ORTHOGONAL DOUBLE COVERS OF COMPLETE BIPARTITE GRAPHS BY PATHS AND CYCLES

M. Higazy* **and R. Scapellato**

Department of Physics and Engineering Mathematics
Faculty of Electronic Engineering
Menoufiya University
32952 Menouf, Menoufiya, Egypt
e-mail: mahmoudhegazy380@hotmail.com

Dipartimento di Matematica "Francesco Brioschi"
Politecnico di Milano
Piazza Leonardo da Vinci, 32 - 20133
Milano, Italy

Received: May 24, 2014; Accepted: July 25, 2014

*Corresponding author
Communicated by K. K. Azad

1. Introduction

Let H be a (finite, undirected) graph of order n. An orthogonal double cover (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of subgraphs of H such that

(i) every edge of H is contained in exactly two members of $\mathcal{G}$, and

(ii) for any two different members G_u and G_v in $\mathcal{G}$, $|E(G_u) \cap E(G_v)|$ is 1 if u and v are adjacent in H and is 0 otherwise.

If $G_i \cong G$ for all $i \in \{0, 1, ..., n-1\}$, then $\mathcal{G}$ is called an *ODC* of H by G.

A collection $\mathcal{G} = \{G_0, G_1, ..., G_{s-1}\}$ of subgraphs of H is called a *suborthogonal double cover* (*SODC*) of H if

(i) every edge of H is contained in exactly two members of $\mathcal{G}$, and

(ii)′ any two subgraphs G_i and G_j with $i \neq j$ have at most one edge in common.

If all of G_i are isomorphic to a graph G, then $\mathcal{G}$ is called an *SODC* of H by G. Note that (ii) implies (ii)′, hence every ODC is a special case of SODC.

As most of the researches on this subject, including that of the present paper, deal with ODCs and SODCs by a given graph G, all ODCs and SODCs considered here will always be of this type.

Throughout this paper, we shall denote by K_n the complete graph with n vertices, by $K_{n,n}$ the complete bipartite graph with colour classes of order n, by P_n the path on n vertices and by C_n the cycle on n vertices.

The concept of ODC was originally defined for the case where H is a complete graph. While in principle any regular graph H is worth considering (e.g., hypercubes have been investigated in [10]), the choice of $H = K_{n,n}$ is

natural, besides, results on those can sometimes be applied to find new ODCs of K_n (see [3, p. 48]).

An algebraic construction of ODCs via "symmetric starters" (see Section 2) has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, all suitable G are found this way (see [3], Table 1). This method has been applied in [3, 11] to detect some infinite classes of graphs G for which there is an ODC of $K_{n,n}$ by G.

Cayley graphs that have ODCs have been studied from different viewpoints in [12] (4-regular circulants) and [13] (3-regular Cayley graphs).

For further results on ODCs, see the survey [7] and the updates cited in [11]. For graph-theoretical terminology not defined here see [1].

Throughout the whole paper, the vertices of $K_{n,n}$ are labelled by the elements of $\Gamma \times \mathbb{Z}_2$, where Γ is an additive abelian group of order n, so that $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$ are the colour classes.

From now on, the elements (u, i) of the aforementioned set will be written as u_i. Moreover, if there is no danger of confusion, then the edge joining the vertices u_o and w_1 will be denoted by uw.

This paper will deal with both ODCs and SODCs of complete bipartite graphs with colour classes of equal order.

In Section 2, after introducing the concept of symmetric starter of an ODC of $K_{n,n}$, we shall prove that if p is a prime, then there exists an ODC of $K_{p,p}$ by P_{p+1} (Theorem 2.4), so settling a conjecture stated by El-Shanawany [2].

