

The Kepler problem primigenial sphere

Maria Dina Vivarelli

Received: 22 October 2013 / Accepted: 29 October 2014 / Published online: 13 November 2014

Abstract The unitary **S**-code description of the one-body Kepler problem is shown to unfold naturally from a primigenial sphere (with centre at the attracting gravitational centre and radius related to the universal constant of gravitation). In this spherical scenery all the Kepler problem fundamental elements are recovered by simple linear vector combinations.

Keywords Kepler problem · Cone of the Kepler conic sections · Regularization · Laplace–Runge–Lenz vector · General mechanics

1 Introduction

Our unitary **S**-description [5–10] of the classical one-body Kepler problem is revisited and shown to find its *primeval* source in a sphere, which we denote by $S_{p^{-1}}$, and which has centre at the attractive centre of the gravitational force acting on the body and radius p^{-1} related to the universal constant of gravitation. This *primigenial sphere* unravels, in a sort of *inbred order of its elements*, not only the cone of the orbital Kepler conic sections, but also the various, fundamental elements of the Kepler problem.

M. D. Vivarelli (✉)
Dipartimento di Matematica F. Brioschi, Politecnico di
Milan, Piazza L. da Vinci 32, 20133 Milan, Italy
e-mail: maria.vivarelli@polimi.it

To begin with, we recall that the classical three-dimensional *Kepler problem* describes the motion of a particle in a central, attractive, inverse square law field. The Kepler motion occurs in a *fixed plane*, orthogonal to the *constant angular momentum vector* Γ per unit of mass

$$\Gamma = \mathbf{x} \wedge \dot{\mathbf{x}} \quad (\cdot \equiv d/dt) \quad (1)$$

where \mathbf{x} represents the particle position vector with respect to the attractive centre and \wedge denotes the wedge product. The Kepler orbit is a conic, commonly expressed by the vector *polar equation*

$$\mathbf{x} = \frac{p}{1 + e \cos \theta} \boldsymbol{\rho} \quad (\mathbf{x} = r\boldsymbol{\rho}) \quad (2)$$

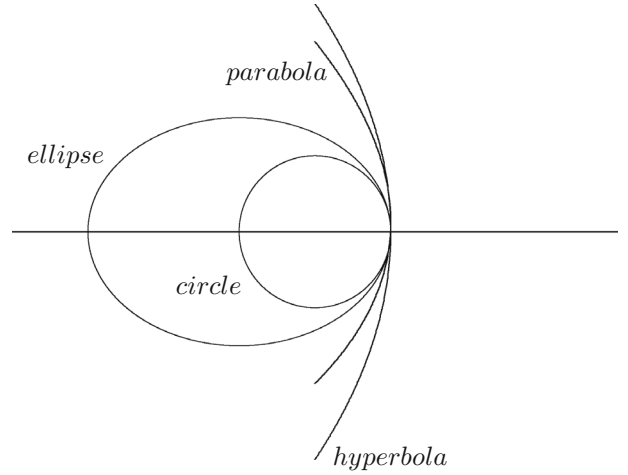
in the plane polar coordinate system (r, θ) , with the origin at the fixed attracting centre and with the two orthogonal unit vectors $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\theta)$ pointing in the direction of increasing r and θ .

The two scalar parameters p and e represent, respectively, the *semi-latus rectum* and the *eccentricity* of the orbit, which is a circle for $e = 0$, an ellipse for $0 < e < 1$, a parabola for $e = 1$, the left branch of an hyperbola for $e > 1$ (the right branch being excluded by the condition $r > 0$, see Fig. 1).

In our previous works [5–10] we introduced the peculiar *sum vector* **S** which encompasses (in a sort of genetic code) all the geometrical and dynamical information about the Kepler problem.

For instance, the vector **S**:

Fig. 1 The family of Kepler confocal conics



1. Allows to express the equation (2) of the whole family of the Kepler orbits as the simple scalar product

$$\mathbf{S} \cdot \mathbf{x} = 1; \quad (3)$$
2. ‘Conceals’ a *constant vector* \mathbf{N} , unraveling the *cone structure* in R^3 which generates all the Kepler orbits as *conic sections*;
3. Discloses a peculiar genesis of the celebrated *regularizing KS-map*, devised to regularize at the origin (collision) the plane Kepler problem.

In this paper, we show how the two vectors \mathbf{S} and \mathbf{N} (together with the cone structure, the regularizing map and other features of the Kepler problem) turn up to be much more entangled and organized than what appeared in our previous papers on the \mathbf{S} -encoding: they are all originated by simple, linear combinations of the primitive elements that define the primigenial sphere S_{p-1} .

This result ensues, first, from an *autonomous and self-consistent review* of the well known governing expression for the Newtonian law of gravitation

$$\mathbf{F} = -\frac{K^2}{r^3} \mathbf{x}, \quad (4)$$

and, secondly, by *enlightening the role of both the spherical angles which parameterize a sphere*. In this spherical scenario the well-known eccentricity vector \mathbf{e} , the Runge–Lenz vector and the Kepler mechanical energy E acquire a particular interpretation.

