A Youla–Kučera Parameterization Approach to Output Feedback Relatively Optimal Control

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1. Introduction

Optimal control under constraints can result in compensators hard to implement especially in the presence of constraints [1]. The most popular approach to solve constrained control problems is the well known model predictive control [2, 3, 4], which requires considerable computational effort even in its explicit version [5].

An optimal control typically is designed to produce optimal transients for any initial condition. On the other hand, there are many examples of systems whose main goal is performing specific tasks or "nominal operations". This is the case of elevators, bascule bridges, automatic gates, cranes and robots. In these cases the control system has to assure stability in any condition, while optimality is important only for the nominal operation.

This observation motivated the introduction of the concept of relatively optimal control [6, 7]. Basically, the relatively optimal control problem is to design a control which is optimal for a specific initial condition and is stabilizing for all other initial states. This problem has been solved in discrete-time [6, 7] and continuous-time [8] where state-feedback solutions have been proposed. Actually in [7] it has been shown that an output feedback solution is possible provided that an observer is adopted which has to be suitably initialized.

Observer initialization is significant in several situations in which the system has a precise (and known) starting time. It is not suitable in other circumstances. For instance, one could be interested in optimizing a certain impulse response associated with an input matrix E. This is obviously equivalent to the optimization of the transient with initial condition $x(0) = E$. But the idea of observer initialization becomes questionable if one wishes to optimize the impulse response.

The main idea of this work is to propose a control scheme with two fundamental steps.

- **Open-loop:** A trajectory parameterization in terms of modal function is introduced. The modes are assigned by means of a stable matrix P whose eigenvalues are fixed. An optimal open-loop trajectory with the assigned initial condition is designed which is a linear combination of the modes. No restrictions on the type of optimality criterion. Any of such parameterized trajectories can be considered as "optimal".

- **Closed-loop:** An output feedback compensator is designed which has to be stabilizing and it must produce the "optimal trajectory" for the nominal initial condition without observer initialization (i.e., the observer initial state is

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0). Such a compensator is "relatively optimal".

The essential features of the proposed framework are that:

1. If the considered constraints and cost functional are convex, the open-loop design requires convex optimization.
2. The existence of the relatively optimal output feedback compensator can be checked by solving a set of linear algebraic equations which, under some assumptions on the number of measured outputs, are generically solvable.

The proposed solution is based on the Youla–Kučera [9, 10] parameterization of all stabilizing compensators. The essential difference with respect the standard convex-optimization-based control synthesis [11, 12, 13] is that there is a full separation between the optimal trajectory design, which can be any open loop trajectory computed regardless of the specific measured output (i.e., the C matrix) and with no further constraint than being an open-loop feasible system trajectory, function of the assigned modes and the nominal initial condition. For instance it could be the trajectory produced by the optimal state-feedback LQ regulator. Then, if the equations are solvable, the provided output feedback compensator produces *exactly the optimal state feedback trajectory for the nominal initial condition*.

We stress that LQ is a possibility, but the optimization criterion is extremely general. It is possible to consider different types of constraints: pointwise-in-time output or input constraints, frequency domain constraints or integral constraints. The objective function can be any convex linear or quadratic cost. Specific problems such as minimal arrival time to an assigned neighbourhood and model matching can also be addressed.

2. Problem Statement

Let us consider a continuous-time linear time-invariant plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the plant, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the controlled output, and A, B, C are real constant matrices of appropriate dimensions. We assume that the initial state x_0 is non-zero and given, and the system is in minimal form (reachable and observable).

In order to handle the specification on the closed-loop poles in a compact way, let us introduce a matrix $P \in \mathbb{R}^{(n+q)\times(n+q)}$ whose eigenvalues are $\lambda_i \in \mathbb{C}$, $i = 1, 2, \dots, n+q$. We assume that matrix P is "cyclic", i.e., the minimal and characteristic polynomials coincide. It is well known that this choice of P is always possible (for instance P can be chosen in companion form), and implies that there always exists a vector $\xi_0 \in \mathbb{R}^{(n+q)}$ such that the pair (P, ξ_0) is reachable.

Then, we consider a behaviour generated by

$$\dot{\xi}(t) = P\xi(t), \qquad \xi(0) = \xi_0 \tag{2}$$

where $\xi(t) \in \mathbb{R}^{(n+q)}$. That is,

$$\xi(t) = e^{Pt}\xi_0. \tag{3}$$

Since (P, ξ_0) is reachable, $\xi(t)$ contains all behaviours of the modes specified. Therefore we can write an input-state trajectory, say $u^o(t)$, $x^0(t)$, as

$$x^o(t) = X\xi(t) \tag{4}$$

$$u^o(t) = U\xi(t) \tag{5}$$

where $X \in \mathbb{R}^{n\times(n+q)}$ and $U \in \mathbb{R}^{m\times(n+q)}$ are appropriate real constant matrices. Indeed, the representation (4) (5) can be characterized by the coefficient matrices of the plant and P. From (1) and (2), we have

$$\begin{aligned} \dot{x}^o(t) &= AX\xi(t) + BU\xi(t), \\ \dot{x}^o(t) &= X\dot{\xi}(t) = XP\xi(t), \end{aligned}$$

and hence $(AX + BU - XP)e^{Pt}\xi_0 = 0$, $\forall t \geq 0$. Since $\xi(t)$ can take all directions in $\mathbb{R}^{(n+q)}$ we must impose

$$AX + BU = XP, \qquad x_0 = X\xi_0 \tag{6}$$

The equations above is the starting point for the two problems that we aim to tackle. The first one is the optimization problem (high level control problem).

Problem 1. Optimization. Given a stable P with assigned poles, find U, X, ξ_0 satisfying (6), such that the pair $(u^o(t), x^o(t))$ with initial condition $x^o(0) = x_0$ given by equations (4) (5) is optimal under some specified criterion. ∎

The second problem is to find a compensator that realizes the optimal pair, solution of the previous problem, from the knowledge of U, X, ξ_0 satisfying (6). In other word, considering the closed-loop system in Fig. 1, we aim at finding a output feedback compensator such that, assuming initial state of the compensator equal to zero, it is able to impose the optimal pair $(u^o(t), x^o(t))$ relative to x_0.