Section 3 is devoted to new constructions of SODCs of complete bipartite graphs. Results on SODCs have been obtained by Schumacher [14], Hartmann [8], Hartmann and Schumacher [9] and Higazy [11]. We shall prove here that $K_{n,n}$ has an SODC by C_{10} whenever n is a multiple of 5

(Theorem 3.4). In particular, for $n = 10$ this gives immediately an ODC of $K_{10,10}$ by C_{10}. Furthermore, we prove that $K_{10,10}$ has an ODC by $C_6 \cup C_4$. Note that the classification of ODCs of $K_{n,n}$ for small values of n given in [3] only included those with $n \leq 9$.

2. Symmetric Starters

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph G_a with $E(G + a) = \{(u + a)(v + a) : uv \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = uv \in E(G)$ is defined by $d(e) = v - u$.

We call G a *half starter* with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are different, i.e., $\{d(e) : e \in E(G)\} = \Gamma$. The following three results, established in [3], were often used as a tool to construct ODCs (see, e.g., [4-6]). We shall apply them yet another time, to prove Theorem 2.4.

Theorem 2.1. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e.,* $\bigcup_{a \in \Gamma} E(G + a) = E(K_{n,n})$.

Here, a half starter will be represented by the vector: $v(G) = (v_{\gamma_0}, ..., v_{\gamma_{n-1}})$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i.

Two half starters $v(G_0)$ and $v(G_1)$ are said to be *orthogonal* if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2.2. *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $G = \{G_{a,i} : (a, i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

To each of the two edge decompositions we may associate bijectively an $(n \times n)$-square with entries belonging to Γ by $L_i = L_i(k, l)$, $i = 0, 1$;

$k, l \in \Gamma$ with $L_i(k, l) = m$, if and only if the edge $\{k_0, l_1\} \in E(G_{m,i})$. For the squares, the orthogonality condition reads as

$$|\{(L_0(k, l), L_1(k, l)) : k, l \in \Gamma\}| = n^2.$$

For more details, see [2, 3, 7].

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0, v_1\} : \{v_0, u_1\} \in E(G)\}$ is called the *symmetric graph* of *G*. Note that if *G* is a half starter, then G_s is also a half starter.

A half starter *G* is called a *symmetric starter* with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 2.3. *Let n be a positive integer and let G be a half starter represented by* $v(G) = (v_{\gamma_0}, ..., v_{\gamma_{n-1}})$. *Then G is a symmetric starter if and only if* $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.

We are now ready to prove.

Theorem 2.4. *If p is a prime, then there exists an ODC of* $K_{p,p}$ *by* P_{p+1}.

Proof. For all $i \in \mathbb{Z}_p$, define the vector $v(G)$ by: $v_i(G) = -i^2$.

Since $\{v_i - v_{-i} + i = i : i \in \mathbb{Z}_p\} = \mathbb{Z}_p$, the vector

$$v(G) = (0, -1, -4, ..., -i^2, ..., -(p-2)^2, -(p-1)^2)$$

is a symmetric starter of an ODC of $K_{p,p}$ by G (with respect to $\mathbb{Z}_p$).

Let us now prove that *G* is a path.

This is clear for $p = 2$, so assume $p > 2$.

For each $r \in \mathbb{Z}_p$, denote by e_r the edge of length r, namely

$$e_r = ((-(r)^2, 0), (-(r)^2 + r, 1)).$$

Let us fix the edge e_r and consider e_s with $s \neq r$.

Thus, e_r and e_s are incident in the vertex with second coordinate 0 whenever $-(r)^2 = -(s)^2$.

Simplifying the previous equation, we get $(r - s)(r + s) = 0$.

As r and s are different, this means $r + s = 0$, hence $e_s = e_{-r}$.

Therefore, there is exactly one value of s, unless $r = 0$, in which case no such s exists.

Likewise, e_r and e_s are incident in the vertex with second coordinate 1 whenever $-(r)^2 + r = -(s)^2 + s$, that is $(r - s)(r + s - 1) = 0$, hence $e_s = e_{1-r}$.

Therefore, there is exactly one value of s, unless $r = 1/2$, in which case no such s exists.