2 Symmetry and planarity: both in Newton’s gravitation law

Our basic goal is to interlace strictly the two fundamental properties of the Kepler problem: symmetry and planarity. Commonly derived [1–3] from the Newtonian law of gravitation (4), these properties notoriously reveal that the central Kepler motion exhibits:

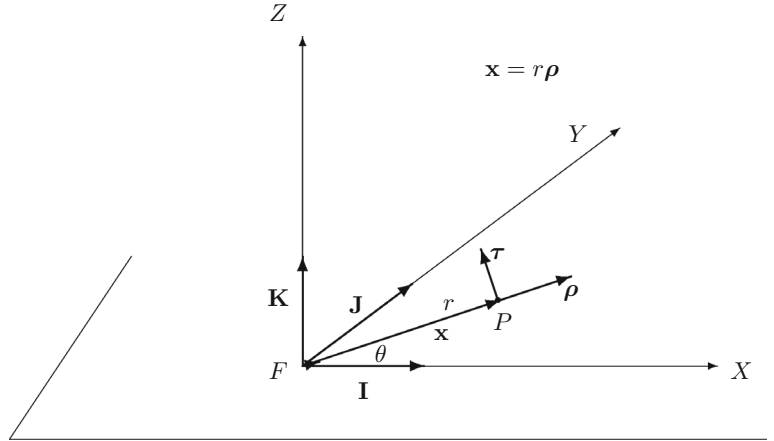
- a. Spherical symmetry (invariance under rotation in R^3 about *any* axis through the fixed attractive point) whence the motion is commonly described by *spherical polar coordinates* (r, θ, ϕ) ;
- b. Rotational symmetry (invariance under rotation about a *given fixed* axis, namely the constant direction of the conserved angular momentum vector $\mathbf{\Gamma}$);
- c. Planarity, so that the motion occurs in the plane orthogonal to $\mathbf{\Gamma}$ and through the fixed attractive centre, whence the motion is simply described by *plane polar coordinates* (r, θ) as in (2).

Our goal is to *embody both symmetry and planarity directly in the classical expression* (4) of the force \mathbf{F} . The result is achieved:

- a. By choosing a fixed right-handed unit system $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ in R^3 , with the origin at the attractive centre F and with \mathbf{K} directed along $\mathbf{\Gamma}$, so that

$$\mathbf{\Gamma} = \Gamma \mathbf{K} \quad (5)$$

Fig. 2 The cartesian $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ and the polar $\{F, \boldsymbol{\rho}, \boldsymbol{\tau}\}$ frames



b. By identifying the \mathbf{I}, \mathbf{J} -plane with the polar $\{\boldsymbol{\rho}, \boldsymbol{\tau}\}$ -plane, the particle position vector \mathbf{x} being given by

$$\mathbf{x} = r \boldsymbol{\rho} \quad (6)$$

(see Fig. 2).

The two choices (5) and (6) enable us to introduce the vector $\boldsymbol{\Gamma}$ in the expression (4), which can now be re-written as

$$\mathbf{F} = -\frac{K^2}{r^2} \boldsymbol{\rho} = -\frac{\Gamma^2}{r^2} \frac{K^2}{\Gamma^2} \boldsymbol{\rho}.$$

If we denote by p the *constant ratio*

$$p = \frac{\Gamma^2}{K^2} \quad (7)$$

so that

$$\mathbf{F} = -\frac{\Gamma^2}{r^2} p^{-1} \boldsymbol{\rho},$$

we finally arrive at the wedge product

$$\mathbf{F} = \frac{\boldsymbol{\Gamma} \wedge (\boldsymbol{\Gamma} \wedge p^{-1} \boldsymbol{\rho})}{r^2}. \quad (8)$$

This is the vector expression sought for. The Newton's gravitational force \mathbf{F} , via the vector $\boldsymbol{\Gamma}$, displays explicitly both the Kepler symmetry and the Kepler planarity.

What is now surprising is that the vector expression (8):

- Winds up with the *parameter* p given by (7), which *coincides exactly with the semi-latus rectum parameter* of the literature;

- Discloses the *existence of the following radial vector*

$$p^{-1} \boldsymbol{\rho}$$

which turns out to be *fundamental for the organization of the entire Kepler problem.*

3 The primigenial sphere $S_{p^{-1}}$. The spherical angles and the rescaling vector ϵ

The radial vector $p^{-1} \boldsymbol{\rho} = p^{-1} \boldsymbol{\rho}(\theta)$ which appears in (8) may be interpreted as the vector which, for each θ , defines the *equator* of a sphere with radius p^{-1} and centre at F . We characterize this sphere by $S_{p^{-1}}$.

