mu(ds, da), \quad 0 \le t \le T, \ a.s. \tag{B.4}$$

Here (Z, L) is the strong (resp. weak) limit of $(Z^m, L^m)_m$ in $\mathbf{L^p(W)} \times \mathbf{L^p}(\tilde{\mu})$, with $p \in [1, 2)$, (resp. in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$). Furthermore, K_t^+ is the weak limit of $(K_t^{m,+})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, and $(K_t^{m,-})_m$ converges strongly up to K_t^- in $\mathbf{L^2}(\mathcal{F}_t)$, for any $0 \le t \le T$.

Proof. *Step* 1. *Limit BSDE.* From the boundedness condition (B.2) and the Hilbert structure of $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{L^2(0,T)}$, there exists a subsequence, $(Z^{m_k}, L^{m_k}, F(\cdot, Y^{m_k}, Z^{m_k}, L^{m_k}))_k$ which converges weakly to some $(Z, L, G) \in \mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{L^2(0,T)}$. Thus, for each stopping time $\tau \leq T$, the following weak convergences hold in $\mathbf{L^2}(\mathcal{F}_\tau)$ as $k \to \infty$:

$$\int_0^\tau F(s, Y_s^{m_k}, Z_s^{m_k}, L_s^{m_k})ds \rightharpoonup \int_0^\tau G(s)ds,$$
$$\int_0^\tau Z_s^{m_k}dW_s \rightharpoonup \int_0^\tau Z_s dW_s,$$
$$\int_0^\tau \int_A L_s^{m_k}(a)\mu(ds, da) \rightharpoonup \int_0^\tau \int_A L_s(a)\mu(ds, da).$$

From (B.3), there exists $K^- \in \mathbf{K^2}$, such that K_t^- is the strong limit of $(K_t^{m_k,-})_k$ in $\mathbf{L^2}(\mathcal{F}_t)$ for all $0 \leq t \leq T$. In particular, $K_\tau^{m_k,-} \rightharpoonup K_\tau^-$. Moreover, since

$$K_\tau^{m_k,+} = Y_0^{m_k} - Y_\tau^{m_k} + K_\tau^{m_k,-} - \int_0^\tau F(s, Y_s^{m_k}, Z_s^{m_k}, L_s^{m_k})ds + \int_0^\tau Z_s^{m_k}dW_s + \int_0^\tau \int_A L_s^{m_k}(a)\mu(ds, da)$$

we also have the weak convergence in $\mathbf{L^2}(\mathcal{F}_\tau)$

$$K_\tau^{m_k,+} \rightharpoonup K_\tau^+ := Y_0 - Y_\tau + K_\tau^- - \int_0^\tau G(s)ds + \int_0^\tau Z_s dW_s + \int_0^\tau \int_A L_s(a)\mu(ds, da),$$

as $k \to \infty$. Note that $\mathbb{E}[(K_T^+)^2] < \infty$ and for any two stopping times $0 \leq \sigma \leq \tau \leq T$, we have $K_\sigma^+ \leq K_\tau^+$ since $K_\sigma^{m,+} \leq K_\tau^{m,+}$. From this it follows that K^+ is an increasing process. Observe now that we have obtained the following decomposition for Y:

$$Y_t = Y_0 - \int_0^\tau G(s)ds - K_t^+ + K_t^- + \int_0^t Z_s dW_s + \int_0^t \int_A L_s(a)\mu(ds, da). \tag{B.5}$$

Since the processes $K^{m_k,+}$ and $K^{m_k,-}$ are predictable, we deduce that K^+ and K^- are also predictable. Besides, by Lemmas 3.1 and 3.2 of [26], K^+, K^- and Y are càdlàg processes. Thus, in the above decomposition of Y in (B.5), the components Z and L are unique. As a matter of fact, the uniqueness of Z follows by identifying the Brownian parts and finite variation parts. The uniqueness of L is then obtained by identifying the predictable parts and by recalling that the jumps of μ are totally inaccessible. From the uniqueness of (Z, L), it follows that the whole sequence $(Z^m, L^m)_m$ converges weakly to (Z, L) in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$.

Step 2. *Properties of the process* K^+. We establish that the contribution of the jumps of K^+ is mainly concentrated within a finite number of intervals with sufficiently small total length. More precisely, we apply Lemma 2.3 in [25] to K^+. Consequently, as in Lemma 2.3 in [25], for any $\delta, \varepsilon > 0$, there exists a finite number of pairs of stopping times (σ_k, τ_k), $k = 0, \ldots, N$, with $0 < \sigma_k \leq \tau_k \leq T$, such that all the intervals $(\sigma_k, \tau_k]$ are disjoint and

$$\mathbb{E}\sum_{k=0}^N (\tau_k - \sigma_k) \geq T - \frac{\varepsilon}{2}, \qquad \mathbb{E}\sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} |\Delta K_t^+|^2 \leq \frac{\varepsilon\delta}{3}. \tag{B.6}$$

We should note that in [25] the filtration is Brownian, therefore it is continuous, and hence each stopping time σ_k can be approximated by a sequence of announceable stopping times. In our case the stopping times σ_k's are constructed as the successive times of jumps of the predictable process K^+ with size bigger than some given positive level, therefore each σ_k is a predictable stopping time and the approximation of σ_k by announceable stopping times is again possible. We can thus argue exactly the same way as in Lemma 2.3 in [25] to derive both estimates in (B.6).