In view of the above discussion, the sequence e_0, e_1, e_{-1}, e_2, e_{-2}, ..., $e_{((p-1)/2)}$, $e_{(-(p-1)/2)}$ is a path P_{p+1}. (Note that $-(p-1)/2 = 1/2$). □

3. Suborthogonal Double Covers of $K_{n,n}$

For any subgraph G of $K_{n,n}$, let $L(G) = \{b - a : (a, b) \in E(G)\}$ be the multiset containing the length of every edge in G. For any two subgraphs G and G' of $K_{n,n}$, let $D(G, G') = -D(G', G) = \{a' - a : (a, b) \in E(G), (a', b') \in E(G'), b - a = b' - a'\}$ be the multiset containing the distance of every pair of equal length edges in G and G'.

Definition 3.1. A collection $\mathcal{P} = \{G_0, ..., G_{g-1}\}$ of subgraphs of $K_{n,n}$ is called an *SODC-starter* if satisfies the following:

(i) Length condition: the union of the multisets $L(G_i)$, $i = 0, 1, ..., g-1$, contains every elements of Γ exactly twice.

(ii) Distance condition: for all pairs i, j with $0 \le i \le j \le g-1$, the multiset $D(G_i, G_j)$ is a set.

The following theorem was proved in [2].

Theorem 3.2. *Let* $\mathcal{P} = \{G_0, ..., G_{g-1}\}$ *be an SODC-starter. Then the collection of all the translates* $G_i + x$, *with* $x \in \mathbb{Z}_n$, *forms an SODC of* $K_{n,n}$.

Lemma 3.3. *There exists an ODC of* $K_{10,10}$ *by* C_{10}.

Proof. Define

$$E(G_0) = \{(0, 0), (0, 2), (5, 3), (5, 4), (6, 0), (6, 3), (8, 4), (8, 9), (9, 2), (9, 9)\}$$

and

$$E(G_1) = \{(0, 1), (0, 2), (1, 8), (1, 9), (2, 1), (2, 8), (5, 0), (5, 9), (7, 2), (7, 0)\}.$$

Then

$$L(G_0) = \{0, 2, 8, 9, 4, 7, 6, 1, 3, 0\},$$

$$L(G_1) = \{1, 2, 7, 8, 9, 6, 5, 4, 5, 3\}$$

and then

$$D(G_0, G_1) = \{1, 9\},$$

$$D(G_0, G_1) = \{0, 6, 7, 9, 5, 4, 2, 8\},$$

$$D(G_1, G_1) = \{2, 8\},$$

which satisfy the length and distance conditions in Definition 3.1, see Figure 1.

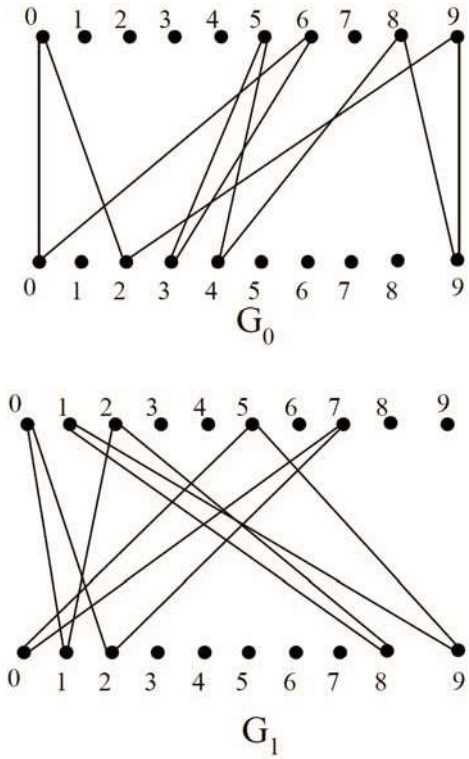

Figure 1. SODC-starter of $K_{10,10}$ by C_{10}.