By adopting spherical coordinates (see Fig. 3)

$$\begin{cases} X &= p^{-1} \sin \phi \cos \theta \\ Y &= p^{-1} \sin \phi \sin \theta \\ Z &= p^{-1} \cos \phi \end{cases}$$

where the *longitude* angle θ and the *colatitude* angle ϕ satisfy

$$\theta \in [0, 2\pi], \quad \phi \in [0, \pi],$$

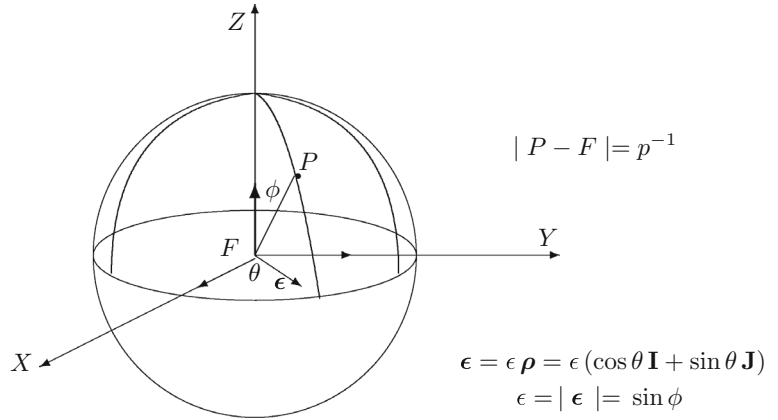
the sphere is notoriously characterized in R^3 by the *set of vectors* $P - F$, given explicitly by

$$P - F = p^{-1} [\sin \phi (\cos \theta \mathbf{I} + \sin \theta \mathbf{J}) + \cos \phi \mathbf{K}]. \quad (9)$$

Now, since in the plane polar description we have that

$$\cos \theta \mathbf{I} + \sin \theta \mathbf{J} = \boldsymbol{\rho},$$

Fig. 3 The sphere $S_{p^{-1}}$ with radius p^{-1} , center F . The rescaling vector ϵ



we rewrite the vector expression (9) of the sphere as

$$P - F = p^{-1} [\sin \phi \boldsymbol{\rho} + \cos \phi \mathbf{K}]. \quad (10)$$

Let us notice that the *longitude* angle θ appears in (10), although indirectly, through $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$.

As for the *colatitude* angle ϕ , we can go further, and try to introduce in (10) its *positive related range*

$$0 \leq \sin \phi \leq 1$$

(obviously satisfied, being $\phi \in [0, \pi]$).

To do so, we define the ‘rescaling vector’ ϵ by the following

Definition 3.1 For each fixed plane unit vector $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$ and for each fixed angle ϕ , the ‘rescaling vector’ is the plane vector

$$\epsilon = \sin \phi \boldsymbol{\rho}(\theta) = \epsilon \boldsymbol{\rho} \quad (11)$$

which belongs to the equatorial plane of the sphere and whose magnitude

$$\epsilon = |\epsilon| = \sin \phi$$

‘rescales’ the unit magnitude of $\boldsymbol{\rho}$ and satisfies the range

$$0 \leq \epsilon \leq 1. \quad (12)$$

Consequently, the introduction of the vector ϵ in (10) leads us to the

Definition 3.2 We call ‘**primigenial sphere**’ of the Kepler problem the sphere $S_{p^{-1}}$ defined by the locus of vectors

$$P - F = p^{-1} (\epsilon + \cos \phi \mathbf{K}) \quad (13)$$

that is the sphere with centre at the attractive origin F of the inertial right-handed frame $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ of R^3 and with radius p^{-1} related to the physical constants of the Kepler problem by (7) (see Fig. 3).

Of course, if $\phi = \frac{\pi}{2}$, we recover, by (13), the equator $P - F = p^{-1} \boldsymbol{\rho}$ of the sphere. And the equator plays a fundamental role, as shown in the following section.

4 Inside and outside the equator. The projection vector

The new rescaling vector ϵ allows to decompose the vectors (13) which define the primigenial sphere $S_{p^{-1}}$ as:

$$\begin{aligned} P - F &= p^{-1} \epsilon + p^{-1} \cos \phi \mathbf{K} \\ &= (P_0 - F) + (P - P_0) \end{aligned} \quad (14)$$

(see Fig. 4).

We call the vector

$$P_0 - F = p^{-1} \epsilon \quad (15)$$

the *projection vector* since it characterizes all the points P of the sphere $S_{p^{-1}}$ by giving their corresponding projections P_0 on the equatorial plane.

In particular, the point P_0 is:

- At the center F of the equator for $\epsilon = 0$;
- Inside the equator for $0 < \epsilon < 1$;
- On the equator for $\epsilon = 1$.

For the future, we find it convenient to extend the range (12) by considering the points outside the equator, thus by adding

- Outside the equator for $\epsilon > 1$.

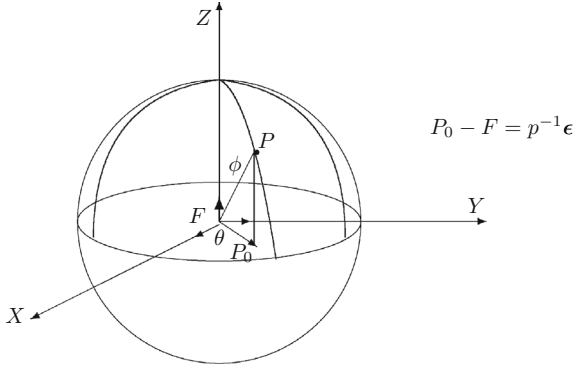


Fig. 4 The sphere and the projection vector $P_0 - F = p^{-1}\epsilon$

5 From the sphere $S_{p^{-1}}$ to the \mathcal{C} -cone

Now we are ready to bring into life the cone which generates the Kepler conic orbits (the famous ‘conic’ sections).