Problem 2. Realization. Suppose that P is Hurwitz stable and X, U, ξ_0 have been chosen in accordance to any performance index. Find a linear stabilizing compensator such that, for $x(0) = x_0$ and *initial conditions of the compensator equal to zero*, produces the optimal transient pair $(x(t), u(t))$ satisfying (4) and (5). ∎

3. Realization of the relatively optimal control

In this section we do not care on how the trajectory has been chosen. Out main goal is to give conditions such that the *assigned* trajectories

$$x^o(t) = X\xi(t), \qquad u^o(t) = U\xi(t), \qquad \xi(t) = e^{Pt}\xi_0$$

under conditions (6) can be achieved by an output feedback compensator with zero compensator initial conditions, in particular *without observer state initialization*. We briefly remind some results proposed in [7, 8] concerning state feedback, and then we propose the main result of the paper concerning output feedback.

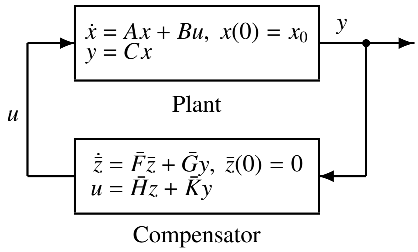

Figure 1: Closed-loop system

3.1. Solution based on state feedback

For the sake of completeness we recall here the solution based on state feedback proposed in [6, 7, 8].

Assume that the compensator which we want to design is a dynamic state feedback (i.e., $C = I$ in Fig. 1)

$$\begin{aligned} \dot{\bar{z}}(t) &= \bar{F}\bar{z}(t) + \bar{G}x(t), \qquad \bar{z}(0) = 0 \\ u(t) &= \bar{H}\bar{z}(t) + \bar{K}x(t) \end{aligned} \tag{7}$$

where $\bar{z}(t) \in \mathbb{R}^q$ is the state of the compensator, and $\bar{F}$, $\bar{G}$, $\bar{H}$, and $\bar{K}$ are real constant matrices of appropriate dimensions.

Theorem 1. *Suppose that matrices (X, U) are given and that X has full row rank. Let Z be such that*

$$\det \begin{bmatrix} X \\ Z \end{bmatrix} \neq 0, \qquad 0 = Z\xi_0 \tag{8}$$

Then a compensator which solves Problem 2 is given by

$$\begin{bmatrix} \bar{K} & \bar{H} \\ \bar{G} & \bar{F} \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1}$$

where $V = ZP$. Moreover, the closed loop matrix is similar to P, i.e.,

$$\begin{bmatrix} A + B\bar{K} & B\bar{H} \\ \bar{G} & \bar{F} \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} P.$$

Remark 1. It has been shown [7] that an output feedback solution can be found by using an observer which can be exactly initialized so that the initial estimation error is equal to zero. Initializing an observer is not always possible. For instance in the case of an impulsive disturbance it would require the a priori knowledge of the time in which the impulse is going to occur.■

3.2. A solution for the output feedback based on pole assignment and Youla–Kučera parameterization

Let us consider the following observer-based Youla–Kučera parameterization of all stabilizing compensator:

$$\dot{w}(t) = Qw(t) - Ly(t) + Bu(t) \tag{9}$$
$$u(t) = Jw(t) + v(t) \tag{10}$$

where $w(t) \in \mathbb{R}^n$ is the observer state,

$$Q = A + LC$$

is an arbitrary stable observer state matrix, J is a matrix such that $A + BJ$ is stable, and $v(t)$ is the output of the Youla–Kučera parameter (to be found) described by

$$\dot{z}(t) = \mathcal{F}z(t) + \mathcal{G}\sigma(t) \tag{11}$$
$$v(t) = \mathcal{H}z(t) + \mathcal{K}\sigma(t) \tag{12}$$

with $z(t) \in \mathbb{R}^s$, for some $s \in \mathbb{N}$ and

$$\sigma(t) = Cw(t) - y(t) = C\left(w(t) - x(t)\right).$$

With these definitions, the compensator state is $\bar{z}(t) = \begin{bmatrix} z^\top(t) & w^\top(t) \end{bmatrix}^\top$ and the compensator matrices (see Fig. 1) are

$$\bar{F} = \begin{bmatrix} \mathcal{F} & \mathcal{G}C \\ B\mathcal{H} & Q + BJ + B\mathcal{K}C \end{bmatrix}, \quad \bar{G} = -\begin{bmatrix} \mathcal{G} \\ L + B\mathcal{K} \end{bmatrix}$$
$$\bar{H} = \begin{bmatrix} \mathcal{H} & J + \mathcal{K}C \end{bmatrix}, \qquad \bar{K} = -\mathcal{K}.$$

Define the state error variable

$$r(t) = w(t) - x(t) \tag{13}$$

so that $\sigma(t) = Cr(t)$. Therefore, the closed-loop system generating the state and input trajectories of the original system is described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \\ \dot{r}(t) \end{bmatrix} = \begin{bmatrix} A + BJ & B\mathcal{H} & B(J + \mathcal{K}C) \\ 0 & \mathcal{F} & \mathcal{G}C \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \\ r(t) \end{bmatrix},$$
$$\begin{bmatrix} x(0) \\ z(0) \\ r(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \\ -x_0 \end{bmatrix} \tag{14}$$

$$u(t) = \begin{bmatrix} J & \mathcal{H} & J + \mathcal{K}C \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \\ r(t) \end{bmatrix}. \tag{15}$$

Notice that Q is fixed so that $r(t) = -e^{Qt}x_0$, $t \geq 0$, is a fixed trajectory of the estimation error.

Remark 2. As commented earlier, if one could choose the initial state of the filter, it would be sufficient to set $w(0) = x_0$, so that $r(0) = 0$ in equation (14). In this way the movement of the state variable would be described by $\dot{x}(t) = (A + BJ)x(t)$ and the "relatively optimal" property $x(t) = X\xi(t)$, $u(t) = U\xi(t)$ would directly follow from (6). ■

Now we specify the rules of the game.

- The compensator must be stabilizing;
- For $x(0) = x_0$ assigned and for $w(0) = 0$ and for $z(0) = 0$ the trajectory has to be (4) and (5) for the given X and U.

