

Effects of molecular complexity and reservoir conditions on the discharge coefficient of adapted planar nozzles

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2015 J. Phys.: Conf. Ser. 633 012092

(<http://iopscience.iop.org/1742-6596/633/1/012092>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 131.175.154.248

This content was downloaded on 13/04/2016 at 10:56

Please note that [terms and conditions apply](#).

Effects of molecular complexity and reservoir conditions on the discharge coefficient of adapted planar nozzles

Alberto Guardone

Department of Aerospace Science and Technology
Politecnico di Milano
Via La Masa 34, 2016 Milano, Italy
E-mail: alberto.guardone@polimi.it

Abstract. The transonic flow at throat section of a convergent-divergent nozzle is studied in adapted conditions to assess the influence of the fluid molecular complexity and total thermodynamic state on the discharge coefficient. The standard Sauer method is applied to solve the transonic perturbation potential equation in the vicinity of the nozzle throat. An analytic expression is derived that allows one to compute the discharge coefficient in terms of the nozzle curvature at the throat section and of the value of the fundamental derivative of gasdynamics at sonic conditions, which depends on the fluid molecular complexity and on the thermodynamic state in the reservoir. A linear dependence of the discharge coefficient on the sonic value of the fundamental derivative of gasdynamics is exposed.

1. Introduction

The flow expansion through a nozzle from a reservoir into a constant pressure ambient is a fundamental problem of fluid mechanics. In the case where the exit conditions are supersonic, namely, if the discharge velocity is supersonic, a convergent-divergent nozzle geometry is implemented. This is the case for example of rocket exhaust nozzle, supersonic wind tunnel and supersonic blade turbine passages. In most relevant conditions, the fluid thermodynamic properties can be computed according to the ideal dilute gas model. However, for operating conditions in the close proximity of the liquid-vapor saturation curve, see figure 1, so-called non-ideal compressible-fluid effects are observed and non-ideal thermodynamics is to be accounted for [1, 2, 3].

In the present work, the well known transonic flow solution of Sauer [4] is applied to the case of a planar nozzle operating in adapted conditions in the non-ideal compressible-fluid regime to evaluate the influence of the fluid molecular complexity and of the reservoir thermodynamic state on the discharge coefficient, namely, on the maximum mass flow that can be processed by the nozzle itself [5].

The paper structure is as follows. In section 2, the Sauer theory is extended to non-ideal compressible fluids and an explicit expression of the perturbation velocity is derived. In section 3, a Taylor-series expansion of the density is derived to compute the discharge coefficient in transonic conditions. In section 4, final remarks and comments are given.



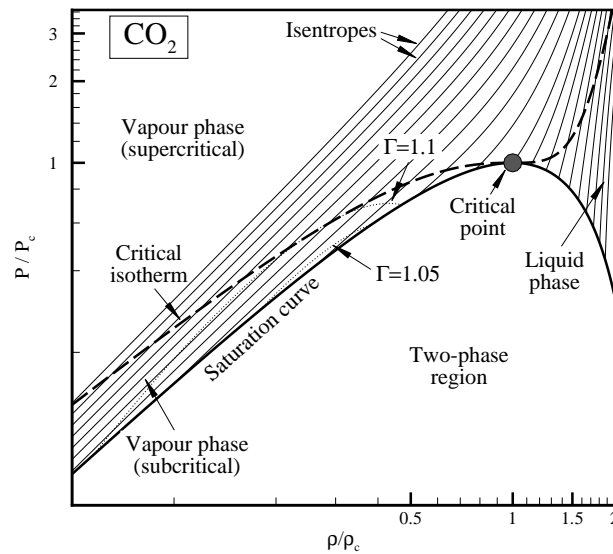


Figure 1. Liquid-vapor saturation curve in the density-pressure thermodynamic plane according to the van der Waals model of carbon dioxide. The critical point and the critical isotherms are also shown, together with a number of isentropes crossing the non-ideal compressible-fluid region. The iso- Γ lines $\Gamma = 1.1$ and $\Gamma = 1.05$ are depicted.

