An Approach to Distributed Predictive Control for Tracking–Theory and Applications

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I. Introduction

MANY decentralized and distributed model predictive control (MPC) algorithms have been recently developed, see the book [12] and the review papers [18], [3]. Most of these methods consider the so-called regulation problem: given a large-scale dynamical system made by a number of (interconnected or independent) subsystems, the problem is to asymptotically steer to zero the state of all the subsystems by coordinating the local control actions with a minimum amount of transmitted information. However, in this setting, the solution of the tracking problem with distributed MPC is much more difficult. In fact, the decentralization constraints do not allow to follow the standard approach, based on the reformulation of the tracking problem as a regulation one by computing, at any set-point change of the output, the corresponding state and control target values. For this reason, and to the best of the authors' knowledge, only the cooperative distributed MPC algorithm described in [7] is nowadays available, while a different approach based on a distributed reference governor has been proposed in [19].

In this brief paper, a new distributed MPC method for the solution of the tracking problem is proposed for systems made by the collection of M subsystems. The algorithm is developed according to the hierarchical structure shown in Fig. 1:

1) at the higher layer, given the required output reference signals $y^{[i]}_{\text{set-point}}$, $i = 1, \ldots, M$, feasible trajectories $\tilde{x}^{[i]}$, $\tilde{u}^{[i]}$, and $\tilde{y}^{[i]}$ of the local state, input and output variables are computed for each subsystem $\mathbf{S}_i$. Notably, these trajectories are computed according to the prescribed distributed information pattern, also adopted at the lower layer;
2) at the lower layer, M regulators $\mathbf{C}_i$ are designed with the distributed algorithm developed in [6] for the solution of the local regulation problems.

Manuscript received June 18, 2013; accepted October 26, 2013. Manuscript received in final form October 28, 2013. Recommended by Associate Editor E. Kerrigan.

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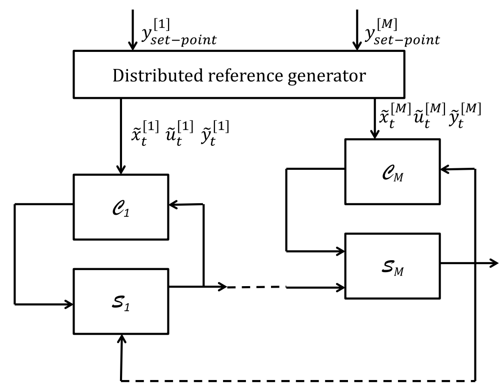

Fig. 1. Overall control architecture.

The overall algorithm guarantees that the controlled outputs reach the prescribed reference values whenever possible, or their nearest feasible value when feasibility problems arise due to the constraints. A preliminary version of this algorithm is presented in [5] where, however, a less general control problem is considered and more restrictive conditions are required.

This brief paper is organized as follows. In Section II, the system's structure and constraints are defined. Section III describes the proposed control system architecture, while the distributed MPC algorithm is presented in Section IV. In Sections V and VI, two benchmark examples are described, namely the control of a small fleet of unicycle robots and the control of a simulated four-tank system. Conclusions are drawn in Section VII. The proofs of the main results are postponed to an Appendix.

Notation: A matrix is Schur if all its eigenvalues lie in the interior of the unit circle. The short-hand $\mathbf{v} = (v_1, \ldots, v_s)$ denotes a column vector with s (not necessarily scalar) components $v_1, \ldots, v_s$. The symbols $\oplus$ and $\ominus$ denote the Minkowski sum and Pontryagin difference, respectively, [16], while $\bigoplus_{i=1}^{M} A_i = A_1 \oplus \cdots \oplus A_M$. A generic q-norm ball centered at the origin in the $\mathbb{R}^{\dim}$ space is defined as follows $\mathscr{B}^{(\dim)}_{q,\varepsilon}(0) := \{x \in \mathbb{R}^{\dim} : \|x\|_q \leq \varepsilon\}$. For a discrete-time signal s_t and $a, b \in \mathbb{N}$, $a \leq b$, $(s_a, s_{a+1}, \ldots, s_b)$ is denoted with $s_{[a:b]}$.