Step 3. Strong convergence. By applying Itô's formula to $|Y_t^m - Y_t|^2$ on a subinterval $(\sigma, \tau]$, with $0 \leq \sigma \leq \tau \leq T$, two stopping times, and recalling that $K^{m,+}$ is continuous, we obtain:

$$\begin{aligned}
&\mathbb{E}\big|Y_\tau^m - Y_\tau\big|^2 \\
&\quad = \mathbb{E}\big|Y_\sigma^m - Y_\sigma\big|^2 + \mathbb{E}\int_\sigma^\tau |Z_s^m - Z_s|^2 ds + \mathbb{E}\int_\sigma^\tau \int_A |L_s^m(a) - L_s(a)|^2 \lambda(da) ds \\
&\qquad + 2\mathbb{E}\int_\sigma^\tau (Y_s^m - Y_s)\big(G(s) - F(s, Y_s^m, Z_s^m, L_s^m)\big) ds \\
&\qquad + \mathbb{E}\sum_{t \in (\sigma,\tau]} |\Delta K_t^+ - \Delta K_t^- + \Delta K_t^{m,-}|^2 \\
&\qquad + 2\mathbb{E}\int_{(\sigma,\tau]} (Y_{s-}^m - Y_{s-}) dK_s^+ - 2\mathbb{E}\int_{(\sigma,\tau]} (Y_{s-}^m - Y_{s-}) dK_s^- \\
&\qquad - 2\mathbb{E}\int_{(\sigma,\tau]} (Y_s^m - Y_s) dK_s^{m,+} + 2\mathbb{E}\int_{(\sigma,\tau]} (Y_{s-}^m - Y_{s-}) dK_s^{m,-} \\
&\qquad + 2\mathbb{E}\int_{(\sigma,\tau]} \int_A \big(Y_s^m - Y_s)(L_s^m(a) - L_s(a))\lambda(da) ds.
\end{aligned} \tag{B.7}$$

Now, let us write

$$\begin{aligned}
\int_{(\sigma,\tau]} (Y_{s-}^m - Y_{s-}) dK_s^+ = &\int_{(\sigma,\tau]} \big(Y_{s-}^m + \Delta K_s^{m,-} - Y_{s-} + \Delta K_t^+ - \Delta K_s^-\big) dK_s^+ \\
&- \sum_{t \in (\sigma,\tau]} (\Delta K_t^+)^2 + \sum_{t \in (\sigma,\tau]} \Delta K_t^+ \Delta(K_s^- - K_s^{m,-}),
\end{aligned}$$

and observe that

$$\int_{(\sigma,\tau]} (Y_{s-}^m - Y_{s-}) d(K_s^- - K_s^{m,-}) \leq 0, \quad \text{and} \quad \int_{(\sigma,\tau]} (Y_s^m - Y_s) dK_s^{m,+} \leq 0.$$

Therefore, by using the inequality $2ab \geq -2b^2 - a^2/2$, we obtain from (B.7)

$$\begin{aligned}
&\mathbb{E}\int_\sigma^\tau |Z_s^m - Z_s|^2 ds + \frac{1}{2}\mathbb{E}\int_\sigma^\tau \int_A |L_s^m(a) - L_s(a)|^2 \lambda(da) ds \\
&\quad \leq \mathbb{E}\big|Y_\tau^m - Y_\tau\big|^2 + 2\lambda(A)\mathbb{E}\int_\sigma^\tau \big|Y_s^m - Y_s\big|^2 ds \\
&\qquad + 2\mathbb{E}\int_\sigma^\tau \big|Y_s^m - Y_s\big|\big|G(s) - F(s, Y_s^m, Z_s^m, L_s^m)\big| ds \\
&\qquad - 2\mathbb{E}\int_{(\sigma,\tau]} \big(Y_{s-}^m + \Delta K_s^{m,-} - Y_{s-} + \Delta K_s^+ - \Delta K_s^-\big) dK_s^+ + 2\mathbb{E}\sum_{t \in (\sigma,\tau]} |\Delta K_t^+|^2 \\
&\qquad - 2\mathbb{E}\sum_{t \in (\sigma,\tau]} \Delta K_t^+ \Delta(K_s^- - K_s^{m,-}) - \mathbb{E}\sum_{t \in (\sigma,\tau]} |\Delta K_t^+ - \Delta K_t^- + \Delta K_t^{m,-}|^2,
\end{aligned}$$