2. Transonic perturbation-potential flows at the throat section

The governing equations describing planar nozzle flows of interest here are now specialized to the case of non-ideal compressible fluids. Two-dimensional flow fields with uniform total enthalpy h^t and entropy s per unit mass, namely, $h^t \equiv h_r$ and $s \equiv c_r$, with the subscript r indicating reservoir conditions, are irrotational by virtue of the well-known Crocco theorem [6]. Therefore, a (scalar) velocity potential ϕ , with $\mathbf{v} = (u, v)^T = \nabla\phi$ velocity vector, can be introduced to obtain the well-known velocity potential equation [7]

$$(\phi_x^2 - c^2) \phi_{xx} + 2 \phi_x \phi_y \phi_{xy} + (\phi_y^2 - c^2) \phi_{yy} = 0, \quad (1)$$

where x and y are the spatial coordinates and where the subscripts indicate partial derivatives and where c is the speed of sound, $c^2(s, \nu) = -\nu^2(\partial P/\partial \nu)_s$, with ν the specific volume, P the pressure and where, according to standard thermodynamic nomenclature, the subscript s indicates that the pressure derivative is computed at constant entropy per unit mass.

It is remarkable that the thermodynamic description of the fluid enters Eq. (1) only via the definition of the speed of sound function. Choosing the entropy s and the enthalpy $h = h(s, \nu)$ per unit mass as independent thermodynamic variables, $c^2 = c^2(s, h)$ and one immediately obtains $h(h_r, w^2) = h_r - \frac{1}{2}w^2$, with $w^2 = u^2 + v^2 = |\nabla\phi|^2$. Therefore, the speed of sound in Eq. (1)—and all thermodynamic quantities—is a function of the velocity module only, or $c^2 = c^2(w^2)$. For example, for an ideal gas with constant isochoric specific heat, $c^2(w^2) = (\gamma - 1)(h_r - \frac{1}{2}w^2)$, where γ is a constant equal to the ratio of the isobaric to the isochoric specific heat.

A perturbation velocity potential $\tilde{\phi}$, $\phi = \bar{w} L(x + \tilde{\phi})$, with \bar{w} velocity module at the reference state, where the flow is parallel to the x axis, and L reference length, is now introduced to study the small-perturbation flow occurring in the vicinity of the nozzle throat. In the following, the

symbols $\bar{}$ and $\tilde{}$ indicate quantities evaluated at the (constant) reference state and perturbation quantities, respectively. Under the assumption of small perturbation velocities, i.e. $\|\tilde{\mathbf{v}}\| \ll 1$, with $\tilde{\mathbf{v}} = \nabla \tilde{\phi}$ perturbation velocity vector, the equation for the perturbation velocity potential in the transonic, i.e. $\bar{M} \sim 1$, regime is then obtained from (1) as

$$[\bar{M}^2 - 1 + 2\bar{\Gamma}\bar{M}^2 \tilde{\phi}_x] \tilde{\phi}_{xx} - \tilde{\phi}_{yy} = 0, \quad (2)$$

where the thermodynamic quantity Γ is the so-called fundamental derivative of gasdynamics introduced by Thompson [8]

$$\Gamma(s, v) \equiv 1 - \frac{\nu}{c} \left(\frac{\partial c}{\partial \nu} \right)_s. \quad (3)$$

In deriving (2), $\bar{\Gamma}$ has been assumed to be $\mathcal{O}(1)$. This assumption, that is valid for most typical fluids, see [9], leads to a simpler equation for the perturbation velocity potential with respect to those derived by [10, 11, 12, 13], which are valid also for the $\Gamma < 0$ region of so-called BZT fluids, see e.g. [10]. For an ideal polytropic gas, Eq. (2) assumes the familiar form $[\bar{M}^2 - 1 + (\gamma + 1)\bar{M}^2 \tilde{\phi}_x] \tilde{\phi}_{xx} - \tilde{\phi}_{yy} = 0$, see e.g. [7].

