A Survey on Blake and Zisserman Functional

Michele Carriero, Antonio Leaci and Franco Tomarelli

Abstract. The aim of this work is to provide a concise survey of results about Blake–Zisserman functional for image segmentation and inpainting. Moreover a refinement of the Almansi decomposition is shown for biharmonic functions in 2-dimensional open disks with crack-tip at the origin.

Mathematics Subject Classification (2010). Primary 49J45; Secondary 49K20.

Keywords. Calculus of variations, image segmentation, inpainting, free discontinuity, Almansi decomposition, crack-tip.

1. Introduction

In this paper we outline the state-of-the-art of the analysis for Blake–Zisserman functional for image segmentation and inpainting.
Image segmentation plays an important role in medical imaging and in the understanding of biological vision, since it provides partitioning of an image in such a way that in each subregion the colors intensity varies as slow as possible and the partition boundaries have length as short as possible. Inpainting is relevant in computer vision and image restoration: it refers to the process of recovering the missing information over a small area where a given image is damaged; this area may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or aging of canvas and colors in a painting ([10], [16], [40], [41], [52], [54], [65]).
Though there is a huge variety of methods available for solving these tasks, no one can be qualified as the best one for every kind of images. The reason is that different methods can "see" different features of the image. Therefore often they are used together (Hybrid Imaging Methods) in medical imaging in order to optimize the relevant issues: safety, cost, contrast and resolution ([10],[58]).

This work is supported by PRIN 2010-2011 M.I.U.R (Progetto *"Calculus of Variations"*).

However variational models performed better than noise filtering in the detection of discontinuities.
The “weak membrane” variational model for image segmentation was proposed by D.Mumford and J.Shah in [61] and starting from the seminal paper [47] was studied by several authors ([5], [6], [7], [43], [44], [46], [48], [56], [57], [60], and more recently [23], [50]).
Blake and Zisserman showed some inconvenient related to Mumford–Shah weak membrane model (mainly the “gradient limit” or over-segmentation of the output when processing images with continuous steep variation of intensity) and introduced an alternative variational approach, the “weak plate” model ([17]) that translates the image segmentation task into a second order variational formulation which was formalized as a free gradient discontinuity problem in [27] and [28]. Further analysis of Blake–Zisserman approach was developed in [4], [22], [26], [29], [30], [31], [32], [34], [36], [42], [66].
Different higher order approachs in image analysis are considered in [14], [15], [25], [55], [64].
The Blake–Zisserman weak plate model faces the segmentation as an energy minimization problem. It takes an image and produces three outputs: two boundary process maps which indicate the location of boundaries (jump and creases of luminance), and a surface attribute map which indicates the smoothed (interpolated) luminance values on the surface of objects in the field.

Here we denote the domain and the color intensity level of a monochromatic image respectively by Ω and g, in order to introduce a strong formulation of Blake–Zisserman functional:

$$\mathfrak{F}(K,v) \ = \ \int_{\Omega\setminus K} \left(|D^2 v|^2 + |v-g|^2\right)\, dx\, dy \ + \ \mathcal{H}^1(K\cap\Omega)\ ; \tag{1.1}$$

in $\mathfrak{F}$ we omit the usual tuning parameters for simplicity: the thorough functional is shown by (5.1) in Section 5.
The three terms in (1.1) are in competition when the functional is minimized. The L^2 norm of the difference $(v-g)$ acts as a fidelity term: it increases if v is not close to the unprocessed image g. The term $\mathcal{H}^1(K\cap\Omega)$ pays for the length of the segmentation K and prevents an excessive partitioning of Ω. The L^2 norm of the hessian in $\Omega\setminus K$ acts as a smoothing of the unprocessed image g outside the segmentation K. In (1.1) and in the sequel $|D^2 v|^2$ denotes $\sum_{i,j=1}^{2} |D_i D_j v|^2$ where D_k is the the distributional partial derivative with respect to the k_{th} variable.

Theorem 1.1. *([28]) Assume*

$$\Omega\subset\mathbb{R}^2 \ \textit{is a bounded open set and} \ \ g\in L^2(\Omega)\cap L^4_{loc}(\Omega)\,, \tag{1.2}$$

then the functional (1.1) achieves a finite minimum among closed subsets K of $\mathbb{R}^2$ and functions v which belong to $C^2(\Omega\setminus K)$.

We remark that nonexistence of minimizers for functional (1.1) may appear if datum g has low integrability: we showed that for any $s<4$ there is g

in $L^2(\Omega) \cap L^s_{loc}(\Omega)$ such that the infimum of (1.1) over closed K and v in $C^2(\Omega \setminus K)$ is not achieved (Counterexample 27.5 in [29]).
Obviously for practical segmentation purposes one can always assume that g belongs to L^∞.
Nevertheless we emphasize that maximum principle fails for minimizers of this second order functional with free discontinuity, as shown in Section 3.

In the present paper we focus on the Blake–Zisserman functional: we show some motivations, recall the main results and mention some open problems; in the last Section we state some new technical tools related to the analysis around a crack-tip. The plan of the paper is as follows.
By referring to the simplified 1-dimensional context, in Section 2, 3 and 4 we shortly recall, respectively: the meaning of the tuning parameters that are usually inserted as weights of the various terms of the Blake–Zisserman functional (denoted by $\mathbf{F}$ in the 1-d case); the lack of maximum principle; the issue of uniqueness. We emphasize that in the 1-d case the strong and weak formulation of the functional coincide, while there is a big gap between them in dimension $n \geq 2$.
Section 5 describes the main results for the Blake–Zisserman functional F for image segmentation in 2-d, as defined by (5.1).
Section 6 describes the main results for the Blake–Zisserman functional G for inpainting in 2-d, as defined by (6.1).
Section 7 recalls some energy density estimates for minimizers of Blake–Zisserman functional for image segmentation and inpainting in 2-d.
Section 8 exhibits some nontrivial admissible and non admissible candidate local minimizers together with a conjecture. Notice that explicit minimizers for second order free discontinuity problems are difficult to find, moreover the rigorous proof of their minimality is still open; whereas some nontrivial local minimizer for Mumford–Shah were exhibited and analyzed ([2],[20]).
In Section 9 we extend some results concerning Almansi decomposition of biharmonic function in presence of a crack-tip and related non-integer power series expansions: these kind of results were useful in the analysis of candidate local minimizers.

2. Contrast and sensitivity parameters

The non convexity of Blake–Zisserman functional leads to several negative results in the analysis: there is neither uniqueness of minimizer nor maximum principle. This happens even in the one-dimensional setting: to clarify these issues we refer to the simpler one-dimensional version $\mathbf{F}$ (defined by (2.1) below) of Blake–Zisserman functional (see (1.1) and (5.1)), since in the 1-d case the strong and weak formulation coincide. So the dependance on the free discontinuity set can be circumvented in the 1-d case and we can play with variations of one single argument, the function, while the closed set is replaced by the singular sets of the function itself and of its derivative.
In this Section we introduce some tuning parameters in the functional, by

limiting our description to the 1-dimensional Blake–Zisserman functional $\mathbf{F}^g_{\alpha,\beta,\mu}$ (shortly denoted by $\mathbf{F}$ when there is no risk of confusion), or *weak rod*: we define $\mathbf{F}^g_{\alpha,\beta,\mu} : \mathbf{H}^2 \to [0,+\infty)$ by setting, $\forall v \in \mathbf{H}^2$,

$$\mathbf{F}^g_{\alpha,\beta,\mu}(v) = \int_I \Big((\ddot{v}(x))^2 + \mu (v(x) - g(x))^2 \Big)\, dx \,+\, \alpha\, \sharp(S_v) \,+\, \beta\, \sharp(S_{\dot{v}}) \,. \tag{2.1}$$

Here and in the sequel $I \subset \mathbb{R}$ is an interval, $\sharp$ denotes the counting measure and

$$\mathbf{H}^2 \;:=\; \big\{ v : I \to \mathbb{R} :\ v \text{ is piece-wise } H^2 \big\} \tag{2.2}$$

denotes the set of $v \in L^2_{loc}(I)$ which are piecewise H^2 Sobolev functions: precisely, $\dot{v}$ denotes the absolutely continuous part of the distributional derivative v' of v; $\ddot{v}$ denotes the absolutely continuous part of $(\dot{v})'$; $S_v \subseteq I$ denotes the *approximate discontinuity set* (or shortly *singular set*) of v ([5],[53]); $S_{\dot{v}} \subseteq I$ denotes the approximate discontinuity set of $\dot{v}$. So $\mathbf{H}^2$ is the space of $v \in L^2_{loc}(I)$ such that S_v and $S_{\dot{v}}$ are finite sets, $v \in H^2(J)$ for any bounded interval $J \subseteq I \setminus (S_v \cup S_{\dot{v}})$ and $v \in H^2_{loc}(J)$ for any unbounded (if any) interval $J \subseteq I \setminus (S_v \cup S_{\dot{v}})$, where "$L^2_{loc}$" and "$H^2_{loc}$" stand respectively for L^2 and H^2 in any bounded (possibly not open) subinterval of I.