II. Interacting Subsystems

Consider M dynamically interacting subsystems, which, according to the notation used in [11], are described by

$$x_{t+1}^{[i]} = A_{ii} x_t^{[i]} + B_{ii} u_t^{[i]} + E_i s_t^{[i]} \tag{1a}$$
$$y_t^{[i]} = C_i x_t^{[i]} \tag{1b}$$
$$z_t^{[i]} = C_{zi} x_t^{[i]} + D_{zi} u_t^{[i]} \tag{1c}$$

where $x_t^{[i]} \in \mathbb{R}^{n_i}$ and $u_t^{[i]} \in \mathbb{R}^{m_i}$ are the states and inputs, respectively, of the ith subsystem, while $y_t^{[i]} \in \mathbb{R}^{p_i}$ is its output, with $p_i \leq m_i$. In line with the interaction-oriented models introduced in [11], the coupling input and output vectors $s_t^{[i]}$ and $z_t^{[i]}$, respectively, are defined to characterize the interconnections among the subsystems, i.e.,

$$s_t^{[i]} = \sum_{j=1}^{M} L_{ij} z_t^{[j]}. \tag{2}$$

We say that subsystem j is a dynamic neighbor of subsystem i if and only if $L_{ij} \neq 0$, and we denote as $\mathscr{N}_i$ the set of dynamic neighbors of subsystem i (which excludes i).

The input and state variables are subject to the local constraints $u_t^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ and $x_t^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, respectively, where the sets $\mathbb{U}_i$ and $\mathbb{X}_i$ are convex. Furthermore, we allow for n_c linear constraints involving the output variables of more than one subsystem: the hth constraint inequality is

$$\sum_{j=1}^{M} H_h^{[j]} y_t^{[j]} \leq l_h \tag{3}$$

where $H_h^{[j]} \in \mathbb{R}^{1 \times p_j}$ are row vectors, which can be possibly equal to zero, for some values of j. Without loss of generality, the outputs involved in coupling constraints are accounted for as coupling outputs, i.e., for all $h = 1, \ldots, n_c$ and for all $j = 1, \ldots, M$, there exists a matrix $H_h^{[j]z}$ such that

$$H_h^{[j]} y_t^{[j]} = H_h^{[j]z} z_t^{[j]} \tag{4}$$

under (1b) and (1c).

We say that the hth inequality is a constraint on subsystem i if $H_h^{[i]} \neq 0$, and we denote the set of constraints on subsystem i as $\mathscr{C}_i = \{h \in \{1, \ldots, n_c\}$: the hth inequality is a constraint on $i\}$, and with $n_c^{[i]}$ the number of elements of $\mathscr{C}_i$. Subsystem $j \neq i$ is a constraint neighbor of subsystem i if there exists $\bar{h} \in \mathscr{C}_i$ such that $H_{\bar{h}}^{[j]} \neq 0$. For all $h = 1, \ldots, n_c$, we finally denote with $\mathscr{S}_h$ the set of subsystems for which the hth inequality is a constraint, i.e., $\mathscr{S}_h = \{i : H_h^{[i]} \neq 0\}$ and with $n_h = |\mathscr{S}_h|$ the cardinality of $\mathscr{S}_h$.