Notice that the first requirements, i.e., asymptotic stability of the closed-loop system, is inherited by the choice of a stable Youla–Kučera parameter. Indeed, as apparent from (14), stability of $\mathcal{F}$ together with the choice of J and L such that $A + BJ$ and $Q = A + LC$ are stable, entail stability of the closed-loop system (and vice versa). As for the second point, it is clear that the time evolution of $x(t)$ and $u(t)$ should depend only on $\xi(t)$. As such, take a new variable

$$\tilde{z}(t) = z(t) - \mathcal{M}\xi(t) - \mathcal{N}r(t)$$

so that

$$\begin{aligned}
\dot{x}(t) &= (A + BJ)x(t) + B\mathcal{H}\mathcal{M}\xi(t) + B\mathcal{H}\tilde{z}(t) \\
&\quad + B(J + \mathcal{K}C + \mathcal{H}\mathcal{N})r(t), \qquad x(0) = x_0 \\
\dot{\tilde{z}}(t) &= \mathcal{F}\tilde{z}(t) + (\mathcal{F}\mathcal{M} - \mathcal{M}P)\xi(t) + (\mathcal{F}\mathcal{N} + \mathcal{G}C - \mathcal{N}Q)r(t), \\
&\qquad \tilde{z}(0) = \mathcal{N}x_0 - \mathcal{M}\xi_0 \\
u(t) &= Jx(t) + \mathcal{H}\tilde{z}(t) + \mathcal{H}\mathcal{M}\xi(t) + (\mathcal{H}\mathcal{N} + J + \mathcal{K}C)r(t)
\end{aligned}$$

Therefore if $\mathcal{F}$ (Hurwitz matrix), $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{M}$, $\mathcal{N}$ satisfy

$$0 = \mathcal{F}\mathcal{N} + \mathcal{G}C - \mathcal{N}Q \tag{16}$$
$$0 = \mathcal{F}\mathcal{M} - \mathcal{M}P \tag{17}$$
$$0 = J + \mathcal{K}C + \mathcal{H}\mathcal{N} \tag{18}$$
$$0 = U - JX - \mathcal{H}\mathcal{M} \tag{19}$$
$$0 = \mathcal{N}x_0 - \mathcal{M}\xi_0 \tag{20}$$

we have that $x^o(t)$, $u^o(t)$ are solutions of (4), (5). Indeed, thanks to (6)

$$\dot{x}^o(t) = (AX + BU)\xi(t) = XP\xi(t) = X\dot{\xi}(t), \qquad X\xi_0 = x_0$$

so that $x^o(t) = X\xi(t)$ and

$$u^o(t) = (JX + HM)\xi(t) = U\xi(t).$$

As apparent, equation (17) just says that in the case $s \geq n + q$ matrix P is a restriction of $\mathcal{F}$ with respect to the subspace given by the columns span of $\mathcal{M}$. We can impose that $\mathcal{M}$ is full column rank. Notice however that all equations above (except (20)) are nonlinear in the 6 unknowns $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$.

3.3. The case $s \geq n + q$

Notice that, if we fix $\mathcal{F} = F$ (Hurwitz stable) and $\mathcal{M} = M$ (full column rank) satisfying (17), it is possible to write down a solution for $\mathcal{H}$ as follows

$$\mathcal{H} = H = (U - JX)(M^\top M)^{-1}M^\top \tag{21}$$

so that the equations to be satisfied becomes

$$0 = F\mathcal{N} + \mathcal{G}C - \mathcal{N}Q \tag{22}$$
$$0 = J + \mathcal{K}C + (U - JX)(M^\top M)^{-1}M^\top\mathcal{N} \tag{23}$$
$$0 = \mathcal{N}x_0 - M\xi_0 \tag{24}$$

These equations are linear in the unknowns $\mathcal{N}$, $\mathcal{G}$, $\mathcal{K}$.

Remark 3. Setting $M = \begin{bmatrix} I_{n+q} \\ 0 \end{bmatrix}$ is not a restriction with respect to considering a generic full rank matrix $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ with invertible M_1 of dimensions $n + q \times n + q$. It is a matter of a state transformation of the Youla–Kučera parameter. Indeed if the original equations have a solution with some full rank $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ with certain matrices F, G, H K, N, then the same equations have a solution with $\hat{M} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\hat{F} = T^{-1}FT$, $\hat{G} = T^{-1}G$, $\hat{H} = HT$, $\hat{N} = T^{-1}N$, $\hat{K} = K$, with $T = \begin{bmatrix} M_1 & 0 \\ M_2 & I \end{bmatrix}$.

Therefore we get a linear system with N_e equations and N_u unknowns with

$$N_e = s \times n + m \times n + s, \qquad N_u = s \times n + s \times p + m \times p.$$

Then the generic solvability is given by $N_u \geq N_e$, i.e.,

$$s \times (p - 1) \geq m \times (n - p) \tag{25}$$

Note that for SISO systems, being $p = m = 1$, we would get $n = 1$. This means that for SISO systems is generically impossible to match a generic trajectory of the state and the input. We conclude the section by stating the main result.

Theorem 2. *For a fixed choice of $\mathcal{F} = F$ stable and $\mathcal{M} = M$ full rank satisfying* (17) *and $s \geq n + q$, if equations* (22)–(24) *have a solution $\mathcal{N} = N$, $\mathcal{G} = G$, $\mathcal{K} = K$, then (F, G, H, K) is a a stable Youla–Kučera parameter and the corresponding compensator of order $n + s$ satisfies Problem 2, i.e., it is relatively optimal.* ∎

It is important to state that the closed-loop system obtained by the realization design is always asymptotically stable, for all possible data X, U and ξ_0 satisfying (6), for which a solution of the linear equations above (22)–(24) exist. Therefore there are no problems of possible cancellations of unstable zeros. The interpretation in terms of *model matching* is relatively simple. Letting $E = x_0$, the choice of P, X, U, ξ_0 satisfying (6) corresponds to finding a stabilizing output-feedback controller such that the transfer function $R_{ref}(s) = \bar{C}X(sI - P)^{-1}\xi_0$ from w to η is matched, where

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\
\eta(t) &= Dx(t).
\end{aligned}$$

This is possible if $R_{ref}(s)$ shares the same structure of unstable zeros of $D(sI - A)^{-1}E$.