Theorem 2.1. *Assume*

$$0 < \beta \le \alpha \le 2\beta, \quad \mu > 0, \quad g \in L^2_{loc}(I), \quad \exists\, \overline{u} \in \mathbf{H}^2 :\ \mathbf{F}^g_{\alpha,\beta,\mu}(\overline{u}) < +\infty \,. \tag{2.3}$$

Then there is $u \in \mathbf{H}^2$ such that $\mathbf{F}^g_{\alpha,\beta,\mu}(u) < +\infty$ and $\mathbf{F}^g_{\alpha,\beta,\mu}(u) \le \mathbf{F}^g_{\alpha,\beta,\mu}(v)$ for all $v \in \mathbf{H}^2$.

Proof. By (2.3) the domain of the nonnegative functional $\mathbf{F}^g_{\alpha,\beta,\mu}$ is not empty. Hence we can select a minimizing sequence and apply the same proof as in [27] that works also in unbounded intervals, by substituting L^p_{loc} to L^p whenever is needed. $\square$

We emphasize that condition $0 < \beta \le \alpha \le 2\beta$, stating that one crease costs not more than one jump and one jump costs no more than two creases, is necessary in 1-d to achieve the infimum. Moreover condition $0 < \beta \le \alpha \le 2\beta$ is essential when looking for minimizers also in dimension $n \ge 2$, since the functional lacks the semicontinuity property when it is missing (see [27]).

The functional (2.1) pays α for each jump in the graph of v, and pays β for each crease (without jump!) in the graph of v and weights μ the fidelity term $\int |v - g|^2$. The whole set of parameters is associated to several quantities that have an interpretation as contrast and sensitivity thresholds for the minimization of Blake–Zisserman functional. By translating the computations of Section 5.2 in [17] (that were deduced by comparison of "broken" minimizers energy with energy minimization restricted to smooth competitors) in terms of the different parameters α, β, μ selected in (2.1), since they are the essential ones from the viewpoint of mathematical analysis, we find

- **contrast threshold:** an isolated step of height h is actually detected as a step if $h > h_0$, where
$$h_0 \;=\; 2^{3/4}\,\alpha^{1/2}\,\mu^{-3/8}\;.$$
- **interaction of two adjacent steps** of mutual distance $a > 0$ and opposite jumps $\pm h$ (top hat): if $a >> \mu^{-1/4}$ then the two steps will be detected, if $a << \mu^{-1/4}$ then the two steps will be detected if $h > h_1$, where the detection threshold is
$$h_1 \;=\; (\alpha^{1/2}\,a^{-1/2}\,\mu^{-1/2})\;h_0 \;=\; 2^{3/4}\,a^{-1/2}\,\alpha\,\mu^{-7/8}\;.$$
- **sensitivity to an isolated crease:** a jump of amplitude $\gamma = \dot{v}_+ - \dot{v}_-$ in the first derivative at a continuity point is actually detected as a crease if $\gamma > \gamma_0$, where the detection threshold γ_0 for isolated creases is
$$\gamma_0 \;=\; (\beta^{1/2}\,\alpha^{-1/2}\,\mu^{1/4})\,h_0 \;=\; 2^{3/4}\,\beta^{1/2}\,\mu^{-1/8}\;.$$

3. No maximum principle

Here we perform some straightforward computations showing that, already in the 1-d case, an L^∞ bound cannot be transferred from datum g to the minimizers: moreover such bound can be exceeded in a subset of infinite measure. We choose

$$I = \mathbb{R}\,, \qquad g(x) = H(x)\ \text{ for }\ x \in \mathbb{R}\,, \qquad \alpha = \beta\,, \qquad \mu = 1\,, \tag{3.1}$$

where H denotes the Heaviside function ($H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x \geq 0$), and define the odd function $u \in C^{3,1}(\mathbb{R})$ as follows (see Figure 1)

$$u(x) = \begin{cases} 1 - (1/2)\,\cos(x/\sqrt{2})\exp(-x/\sqrt{2}) & x \geq 0\,,\\ (1/2)\,\cos(x/\sqrt{2})\exp(x/\sqrt{2}) & x < 0\,; \end{cases} \tag{3.2}$$

then

$$\max u \;\approx\; 1.034\,, \qquad \min u \;\approx\; -0.034\,; \tag{3.3}$$

$$u''''(x) + u(x) \;=\; H(x) \qquad \text{in } \mathcal{D}'(\mathbb{R})\,; \tag{3.4}$$

$$\mathbf{F}^H_{\alpha,\alpha,1}(u) = \int_{\mathbb{R}} \left((u''(x))^2 + (u(x) - H(x))^2\right) dx = \frac{1}{2\sqrt{2}}\;; \tag{3.5}$$

$$\mathbf{F}^H_{\alpha,\alpha,1}(H) = \alpha\;. \tag{3.6}$$

Proposition 3.1. *Assuming* (3.1) *and* (3.2) *we obtain these alternatives.*

- *If* $\alpha > 1/(2\sqrt{2})$ *then* u *is the unique minimizer of* $\mathbf{F}^H_{\alpha,\alpha,1}$
- *If* $\alpha = 1/(2\sqrt{2})$ *then both* u *and* g *are the only minimizers of* $\mathbf{F}^H_{\alpha,\alpha,1}$
- *If* $\alpha < 1/(2\sqrt{2})$ *then* $g = H$ *itself is the unique minimizer of* $\mathbf{F}^H_{\alpha,\alpha,1}$ *.*

Proof - Any "broken" competitor must have at least either a step or a crease, i.e. in both cases an energy not less than α.
On the other hand a smooth competitor has a strictly bigger energy than $\mathbf{F}^{H}_{\alpha,\alpha,1}(u) = 1/(2\sqrt{2})$, unless it coincides with u, due to (3.5) and the strict convexity of the functional $\mathbf{F}^{H}_{\alpha,\alpha,1}$ with domain restricted to $H^2(I)$.
These remarks together with (3.6) easily lead to the claims. □

Remark 3.2. In the above statement the condition (2.3) is useless since H has finite energy and makes the domain of $\mathbf{F}^{H}_{\alpha,\alpha,1}$ nonempty, whereas the minimum is explicitly exhibited without exploiting minimizing sequences.

Remark 3.3. We emphasize that, contrarily to Mumford–Shah functional, the minimizers of Blake–Zisserman are not bounded by datum in general, even in this simplified 1-dimensional framework. Precisely the choice (2.1),(3.1) exhibits a case ($\alpha > 1/(2\sqrt{2})$) where maximum principle fails, that is

$$\| \operatorname{argmin} \mathbf{F}^{g}_{\alpha,\beta,\mu} \|_{L^\infty} > \|g\|_{L^\infty} \ .$$

The example shows also that the positivity of datum g may be not preserved by the minimizer.

Remark 3.4. The choice (2.1),(3.1) shows also that in 1-d the monotonicity of datum g does not entail in general the monotonicity of the minimizer u.

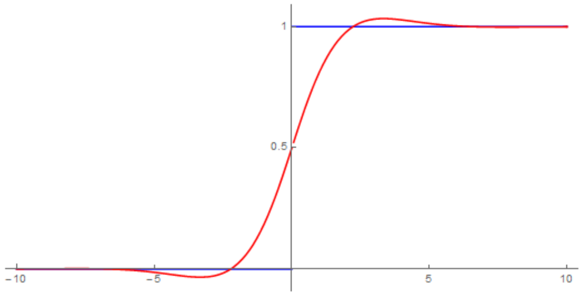

FIGURE 1. Graph of u. Actually there are infinitely many oscillations, though they are numerically negligible due to exponential damping.

Remark 3.5. Since the Euler equation $u'''' + u = g$ is linear and the energy of a minimizer among the restricted domain of $C^2(\mathbb{R})$ functions is quadratic, with the choices $g(x) = k\,H(x)$ and $\mu = 1$ we obtain exactly the same alternatives of Proposition 3.1 where the competitors for minimization of $\mathbf{F}^{kH}_{\alpha,\alpha,1}$ are $k\,H$ and $k\,u$, while the threshold is $k^2/(2\sqrt{2})$.

Actually the minimization of functional (2.1) exhibits many additional kinds of non uniqueness for minimizers, nevertheless in the next Section we clarify that a generic uniqueness property holds true for minimizers.