As before we stick to the *vector* definition (13) of the sphere $S_{p^{-1}}$, considered as the ‘star’ of vectors $P - F$ of the same magnitude p^{-1} issuing from its centre F (and *not* as the equivalent definition, i.e. as the locus of equidistant points).

Accordingly, in the star of vectors (13) we select, for each fixed, constant value ϕ^* of the colatitude ϕ , the related vectors:

$$P^* - F = p^{-1} (\sin \phi^* \boldsymbol{\rho} + \cos \phi^* \mathbf{K}) \quad (16)$$

which define a *circular right cone* with axis \mathbf{K} and semi-aperture ϕ^* (see Fig. 5).

Moreover, among the circular cones given by (16), we fix our attention to the cone corresponding to the particular relation

$$\sin \phi^* = \cos \phi^*. \quad (17)$$

Since $\phi^* = \frac{\pi}{4}$, this particular cone (Fig. 5) is characterized by the vectors

$$P_{\frac{\pi}{4}}^* - F = \frac{\sqrt{2}}{2} p^{-1} (\boldsymbol{\rho} + \mathbf{K}). \quad (18)$$

Of course this circular cone is a ‘limited, finite’ one: the arrowed points $P_{\frac{\pi}{4}}^*$ belong to the sphere of finite radius p^{-1} (the scalar factor $\frac{\sqrt{2}}{2} p^{-1}$ bringing the magnitude $\sqrt{2}$ of the sum vector $\boldsymbol{\rho} + \mathbf{K}$ exactly to the finite value p^{-1} and the magnitude of the projection

vector $P_0^* - F$ corresponding to the finite value $\epsilon = \frac{\sqrt{2}}{2}$ of ϵ).

Thus, with the aim at arriving at the (infinite) cone which generates the Kepler orbits, we rescale (18) and finally give the following

Definition 5.1 The circular \mathcal{C} -cone associated to the primigenous sphere $S_{p^{-1}}$ of the Kepler problem, is the extended circular right cone, with vertex at F , characterized by the vector equation

$$\mathcal{C} - F = \lambda p^{-1} (\boldsymbol{\rho} + \mathbf{K}) \quad (19)$$

with the positive scalar parameter $\lambda \in \mathcal{R}$ and with p^{-1} given by (7).

In cartesian coordinates, being $F(0, 0, 0)$ and $\mathcal{C} = (X_C, Y_C, Z_C)$ where

$$\begin{aligned} X_C &= \lambda p^{-1} \cos \theta, & Y_C &= \lambda p^{-1} \sin \theta, \\ Z_C &= \lambda p^{-1} \end{aligned}$$

the scalar cartesian equation of the right \mathcal{C} -cone is

$$X_C^2 + Y_C^2 - Z_C^2 = 0. \quad (20)$$

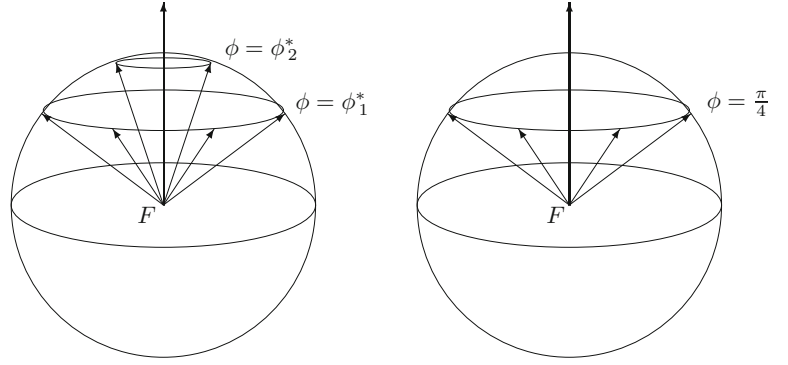
Remarks (1) Of course, the cartesian equation (20) coincides with the standard one for a circular right cone with semi-aperture $\frac{\pi}{4}$. But the \mathcal{C} -cone, through its *vector* definition $\mathcal{C} - F$ given by (19), is *intrinsically and explicitly* related to the Kepler problem via the physical Kepler value p^{-1} and via its vertex, the attractive centre F of the force \mathbf{F} . (2) How the \mathcal{C} -cone generates the Kepler orbits is revealed in the following section.