Remark 4. As it has already been pointed out, the quantities X, U, ξ_0 in Problem 2 are considered fixed and given by a high-level optimization problem, and the problem is the determination of a realization of the optimal input-state pair in closed-loop through an output-feedback dynamic compensator. One can however, consider a global optimization problem where also X, U and ξ_0 are variables to be optimized. This amounts to add to (16)–(20) the congruence equations (6) and solve the nonlinear equations. Notice however that, given x_0, if a solution $(P, X, U, \xi_0, F, G, H, K, M, N)$ exists to equations (16)–(20) and (6), then the same equations are satisfied by $(\tilde P,\tilde X,\tilde U,\tilde\xi_0,\tilde F,\tilde G,\tilde H,\tilde K,\tilde M,\tilde N)$, with $\tilde P = TPT^{-1}$, $\tilde X = XT^{-1}$, $\tilde U = UT^{-1}$, $\tilde\xi_0 = T\xi_0$, $\tilde F = TFT^{-1}$, $\tilde G = TG$, $\tilde H = HT^{-1}$, $\tilde K = K$, $\tilde M = TM$, $\tilde N = TN$ and T any invertible matrix. ■

A fundamental role of the optimization problem is played by the initial condition ξ_0 of the reference model $\dot\xi(t) = P\xi(t)$. In this regard, one might be interested in a *relaxed* optimization problem where one wants to parameterize all output-feedback controllers giving rise to input and state trajectories $x(t) = X\xi(t)$, $u(t) = U\xi(t)$ compatible with $\dot\xi(t) = P\xi(t)$. In such a case ξ_0 is a variable not fixed a-priori. In (22)–(24) one can then add the condition $X\xi_0 = x_0$ and look for a solution of the associated linear equation. The generic solvability condition turns out to be $s\times(p-1) \geq m\times(n-p)-q$ that can be satisfied even in the SISO case if $q \geq n-1$. The following simple example shows exactly this case.

Example 1. Consider a second order system with transfer function $G(s) = (1-s)/s^2$, characterized by $x_0 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top$ and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let $q = 1$ with

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}.$$

Finally consider X, U (satisfying $AX + BU = XP$) as follows:

$$X = \begin{bmatrix} 0.1744 & -0.1555 & 0.0106 \\ -0.8532 & -0.3572 & -0.1850 \end{bmatrix},$$
$$U = \begin{bmatrix} 0.0106 & -0.1426 & 0.1872 \end{bmatrix}.$$

Now, define the observer matrices as

$$L = \begin{bmatrix} -5 \\ -7 \end{bmatrix}, \qquad J = \begin{bmatrix} 4 & 2 \end{bmatrix}.$$

Once X, U and P are given, system (22)–(24), (6) is linear, characterized by 13 unknowns and 13 equation. The following solution exists:

$$M = I_3, \qquad F = P,$$
$$N = \begin{bmatrix} 2.3632 & -3.9976 \\ -3.9976 & -3.1791 \\ -3.1791 & 19.1666 \end{bmatrix}, \qquad G = \begin{bmatrix} 19.1666 \\ 23.0745 \\ -74.3057 \end{bmatrix}$$
$$H = \begin{bmatrix} 1.0192 & 1.1937 & 0.5149 \end{bmatrix}, \qquad K = -4$$
$$\xi_0 = \begin{bmatrix} 10.3583 & 2.3605 & -41.5122 \end{bmatrix}^\top.$$

The compensator order is 5, with eigenvalues -7, -6.1926, $-1\pm j$, -0.8074. Its transfer function is $C(s) = 4(s+55)/(s+7)$. Fig. 2 gives the behaviors of the two state variables obtained from the closed-loop system by taking zero initial state for the compensator and x_0 for the system's state. As apparent, these variables perfectly coincide with the components of $X\xi(t)$ obtained by taking the given ξ_0. ■

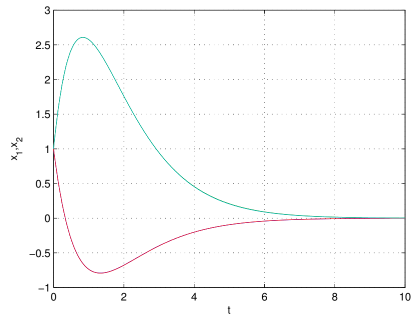

Figure 2: State variables $x_i(t)$ and $(X\xi(t))_i$, $i = 1, 2$

Remark 5. Model matching.
Given a linear system with transfer function $G(s)$ and state-space description

$$\dot x(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t),$$

the classical model matching problem [14] consists in finding a *state-space* feedback

$$u(t) = \Xi x(t) + \Psi v(t)$$

such that the closed loop system transfer function from v to y matches a given transfer function $G_M(s)$, i.e., $G_M(s) = (C + D\Xi)(sI - A - B\Xi)^{-1}B\Psi + D\Psi$. A quick though shows that this equation can be written as

$$G_M(s) = G(s)(I - \Xi(sI - A)^{-1}B)^{-1}\Psi.$$

It is well known that Ξ and Ψ solving this problem exist and $A + B\Xi$ is Hurwitz if and only if

(i) $\begin{bmatrix} G(s) & G_M(s) \end{bmatrix}$ and $G(s)$ share the same infinite zero structure,
(ii) $\begin{bmatrix} G(s) & G_M(s) \end{bmatrix}$ and $G_M(s)$ share the same infinite zero structure,
(iii) $\begin{bmatrix} G(s) & G_M(s) \end{bmatrix}$ and $G_M(s)$ share the same finite zero structure,
(iv) $\begin{bmatrix} G(s) & G_M(s) \end{bmatrix}$ and $G(s)$ share the same unstable zero structure.

The algorithm follows these steps. First find the square system $R(s)$ satisfying $G_M(s) = G(s)R(s)$. Moreover find a constant matrix H such that $R(s)^{-1} - R(\infty)^{-1} = H(sI - A)^{-1}B$. Finally set $\Psi = R(\infty)$ and $\Xi = -\Psi H$. In the relatively optimal approach we have that one has to realize with output feedback the output movement $y(t) = Cx(x)$ with dynamics $\dot x(t) = (A + BK)x(t)$ and initial state $x_0 = BG$. The model transfer function is $G_m(s) = CX(sI - P)^{-1}\xi_0$, where X, P, ξ_0 satisfy $AX + BU = XP$. ■

4. Open loop profile design via convex optimization

In this section we consider the first part of the problem, namely the optimization of the open-loop trajectory by means of convex optimization.

The constraints of our problems are

$$AX + BU = XP, \qquad x_0 = X\xi_0 \tag{6}$$

with (P, ξ_0) reachable. Note that this offers a complete parameterization in terms of modes.