4. Generic uniqueness

In this Section, by restricting the analysis to the 1-dimensional case with the choice $I = (0,1)$, we focus our attention on uniqueness of minimizer for Blake–Zisserman functional $\mathbf{F}^g_{\alpha,\beta,\mu}$, or *weak rod*, defined for all $v \in \mathbf{H}^2$ by

$$\mathbf{F}^g_{\alpha,\beta,\mu}(v) = \int_0^1 \Big((\ddot{v}(x))^2 + \mu (v(x) - g(x))^2 \Big)\, dx + \alpha \sharp(S_v) + \beta \sharp(S_{\dot v})\,. \quad (4.1)$$

Moreover we denote

$$\begin{aligned} m^g(\alpha,\beta,\mu) &= \inf\{\mathbf{F}^g_{\alpha,\beta,\mu}(v) \,:\, v \in \mathbf{H}^2\}, \\ \operatorname{argmin} \mathbf{F}^g_{\alpha,\beta,\mu} &= \{\, v \in \mathbf{H}^2 \,:\, \mathbf{F}^g_{\alpha,\beta,\mu}(v) = m^g(\alpha,\beta,\mu)\,\}. \end{aligned}$$

By Theorem 2.1, $\operatorname{argmin} \mathbf{F}^g_{\alpha,\beta,\mu} \neq \emptyset$ whenever the two following conditions are satisfied:

$$0 < \beta \le \alpha \le 2\beta\,, \quad \mu > 0\,, \quad (4.2)$$

$$g \in L^2(0,1). \quad (4.3)$$

Nevertheless minimizers are not unique in general, even if g is piecewise affine. Section 3 of [18] shows examples of $g \in L^2(0,1)$ and α, β fulfilling (4.2) such that $\mathbf{F}^g_{\alpha,\beta,\mu}$ has more than one minimizer: if $g = \chi_{[1/2,1]}$ there is $\alpha > 0$ such that $\mathbf{F}^g_{\alpha,\alpha,\mu}$ has exactly two minimizers (Counterexample 3.1 in [18]). There are $\alpha > 0$ and $g \in L^2(0,1)$ such that uniqueness fails for any β belonging to a non empty interval $(\alpha - \varepsilon, \alpha]$ (Counterexample 3.2 in [18]). For any α and β fulfilling (4.2) there is $g \in L^2(0,1)$ with $\sharp(\operatorname{argmin} \mathbf{F}^g_{\alpha,\beta,\mu}) > 1$ (Counterexample 3.3 in [18]). Moreover there exists an example of a non empty open subset $\mathcal{N} \subseteq L^2(0,1)$ such that for any $g \in \mathcal{N}$ there are α, β fulfilling (4.2) and $\sharp(\operatorname{argmin} \mathbf{F}^g_{\alpha,\beta,\mu}) \ge 2$ (Counterexample 3.4 in [18]).
However the minimum value $m^g(\alpha,\beta,\mu)$ of Blake–Zisserman functional depends continuously on g, α, β, μ in the region defined by (4.2),(4.3): see Theorem 15 and 16 in [19] respectively for the 1-d and n-dimensional case.
The main result concerning generic uniqueness is the following statement (Theorem 2 in [19]), where dense G_δ set denotes the intersection of at most countably many dense open sets:

Theorem 4.1. *For any α, β and μ with $0 < \beta \le \alpha \le 2\beta$, $\mu > 0$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$, there is a dense G_δ set $E_{\alpha,\beta,\mu} \subseteq L^2(0,1)$ such that for any $g \in E_{\alpha,\beta,\mu}$ we have $\sharp(\operatorname{argmin} \mathbf{F}^g_{\alpha,\beta,\mu}) = 1$.*

Since both the complement in $L^2(0,1)$ of a dense G_δ set and the complement in $\mathbb{R}^2$ of the set $\{(\alpha,\beta) \in \mathbb{R}^2 :\ \alpha/\beta \notin \mathbb{Q}\}$ are sets of first category, Theorem 4.1 says that uniqueness for minimizers of $\mathbf{F}^g_{\alpha,\beta,\mu}$ is a generic property. This is a remarkable property since for variational problems there are few uniqueness results beyond the case of strictly convex functionals, even in the 1-dimensional case.
Hence the whole picture about generic uniqueness and counterexamples is coherent with the presence of instable patterns, each of them corresponding to a bifurcation of optimal segmentation under variation of parameters: this

fact is natural since suitable ratios involving parameters α and β are related to contrast threshold, crease detection, luminance sensitivity, resistance to noise and double-step detection (see Section 2).
We emphasize that jump and crease points of minimizers are not necessarily localized among those of datum g, even if g is continuous piecewise-affine (see Section 4 of [18]): hence the techniques used for proving the generic uniqueness for Mumford–Shah functional in [3] cannot be plainly applied to Blake–Zisserman functional. For this reason a different strategy is used in the proof of Theorem 4.1 consisting in careful exploitation of intersection properties between real analytic varieties.

5. Blake–Zisserman functional for image segmentation

In Section 2 we have shown the relevance of parameters that provide the possibility of tuning the contrast threshold and sensitivity to steps and creases, both isolated and interacting in the simplified 1-d context. Here we formalize the relevant 2-d case when those parameters are taken into account.
Precisely we define the strong formulation F of Blake–Zisserman functional with tuning contrast parameters for segmentation of 2-d images, according to [28] and [31].
We denote respectively by Ω and g the domain and the color intensity level of a monochromatic image. The strong formulation of Blake–Zisserman functional where the tuning contrast parameters are taken into account is given by:

$$\begin{aligned} F(K_0, K_1, v) \;&=\; F^g_{\alpha,\beta,\mu}(K_0, K_1, v) \;:= \\ &=\; \int_{\Omega\setminus(K_0\cup K_1)} |D^2 v|^2\, dx\, dy \;+\; \mu \int_\Omega |v-g|^2\, dx\, dy \;+ \\ &\qquad +\; \alpha\mathcal{H}^1(K_0\cap\Omega) \;+\; \beta\mathcal{H}^1(K_1\cap\Omega)\,. \end{aligned} \tag{5.1}$$

All terms are positive and play in competition when functional minimization is performed: better than for (1.1), here it is possible a suitable tuning of these terms interaction for the images to be processed, acting on the parameters α, β, μ.

Theorem 5.1. *([28]) Assume (1.2), $\mu > 0$ and*

$$0 < \beta \le \alpha \le 2\beta\,, \tag{5.2}$$

then the functional (5.1) achieves a finite minimum among the essential triplets (K_0, K_1, v), say the triplets fulfilling:

$$\begin{cases} K_0,\, K_1 & \textit{are disjoint Borel subsets of } \mathbb{R}^2, \\ K_0\cup K_1 & \textit{is the smallest closed set s.t. } v\in C^2(\Omega\setminus(K_0\cup K_1)) \\ & \textit{and } v \textit{ is appproximately continuous in } \Omega\setminus K_0\,. \end{cases} \tag{5.3}$$

Moreover, for any locally minimizing triplet (K_0, K_1, v) the set (K_0, K_1) provides the required segmentation of the given raw image g, and v fulfils the following partial differential equation:

$$\Delta^2 v + \mu (v - g) = 0 \qquad \text{in } \Omega \setminus \big(K_0 \cup K_1\big) . \tag{5.4}$$

We emphasize that the unconstrained minimization of F corresponds to the implicit assumption of natural Neumann boundary condition on $\partial\Omega$ and at least formally on $K_0 \cup K_1$ (see [32]).
As we already noticed condition (5.2) is necessary for the lower semicontinuity of the functional.

We briefly recall the main difficulties in the analysis of the Blake–Zisserman functional.

- The presence of both sets and functions in competition for the minimization leads to a lack of convexity; this difficulty is circumvented by a suitable weak formulation ([27]) for which compactness and semicontinuity are proved.
- The finite energy set associated to the weak formulation of the functional is not a subset of the space of distributions, precisely it requires the introductions of vector-valued functions with generalized bounded variation ([45],[27],[62]).
- The lack of convexity leads to examples of non-uniqueness of minimizers; this is not surprising if one recalls inherent ambiguity of images interpretation; nevertheless a reasonable kind of well posedness can be recovered in term of generic uniqueness ([18],[19]).
- Typical features of second order problems are the lack of maximum principle (see Section 3) to be exploited in the regularization procedure and the fact that rough truncations of competing functions are not cost-free ([63]); this fact is circumvented in [27] by introducing a suitable smooth tapering in place of truncation.
- There is a substantial lack of coercivity, which prevents even the use of sequential or topological recession functional ([12],[24]):
  - there is no control on the gradient as shown by the example in Remark 5.2 of [21];
  - we emphasize the fact that introducing a small penalization (of the kind $\varepsilon \int_{\Omega\setminus(K_0\cup K_1)} |Dv|^2$) to keep under control the first gradient, would re-introduce the over-segmentation phenomenon and the other inconveniences associated to the Mumford–Shah functional by mixing the weak plate and weak membrane response; moreover, it would make numerical computations even more cumbersome (see [17], pag.103);
  - actually the fidelity term $\int |v-g|^2$ cannot be dealt with as a lower order term due to the lack of control for the first gradient, so the mimimizers of the functional are not quasi minimizers of the main part of the functional (say the functional itself without the fidelity term) so that the regularization techniques developed in [48] must be adapted accordingly.