Collecting the subsystems (1) for all $i = 1, \ldots, M$, we obtain the collective dynamical model

$$\mathbf{x}_{t+1} = \mathbf{A}\,\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \tag{5a}$$
$$\mathbf{y}_t = \mathbf{C}\,\mathbf{x}_t \tag{5b}$$

where $\mathbf{x}_t = (x_t^{[1]}, \ldots, x_t^{[M]}) \in \mathbb{R}^n$, $n = \sum_{i=1}^{M} n_i$, $\mathbf{u}_t = (u_t^{[1]}, \ldots, u_t^{[M]}) \in \mathbb{R}^m$, $m = \sum_{i=1}^{M} m_i$, and $\mathbf{y}_t = (y_t^{[1]}, \ldots, y_t^{[M]}) \in \mathbb{R}^p$, $p = \sum_{i=1}^{M} p_i$, are the collective state, input, and output vectors, respectively. The state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \ldots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are the diagonal blocks of $\mathbf{A}$, whereas the dynamic coupling terms between the subsystems correspond to the nondiagonal blocks of $\mathbf{A}$, i.e., $A_{ij} = E_i L_{ij} C_{zj}$, with $j \neq i$. Correspondingly, $B_{ii}, i = 1, \ldots, M$, are the diagonal blocks of $\mathbf{B}$, whereas the influence of the input of a subsystem upon the state of different subsystems is represented by the off-diagonal terms of $\mathbf{B}$, i.e., $B_{ij} = E_i L_{ij} D_{zj}$, with $j \neq i$. The collective output matrix is defined as $\mathbf{C} = \mathrm{diag}(C_{11}, \ldots, C_{MM})$.

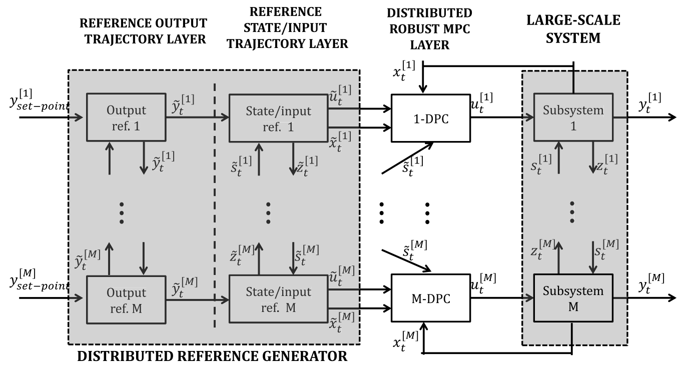

Fig. 2. Distributed architecture for tracking reference signals.

Concerning system (5a) and its partition, the following main assumption on decentralized stabilizability is introduced.

Assumption 1: There exists a block-diagonal matrix $\mathbf{K} = \mathrm{diag}(K_1, \ldots, K_M)$, with $K_i \in R^{m_i \times n_i}$, $i = 1, \ldots, M$ such that: 1) $\mathbf{F} = \mathbf{A} + \mathbf{BK}$ is Schur and 2) $F_{ii} = (A_{ii} + B_{ii} K_i)$ is Schur, $i = 1, \ldots, M$.

Remark 1: The design of the stabilizing matrix $\mathbf{K}$ can be performed according to the procedure proposed in [6] or by resorting to an linear matrix inequality (LMI) formulation, see [2], based on well-known results in decentralized control, see [20].

The following standard assumption is made.

Assumption 2:

$$\mathrm{rank}\left(\begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}\right) = n + p.$$

III. Control System Architecture

The higher layer of the hierarchical scheme of Fig. 1 is itself made by two sub-layers, as shown in Fig. 2: a reference output trajectory layer computes in a distributed way the output reference trajectories $\tilde{y}^{[i]}$ given the ideal set-points $y_{\mathrm{set-point}}^{[i]}$, while a reference state and input trajectory layer determines the corresponding state and control trajectories $\tilde{x}^{[i]}$ and $\tilde{u}^{[i]}$. At the lower layer of the structure of Fig. 1, a distributed robust MPC layer is designed to drive the real state and input trajectories $x_t^{[i]}$ and $u_t^{[i]}$ of the subsystems as close as possible to $\tilde{x}_t^{[i]}$, $\tilde{u}_t^{[i]}$, while satisfying the constraints. Notably, at each level, information is required to be transmitted only among the neighboring subsystems.