Proposition 1. *Let (P, ξ_0) a reachable pair. Then any other reachable pair $(\hat{P}, \hat{\xi}_0)$ with P and $\hat{P}$ similar provide an equivalent parameterization.*

Proof. Since (P, ξ_0) and $(\hat{P}, \hat{\xi}_0)$ are reachable pairs, there exist non-singular matrices T_1 and T_2 such that $(T_1 P T_1^{-1}, T_1\xi_0)$ and $(T_2\hat{P}T_2^{-1}, T_2\hat{\xi}_0)$ are the reachable canonical forms. That is, $T_1\xi_0 = T_2\hat{\xi}_0$. Since P and $\hat{P}$ are similar, $T_1PT_1^{-1} = T_2\hat{P}T_2^{-1}$. That is, there always exists a non-singular matrix $T = T_1^{-1}T_2$ such that $\hat{P} = T^{-1}PT$ and $\hat{\xi}_0 = T^{-1}\xi_0$. So we would have

$$AXT + BUT = XTT^{-1}PT$$

namely we would have the equivalent constraints

$$A\hat{X} + B\hat{U} = \hat{X}\hat{P}, \qquad \hat{\xi}_0 = T^{-1}\xi_0$$

and denoting by $\hat{\xi}(t) = T^{-1}\xi(t)$, we get the equivalent parameterization

$$x(t) = \hat{X}\hat{\xi}(t), \qquad u(t) = \hat{U}\hat{\xi}(t).$$

In simple words, only the eigenvalues of P have a role. In practice we are choosing a set of modes $e^{\lambda_i t}$ as basis for the solution. Once this basis is fixed, it is possible to optimize the transient from the initial condition x_0. Here we have no restrictions in terms of objective functions and type of constraints.

We present next a summary of possible objective function and constraints which can be dealt with. We are sure that the reader can find a new one not included in this list.

1) *Quadratic performance index.* Given P and ξ_0, find X, U satisfying (6) and minimizing

$$\begin{aligned} J(\xi_0) &= \int_0^\infty \left(x^\top(t)\bar{Q}x(t) + u^\top(t)\bar{R}u(t)\right) \mathrm{d}t \\ &= \xi_0^\top \left(\int_0^\infty e^{P^\top t}\left(X^\top \bar{Q} X + U^\top \bar{R} U\right) e^{Pt}\, \mathrm{d}t\right)\xi_0 \end{aligned}$$

with $\bar{Q} \geq 0$ and $\bar{R} > 0$. It turns out that

$$J(\xi_0) = \xi_0^\top W \xi_0$$

where $W \geq 0$ solves the Lyapunov equation

$$WP + P^\top W = -X^\top \bar{Q} X - U^\top \bar{R}$$

Note here that we can minimize $\alpha > 0$ such that

$$\begin{bmatrix} WP + P^\top W & X^\top & U^\top \\ X & -\bar{Q} & 0 \\ U & 0 & -\bar{R} \end{bmatrix} < 0, \quad \begin{bmatrix} \alpha & \xi_0^\top \\ \xi_0 & W \end{bmatrix} > 0, \quad W \geq 0$$

which describes a convex constraint in W, X, and U.

2) $\mathcal{L}_1$ *norm.* Assume that $x_0 = E$ where

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \qquad \eta(t) = Dx(t)$$

and η scalar. If we wish to optimize the $\mathcal{L}_1$ of the impulse response from w to η, then we can look for the minimum of

$$J(\xi_0) = \int_0^\infty |DXe^{Pt}\xi_0|\, \mathrm{d}t.$$

Again, one can minimize $\alpha > 0$ such that $J(\xi_0) < \alpha$ that is a convex function in X.

3) $\mathcal{L}_2$ *norm.* Assume again that $x_0 = E$ as above and that we wish to optimize the $\mathcal{L}_2$ of the impulse response of the system $D(sI - A)^{-1}E$. Then

$$J(\xi_0) = \xi_0^\top \left(\int_0^\infty e^{P^\top t} X^\top D^\top D X e^{Pt}\, \mathrm{d}t\right)\xi_0.$$

This is a convex function in X. The rationale follows the same lines as in point 1).

4) $\mathcal{H}_\infty$ *norm.* Assume we wish to optimize (or attenuate) the transfer H_∞ norm of the transfer function from w to η. Then we can define

$$J(\xi_0) = \sup_\omega |DX(j\omega I - P)^{-1}\xi_0|$$

and consider the convex problem of finding X such that $J(\xi_0) < \alpha$.

5) Minimum time arrival. Assume that we wish to arrive in minimum time to a (possibly controlled invariant) given ellipsoid $x^\top W x \leq \epsilon$. This correspond to solving

$$\min \ t_f > 0 : \qquad \begin{bmatrix} \epsilon & \xi_0^\top e^{P^\top t_f} X^\top \\ X e^{P t_f}\xi_0 & W^{-1} \end{bmatrix} > 0.$$

This is a convex constraint which typically requires to iterate over t_f.

6) *Smallest ellipsoid at fixed time.* Assume t_f is given and we wish to arrive to the smallest ellipsoid $x^\top W x \leq \epsilon$. This corresponds to solving

$$\min \ \epsilon > 0 : \qquad \begin{bmatrix} \epsilon & \xi_0^\top e^{P^\top t_f} X^\top \\ X e^{P t_f}\xi_0 & W^{-1} \end{bmatrix} > 0.$$

A possible set of constraints can be also considered.

7) *Control pointwise constraints.* We want to impose hard bounds on the input function, i.e., $\|u(t)\| < \gamma$, for each $t \geq 0$. This is cast by imposing

$$\max_{t \geq 0} \|U e^{Pt}\xi_0\| \leq \gamma.$$

8) *Control energy constraints.* We want to impose a soft energy constraint to the input, i.e., $\int_0^\infty \|u(t)\|^2\, \mathrm{d}t \leq \gamma$. This means

$$\int_0^\infty \|U e^{Pt}\xi_0\|^2\, \mathrm{d}t \leq \gamma.$$

9) *Soft starting.* We want to impose that the performance output $\eta(t) = Dx(t)$ has zero derivative at time zero. This can be achieved by imposing the linear constraint

$$DXP\xi_0 = 0.$$

10) *Output shaping.* We may impose $e^-(t) \le e(t) = Dx(t) \le e^+(t)$, for each t, that is

$$e^-(t) \le DXe^{Pt}\xi_0 \le e^+(t), \qquad \forall t \ge 0.$$

All above points 1)–10) represent convex problems and as such can be solved by means of efficient algorithms.