The proof of Theorem 5.1 was achieved by the direct methods in Calculus of Variations. The strong functional F depends on triplets; we introduced a weak version of the strong functional F: say a new functional $\mathcal{F}$ depending only on the function v whereas the sets K_0 and K_1 are taken into account respectively through the discontinuity set of v and of its approximate gradient. The minimization of the functional $\mathcal{F}$ was achieved by showing semicontinuity and compactness in the functional space of images with finite energy ([27],[26]), which turned out to be a broad class of functions whose derivatives are special bounded measure in the sense of De Giorgi, and they are not even contained in the space of distributions (denoted by $GSBV^2(\Omega)$, see [27]). Then the triplet solving the minimization problem for functional (5.1) over (5.3) is recovered by showing additional regularity of weak minimizers, by a blow-up technique ([28]). The corresponding optimal segmentation is provided by the pair (K_0, K_1) related to the minimizing triplet.
An essential step in the previous argument is a Poincaré-Wirtinger type inequality in the class $GSBV^2(\Omega)$ which was proven in [28] allowing surgical truncations of non integrable functions of several variables.
Then we proved regularity properties for optimal segmentation in [27], [28], [34], [35], and energy density estimates in [29], [33]; the main estimates are summarized in Section 7.
Approximation properties of the functional were studied in [4] and [37]. The framework and techniques adopted to solve this problem are provided by the Γ-convergence theory ([49]).
In [32] we derived many necessary conditions about weak extremals by performing various kinds of first variations: these delicate computations were performed by taking into account the differential geometry of free discontinuity set in any dimension $n \geq 2$; in particular we developed the full analysis of crack-tip and crease-tip (boundaries of free discontinuity set).

The issue of a sharp separation between the jump set K_0 and the gradient discontinuity set K_1 for an optimal segmentation is still open, though some information is available via numerical experiments.

6. Blake–Zisserman functional for inpainting G

In image restoration the term inpainting denotes the process of retrieving the missing information in small subdomains where a given image is damaged: these subdomains may correspond to scratches in a camera picture, occlusions, blotches in an old movie film or aging of canvas and colors in a painting ([10], [40], [41], [52], [65]). The Mumford–Shah model has been adapted by several authors to the inpainting problem, imposing a Dirichlet-type condition on the preserved part of the image, but some inconvenient has been detected in this approach (see [41] and [52]). In the Mumford–Shah approach ([48], [61]), the preferable edge curves are the ones with shortest length, therefore the model fosters straight edges and may produce artificial corners. In [37] we faced the inpainting problem for a monochromatic image

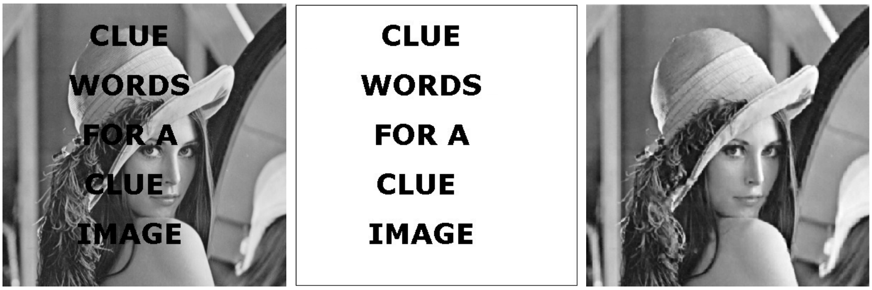

FIGURE 2. Left: Image with occlusion. Center: occluding mask Ω (the black subregion) to be inpainted in the rectangular image domain $\widetilde{\Omega}$. Right: inpainted image.

with a higher order variational approach: minimizing Blake–Zisserman or weak plate functional under Dirichlet-type boundary conditions ([34]). This weak plate functional smooths artificial corners, due to the cost of second derivatives in the functional (see Figure 3).

FIGURE 3. Inpainting and segmentation of a circle obtained by minimization of (6.1).

In the present Section this approach is shortly outlined: we define the second order functional G to deal with monochromatic images with gray levels between 0 and 1 (see (6.1)), aiming to image inpainting in the case of complete loss of information in the subregion Ω (see Figure 2).
The functional G is defined as follows:

$$\begin{aligned} G(K_0, K_1, v) = & \int_{\widetilde{\Omega}\setminus(K_0\cup K_1)} \left|D^2 v\right|^2 \, dx\, dy + \delta \int_{\widetilde{\Omega}} (v-1/2)^2 \, dx\, dy \\ & + \alpha \mathcal{H}^1\big(K_0 \cap \widetilde{\Omega}\big) + \beta \mathcal{H}^1\big(K_1 \cap \widetilde{\Omega}\big) \, . \end{aligned} \tag{6.1}$$

Here $\widetilde{\Omega}$ is an open set, which represents the image domain, with $\widetilde{\Omega} \subset\subset \mathbb{R}^2$, while α, β, μ, δ are positive tuning parameters. The integral of $(v-1/2)^2$, weighted with a fixed positive constant δ, penalizes deviation of gray levels in the output from the central gray level in the raw image and it entails the coerciveness of G.

Let Ω be an open set with $\Omega \subset\subset \widetilde{\Omega}$ and $\partial\Omega$ Lipschitz, let w be a given $L^\infty(\widetilde{\Omega} \setminus \overline{\Omega})$ function with $0 \le w \le 1$, representing the gray level intensity of the raw image under processing, which is damaged due to the presence of blotches in the set $\overline{\Omega}$: the noiseless intensity w is known in $\widetilde{\Omega} \setminus \overline{\Omega}$ while is completely lost in the possibly disconnected set Ω.
To face the inpainting problem for w, we look for minimizers of G among triplets (K_0, K_1, v) which fulfill the Dirichlet condition

$$v = w \quad \text{a.e. on } \; \widetilde{\Omega} \setminus \overline{\Omega} . \tag{6.2}$$

and are **essential triplets**, say they fulfill (5.3).

The main Theorem 6.1 below, concerning the second order variational model for image inpainting via minimization of functional G, was proved in [37], is a general tool since it deals both with discontinuity and gradient discontinuity in the raw image w which is given in $\widetilde{\Omega} \setminus \overline{\Omega}$ and must be processed in Ω. Without loss of generality we can assume that the raw image w is everywhere defined and with constant value equal to $1/2$ in $\overline{\Omega}$.

Theorem 6.1. *([37]) Assume*

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2 , \tag{6.3}$$

$$\Omega \textit{ Lipschitz open set with piece-wise } C^2 \textit{ boundary, } \; \widetilde{\Omega} \textit{ open set,} \tag{6.4}$$

$$0 < \beta \le \alpha \le 2\beta, \quad \mu > 0, \quad \delta > 0, \tag{6.5}$$

$$w : \widetilde{\Omega} \to [0,1] , \quad D^2 w \in L^\infty\big(\widetilde{\Omega} \setminus (T_0 \cup T_1 \cup \partial\Omega)\big). \tag{6.6}$$

$$(T_0, T_1, w) \textit{ is an essential triplet in the sense of } (5.3) , \tag{6.7}$$

$$T_0 \cup T_1 \; \textit{ is a finite union of } C^1 \textit{ curves,} \tag{6.8}$$

$$\mathcal{H}^1\big((T_0 \cup T_1) \cap \widetilde{\Omega}\big) < +\infty , \quad (T_0 \cup T_1) \cap \partial\Omega \; \textit{ is a finite set}. \tag{6.9}$$

Then there exists a triplet (K_0, K_1, v) minimizing the functional G defined by (6.1) among triplets fulfilling (6.2) and (5.3) with $G(K_0, K_1, v) < +\infty$. Moreover any triplet (K_0, K_1, v) which minimizes the functional G among triplets fulfilling (6.2) and (5.3), verifies:

$$K_0 \cap \widetilde{\Omega} \textit{ and } K_1 \cap \widetilde{\Omega} \textit{ are } (\mathcal{H}^1, 1) \textit{ rectifiable sets,} \tag{6.10}$$

$$v \textit{ and } Dv \textit{ have well defined two-sided traces, finite } \mathcal{H}^1 \textit{a.e.on } K_0 \cup K_1 , \tag{6.11}$$

where Dv is the distributional gradient of v in $\Omega \setminus (K_0 \cup K_1)$.

Remark 6.2. Hypotheses (6.6), (6.7), (6.8), (6.9) may seem technical assumptions. Actually they are natural requirements for data whose gray levels correspond to a piece-wise smooth cartoon outside the inpainting region: precisely we ask for data expressed by an essential triplet.

Several numerical experiments were implemented in [4] for segmentation problem and in [37] for segmentation and inpainting problem: they are all based on approximation of F, respectively G, in terms of elliptic functionals (see also [8],[9] for Mumford–Shah and [13] for Blake–Zisserman) in the framework of Γ-convergence ([49],[59]). A different approach based on finite elements schemes was introduced and implemented in [22]. The inpainted output images in Figures 2, 3 and 4 are obtained by the scheme introduced in [37].