6 The primigenous role of the sphere $S_{p^{-1}}$

The vector expression (13) of the sphere $S_{p^{-1}}$ relies, at the core, on the two spherical coordinates θ and ϕ : but, whereas the fixed unit vector $\mathbf{K} = \mathbf{K}(0)$, corresponding to the angle $\phi = 0$, appears *explicitly*, the fixed unit vector $\mathbf{I} = \boldsymbol{\rho}(0)$, corresponding to the angle $\theta = 0$, appears only *implicitly* through the particular rescaling vector $\epsilon = \epsilon \boldsymbol{\rho}(\theta)$ corresponding to $\theta = 0$.

We recover this (implicit) fixed direction $\mathbf{I} = \boldsymbol{\rho}(0)$ by considering the *particular, constant rescaling vector* ϵ_0 defined by setting $\theta = 0$ and $\phi = \phi_0$ in the definition of ϵ given by (11) (*the fixed, constant*

Fig. 5 The Sphere and its cones for $\phi = \text{cost}$. The \mathcal{C} -cone for $\phi = \frac{\pi}{4}$



colatitude value ϕ_0 and its physical meaning will be explored in Section 9).

The new constant vector ϵ_0 is thus defined by:

$$\epsilon_0 = \sin \phi_0 \rho(0) = \sin \phi_0 \mathbf{I} = \epsilon_0 \mathbf{I}$$

and lies on the \mathbf{I} -axis.

As a consequence, the sphere $S_{p^{-1}}$, via its $(P - F)$ -vector definition (13) in the $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ -frame, is characterized by the following four **primitive elements**

$$p^{-1}, \rho, \mathbf{K}, \epsilon_0 = \epsilon_0 \mathbf{I}, \quad (21)$$

which are: the radius, two unit orthogonal vectors (the first one related to the longitude angle θ , measured starting from \mathbf{I}) and the rescaling vector ϵ_0 (related by $\epsilon_0 = \sin \phi_0$ to the colatitude angle ϕ_0 , measured starting from \mathbf{K}).

Now, *what is surprising* is that *simple combinations* of these primitive elements, such as the following simple linear combinations:

$$p^{-1}(\rho + \mathbf{K}) \quad (22)$$

$$p^{-1}(\epsilon_0 + \mathbf{K}) \quad (23)$$

$$p^{-1}(\epsilon_0 + \rho) \quad (24)$$

originate *immediately* all the fundamental elements of the Kepler problem.

That is why we called the sphere 'primigential'.

The combined vectors examined (Table 1).

We are now examining in detail the Table 1 which reassumes the combined vectors (22)-(24) (obtained exclusively by following our $S_{p^{-1}}$ spherical scheme) and compares them with the fundamental vectors we obtained by different procedures in [5-9].

a. **The \mathcal{C} -cone: a 3-dimensional characterization of Kepler orbits.** The vectors (22), rescaled by λ , give,

Table 1 The Combined primitive elements

Combined vector	Comparison	Name
$p^{-1}(\rho + \mathbf{K})$	$p^{-1}(\rho + \mathbf{K})$	\mathcal{C} -cone = N-cone
$p^{-1}(\epsilon_0 + \mathbf{K})$	$p^{-1}(\mathbf{e} + \mathbf{K}) = \mathbf{N}$	\mathbf{N} vector
$p^{-1}(\epsilon_0 + \rho)$	$p^{-1}(\mathbf{e} + \rho) = \mathbf{S}$	Sum vector \mathbf{S}

for each θ , the whole \mathcal{C} -cone structure (19) or (20), strictly related to the primigential sphere $S_{p^{-1}}$.

We now compare the scalar equation (20) of the \mathcal{C} -cone with a scalar one obtained by considering the second combined vector (23). For a general representation in R^3 , we rewrite this vector (which lies in the $\{\mathbf{I}, \mathbf{K}\}$ -plane) by relaxing the restriction $\epsilon_0 = \epsilon_0 \mathbf{I}$ so that this constant vector is $\epsilon_0 = \epsilon_X \mathbf{I} + \epsilon_Y \mathbf{J}$ whence the vector (23) becomes

$$p^{-1}(\epsilon_0 + \mathbf{K}) = \frac{\epsilon_X}{p} \mathbf{I} + \frac{\epsilon_Y}{p} \mathbf{J} + \frac{1}{p} \mathbf{K} \quad (25)$$

which has *tip point*, say N , with coordinates

$$X = p^{-1} \epsilon_X, \quad Y = p^{-1} \epsilon_Y, \quad Z = p^{-1} \quad (26)$$

which satisfy the scalar equation sought for

$$X^2 + Y^2 - Z^2 = (\epsilon_0^2 - 1)p^{-2} \quad (27)$$

where $\epsilon_0 = |\epsilon_0| = \sqrt{\epsilon_X^2 + \epsilon_Y^2}$ and $Z > 0$.

In this equation, the sign of the term $(\epsilon_0^2 - 1)$ depends on the *rescaling* values $0 \leq \epsilon_0 \leq 1$ and $\epsilon_0 > 1$.

But these ranges remind us that an ellipse, a parabola and the left branch of the hyperbola correspond to the same ranges of the well known *eccentricity* $0 \leq e \leq 1$ and $e > 1$.