A. Reference Output Trajectory Layer

At any time t, the reference trajectories $\tilde{y}_{t+k}^{[i]}$, $k = 1, \ldots, N-1$, are regarded as an argument of an optimization problem itself (see Section IV-B) rather than a fixed parameter, similarly to the approach in [10]. However, in the considered distributed context, too rapid changes of the output reference trajectory of a given subsystem could greatly affect the performance and the behavior of the other subsystems. Therefore, to limit the rate of variation, it is required that, for all $i = 1, \dots, M$, for all $t \geq 0$

$$\tilde{y}_{t+1}^{[i]} \in \tilde{y}_t^{[i]} \oplus \mathscr{B}_{q,\varepsilon}^{(p_i)}(0). \tag{6}$$

The reference output trajectory management layer is also committed to defining suitable update laws for $\tilde{y}_t^{[i]}$ in such a way that, for all time steps, (3) is verified for all $h = 1, \dots, n_c$.

B. Reference State and Input Trajectory Layers

Given, at any time step t, the future reference trajectories $\tilde{y}_k^{[i]}$, $k = t, \dots, t+N-1$, to define the state and control reference trajectories $(\tilde{x}_t^{[i]}, \tilde{u}_t^{[i]})$, consider the systems

$$\tilde{x}_{t+1}^{[i]} = A_{ii}\tilde{x}_t^{[i]} + B_{ii}\tilde{u}_t^{[i]} + E_i\tilde{s}_t^{[i]} \tag{7a}$$

$$\tilde{e}_{t+1}^{[i]} = \tilde{e}_t^{[i]} + \tilde{y}_{t+1}^{[i]} - C_i\tilde{x}_t^{[i]} \tag{7b}$$

where similarly to (1c) and (2)

$$\tilde{z}_t^{[i]} = C_{zi}\tilde{x}_t^{[i]} + D_{zi}\tilde{u}_t^{[i]} \tag{7c}$$

$$\tilde{s}_t^{[i]} = \sum_{j\in\mathscr{N}_i} L_{ij}\tilde{z}_t^{[j]}. \tag{7d}$$

Define $\chi_t^{[i]} = (\tilde{x}_t^{[i]}, \tilde{e}_t^{[i]})$

$$\mathscr{A}_{ij} = \begin{cases} \begin{bmatrix} A_{ii} & 0 \\ -C_i & I_{p_i} \end{bmatrix} & \text{if } j = i \\ \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix} & \text{if } j \neq i \end{cases}, \mathscr{B}_{ij} = \begin{bmatrix} B_{ij} \\ 0 \end{bmatrix}, \mathscr{G}_i = \begin{bmatrix} 0 \\ I_{p_i} \end{bmatrix} \tag{7e}$$

and consider the control law

$$\tilde{u}_t^{[i]} = \mathscr{K}_i\chi_t^{[i]} \tag{7f}$$

where $\mathscr{K}_i = \begin{bmatrix} K_i^x & K_i^e \end{bmatrix}$. Letting $\mathscr{F}_{ij} = \mathscr{A}_{ij} + \mathscr{B}_{ij}\mathscr{K}_j$, the dynamics of $\chi_t^{[i]}$ is therefore defined by

$$\chi_{t+1}^{[i]} = \mathscr{F}_{ii}\chi_t^{[i]} + \sum_{j\in\mathscr{N}_i} \mathscr{F}_{ij}\chi_t^{[j]} + \mathscr{G}_i\tilde{y}_{t+1}^{[i]}. \tag{8}$$

The gain matrix $\mathscr{K}_i$ is to be determined as follows: denoting by $\mathscr{A}$ and $\mathscr{B}$, the matrices whose block elements are $\mathscr{A}_{ij}$ and $\mathscr{B}_{ij}$, respectively, and $\mathscr{K}$ =diag$(\mathscr{K}_1, \dots, \mathscr{K}_M)$, the following assumption must be fulfilled.