Remark 6. Concerning the possibility of imposing input and state l_∞ constraints, let us consider a matrix P with an adequate number of modes and such that $\|e^{Pt}\|_1 \le 1$. This is possible, for instance by taking $A = \text{block diag}\{A_i\}$ with A_i either negative scalars or 2×2 matrices $A_i = \begin{bmatrix} -\xi & \omega \\ -\omega & -\xi \end{bmatrix}$ with $0 < \omega \le \xi$ [15]. Now, if ξ_0 is chosen in such a way that (P, ξ_0) is reachable, $\|\xi_0\|_1 \le 1$, and the following convex constraints on the X and U matrix are imposed

$$\|X\|_1 \le x_{max} \qquad \text{and} \qquad \|U\|_1 \le u_{max}$$

then, the optimal value of any of the convex minimization problems just enumerated will automatically result in

$$\begin{aligned} \|x(t)\|_1 &= \|Xe^{Pt}\xi_0\|_1 \le \|X\|_1\|e^{Pt}\|_1\|\xi_0\|_1 \le x_{max} \\ \|u(t)\|_1 &= \|Ue^{Pt}\xi_0\|_1 \le \|U\|_1\|e^{Pt}\|_1\|\xi_0\|_1 \le u_{max}. \end{aligned}$$

5. Output feedback realization of state-feedback controllers

In some case we can exploit the previous results without imposing the poles. An idea is to solve a standard optimization problem, and use the closed-loop matrix as P. To this aim we have a surprising result. Let $u = K_{LQ}x$ the optimal LQ controller. Then we may set

$$P = A + BK_{LQ}, \qquad X = I, \qquad U = K_{LQ}.$$

Then assume that $x_0 = \xi_0$ is such that (P, x_0) is reachable. Taking $s = n$.

Theorem 3. *If the equations* (22)–(24) *are solvable, then there exist an output feedback stabilizing compensator which achieves optimality for* $x(0) = x_0$.

Note that in general we must assure that $J \ne K_{LQ}$ otherwise we would have a singularity in view of the condition

$$H = U - JX = K_{LQ} - J.$$

As already said, for the SISO case equations (22)–(24) can be hardly solved, unless the initial state ξ_0 is not fixed a priori. In the present context this means not to fix the initial state x_0 (relaxed optimal control). To investigates the solvability condition, let, without any loss of generality,

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Then the only solutions of (22)–(24) are such that

$$\begin{aligned} G &= NQe_n - P^{n-1}\alpha, \\ N &= \begin{bmatrix} \alpha & P\alpha & \cdots & P^{n-2}\alpha & P^{n-1}\alpha \end{bmatrix}, \\ \alpha &= \left(\sum_{k=0}^{n} P^k e_k^\top x_0 \right)^{-1} x_0. \end{aligned}$$

Moreover, J and K can be easily found satisfying $J + KC + (U - J)N = 0$. However it is likely that all possible congruent J are not such that $A + BJ$ is Hurwitz.

Example 2. As in the previous example consider a second order system with transfer function $G(s) = (1 - s)/s^2$, characterized by $x_0 = \begin{bmatrix} -0.3568 & -0.9342 \end{bmatrix}^\top$ and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Consider the optimal control law $u = K_{LQ}x$ that minimizes $J = \int_0^\infty x^\top(t)x(t) + u(t)^2 dt$. It results in

$$K_{LQ} = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

so that the closed-loop matrix is

$$A_{LQ} = A + BK_{LQ} = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix}.$$

Define $P = A_{LQ}$ and take

$$L = \begin{bmatrix} -5 \\ -7 \end{bmatrix}, \qquad J = \begin{bmatrix} 4 & 1 \end{bmatrix}.$$

Moreover, let

$$M = I_2, \qquad N = \begin{bmatrix} 4 & -1.1459 \\ -10.8541 & 5.1459 \end{bmatrix}.$$

The Y-K matrices that satisfy (22)–(24) are

$$\begin{aligned} F &= P, & G &= \begin{bmatrix} -10.2705 \\ 17.6869 \end{bmatrix}, \\ H &= \begin{bmatrix} -1 & 0 \end{bmatrix}, & K &= -2.1459 \end{aligned}$$

The compensator transfer function is $C(s) = (0.382 + 2.146s)/(s + 4.528)$. It can be easily checked that if the initial state x_0 is the same as in Example 1, no solution exists. As a matter of fact J should be such that $J_1 + J_2 = -0.5$ whereas for stability of $A + BJ$ we need $J_1 > J_2 > 0$. ■

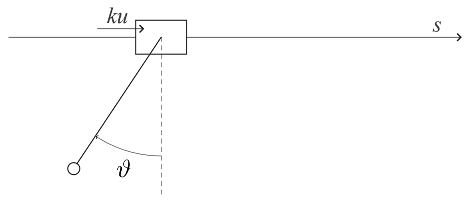

Figure 3: Scheme of the cart-pole system.

We can extend the results to a more general class of performance indices:

$$\mu_Q = \int_0^{\infty} \left(x^\top(t) Q x(t) + u^\top(t) R u(t) + \dot{u}^\top(t) R \dot{u}(t) \right) \mathrm{d}t.$$

This problem can be faced by considering an augmented system

$$A_{aug} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad B_{aug} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Then let $[\ K_1 \ \ K_2\ ]$ be the optimal gain and take

$$P = \begin{bmatrix} A & B \\ K_1 & K_2 \end{bmatrix}, \quad X = \begin{bmatrix} I & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & I \end{bmatrix}.$$

The equation $AX + BU = XP$ is trivially satisfied. Take ξ_0 such that $x_0 = \begin{bmatrix} I & 0 \end{bmatrix} \xi_0$ Then, again, we can claim optimality for the initial condition x_0 of the YK based compensator.

Theorem 4. *If the equations* (22)–(24) *are solvable, then there exists an output feedback stabilizing compensator which achieves optimality for* $x(0) = x_0$.

We ran numerical experiments on randomly generated matrices, which have shown that the equations are generically satisfied.

6. Example

Consider the cart-pole system described in [16], whose scheme is reported in Fig. 3. Let the continuous-time model, linearized around a stable equilibrium point be:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where $x(t) = \begin{bmatrix} \vartheta(t) & \dot{\vartheta}(t) & s(t) & \dot{s}(t) \end{bmatrix}^\top$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -19.62 & -0.125 & 0 & -9.886 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4.943 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 11.53 \\ 0 \\ 5.767 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Take the initial state as

$$x(0) = x_0 = \begin{bmatrix} 0 & 0 & -0.75 & 0 \end{bmatrix}^\top$$

and consider the problem of reaching the sphere $x^\top x \leq \epsilon_0 = 0.0049$ in minimum time t_f, subject to the input and state constraints:

$$|u(t)| \leq u_{max} = 1, \qquad \forall t \in \left[0, t_f\right] \tag{26}$$

$$|x_1(t)| \leq \vartheta_{max} = 0.23, \qquad \forall t \in [0, t_f]. \tag{27}$$