Thus, by comparing the scalar Eq. (27) with (20), we may state that:

Proposition 6.1 The tip points N of the combined vectors (23) which lie inside the \mathcal{C} -cone correspond to elliptical orbits, those on the cone to parabolic ones and those outside the cone to hyperbolic orbits (Fig. 6).

b. **The cone structure.** The \mathcal{C} -cone coincides exactly with what we have defined and called (for other reasons) N -cone in [9]. For uniformity we will call it here \mathcal{C} -cone.

c. **The Rescaling vector ϵ_0 and the Eccentricity vector \mathbf{e} .** As a consequence of the results obtained in a. we may set

$$\epsilon_0 = e \quad \Rightarrow \quad \epsilon_0 = \mathbf{e}$$

so that:

Proposition 6.2 The constant rescaling vector $\epsilon_0 = \epsilon_0 \mathbf{I}$ corresponding to $\theta = 0$ and to the particular angle $\phi = \phi_0$, turns out to be the *eccentricity vector* $\mathbf{e} = e \mathbf{I}$ of the literature which characterizes each type of the orbits ($e = 0$ for circles, $0 < e < 1$ for ellipses, $e = 1$ for parabolas, $e > 1$ for hyperbolas).

Remark The important fact is that in the literature the eccentricity vector is shown to rely on the concept of geometrical ratio (see [2, 3]), whereas in our description it comes from a completely different spherical approach.

d. **The combined vector \mathbf{N} .** The vector defined in (23) has the same coordinate $Z = p^{-1}$ of the North pole of the sphere $S_{p^{-1}}$: that is why we denoted its tip point by N . This vector, deeply rooted into the spherical scheme of this work and strictly related to the Kepler conic orbits via the previous Proposition 5.1, coincides, being $\epsilon_0 = \mathbf{e}$, with the vector \mathbf{N}

$$\mathbf{N} \equiv N - F \equiv p^{-1}(\mathbf{e} + \mathbf{K}),$$

we introduced in [9] in a completely different way (see Fig. 7a) where $\mathbf{e} = e \mathbf{I}$.

(e) **The Kepler orbits as conic sections. The polar plane.** The sphere $S_{p^{-1}}$ may be considered as an extension of the unit sphere with centre at F , that is

$$X^2 + Y^2 + Z^2 = 1. \quad (28)$$

It follows that (with respect to the unit sphere) each tip point N of the vector \mathbf{N} defines a *polar plane*, that is the plane (orthogonal to the line FN) which passes through the inverse point N^* of N so that $|N^* - F| = \frac{1}{|N - F|}$. The points (X, Y, Z) belonging to

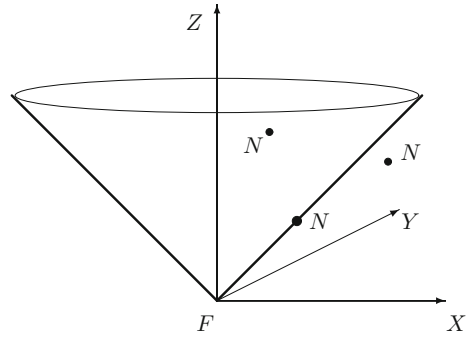


Fig. 6 The \mathcal{C} -cone structure and the tip points N

the polar plane satisfy, recalling (26), (27) and (28), the equation

$$p^{-1}e_X X + p^{-1}e_Y Y + p^{-1}Z - 1 = 0 \quad (29)$$

As a consequence:

Proposition 6.3 The polar plane (29) intersects the \mathcal{C} -cone in the locus given by the equation

$$(1 - e_X^2)X^2 + (1 - e_Y^2)Y^2 + 2pe_X X + 2pe_Y Y - 2e_X e_Y XY - p^2 = 0$$

which is a conic section, which, projected orthogonally onto the $\{\mathbf{I}, \mathbf{J}\}$ -plane gives the Kepler conic section with focus at the vertex $F(0, 0, 0)$ of the \mathcal{C} -cone, eccentricity e and parameter p . (The Fig. 8 shows an elliptic conic section).

Proposition 6.4 The polar plane makes an angle β with the \mathbf{I} -axis such that

$$\tan \beta = e$$

which is exactly the eccentricity of the Kepler orbit (see Fig. 8).

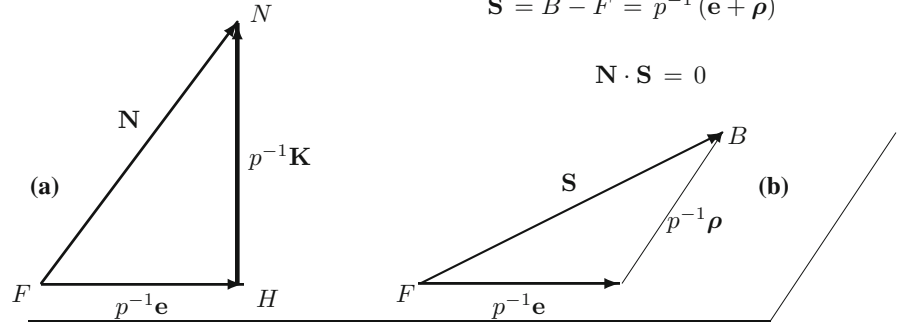
f. **The combined vector \mathbf{S} .** The combined vector (24), which is orthogonal to \mathbf{N} , coincides with the vector we defined in [6, 7] as the *sum vector*

$$\mathbf{S} \equiv B - F \equiv p^{-1}(\mathbf{e} + \boldsymbol{\rho})$$

(see Fig. 7b)).