FIGURE 4. An input image with overlapping text mask and the corresponding inpainted output.

The functional F in Section 5 for segmentation implicitly refers to Neumann boundary condition, while here functional G for segmentation and inpainting refers to a relaxed Dirichlet condition on the boundary of the inpainting region: the datum may be assumed or not. In the second possibility the segmentation appears on the boundary: when this phenomenon occurs, it is described by energy contribution of a length-type term $\alpha\mathcal{H}^1(K_0\cap\partial\Omega)+\beta\mathcal{H}^1(K_1\cap\partial\Omega)$. As in the case of segmentation, the proof of Theorem 6.1 relies on regularity properties of a minimizer of a weak formulation. In this case the main difficulties are related to the behavior at the boundary points.

The strategy for proving Theorem 6.1 in [37] consists in showing partial regularity at interior points, at boundary points, and close to the boundary of the weak minimizers under Dirichlet condition at the boundary of the inpainting region.
The interior regularity can be recovered by the previous results concerning Neumann problem; here new tools aiming to regularity up to the boundary are necessary:

- a Poincaré-Wirtinger inequality for $GSBV^2$ functions vanishing in a sector;
- an L^2-hessian decay of biharmonic functions in half-disk vanishing together with its normal derivative on the diameter; in [34] we proved that any function which is biharmonic in a half-disk and vanishes together with its normal derivative on the diameter has this hessian decay:

$$\|D^2 z\|^2_{L^2(B_r^+)} \;\leq\; r^2\;\|D^2 z\|^2_{L^2(B_1^+)} \tag{6.12}$$

for all

$$z\in H^2(B_1^+):\;\; \Delta^2 z=0 \text{ on } B_1^+\,,\;\; z=\partial z/\partial y=0 \;\text{ on } \Sigma\,,\quad \text{and } r\leq 1\,,$$

where

$$B_1^+ = \{(x,y)\in\mathbb{R}^2:\ x^2+y^2<1,\ y>0\}\subset\mathbb{R}^2\,,$$
$$\Sigma = B_1(\mathbf{0})\cap\{(x,y)\in\mathbb{R}^2: y=0\}\,.$$

These tools are exploited to perform a blow-up argument on minimizers of the weak functional and deduce a decay estimate for the energy functional.
The study of regularity at boundary points usually requires a smooth extension with suitable estimates of the blown-up solution. The extension of biharmonic functions is quite different from extension of an harmonic function vanishing at the diameter, the latter being based on classical Schwarz reflection principle that doubles L^2 norm of the gradient in the whole disk. This doubling property was exploited to prove energy decay property for minimizers of a Dirichlet problem for the Munford–Shah functional. Unfortunately biharmonic extension lacks this doubling property: the biharmonic extension to the whole disk may increase a lot the L^2 norm of the hessian in the complementary half-disk (see Remark 5 in [33]).
We overcome this difficulty by a careful application of an H^2 Almansi decomposition of biharmonic functions in a disk with a crack (see Section 9) together with the following extension formula, due to R.J.Duffin ([51]), for any biharmonic function in half-disk and vanishing together with its normal derivative on the diameter:
if z is biharmonic in $B^+(\mathbf{0}) := B(\mathbf{0})\cap\{\,y>0\,\}$ and one sets

$$\begin{cases} Z(x,y) = z(x,y)\,, & \forall\,(x,y)\in B^+(\mathbf{0})\,,\\ Z(x,-y) = -z(x,y) + 2\,y\,z_y(x,y) - y^2\Delta z(x,y)\,, & \forall\,(x,-y)\in B^-(\mathbf{0})\,,\end{cases}$$

then Z is biharmonic in the whole disk $B(\mathbf{0})$.

7. Energy density estimates for minimizers of F

To describe some properties of minimizers for functional F defined by (5.1) in Section 5, we refer to the usual notation for balls: for every $\mathbf{x}\in\mathbb{R}^2$ and for every real number $r>0$ we denote by $B_r(\mathbf{x})$ the disk with radius r and center at $\mathbf{x}$, namely $B_r(\mathbf{x}) := \{\,\mathbf{y}\in\mathbb{R}^2\ :\ \|\mathbf{y}-\mathbf{x}\|<r\,\}$. We write shortly B_r in place of $B_r(\mathbf{0})$.
Moreover we introduce a technical construction for removing inessential details from generic triplets.

Definition 7.1. Given a triplet (U_0, U_1, u) (not necessarily a minimizing triplet) with U_0, $U_1\subset\mathbb{R}^2$ Borel sets, $U_0\cup U_1$ a closed set, $u\in C^2(\Omega\setminus(U_0\cup U_1))$ and u approximately continuous in $\Omega\setminus U_0$, then there is an essential triplet (K_0, K_1, v) associated to triplet (U_0, U_1, u), uniquely defined by this procedure:

$$\begin{aligned} v &= \tilde{u}\\ K_0 &= \overline{U_0\cap K}\setminus(U_1\setminus U_0)\\ K_1 &= \overline{U_1\cap K}\setminus U_0\end{aligned}$$

where K is the smallest closed subset of $U_0 \cup U_1$ such that $\widetilde{u} \in C^2(\Omega \setminus K)$.

We emphasize that the above construction entails

$$F(K_0, K_1, v) = F(U_0, U_1, u)$$

and preserves all the relevant quantities as clarified by the next statements, proved in [33].

Theorem 7.2. *Assume (U_0, U_1, u) is any minimizing essential triplet of F, then, if the construction of Definition 7.1 is applied to such (U_0, U_1, u) to define the triplet (K_0, K_1, v), then we obtain*

$$(K_0, K_1, v) = (U_0, U_1, u).$$

Theorem 7.3. *Assume (K_0, K_1, v) is a minimizing essential triplet of F and denote by S_v, $S_{\nabla v}$ the sets where the approximate continuity ([5]) of respectively v and ∇v fails. Then*

$$K_0 \cap K_1 = \emptyset\,, \quad K_0 = \overline{K_0} \setminus K_1 = K_0 \setminus K_1\,, \tag{7.1}$$

$$K_1 = \overline{K_1} \setminus K_0\,, \quad \overline{K_1} \setminus K_1 \subset K_0\,, \quad K_1 \setminus K_0 = K_1\,, \tag{7.2}$$

$$\mathcal{H}^1(S_v \triangle K_0) = 0\,, \qquad \mathcal{H}^1\big((S_{\nabla v} \setminus S_v) \triangle K_1\big) = 0\,, \tag{7.3}$$

$$\begin{aligned} F(K_0, K_1, v) \;=\; & \int_{\Omega \setminus (K_0 \cup K_1)} \left| D^2 v \right|^2 dx\,dy + \int_\Omega |v - g|^2\,dx\,dy \\ & + \alpha\,\mathcal{H}^1\left(K_0 \cap \Omega\right) + \beta\,\mathcal{H}^1\left(K_1 \cap \Omega\right)\,. \end{aligned} \tag{7.4}$$

We mention four energy density estimates for essential minimizing triplets.

The optimal segmentation (K_0, K_1) satisfies an upper bound on the "mean density" on every disk $B_r(\mathbf{x})$, i.e. the ratio between the measure of the portion of (K_0, K_1) in $B_r(\mathbf{x})$ and r (modulo a normalization constant).

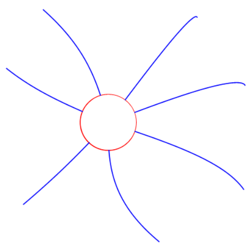

FIGURE 5. Density Upper Bound: if $\alpha + (\mu/2)\|g\|^2_{L^\infty} \leq 1$ then an optimal segmentation cannot have more than six lines reaching a single point.

Density Upper Bound
Assume $\Omega \subset \mathbb{R}^2$ is a bounded open set and g belongs to $L^\infty(\Omega)$. Then the estimate

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_r(\mathbf{x})\right) \leq (\pi\mu\|g\|^2_{L^\infty} + 2\pi\alpha) r$$

holds true for every $\mathbf{x} \in \Omega$ *and every* $0 < r \leq 1$ *such that* $B_r(\mathbf{x}) \subset \Omega$.
The next property is the crucial point in the proof of the existence theorem for F, arguing on a minimizer for the weak formulation by $\mathcal{F}$.

Density Lower Bound
Assume $\Omega \subset \mathbb{R}^2$ *is a bounded open set and* g *belongs to* $L^\infty(\Omega)$.
Then there exist $\varepsilon_1 > 0, \varrho_1 > 0$ *such that the estimate*

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_r(\mathbf{x})\right) \quad \geq \quad \varepsilon_1 r$$

holds true for every $\mathbf{x} \in K_0 \cup K_1$ *and every* $0 < r \leq \varrho_1$ *with* $B_r(\mathbf{x}) \subset \Omega$.