To find the optimal open-loop trajectory we take $n + q = 8$ and the reachable pair (P, ξ_0) as

$$P = \begin{bmatrix} -2 & 1 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0.2 & 0.1 & 0 & 0 & 0 & 0 \\ 0.1 & 0.3 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.1 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 1 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2.5 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0 & 0.1 & 0.3 & -3.5 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.1 & 0 & -5.5 \end{bmatrix},$$

$$\xi_0 = \begin{bmatrix} 45 & -45 & -45 & 0 & 45 & 45 & 0 & 0 \end{bmatrix}^\top.$$

The choice of such a P amounts to imposing the (stable) closed-loop poles

$$\begin{aligned} \sigma(P) = \{ & -1.95 - 0.998j, \quad -1.95 + 0.998j, \quad -3.09, \quad -5, \\ & -2.45 - 0.998j, \quad -2.45 + 0.998j, \quad -3.59, \quad -5.5 \}. \end{aligned}$$

Needless to say, the optimality of the open-loop trajectory is restricted to the behaviours achievable by those modes. The minimum time trajectory may be found iteratively, by solving a sequence of optimization problems with fixed t_f:

$$\min_{X,U,\epsilon} \ \epsilon \tag{28}$$

$$\text{s.t.} \quad \epsilon \geq 0 \tag{29}$$

$$\begin{bmatrix} \epsilon & \xi_0^\top e^{P^\top t_f} X^\top \\ X e^{P t_f} \xi_0 & I \end{bmatrix} > 0 \tag{30}$$

$$AX + BU - XP = 0 \tag{31}$$

$$X\xi_0 - x_0 = 0 \tag{32}$$

$$\left| U e^{Pt} \xi_0 \right| \leq 1, \quad \forall t = \frac{k}{100} t_f, \quad k = 0, 1, \ldots, 100 \tag{33}$$

$$\left| \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} X e^{Pt} \xi_0 \right| \leq 0.23, \quad \forall t = \frac{k}{100} t_f, \quad k = 0, 1, \ldots, 100. \tag{34}$$

Notice that (33) and (34) represent a finite number of pointwise constraints that are not equivalent to (26) and (27). This is motivated by the need of avoiding the intrinsic conservativeness of the approach of Remark 6. From a practical point of view, the fulfillment of (26) and (27) by the whole optimal trajectory may be checked after its computation; if needed, satisfaction of (26) and (27) may be enforced by slightly tightening the constraints (33) and (34).

As far as the iteration scheme is considered, the algorithm is as follows (where λ is a tolerance):

1. Take t^- sufficiently small and t^+ sufficiently large[1];
2. $t_f \leftarrow (t^+ + t^-)/2$;
3. If $(t^+ - t^-)/t^+ < \lambda$ and $\epsilon \leq \epsilon_0$ then exit;
4. Solve (28)–(34);
5. If $\epsilon > \epsilon_0$ then set $t^- \leftarrow t_f$ and go to step 2;
6. Set $t^+ \leftarrow t_f$ and go to step 2.

To solve problem (28)–(34) we used CVX, a package for specifying and solving convex programs [17, 18]. By choosing $t^- = 0.1$, $t^+ = 4$, $\lambda = 0.01$ we found, in 9 iterations, a minimum time $t_f = 1.95$, achieved by:

$$X = \left[\begin{array}{cccc} 28.7 & 6.175 & -43.83 & -297.6 \\ -127.5 & -26.56 & 144.2 & 1489 \\ 31.97 & 7.537 & -62.74 & -261.7 \\ -130.1 & -28.1 & 202.5 & 1309 \end{array}\right.$$

$$\left.\begin{array}{cccc} -48.61 & -17.74 & 178.4 & 129.1 \\ 182.9 & 62.17 & -647.5 & -712 \\ -59.72 & -27.48 & 218.6 & 104.5 \\ 219.5 & 85 & -794.5 & -577.3 \end{array}\right],$$

$$U = \left[\begin{array}{cccc} -12.6 & -3.658 & 58.24 & -13.43 \end{array}\right.$$

$$\left.\begin{array}{cccc} 44.46 & 22.74 & -180.6 & 57.23 \end{array}\right].$$

The optimal trajectories of the state and the input (along with the state and input constraints) are reported, respectively in Fig. 4 and Fig. 5. The phase portraits for the pairs (x_1, x_2) and (x_3, x_4) are shown in Fig. 6 and Fig. 7.

Now it is necessary to assign stable eigenvalues to $A + BJ$ and $A + LC$ by means of a choice of J and L. Such a choice is arbitrary, provided that the stability is guaranteed. By solving a standard LQR problem with performance index $\int_0^\infty \left(10x_1^2(t) + x_3^2(t) + u^2(t)\right) \mathrm{d}t$, we get a feedback matrix

$$J = \left[\begin{array}{cccc} -2.902 & -0.2297 & -1 & -0.4368 \end{array}\right]$$

such that

$$\sigma(A + BJ) = \{ -3.128 - 4.635j, \quad -3.128 + 4.635j, \quad -2.574, \quad -1.406 \}.$$

Now, L is taken as:

$$L = \left[\begin{array}{cc} -31.1 & -16.9 \\ -481.3 & -169.2 \\ 17.5 & -15.01 \\ 61.05 & 3.255 \end{array}\right],$$

in order to assign to the observer a dynamics that is five times faster than that of the controller. Finally, we take the order of the YK parameter $s = n + q = 8$ and impose $M = I$. Thus, from (17) and (19) we get $F = P$ and

$$H = U - JX$$

[1]Meaning that problem (28)–(34) has a solution $\epsilon > \epsilon_0$ for $t_f = t^-$ and $\epsilon \leq \epsilon_0$ for $t_f = t^+$.

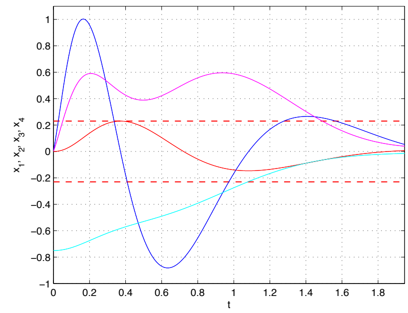

Figure 4: State variables $x_i(t)$ and $(X\xi(t))_i$, $i = 1, 2, 3, 4$ for the cart-pole system. The dash lines represent the constraints on the pole angle (variable x_1.)

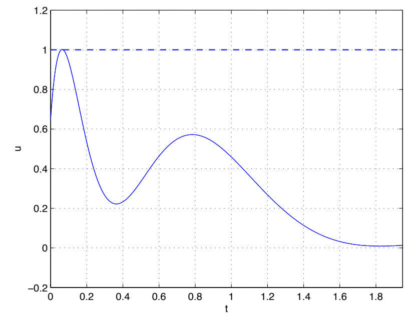

Figure 5: Input $u(t)$ and $U\xi(t)$ for the cart-pole system. The dash line represents the maximum allowed input.