The non-ideal fluid equation (2) differs from the ideal gas one considered by Sauer [4] only for the constant multiplying the first term, whose value is of the same order of the ideal-gas constant $\gamma + 1$, since $\bar{\Gamma} \sim 1$ in the region of interest. The Sauer technique can therefore be applied without modifications to obtain the perturbation potential as [7]

$$\tilde{\phi}(\hat{x}, y) = \frac{1}{16} \alpha [8(\hat{x}^2 + \alpha \Gamma \hat{x} y^2) + (\alpha \Gamma y^2)^2] \quad (4)$$

where $\alpha = \sqrt{1/(\bar{\Gamma} r_t y_t)}$, with r_t and y_t the throat radius of curvature and the height of the wall at $x = x_t$, namely, at the throat section. In the expression above, $\hat{x} = x - x_t - \frac{1}{4} \Gamma \alpha y_t^2$. The x and y component of the perturbation velocity field are then computed from the perturbation potential (4) as

$$\tilde{u}(\hat{x}, y) = \tilde{\phi}_{\hat{x}} = \alpha \hat{x} + \bar{\Gamma}(\alpha y)^2 \quad \text{and} \quad \tilde{v}(\hat{x}, y) = \tilde{\phi}_y = 2\bar{\Gamma} \alpha^2 y [\hat{x} + \frac{1}{3} \bar{\Gamma} \alpha y^2] \quad (5)$$

3. Discharge coefficient

The discharge coefficient of a convergent-divergent planar nozzle in choked conditions is now computed by integrating the momentum density at the nozzle section, using the Sauer solution of the perturbation potential (4) for non-ideal compressible-fluid flows.

The mass flux per unit height across the throat section ($x_t = 0$) is defined as

$$\dot{m} = \int_{-\frac{y_t}{2}}^{\frac{y_t}{2}} \rho(0, y) u(0, y) \frac{dy}{y_t} \quad (6)$$

The momentum along the x direction is $\rho u = \bar{\rho} \bar{u} (1 + \tilde{\rho}) (1 + \tilde{u}) = \bar{\rho} \bar{u} (1 + \tilde{\rho} + \tilde{u} + \tilde{\rho} \tilde{u})$ and therefore the discharge coefficient per unit height is computed from (6) as

$$\frac{\dot{m}}{\dot{m}_{1D}} - 1 = \int_{-\frac{y_t}{2}}^{\frac{y_t}{2}} \frac{\rho(0, y) u(0, y)}{\bar{\rho} \bar{u}} \frac{dy}{y_t} - 1 = \int_{-\frac{y_t}{2}}^{\frac{y_t}{2}} [\tilde{\rho}(0, y) + \tilde{u}(0, y) + \tilde{\rho}(0, y) \tilde{u}(0, y)] \frac{dy}{y_t} \quad (7)$$

where $\dot{m}_{1D} = \bar{\rho} \bar{u}$ is the mass flux per unit height in one-dimensional conditions.

To compute the integral (7), an approximate expression of the function $\tilde{\rho} = \tilde{\rho}(w)$ is now derived. Expansions in the vicinity of the reference state gives

$$\rho = \bar{\rho} + \frac{d\rho}{dw^2} \Big|_{\bar{w}} [w^2 - \bar{w}^2] + \frac{1}{2} \frac{d^2\rho}{d^2w^2} \Big|_{\bar{w}} [w^2 - \bar{w}^2]^2 + \mathcal{O}([\Delta w^2]^3) \quad (8)$$