The density lower bound entails, among other things, that very small isolated objects are filtered out by the segmentation achieved through minimization of the functional F.

Elimination Property
Assume $\Omega \subset \mathbb{R}^2$ *is a bounded open set and* $g \in L^\infty(\Omega)$. *Choose* $\varepsilon_1 > 0, \varrho_1 > 0$ *as in the previous statement and* $0 < r \leq \varrho_1$ *such that* $B_r(\mathbf{x}) \subset \Omega$.
If

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_r(\mathbf{x})\right) \quad < \quad \frac{\varepsilon_1}{2} r$$

then

$$(K_0 \cup K_1) \cap B_{r/2}(\mathbf{x}) = \emptyset.$$

The elimination property states that, when an optimal segmentation in a small disk has length less than an absolute constant times the radius of the disk, then such segmentation does not intersect the disk with same center but halved radius. This is a useful information for the numerical analysis of the problem, in the sense that a suitable algorithm can eliminate such isolated parts of $K_0 \cup K_1$ because they are "needless energy" for the optimal segmentation.

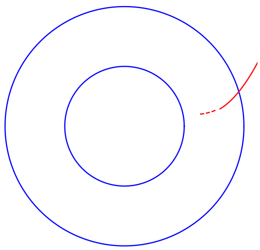

FIGURE 6. Elimination property.

These properties altogether drive the optimal segmentation (K_0, K_1) to reshape an essential pattern of basic lines partitioning the given image in homogeneous subregions. In this way the optimal segmentation brings more clearly to the fore the semantic meaning concealed in g.

Minkowski content of the segmentation
Assume $\Omega \subset \mathbb{R}^2$ is a bounded open set, g belongs to $L^\infty(\Omega)$ and (K_0, K_1, u) is an essential minimizing triplet for the functional F. Then

(i) *$K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable;*
(ii) *for every $\Omega' \subset\subset \Omega$ the following equality holds*

$$\lim_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \Omega\,;\; dist(\mathbf{x}, (K_0 \cup K_1) \cap \Omega') < \varrho\,\}|}{2\varrho} = \mathcal{H}^1\left((K_0 \cup K_1) \cap \Omega'\right)\ .$$

This result expresses the agreement between the Hausdorff one dimensional measure and the Minkowski content of the optimal segmentation $K_0 \cup K_1$: roughly speaking, the property states that (far away from the boundary of the image) a uniform fattening of an optimal segmentation is a reasonable approximation of the segmentation itself. Even this information is important for numerical computations of minimizers, in order to approximate F by elliptic functionals for which efficient numerical algorithms can be found: see [4] and [37].

The above interior estimates were proved in [29] for minimizing triplets of functional F, their technical refinement up to the boundary were proved in [33] under Dirichlet boundary conditions, that is the case of functional G in Section 6 concerning image inpainting. All the statements of the present Section hold true for functional G too, provided Ω is replaced by $\widetilde{\Omega}$ in Definition 7.1 and formula (7.4) and $g \equiv 1/2$ is assumed everywhere in $\widetilde{\Omega}$.

8. Admissible and not admissible nontrivial candidates

The main part E of the Blake–Zisserman functional F is defined as follows.

$$E(K_0, K_1, u) = \int_{\Omega \setminus (K_0 \cup K_1)} \left|D^2 u\right|^2 dx\, dy\ + \alpha\, \mathcal{H}^1(K_0 \cap \Omega) + \beta\, \mathcal{H}^1(K_1 \cap \Omega)\,. \tag{8.1}$$

In the study of regularity of minimizers of F via blow-up procedure it is important the study of locally minimizing triplets for E, with the choice $\Omega = \mathbb{R}^2$. We recall the definition of locally minimizing triplets of E and the notation for the localized functional.

Definition 8.1. By E_A we denote the functional defined as E with Ω replaced by an open subset $A \subset \Omega$.

Definition 8.2. **(Locally minimizing triplet of E)**
An admissible triplet (K_0, K_1, v) is a locally minimizing triplet of the functional E if

$$E_A(K_0, K_1, v) < +\infty \tag{8.2}$$

$$E_A(K_0, K_1, v) \leq E_A(U_0, U_1, u) \tag{8.3}$$

for every open subset $A \subset\subset \Omega$ and for every admissible triplet (U_0, U_1, u) such that

$$\mathrm{spt}(v-u) \quad \text{and} \quad \overline{(U_0 \cup U_1) \triangle (K_0 \cup K_1)} \quad \text{are subsets of } A.$$

We emphasize some facts about locally minimizing triplets of the main part of the Blake–Zisserman functional E in $\Omega = \mathbb{R}^2$ (see [32],[35]):

- any locally minimizing triplet (K_0, K_1, u) is transformed in another locally minimizing triplet by all natural re-scaling (centered at $\mathbf{x}_0 \in \Omega$), which maps
$u(\mathbf{x}) \mapsto \varrho^{-3/2} u(\mathbf{x}_0 + \varrho \mathbf{x})\,, \quad K_j \mapsto \varrho^{-1}(K_j - \mathbf{x}_0)\,, \quad \text{for } \varrho > 0,\ j = 0, 1\,.$
- any locally minimizing triplet (K_0, K_1, u) with compact segmentation $K_0 \cup K_1$ and finite energy actually corresponds to an affine function with empty singular set;
- neither a straight infinite jump nor a straight infinite wedge can be the third element of a locally minimizing triplet.

Therefore the above examples cannot be nontrivial admissible candidate local minimizers.
Proving the minimality of a given candidate for a free discontinuity problem is a difficult task in general.
Moreover, for the case of Blake–Zisserman functional one has to take into account the long list of necessary conditions fulfilled by strong extremals in 2-dimensional case proved in [32] by performing several kinds of variations of the admissible triplets: this leads to severe qualitative and quantitative restrictions on the behavior of extremals. Notwithstanding this huge amount of constraints, we exhibited explicitly a nontrivial candidate for local minimality which fulfills all Euler conditions in the list and the energy equipartition between bulk energy and segmentation length ([35]).
We consider the following nontrivial function, with jump discontinuity along the negative real axis and empty discontinuity set of the gradient:

$$\pm \sqrt{\frac{\alpha}{193\,\pi}}\, r^{3/2} \left(\sqrt{21}\, \omega(\vartheta) \pm w(\vartheta) \right), \qquad -\pi < \vartheta < \pi\,, \tag{8.4}$$

where, by making explicit the *modes* ω and w,

$$\pm \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2} \left(\sqrt{21} \left(\sin\frac{\theta}{2} - \frac{5}{3} \sin\left(\frac{3}{2}\theta\right) \right) \pm \left(\cos\frac{\theta}{2} - \frac{7}{3} \cos\left(\frac{3}{2}\theta\right) \right) \right),$$

Function (8.4), together with $K_0 = \{\text{negative real axis}\}$ and $K_1 = \emptyset$, satisfies all extremality conditions proved for locally minimizing triplets of functional E in $\mathbb{R}^2$: hence such function is a natural candidate to be a local minimizer of Blake–Zisserman functional, as conjectured in [35].

9. Almansi decomposition around a crack-tip

In this Section we collect some results of wider interest than the application to image analysis, nevertheless some of them were exploited in [35] and [38] in the analysis of nontrivial minimizing triplets for Blake–Zisserman functional. See also [11] for an alternative approach to crack tips in planar elasticity.
Here we look for a description of all functions which are defined almost everywhere in $B_\varrho \subset \mathbb{R}^2$ and are biharmonic in $B_\varrho \setminus \Gamma$ where $0 < \varrho < +\infty$ and Γ is the closed negative real axis.

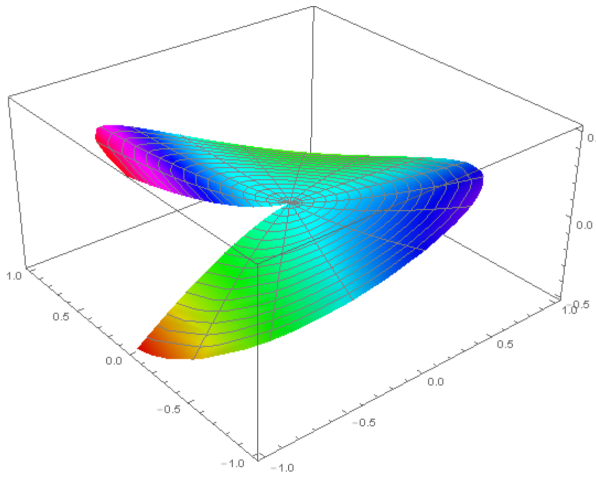

FIGURE 7. Graph of one function in the family (8.4), with $\alpha = 1$ and plus signs: $r^{3/2}\left(\sqrt{21}\,\omega(\vartheta) + w(\vartheta)\right)/\sqrt{193\,\pi}$.