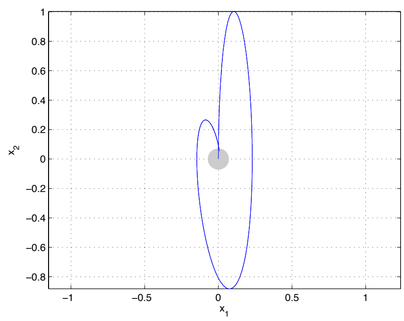

Figure 6: Pole trajectory and intersection of the arrival sphere with the x_1–x_2 plane.

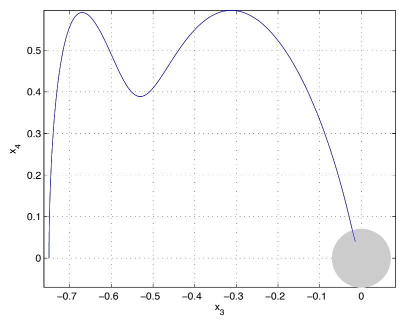

Figure 7: Cart trajectory and intersection of the arrival sphere with the x_3–x_4 plane.

$$= \left[\begin{array}{cccc} 16.56 & 3.429 & -10.12 & -225 \end{array} \right.$$
$$\left. \begin{array}{cccc} -18.44 & -4.815 & 60.09 & 120.8 \end{array} \right].$$

Then, by solving the linear system:

$$\begin{aligned} 0 &= FN + GC - N(A + LC) \\ 0 &= J + HN + KC \\ 0 &= Nx_0 - \xi_0 \end{aligned}$$

in the unknowns N, G and K we get:

$$N = \begin{bmatrix} 7.333 & -1.374 & -60 & -18.07 \\ -2.464 & 2.63 & 60 & 5.162 \\ -14.63 & 5.314 & 60 & 3.975 \\ -1.523 & 0.3099 & 0 & -0.5759 \\ 8.683 & -4.357 & -60 & -0.001261 \\ 8.79 & -1.825 & -60 & -17.27 \\ -2.33 & 0.3992 & 0 & 0.8551 \\ -2.138 & 0.2018 & 0 & 0.4809 \end{bmatrix},$$

$$G = \begin{bmatrix} -1669 & 746.7 \\ 129.6 & -1239 \\ -958.2 & -1372 \\ -151.9 & -22.58 \\ 876.2 & 1401 \\ -1432 & 794.8 \\ -86.94 & -1.374 \\ -19.59 & 21.56 \end{bmatrix}, \qquad K = \begin{bmatrix} -0.05003 & 0.8601 \end{bmatrix}.$$

The output-feedback state and input trajectories from x_0 and zero initial state for the observer and the YK parameter are reported in Figures 4, 5, 6, and 7. They are indistinguishable from the open-loop optimal trajectories.

Finally, Figs. 8, 9, 10, and 11 show some perturbed trajectories, i.e., trajectories obtained from some randomly chosen $x(0) : \|x(0) - x_0\| \leq 0.05$. It is apparent that the control system exhibits a small sensitivity to perturbations of the initial state.

7. Concluding Remarks

In this paper, the relatively optimal control for continuous-time systems via output feedback has been investigated. The key idea is to fix the closed loop system modes (once again, the larger the number of chosen modes, the greater the flexibility in the optimization stage) and parameterize all the possible trajectories of the closed-loop system by two matrices X and U. This in turn enables to perform the nominal trajectory optimization in a finite-dimensional space and derive a finite dimensional compensator which achieves the resultant optimal trajectory for given initial states x_0 and 0 of the plant and the compensator. The compensator matrices have been given by explicit formulas.

Although in this paper the trajectory parameterization by means X and U has been employed, it is definitely worth recalling that there exists a different approach to the constraint optimization. One example is a *transfer function* approach [11] based on Youla–Kučera parameterization [9, 10].

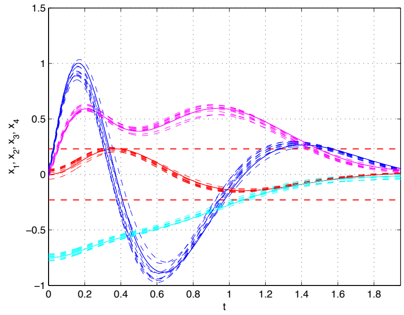

Figure 8: State trajectories for some non-nominal initial conditions.

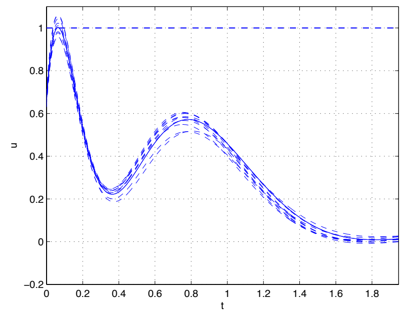

Figure 9: Input trajectories for some non-nominal initial conditions.

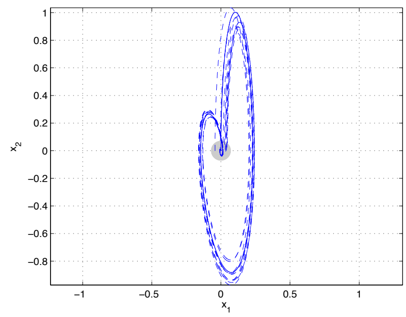

Figure 10: Pole phase portraits for some non-nominal initial conditions.

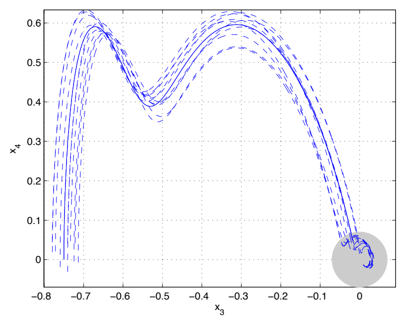

Figure 11: Cart phase portraits for some non-nominal initial conditions.

An advantage of the present method over the transfer function approach is that the method is formulated in the *state space* and thus enjoys several flexibilities regarding the representation of control problems and allows to use well-established computational tools for the solution of convex minimization problems, e.g., all those widely adopted in the linear matrix inequality framework. For instance, it is possible to extend the presented method to the case in which multiple initial states of the plant are given, similarly to what has been proposed in [6].

A few related extensions are currently under investigation.

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