Assumption 3: The matrix $\mathscr{F} = \mathscr{A} + \mathscr{B}\mathscr{K}$ is Schur stable. □

The synthesis of the $\mathscr{K}_i$'s can be performed provided that Assumption 2 is verified and according to the procedures proposed in Remark 1. Moreover, when $p_i > m_i$ for some i, the available degrees of freedom can be used to select $\mathscr{K}_i$'s fulfilling some additional optimization criteria.

Define, for all $i = 1, \dots, M$ and for all $t \geq 0$, $\chi_t^{[i]ss} = (x_t^{[i]ss}, e_t^{[i]ss})$, as the steady-state condition for (8) corresponding to the reference outputs $\tilde{y}_t^{[i]}$ assumed constant, i.e., $\tilde{y}_{t+1}^{[i]} = \tilde{y}_t^{[i]}$, and satisfying for all $i = 1, \dots, M$

$$\chi_t^{[i]ss} = \mathscr{F}_{ii}\chi_t^{[i]ss} + \sum_{j\in\mathscr{N}_i} \mathscr{F}_{ij}\chi_t^{[j]ss} + \mathscr{G}_i\tilde{y}_t^{[i]}. \tag{9}$$

In view of (7) and Assumption 3, $C_i x_t^{[i]ss} = \tilde{y}_{t+1}^{[i]}$ and $\mathscr{F}$ is Schur stable. Then, a solution to the system (9) exists and is unique. Collectively define $\boldsymbol{\chi}_t^{ss} = (\chi_t^{[1]ss}, \dots, \chi_t^{[M]ss})$, $\boldsymbol{\chi}_t = (\chi_t^{[1]}, \dots, \chi_t^{[M]})$, and $\tilde{\mathbf{y}}_t = (\tilde{y}_t^{[1]}, \dots, \tilde{y}_t^{[M]})$. From (7e)–(9), we can collectively write

$$\boldsymbol{\chi}_{t+1}^{ss} - \boldsymbol{\chi}_t^{ss} = (I_{n+p} - \mathscr{F})^{-1}\mathscr{G}(\tilde{\mathbf{y}}_{t+1} - \tilde{\mathbf{y}}_t) \tag{10}$$

where $\mathscr{G} = \text{diag}(\mathscr{G}_1, \dots, \mathscr{G}_M)$. Therefore

$$\boldsymbol{\chi}_{t+1} - \boldsymbol{\chi}_{t+1}^{ss} = \mathscr{F}(\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss}) - (I_{n+p} - \mathscr{F})^{-1}\mathscr{F}\mathscr{G}(\tilde{\mathbf{y}}_{t+1} - \tilde{\mathbf{y}}_t) \tag{11}$$

which can be rewritten as

$$\boldsymbol{\chi}_{t+1} - \boldsymbol{\chi}_{t+1}^{ss} = \mathscr{F}(\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss}) + \tilde{\mathbf{w}}_t \tag{12}$$

where $\tilde{\mathbf{w}}_t$ can be seen as a bounded disturbance. In fact, in view of (6), $\tilde{\mathbf{w}}_t \in \tilde{\mathbb{W}} = -(I_{n+p} - \mathscr{F})^{-1}\mathscr{F}\mathscr{G}\prod_{i=1}^M \mathscr{B}_{q,\varepsilon}^{(p_i)}(0)$. Under Assumption 3, for (12) there exists a possibly non-rectangular robust positive invariant (RPI) set $\boldsymbol{\Delta}^\chi$ such that, if $\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss} \in \boldsymbol{\Delta}^\chi$, then it is guaranteed that $\boldsymbol{\chi}_{t+k} - \boldsymbol{\chi}_{t+k}^{ss} \in \boldsymbol{\Delta}^\chi$ for all $k \geq 0$. This, in turn, implies that the convex sets Δ_i^χ exist and can be defined in such a way that $\boldsymbol{\Delta}^\chi \subseteq \prod_{i=1}^M \Delta_i^\chi$, so that, for any initial condition $\boldsymbol{\chi}_0 - \boldsymbol{\chi}_0^{ss} \in \boldsymbol{\Delta}^\chi$, for all $t \geq 0$, it holds that