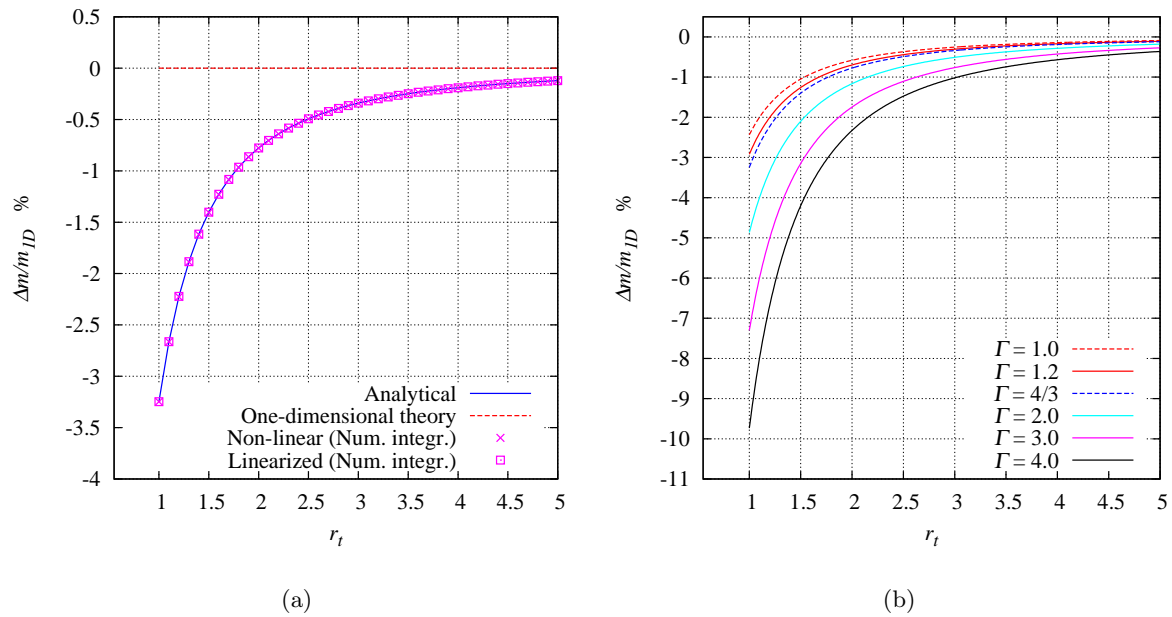


Figure 2. a) Discharge coefficient computed using the Sauer [4] method for non-ideal compressible-fluid flows. a) Comparison of the results from the analytic expression (12) to numerical integration of the Sauer solution including all non-linear terms and of the linearized form (11). b) Mass flow defect from the analytic expression (12) for different values of Γ .

where the first and second order derivative of the density function reads

$$\frac{d\rho}{dw^2} = \left(\frac{\partial \rho}{\partial h} \right)_s \frac{dh}{dw^2} = -\frac{1}{2} \frac{\rho}{c^2} \quad \text{and} \quad \frac{d^2 \rho}{d^2 w^2} = \frac{3 - 2\Gamma}{4} \frac{\rho}{c^4}. \quad (9)$$

By retaining first and second order terms of the velocity potential, the perturbation density is finally obtained as follows

$$\tilde{\rho} = \frac{\rho}{\bar{\rho}} - 1 = -\frac{\bar{M}^2}{2} (2\tilde{u} + \tilde{u}^2 + \tilde{v}^2) + \frac{\bar{M}^4}{4} (3 - 2\Gamma) \tilde{u}^2 + \mathcal{O}(\tilde{v}^3, \tilde{v}^3). \quad (10)$$