$$\leq \mathbb{E}\big|Y_\tau^m - Y_\tau\big|^2 + 2\lambda(A)\mathbb{E}\int_\sigma^\tau \big|Y_s^m - Y_s\big|^2 ds$$
$$+ 2\mathbb{E}\int_\sigma^\tau \big|Y_s^m - Y_s\big|\big|G(s) - F(s, Y_s^m, Z_s^m, L_s^m)\big|ds$$
$$- 2\mathbb{E}\int_{(\sigma,\tau]} \big(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-\big)dK_s^+ + \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K_t^+|^2$$

by using the inequality $2a^2 - 2ab - (a-b)^2 \leq a^2$. We know that the first two terms on the right-hand side of (B.7) converge to zero as $m \to \infty$. The third term also tends to zero since $(G(\cdot) - F(\cdot, Y^m, Z^m, L^m))_m$ is bounded in $\mathbf{L^2(0, T)}$, and so by Cauchy–Schwarz inequality

$$\mathbb{E}\int_0^T \big|Y_s^m - Y_s\big|\,\big|G(s) - F(s, Y_s^m, Z_s^m, L_s^m)\big|ds \to 0, \quad \text{as } m \to \infty.$$

For the fourth term, since $K^{m,-}$ is predictable, the predictable projection of Y^m is ${}^pY_t^m = Y_{t^-}^m + \Delta K_t^{m,-}$. Similarly, from (B.5) and since K^+ and K^- are predictable processes, we see that ${}^pY_t = Y_{t^-} - \Delta K_t^+ + \Delta K_t^-$. By the dominated convergence theorem, we obtain

$$\lim_{m\to\infty} \mathbb{E}\int_{(\sigma,\tau]} \big(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-\big)dK_s^+ = 0.$$

For the last term in (B.7), we exploit the results in (B.6), regarding the contribution of the jumps of K^+. More precisely, we apply estimate (B.7) for each $\sigma = \sigma_k$ and $\tau = \tau_k$, with σ_k, τ_k defined in Step 2, and then take the sum over $k = 0, \ldots, N$. It follows that

$$\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k} |Z_s^m - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}\int_A |L_s^m(a) - L_s(a)|^2 \lambda(da)ds$$
$$\leq \sum_{k=0}^N \mathbb{E}\big|Y_{\tau_k}^m - Y_{\tau_k}\big|^2 + 2\lambda(A)\mathbb{E}\int_0^T \big|Y_s^m - Y_s\big|^2 ds$$
$$+ 2\mathbb{E}\int_0^T \big|Y_s^m - Y_s\big|\,\big|G(s) - F(s, Y_s^m, Z_s^m, L_s^m)\big|ds + \sum_{k=0}^N \mathbb{E}\sum_{t\in(\sigma_k,\tau_k]} |\Delta K_t^+|^2$$
$$- 2\sum_{k=0}^N \mathbb{E}\int_{(\sigma_k,\tau_k]} \big(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-\big)dK_s^+.$$

From the above convergence results, we deduce that

$$\limsup_{m\to\infty}\Big(\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k} |Z_s^m - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}\int_A |L_s^m(a) - L_s(a)|^2 \lambda(da)ds\Big)$$
$$\leq \sum_{k=0}^N \mathbb{E}\sum_{t\in(\sigma_k,\tau_k]} |\Delta K_t^+|^2 \leq \frac{\varepsilon\delta}{3}.$$

Therefore, following the same steps as in the proof of Theorem 2.1 in [25], we deduce that the sequences $(Z^m)_m$ and $(L^m)_m$ converge in measure, respectively, to Z and L. Since they are bounded, respectively, in $\mathbf{L^2(W)}$ and $\mathbf{L^2}(\tilde{\mu})$, they are uniformly integrable in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$, for any $p \in [1, 2)$. Thus, $(Z^m)_m$ and $(L^m)_m$ converge strongly to Z and L in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$, respectively.

By the Lipschitz condition on F, we also have the strong convergence in $\mathbf{L^p(0,T)}$ of $(F(\cdot, Y^m, Z^m, L^m))_m$ to $F(\cdot, Y, Z, L)$. Since $G(\cdot)$ is the weak limit of $(F(\cdot, Y^m, Z^m, L^m))_m$ in $\mathbf{L^2(0,T)}$, we deduce that $G(\cdot) = F(\cdot, Y, Z, L)$. Therefore we obtain that (Y, Z, L, K^+, K^-) satisfies the BSDE (B.4). □

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