$$\chi_t^{[i]} - \chi_t^{[i]ss} \in \Delta_i^\chi. \tag{13}$$

C. Robust Distributed MPC Layer

The DPC algorithm described in [6] is used to drive the real state and input trajectories $x_t^{[i]}$ and $u_t^{[i]}$ as close as possible to their references $\tilde{x}_t^{[i]}$, $\tilde{u}_t^{[i]}$. Specifically, by adding suitable constraints to the distributed MPC problem formulation, for each subsystem and for all $t \geq 0$ it is possible to guarantee that the actual coupling output trajectories lie in specified time-invariant neighborhoods of their reference trajectories. More formally, if $z_t^{[i]} \in \tilde{z}_t^{[i]} \oplus \mathscr{Z}_i$, where $\mathscr{Z}_i$ is compact, convex and $0 \in \mathscr{Z}_i$, in view of (7d) it is guaranteed that $s_t^{[i]} \in \tilde{s}_t^{[i]} \oplus \mathscr{S}_i$, where $\mathscr{S}_i = \bigoplus_{j\in\mathscr{N}_i} L_{ij}\mathscr{Z}_j$. In this way, (1a) can be written as

$$x_{t+1}^{[i]} = A_{ii}x_t^{[i]} + B_{ii}u_t^{[i]} + E_i\tilde{s}_t^{[i]} + E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) \tag{14}$$

where $E_i(s_t^{[i]} - \tilde{s}_t^{[i]})$ can be seen as a bounded disturbance, while $E_i\tilde{s}_{t+k}^{[i]}$ can be interpreted as an input, known in advance over the prediction horizon $k = 0, \dots, N-1$.

For the statement of the individual MPC sub-problems, henceforth called i-DPC problems, define the ith subsystem nominal model associated to equation (14)

$$\hat{x}_{t+1}^{[i]} = A_{ii}\hat{x}_t^{[i]} + B_{ii}\hat{u}_t^{[i]} + E_i\tilde{s}_t^{[i]} \tag{15}$$

and let

$$\hat{z}_t^{[i]} = C_{zi}\hat{x}_t^{[i]} + D_{zi}\hat{u}_t^{[i]}. \tag{16}$$

The control law for the ith subsystem (14), for all $t \geq 0$, is assumed to be given by

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i(x_t^{[i]} - \hat{x}_t^{[i]}) \tag{17}$$

where K_i satisfies Assumption 1. Letting $\varepsilon_t^{[i]} = x_t^{[i]} - \hat{x}_t^{[i]}$ from (14), (15), and (17) it follows that:

$$\varepsilon_{t+1}^{[i]} = F_{ii}\varepsilon_t^{[i]} + w_t^{[i]} \tag{18}$$

where

$$w_t^{[i]} = E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) \tag{19}$$

is a bounded disturbance since $s_t^{[i]} - \tilde{s}_t^{[i]} \in \mathscr{S}_i$. It follows that

$$w_t^{[i]} \in \mathbb{W}_i = E_i\mathscr{S}_i. \tag{20}$$

Since $w_t^{[i]}$ is bounded and F_{ii} is Schur, there exists an RPI $\mathscr{E}_i$ for (18) such that, for all $\varepsilon_t^{[i]} \in \mathscr{E}_i$, then $\varepsilon_{t+1}^{[i]} \in \mathscr{E}_i$. Therefore, at time $t+1$, in view of (1c) and (16), it holds that $z_{t+1}^{[i]} - \hat{z}_{t+1}^{[i]} = (C_{zi} + D_{zi}K_i)\varepsilon_{t+1}^{[i]} \in (C_{zi} + D_{zi}K_i)\mathscr{E}_i$.

To guarantee that, at time $t+1$, $z_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in \mathscr{Z}_i$ can be still verified by adding suitable constraints to the optimization problems, the following assumption must be fulfilled.