The momentum defect in the x direction for $\bar{M} = 1$ therefore reads

$$\frac{\rho u}{\bar{\rho} \bar{u}} - 1 = \tilde{\rho} + \tilde{u} + \tilde{\rho} \tilde{u} \simeq -\bar{\Gamma} \tilde{u}^2 - \frac{1}{2} \tilde{v}^2, \quad (11)$$

an expression that exposes the second-order nature of the mass flux defect correction within the potential perturbation theory. Substituting the expressions of the perturbation velocity components (5) evaluated at the nozzle throat ($x = 0$, i.e., $\hat{x} = \epsilon$), one finally obtains

$$\frac{\dot{m}}{\dot{m}_{1D}} - 1 = \frac{1}{45} \frac{\bar{\Gamma}}{r_t^3} \left(r_t + \frac{2}{21} \right). \quad (12)$$

Retaining higher-order terms in the expansion (10) was found not to improve the accuracy of expression if compared to the numerical integration of (7), as it can be appreciated from figure

2a, where the results from the analytic expression (12) are compared to numerical integration of the Sauer solution including all non-linear terms and of the linearized form (11). If figure 2b, the mass flow defect from the analytic expression (12) is reported for different values of $\Gamma \geq 1$. Value of Γ in between 1 and $4/3$ are attainable by ideal gases with constant specific heats: $\Gamma = 1$ corresponds to the limiting case of an infinite value of the isochoric specific heat, $\Gamma = 1.2$ to standard air, $\Gamma = 4/3$ to a monoatomic gas. Values of Γ outside this range can be attained only in the non-ideal compressible-fluid regime. In particular, large values of Γ are expected in the liquid phase and in the close proximity of the critical point, where $\Gamma \rightarrow \infty$ [14].

4. Conclusions

The Sauer method for solving the perturbation potential equation in the transonic regime was applied to non-ideal compressible-fluid flows in the close proximity of the throat of convergent-divergent planar nozzles. The analytic expression of the perturbation velocity was derived as a function of the curvature of the nozzle geometry at the throat section and of the sonic value of the fundamental derivative of gasdynamics Γ , which embeds non-ideal fluid behavior and its dependence on the reservoir conditions. An analytic expression of the discharge coefficient was devised for the first time which exposes its linear dependence on Γ . For large values of Γ , which are expected to be observed in the liquid phase and in the close proximity of the liquid-vapor saturation curve, the discharge coefficient is as low as 90% (for $\Gamma = 4$), to be confronted to a minimum value of 97% corresponding to mono-atomic ideal gases. The discharge coefficient depends also on the inverse of the second and third powers of the curvature radius at the throat section. Experimental and numerical activities are planned to verify the present findings.

Acknowledgments

The research is partially funded by the European Research Council under Grant ERC Consolidator 2013, project NSHOCK 617603.

References

- [1] Harinck J, Colonna P, Guardone A and Rebay S 2010 *Journal of turbomachinery* **132** 011001
- [2] Guardone A 2011 *8th International Conference on Flow Dynamics* (Sendai)
- [3] Guardone A, Spinelli A and Dossena V 2013 *ASME Journal of Engineering for Gas Turbines and Power* **135** 042307
- [4] Sauer R 1947 General characteristics of the flow through nozzles at near critical speeds TM 1147 NACA
- [5] Nakao S i 2007 **122** 730–734
- [6] Crocco L 1937 *Z. Angew. Math. Mech.* **17** 1–7
- [7] Zucrow M H and Hoffman J D 1976 *Gas dynamics* vol 1 (Wiley, Jhon & Son)
- [8] Thompson P A 1971 *Phys. Fluids* **14** 1843–1849
- [9] Thompson P A 1988 *Compressible Fluid Dynamics* (McGraw-Hill)
- [10] Kluwick A 1993 *J. Fluid Mech.* **247** 661–688
- [11] Schnerr G H and Molokov S 1994 *Phys. Fluids* **6** 3465
- [12] Schnerr G H and Molokov S 1995 *Phys. Fluids* **7** 2867
- [13] Rusak Z and Wang C W C W 1997 *Journal of Fluid Mechanics* **346** 1–21
- [14] Nannan N R, Guardone A and Colonna P 2012 *Fluid Phase Equilib.* **337** 259–273