Then the collection $\mathcal{P} = \{G_0, G_1\}$ is an SODC-starter of $(K_{10,10}, C_{10})$ and the collection of all of its translates is an ODC of $K_{10,10}$ by C_{10}. □

Theorem 3.4. *For every positive integer* $n \equiv 0 \pmod 5$, *there exists an SODC of* $K_{n,n}$ *by* C_{10}.

Proof. For $n = 5$, the result is known (see [3]). Concerning $n = 10$, it follows from the proof of Lemma 3.3. Let us prove it for the remaining values of n; put $g = n/5$.

Define $\mathcal{P} = \{G_k : 0 \le k \le g-1\}$ and

$$E(G_k) = \{(0, 5k), (0, 5k+2), (5, 5k+3), (5, 5k+4), (6, 5k), (6, 5k+3),$$
$$(8, 5k+4), (8, 5k+9), (9, 5k+2), (9, 5k+9)\}.$$

With abuse of notation, we will consider the integers from 0 to $g-1$ as residue classes modulo g whenever using them as subscripts. Then

$$L(G_k) = \{5k, 5k+2, 5k-2, 5k-1, 5k-6, 5k-3, 5k-4, 5k+1, 5k-7, 5k\}.$$

Likewise:

$$L(G_{k+1}) = \{5k+5, 5k+7, 5k+3, 5k+4, 5k-1, 5k+2, 5k+1,$$
$$5k+6, 5k-2, 5k+5\},$$
$$L(G_{k-1}) = \{5k-5, 5k-3, 5k-7, 5k-6, 5k-11, 5k-8, 5k-9,$$
$$5k-4, 5k-12, 5k-5\}.$$

$L(G_k)$ is the disjoint union of the multiset $\{5k, 5k\}$ and the sets $L(G_k) \cap L(G_{k+1})$, $L(G_k) \cap L(G_{k-1})$.

Hence, the multiples of 5 appear twice in the same set of lengths, while the other elements of $\mathbb{Z}_n$ appear in exactly two sets of lengths. Therefore, the length condition is satisfied by $\mathcal{P}$.

Furthermore, the above argument implies that $L(G_k) \cap L(G_h)$ is empty whenever $h-k \notin \{0, 1, -1\}$, so $D(G_k, G_h)$ is empty in all such cases.

To check the distance condition, we can then assume that $h = k$ or $h = k+1$. In the former, we have $D(G_k, G_k) = \{-9, 9\}$, in the latter, $D(G_k, G_{k+1}) = \{0, 1, 4, 6\}$.

Therefore, the distance condition is satisfied by $\mathcal{P}$. Then $\mathcal{P}$ is an SODC-starter of $K_{n,n}$ by C_{10}. The statement now follows from Theorem 3.2.

Example 3.5. The collection $\mathcal{P} = \{G_k : 0 \le k \le 2\}$ is an SODC-starter of $K_{15,15}$ by C_{10}, where

$$E(G_0) = \{(0, 0), (0, 2), (5, 3), (5, 4), (6, 0), (6, 3), (8, 4), (8, 9), (9, 2), (9, 9)\},$$

$$E(G_1) = \{(0, 5), (0, 7), (5, 8), (5, 9), (6, 5), (6, 8), (8, 9), (8, 14), (9, 7), (9, 14)\},$$

$$E(G_2) = \{(0, 10), (0, 12), (5, 13), (5, 14), (6, 10), (6, 13), (8, 14), (8, 4), (9, 12), (9, 4)\},$$

then

$$L(G_0) = \{0, 2, 13, 14, 9, 12, 11, 1, 8, 0\},$$

$$L(G_1) = \{5, 7, 3, 4, 14, 3, 1, 6, 13, 5\},$$

$$L(G_2) = \{10, 12, 8, 9, 4, 7, 6, 11, 3, 10\}, \text{ and then}$$

$$D(G_0, G_0) = D(G_1, G_1) = D(G_2, G_2) = \{6, 9\},$$

$$D(G_0, G_1) = D(G_1, G_2) = D(G_2, G_0) = \{6, 4, 1, 0\},$$

which satisfy the length and distance conditions in Definition 3.1.