While the vector \mathbf{N} gives a 3-dimensional picture of the conic orbits, the sum vector \mathbf{S} gives a 2-dimensional one, for it allows to express the standard plane polar equation of the whole family of Kepler orbits

Fig. 7 The constant vector \mathbf{N} . The plane sum vector \mathbf{S} .



$$\mathbf{N} = N - F = p^{-1}(\mathbf{e} + \mathbf{K})$$

$$\mathbf{S} = B - F = p^{-1}(\mathbf{e} + \boldsymbol{\rho})$$

$$\mathbf{N} \cdot \mathbf{S} = 0$$

$$r = \frac{p}{1 + e \cos \theta}$$

as the *simple equation* given by the scalar product

$$\mathbf{S} \cdot \mathbf{x} = 1.$$

7 Outburst of the Laplace–Runge–Lenz vector

A remarkable, well-known feature of the Kepler problem is the existence (beyond the constant angular momentum vector $\boldsymbol{\Gamma}$) of an additional constant vector, the so called Laplace–Runge–Lenz vector, that is

$$\mathbf{V} = K^2 \mathbf{e}.$$

Now we notice that both our two important vectors \mathbf{N} and \mathbf{S} have in common the vector

$$p^{-1} \mathbf{e} = \frac{K^2}{\Gamma^2} \mathbf{e}$$

(Fig. 7).

This is not a coincidence: in the spirit of the combined vectors generated by the sphere $S_{p^{-1}}$, if we simply rescale this vector by Γ^2 we immediately recover the celebrated Laplace–Runge–Lenz vector

$$\mathbf{V} = K^2 \mathbf{e}$$

which satisfies

$$\mathbf{V} \cdot \boldsymbol{\Gamma} = 0$$

and which, being a constant vector, notoriously expresses the fact that a Kepler orbit does not precess in the plane of motion given by $\boldsymbol{\Gamma}$.

Remark The ‘origin’ of this extra conserved vector is related in the literature to the so called ‘*hidden symmetry*’ of the Kepler problem, for it arises from the invariance of the Hamiltonian function of the Kepler problem under the symmetry group of a four-dimensional real rotation group in the four-dimensional Euclidean space R^4 .

For us the Laplace–Runge–Lenz vector is not completely hidden, but it already pops up by itself as an outstanding vector in the three-dimensional Euclidean arena: it is shared by the two fundamental vectors \mathbf{N} and \mathbf{S} .

8 The primigenial sphere and the regularizing KS-map

The well known singularity for $r = 0$ in the equation of the Kepler motion

$$\ddot{\mathbf{x}} + \frac{K^2}{r^3} \mathbf{x} = 0 \quad (. \equiv d/dt)$$

was removed by Kustaanheimo and Stiefel ([4]) by means of the so called KS-regularization in real form, which relies on both a time transformation used by Levi–Civita and a peculiar coordinate transformation (briefly *KS-map*) given by

$$(R^4 - \{0\}) \longrightarrow (R^3 - \{0\}) : \quad \mathbf{u} \longrightarrow \mathbf{x} \quad (30)$$

which maps a parametric four-dimensional Euclidean space R^4 of real vectors $\mathbf{u} = (u_1, u_2, u_3, u_4)$ onto the ordinary three-dimensional Euclidean space or real vectors $\mathbf{x} = (x_1, x_2, x_3)$.

Fig. 9 The regularizing KS-map
 pictured as a roto-dilation
 $\mathbf{K} \rightarrow \boldsymbol{\rho} \rightarrow r\boldsymbol{\rho}$