First, inspired by classical ideas, we rewrite in modern language a statement by E.Almansi, Theorem 9.1 below, concerning a decomposition of biharmonic functions. Then we relax both assumptions on domain topology and the function regularity and we make explicit the related decomposition operators (Theorem 9.2).

Theorem 9.1. *([1])*
Classical Almansi decomposition of a biharmonic function in a disk.
Assume $\mathbf{0} \in \Omega \subset \mathbb{R}^2$ *open set,* Ω *star-shaped with respect to the origin and* $u \in C^4(\Omega)$, *then* ${\Delta_{\mathbf{x}}}^2 u = 0$ *in* Ω *iff*

$$\exists\, \varphi, \psi : \quad u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x}), \quad \Delta_{\mathbf{x}}\,\varphi(\mathbf{x}) = \Delta_{\mathbf{x}}\,\psi(\mathbf{x}) \equiv 0 \quad \forall\, \mathbf{x} \in \Omega\,.$$

In this section we denote the Laplacean operator in cartesian coordinates by $\Delta_{\mathbf{x}}$, to avoid confusion with polar coordinates used in the sequel.
For our purposes we adapt the previous classical statement to open sets not containing the origin, in particular to 2-dimensional disks with a straight cut. This geometry introduces several difficulties since the crack-tip allows some kind of singularities at the origin.

Theorem 9.2. *([35])*
Decomposition of a biharmonic H^2 function around a crack-tip.
Let $u \in H^2(B_\varrho \setminus \Gamma)$, $0 < \varrho \leq +\infty$. *Then*

$$\Delta_{\mathbf{x}}^2\, u = 0 \quad \textit{in } B_\varrho \setminus \Gamma \tag{9.1}$$

if and only if

$$\exists\, \varphi, \psi : \, u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x}), \;\; \Delta_{\mathbf{x}}\varphi(\mathbf{x}) = \Delta_{\mathbf{x}}\psi(\mathbf{x}) \equiv 0\,, \forall\, \mathbf{x} \in B_\varrho \setminus \Gamma. \tag{9.2}$$

Decomposition (9.2) is unique up to possible linear terms in ψ*: say* $A\varrho\cos\vartheta = Ax$ *and* $B\varrho\sin\vartheta = By$ *that can switch indifferently between* $A\varrho^{-1}\cos\vartheta$ *and* $B\varrho^{-1}\sin\vartheta$ *in* φ.
By denoting $A_\varrho := \{v \in L^2(B_\varrho) \text{ s.t. } \Delta_{\mathbf{x}} v = 0 \text{ in } B_\varrho \setminus \Gamma\ \}$ *the space of* L^2 *functions*

which are harmonic outside the crack, we can make explicit the decomposition by introducing the operators Φ *and* Ψ*, that act on every* $u \in H^2(B_\varrho\backslash\Gamma)$ *expressed in polar coordinates as follows:*

$$\Phi : H^2(B_\varrho\backslash\Gamma) \to H^2(B_\varrho\backslash\Gamma)\cap A_\varrho, \quad \Phi[u](r,\vartheta) = r^{-1}\int_0^r \frac{1}{4}\Delta_{\mathbf{x}} u(t,\vartheta)\,dt\,, \tag{9.3}$$

$$\Psi : H^2(B_\varrho\backslash\Gamma) \to H^2(B_\varrho\backslash\Gamma)\cap A_\varrho, \quad \Psi[u](r,\vartheta) \;=\; u(r,\vartheta) - r^2\,\Phi[u](r,\vartheta)\,, \tag{9.4}$$

$$u \;=\; \Psi[u] + r^2\Phi[u]\,, \tag{9.5}$$

The claim in Theorem 9.2 that decomposition (9.2) entails u is biharmonic is a straightforward computation. The proof of reversed inference is straightforward if in addition $u \in C^4(B_\varrho \setminus \Gamma)$ and $\Delta_{\mathbf{x}} u$ is continuous up to the origin, even without assuming $u \in H^2(B_\varrho \setminus \Gamma)$. Actually the statement of Theorem 9.2 allows to deal also with the case when u has a singularity at $\mathbf{0}$. For the detailed proof and related results under weaker regularity assumptions we refer to [39].

Notice that the candidate minimizer (8.4) in Section 8 fulfils the assumption of Theorem 9.2: for any $\varrho < \infty$, it belongs to $H^2(B_\varrho\backslash\Gamma)$ and is biharmonic in $B_\varrho\backslash\Gamma$.

Acknowledgments
The content of this paper was partly presented in the talk given by the second Author at the “Seminario Matematico e Fisico di Milano” on February, 16^{th} 2015.

References

[1] E.Almansi, *Sull'integrazione dell'equazione differenziale* $\Delta^{2n} = 0$*,* Ann. Mat. Pura Appl., **III**, 1-51 (1899)

[2] G.Alberti, G.Bouchitté & G.Dal Maso, *The calibration method for the Mumford-Shah functional and free-discontinuity problems,* Calc. Var. Partial Differential Equations, **16**, 299–333 (2003).

[3] M.Amar, V.De Cicco, *The uniqueness as a generic property for some one dimensional segmentation problems,* Rend. Sem. Univ. Padova, **88** (1992), 151–173.

[4] L.Ambrosio, L.Faina, R.March, *Variational approximation of a second order free discontinuity problem in computer vision,* SIAM J. Math. Anal., **32** (2001), 1171–1197.

[5] L.Ambrosio, N.Fusco, D.Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[6] L. Ambrosio, D. Pallara, *Partial regularity of free discontinuity sets,*Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** 1, (1997) 1–38.

[7] L.Ambrosio, N.Fusco, D.Pallara, *Partial regularity of free discontinuity sets II,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** 1, (1997) 39–62.

[8] L.Ambrosio, V.M.Tortorelli, *Approximation of functionals depending on jumps by elliptic functionals via* Γ*-convergence,* Comm. Pure Appl. Math., **43** (1990), pp. 999–1036.

[9] L.Ambrosio, V.M.Tortorelli, *On the approximation of free discontinuity problems,* Boll. Un. Mat. Ital. B (7), **6** (1992), pp. 105–123.

[10] G.Aubert, P.Kornprobst, Mathematical problems in image processing, Partial Differential Equations and the Calculus of Variations, Applied Mathematical Sciences, **147** 2nd ed., Springer, New York, 2006.

[11] J.-F.Babadjian, A.Chambolle & A.Lemenant, *Energy release rate for non smooth cracks in planar elasticity,* Journal de l'Ecole Polytechnique - Mathématiques, **2** (2015), 117-152.

[12] C.Baiocchi, G.Buttazzo, F.Gastaldi, F.Tomarelli *General Existence Theorems For Unilateral Problems in Continuum-Mechanics* Arch. Rational Mech. Anal., **100** 2, (1988) 149–189.

[13] G.Bellettini, A.Coscia, *Approximation of a functional depending on jumps and corners,* Boll. Un. Mat. Ital. B (7), **8** (1994), pp. 151–181.

[14] M.Bergounioux, L.Piffet, *A full second order variational model for multiscale texture analysis,* Comput.Optim.Appl., DOI: 10.1007/s10589-012-9484-9, (2013).

[15] M.Bergounioux, *Mathematical Analysis of a Inf-Convolution Model for Image Processing,* J. Optim. Theory Appl., DOI: 10.1007/s10957-015-0734-8, (2015).

[16] M.Bertalmío, V.Caselles, S.Masnou, G.Sapiro, *Inpainting,* in "Encyclopedia of Computer Vision", Springer, 2011.

[17] A.Blake, A.Zisserman, *Visual Reconstruction,* The MIT Press, Cambridge, 1987.

[18] T.Boccellari, F.Tomarelli, *About well-posedness of optimal segmentation for Blake & Zisserman functional,* Istituto Lombardo (Rend. Scienze), **142** (2008), 237–266.

[19] T.Boccellari, F.Tomarelli, *Generic uniqueness of minimizer for Blake & Zisserman functional,* Revista Matematica Complutense, **26**, (2013) 361–408.

[20] A.Bonnet & G.David, *Crack-tip is a global Mumford-Shah minimizer,* Astérisque, **274** (2001).

[21] A.Braides, *Lower semicontinuity conditions for functionals on jumps and creases,* SIAM J. Math. Anal., **26** (1995), 1184-1198.

[22] A.Braides, A.De Franceschi, E.Vitali, *A compactness result for a second-order variational discrete model,* ESAIM: Mathematical Modelling and Numerical Analysis, **46** (2012), 389-410.

[23] D.Bucur, S.Luckhaus, *Monotonicity Formula and Regularity for General Free Discontinuity Problems,* Arch. Rational Mech. Anal., **211** 2, (2014) 489–511.

[24] G.Buttazzo, F.Tomarelli, *Compatibility Conditions For Nonlinear Neumann Problems* Advances In Mathematics **89** 2, (1991) 127–143.

[25] L.Calatroni, B.During, C.B.Schonlieb, *ADI splitting schemes for a 4th order nonlinear PDE from image processing,* Discr.Cont.Dynamical Systems, Series A, Special Issue for Arieh Iserles 65th birthday, 34(3), March (2014) 931–957.