Proposition 3.6. *There exists an ODC of* $K_{10,10}$ *by* $C_6 \cup C_4$.

Proof. Define

$$E(G_0) = \{(0, 0), (0, 1), (8, 1), (9, 0), (8, 8), (9, 8), (1, 3), (1, 7), (5, 3), (5, 7)\}$$

and

$$E(G_1) = \{(0, 3), (0, 6), (2, 6), (2, 7), (5, 2), (5, 4), (7, 2), (7, 4), (9, 3), (9, 7)\},$$

then

$$L(G_0) = \{0, 0, 1, 1, 2, 2, 3, 6, 8, 9\},$$

$$L(G_1) = \{4, 4, 5, 5, 7, 7, 3, 6, 8, 9\},$$

and then

$$D(G_0, G_0) = \{\pm 8, \pm 1, \pm 4\},$$

$$D(G_0, G_1) = \{9, 6, 4, 2\},$$

$$D(G_1, G_1) = \{\pm 7, \pm 5, \pm 2\},$$

which satisfy the length and distance conditions in Definition 3.1. Then the collection $\mathcal{P} = \{G_0, G_1\}$ is an SODC-starter of $(K_{10,10}, C_6 \cup C_4)$ and the collection of all of its translates is an ODC of $K_{10,10}$ by C_{10}. □

References

[1] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Springer, Berlin, 2000.

[2] R. El-Shanawany, Orthogonal double covers of complete bipartite graphs, Ph.D. Thesis, Universität Rostock, 2002.

[3] R. El-Shanawany, H.-D. O. F. Gronau and M. Grüttmüller, Orthogonal double covers of $K_{n,n}$ by small graphs, Discrete Appl. Math. 138 (2004), 47-63.

[4] R. A. El-Shanawany, M. S. Higazy and R. Scapellato, Orthogonal double covers of complete bipartite graphs by the union of a cycle and a star, Australas. J. Combin. 43 (2009), 281-293.

[5] R. A. El-Shanawany, M. Higazy and R. Scapellato, A note on orthogonal double covers of complete bipartite graphs by a special class of six caterpillars, AKCE J. Graphs Combin. 7(1) (2010), 1-4.

[6] R. A. El-Shanawany and M. Sh. Higazy, General symmetric starter of orthogonal double covers of complete bipartite graph, Int. J. Math. Math. Sci. 2007, Art. ID 42892, 8 pp.

[7] H.-D. O. F. Gronau, M. Grüttmüller, S. Hartmann, U. Leck and V. Leck, On orthogonal double covers of graphs, Des. Codes Cryptogr. 27 (2002), 49-91.

[8] S. Hartmann, Asymptotic results on suborthogonal $\vec{G}$-decompositions of complete digraphs, Discrete Appl. Math. 95 (1999), 311-320.

[9] S. Hartmann and U. Schumacher, Suborthogonal double covers of complete graphs, Congr. Numer. 147 (2000), 33-40.

[10] S. Hartmann and U. Schumacher, Orthogonal double covers of general graphs, Discrete Appl. Math. 138 (2004), 107-116.

[11] M. Higazy, A study of the suborthogonal double covers of complete bipartite graphs, Ph.D. Thesis, Menoufiya University, 2009.

[12] R. Sampathkumar and S. Srinivasan, Cyclic orthogonal double covers of 4-regular circulant graphs, Discrete Math. 311 (2011), 2417-2422.

[13] R. Scapellato, R. El-Shanawany and M. Higazy, Orthogonal double covers of Cayley graphs, Discrete Appl. Math. 157 (2009), 3111-3118.

[14] U. Schumacher, Suborthogonal double covers of complete graphs by stars, Discrete Appl. Math. 95 (1999), 439-444.