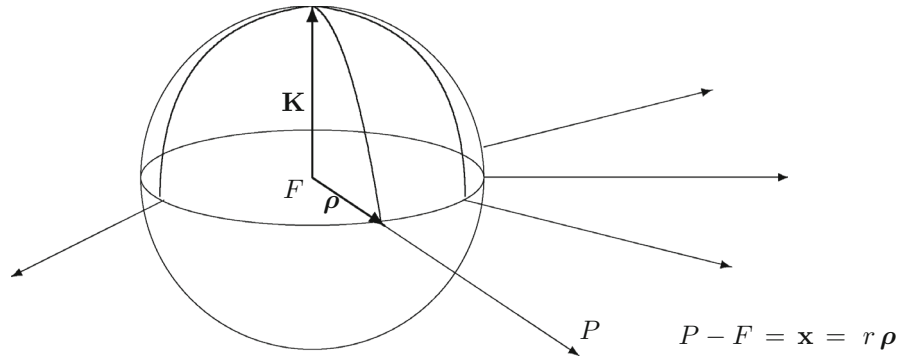
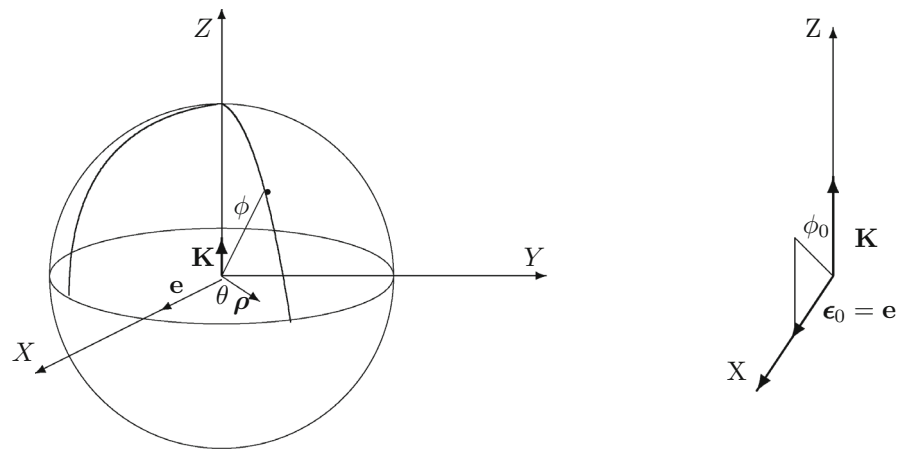


Fig. 10 The primigenial
 sphere S_{p-1} and its primitive
 elements $\boldsymbol{\rho}, \mathbf{K}, \epsilon_0 = \mathbf{e}$



of a primeval structure, the ‘primigenial’ sphere S_{p-1} (see Fig. 10).

Essentially, we have:

- Rewritten the governing Newtonian attractive force \mathbf{F} so to keep track not only of the symmetries but also of the planarity of the Kepler motion in R^3 ;
- Emphasized the role of both the two spherical coordinates, the longitude θ and the colatitude ϕ of a sphere;
- Considered a sphere as a *starry* sphere, that is as a locus of *vectors*, whence it is precisely the vector equation of the primigenial sphere S_{p-1}

$$P-F = p^{-1}(\epsilon + \cos\phi\mathbf{K}) \quad (31)$$

which not only generates directly in R^3 the \mathcal{C} -cone structure which defines the ‘conic’ orbits, but

which also embodies in a natural way the geometrical and physical elements of the orbits: the centre of attraction F , the semilatus-rectum parameter p and the eccentricity $e = \epsilon_0$ encapsulated in the extension vector ϵ_0 .

To further highlight the significance of the primigenial sphere description, let us add that there is more than meets the eye, since *the colatitude angle ϕ_0 is strictly related to the physical mechanical energy E of the Kepler problem.*

In fact let us rotate the unit vector \mathbf{K} in the (\mathbf{K}, \mathbf{I}) -plane around F through the particular angle ϕ_0 defined by

$$\sin \phi_0 = \sqrt{2EpK^{-2} + 1} \quad (32)$$

where E denotes the constant energy of the Kepler orbits. Since E is notoriously related to e by

$$e^2 - 1 = 2E \Gamma^2 K^{-4}$$

we find exactly that $\sin \phi_0 = e$, which now has both a geometrical interpretation (see Fig. 10 where the rotating vector \mathbf{K} is a unit one) and a physical one by (32).

This last result shows that the finding of a primigenial structure (such as the sphere S_{p-1} of the Kepler problem) is more than a fortuitous invention: suitably developed and extended, a primigenial structure may help in suggesting and obtaining the main features and the evolution of other different dynamical theories.

Acknowledgments Work supported by the Italian Ministry for University and Scientific, Technological Research MIUR.

References

1. Boccaletti D, Pucacco G (1996) Theory of orbits. Springer-

- Verlag, Berlin
2. Goldstein H (1980) Classical mechanics, 2nd edn. Addison-Wesley, Reading, MA
 3. Grossman N (1996) The sheer joy of celestial mechanics. Birkhauser, Boston
 4. Kustaanheimo P, Stiefel E (1965) Perturbation theory of Kepler motion based on spinor regularization. *J Reine Angew Math* 218:204–219
 5. Vivarelli MD (1994) The Kepler problem: a unifying view. *Celest Mech Dyn Astron* 60:291–305
 6. Vivarelli MD (2000) The sum vector \mathbf{S} and the fictitious time s in the Kepler problem. *Meccanica* 35:55–67
 7. Vivarelli MD (2005) The amazing \mathbf{S} -code of the conic sections and the Kepler problem. Polipress, Milano
 8. Vivarelli MD (2007) A Julia set for the Kepler problem. *Meccanica* 42:365–374
 9. Vivarelli MD (2010) The Kepler problem: a concealed vector. *Meccanica* 45:331–340
 10. Vivarelli MD (2012) Kepler conics \mathbf{S} -code: golden ratio, Dandelin spheres, fibonacci sequence. *Meccanica* 47: 245–256