[26] M.Carriero, A.Leaci, F.Tomarelli, *Free gradient discontinuities,* in "Calculus of Variations, Homogeneization and Continuum Mechanics", (Marseille 1993), 131-147, Ser.Adv.Math Appl.Sci., **18**, World Sci. Publishing, River Edge, NJ, 1994.

[27] M.Carriero, A.Leaci, F.Tomarelli, *A second order model in image segmentation: Blake & Zisserman functional,* in: *Variational Methods for Discontinuous Structures* (Como, 1994), 57–72, Progr. Nonlinear Differential Equations Appl., **25,** Birkhäuser, Basel, 1996.

[28] M.Carriero, A.Leaci, F.Tomarelli, *Strong minimizers of Blake & Zisserman functional,* Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), **25**, n.1-2 (1997), 257–285.

[29] M.Carriero, A.Leaci, F.Tomarelli, *Density estimates and further properties of Blake & Zisserman functional,* in "From Convexity to Nonconvexity", R.Gilbert & Pardalos Eds., Nonconvex Optim. Appl., **55**, Kluwer Acad. Publ., Dordrecht (2001), 381–392.

[30] M.Carriero, A.Leaci, F.Tomarelli, *Necessary conditions for extremals of Blake & Zisserman functional,* C. R. Math. Acad. Sci. Paris, **334**, n.4 (2002), 343–348.

[31] M.Carriero, A.Leaci, F.Tomarelli, *Calculus of Variations and image segmentation,* J. of Physiology, Paris, **97**, n.2-3 (2003), 343–353.

[32] M.Carriero, A.Leaci, F.Tomarelli, *Euler equations for Blake & Zisserman functional,* Calc. Var. Partial Differential Equations, **32**, n.1 (2008), 81–110.

[33] M.Carriero, A.Leaci, F.Tomarelli, *Uniform density estimates for Blake & Zisserman functional,* Discrete Contin. Dyn. Syst. - Series A, **31, (4)** (2011), 1129–1150.

[34] M.Carriero, A.Leaci, F.Tomarelli, *A Dirichlet problem with free gradient discontinuity,* Advances in Mathematical Sciences and Applications, **20**, n.1 (2010), 107–141.

[35] M.Carriero, A.Leaci, F.Tomarelli, *A candidate local minimizer of Blake & Zisserman functional,* J. Math. Pures Appl., **96**, (2011), 58–87.

[36] M.Carriero, A.Leaci, F.Tomarelli, *Free Gradient Discontinuity and Image Inpainting,* J. Math. Sci. (N.Y.), **181**, n.6, (2012) 805–819.

[37] M.Carriero, A.Leaci, F.Tomarelli *Image inpainting via variational approximation of a Dirichlet problem with free discontinuity,* Adv. Calc.Var., **7** (3), 267–295 (2014).

[38] M.Carriero, A.Leaci, F.Tomarelli, *Corrigendum to "A candidate local minimizer of Blake & Zisserman functional" [J.Math.Pures Appl.96(1), (2011) 58–87],* J. Math. Pures Appl., to appear, DOI: 10.1016/j.matpur.2015.03.012

[39] M.Carriero, A.Leaci, F.Tomarelli *Almansi decomposition around the crack-tip and power series expansion for harmonic, biharmonic and polyharmonic functions in open sets with a flat crack,* to appear.

[40] V.Caselles, G.Haro, G.Sapiro, J.Verdera, *On geometric variational models for inpainting surface holes,* Computer Vision and Image Understanding, **111** (2008), 351–373.

[41] T.F.Chan, J.Shen, *Variational image inpainting,* Comm. Pure Appl. Math., **LVIII** (2005), 579–619.

[42] A.Coscia, *Existence result for a new variational problem in one-dimensional segmentation theory,* Ann. Univ. Ferrara - Sez. VII - Sc. Mat., **XXXVII** (1991), 185-203.

[43] G. Dal Maso, J. M. Morel & S. Solimini:*A variational method in image segmentation: existence and approximation results,*Acta Math.,168 (1992), 89–151.

[44] G.David, *Singular sets of minimizers for the Mumford-Shah functional.* Progress in Mathematics, **233**, Birkhäuser, Basel, 2005.

[45] E.De Giorgi, *Free discontinuity problems in calculus of variations,* in "Frontiers in Pure & Appl. Math.", R.Dautray Ed., North–Holland, Amsterdam, (1991), 55–61.

[46] E.De Giorgi, *Selected papers.* Edited by L.Ambrosio, G.Dal Maso, M.Forti, M.Miranda and S.Spagnolo. Reprint of the 2006 edition. Springer Collected Works in Mathematics. Springer, Heidelberg, 2013.

[47] E.De Giorgi, L.Ambrosio, *Un nuovo tipo di funzionale del calcolo delle variazioni,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **82** (1988), 199–210.

[48] E.De Giorgi, M.Carriero, A.Leaci, *Existence theorem for a minimum problem with free discontinuity set,* Arch. Rational Mech. Anal., **108** (1989), 195–218.

[49] E.De Giorgi, T.Franzoni *Su un tipo di convergenza variazionale,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **58**, n.6 (1975), 842–850.

[50] C.De Lellis, M.Focardi *Density lower bound estimates for local minimizers of the 2d Mumford-Shah energy,* Manuscripta Math., **142**, n.1-2 (2013), 215–232.

[51] R.J.Duffin, *Continuation of biharmonic functions by reflection,* Duke Math. J., **22** (1955), 313-324.

[52] S.Esedoglu, J.H.Shen, *Digital inpainting based on the Mumford-Shah-Euler image model,* Eur. J. Appl. Math., **13**, n.4 (2002), 353-370.

[53] H.Federer, Geometric Measure Theory, Springer, Berlin, 1969.

[54] I.Fonseca, G.Leoni, F.Maggi, M.Morini, *Exact reconstruction of damaged color images using a total variation model,* Ann. Inst. H. Poincar Anal. Non Linaire, **27** 5, (2010) 12911331.

[55] M.Fornasier, C.B.Schonlieb, *Subspace correction methods for total variation and ℓ^1 minimization,* SIAM J.Num.An., 47 (5), (2009) 3397–3428

[56] N.Fusco, *An Overview of the Mumford-Shah Problem,* Milan J. Math., **71**, (2003) 95–119.

[57] F.A.Lops, F.Maddalena, S.Solimini, *Hölder continuity conditions for the solvability of Dirichlet problems involving functionals with free discontinuities,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **18** n.6, (2001) 639–673.

[58] P.A.Markovich, *Applied Partial Differential Equations: a Visual Approach,* Springer, New York (2007).

[59] L.Modica, S.Mortola, *Un esempio di* Γ*-convergenza,* Boll. Un. Mat. ltal. **5 14-B** (1977), 285-299.

[60] J.M.Morel, S.Solimini, *Variational Models in Image Segmentation,* Progr. Nonlinear Differential Equations Appl., **14**, Birkhäuser, Basel, 1995.

[61] D.Mumford, J.Shah, *Optimal approximation by piecewise smooth functions and associated variational problems,* Comm. Pure Appl. Math., **XLII** (1989), 577–685.

[62] D.Pallara, *Some new results on functions of bounded variation,* Rend. Accad. Naz. delle Scienze (dei XL), (108), **XIV** (1990), 295–321.

[63] G.Savaré, F.Tomarelli, *Superposition and chain rule for bounded Hessian functions,* Adv.Math., **140**, 2 (1998), 237–281.

[64] C.-B.Schönlieb, A. Bertozzi, *Unconditionally stable schemes for higher order inpainting,* Communications in Mathematical Sciences, **9**, 2 (2011), 413–457.

[65] J.Verdera, V.Caselles, M. Bertalmio, G.Sapiro, *Inpainting Surface Holes,* In: Int. Conference on Image Processing, (2003), 903–906.

[66] M.Zanetti, A.Vitti, *The Blake-Zisserman model for digital surface models segmentation,* In: ISPRS Ann. Photogramm. Remote Sens. Spatial Inf. Sci., II-5/W2, DOI: 10.5194/isprsannals-II-5-W2-355-2013, (2013), 355–360.

Received: June 23, 2015.
Revised: July 30, 2015.

Michele Carriero
Università del Salento
Dipartimento di Matematica e Fisica “Ennio De Giorgi”
I 73100 Lecce - Italy
e-mail: `michele.carriero@unisalento.it`

Antonio Leaci
Università del Salento
Dipartimento di Matematica e Fisica “Ennio De Giorgi”
I 73100 Lecce - Italy
e-mail: `antonio.leaci@unisalento.it`

Franco Tomarelli
Politecnico di Milano
Dipartimento di Matematica
Piazza “Leonardo da Vinci”, 32
I 20133 Milano - Italy
e-mail: `franco.tomarelli@polimi.it`