Assumption 4: For all $i = 1, \ldots, M$, there exists a positive scalar ρ_i such that

$$(C_{zi} + D_{zi}K_i)\mathscr{E}_i \oplus \mathscr{B}_{q,\rho_i}(0) \subseteq \mathscr{Z}_i. \tag{21}$$

If Assumption 4 is fulfilled define, for all $i = 1, \ldots, M$, the convex neighborhood of the origin Δ_i^z satisfying

$$\Delta_i^z \subseteq \mathscr{Z}_i \ominus (C_{zi} + D_{zi}K_i)\mathscr{E}_i \tag{22}$$

and consider the constraint $\hat{z}_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in \Delta_i^z$, in such a way that

$$\begin{aligned} z_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} &= z_{t+1}^{[i]} - \hat{z}_{t+1}^{[i]} + \hat{z}_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \\ &\in (C_{zi} + D_{zi}K_i)\mathscr{E}_i \oplus \Delta_i^z \subseteq \mathscr{Z}_i \end{aligned} \tag{23}$$

as required at all time steps $t \geq 0$.

IV. Distributed Predictive Control Algorithm

The overall design problem is composed by a preliminary centralized off-line design and an on-line solution of the M i-DPC problems, as now detailed.

A. Off-Line Design

The off-line design consists of the following procedure:

1) compute the matrices $\mathbf{K}$ and $\mathscr{K}$ satisfying Assumptions 1 and 3 (see Remark 1);
2) define $\mathscr{B}_{q,\varepsilon}^{(\rho_i)}(0)$, compute $\mathbf{\Delta}^\chi$ [an RPI for (12)] and Δ_i^χ (for the computation of RPIs see [16]);
3) compute the RPI sets $\mathscr{E}_i$ for the subsystems (18) and the sets Δ_i^z satisfying (22) and (23);
4) compute $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus \mathscr{E}_i$, $\hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i\mathscr{E}_i$, the positively invariant set Σ_i for

$$\delta x_{t+1}^{[i]} = F_{ii}\delta x_t^{[i]} \tag{24}$$

such that

$$(C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta_i^z \tag{25a}$$

and the convex sets $\mathbb{Y}_i$ such that

$$\begin{aligned} \begin{bmatrix} I_{n_i} & 0 \\ K_i^x & K_i^e \end{bmatrix} \Big(\Gamma_i(I_{n+p} - \mathscr{F})^{-1}\mathscr{G}\textstyle\prod_{j=1}^M \mathbb{Y}_j \oplus \Delta_i^\chi\Big) \\ \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i \end{aligned} \tag{25b}$$

where Γ_i is the matrix, of suitable dimensions, that selects the subvector $\chi_t^{[i]}$ out of $\boldsymbol{\chi}_t$. Specifically, $\mathbb{Y}_i$ is the set associated to $\tilde{y}_t^{[i]}$ such that the corresponding steady-state state and input satisfy the control and state constraints defined by $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{U}}_i$. Concerning the set-theoretical conditions guaranteeing the design of the control scheme, it is worth mentioning that, at the price of a more conservative scheme and slower settling times (e.g., small parameter ε), (25) can always be verified. On the other hand, it is not always possible to select sets $\mathscr{Z}_i$ such that (21) is verified; as investigated in [6] it consists in a network-wide small gain condition.

B. On-Line Design

The on-line design is based on the solution of the following distributed and independent optimization problems. It is worth remarking that the reference output layer and the MPC problem consist in two independent optimization problems. This enhances the reliability of the approach and reduces its computational load, at the price of limiting the rate of variation of the output reference to the value that guarantees constraint satisfaction in all possible conditions, contrarily to [10].

1) Computation of the Reference Outputs: The output reference trajectories $\tilde{y}_{t+N}^{[i]}$ are computed to minimize the distance from the ideal set-points $y_{\text{set-point}}^{[i]}$ and to fulfill the constraints, including the coupling ones (3). Concerning the latter, define

$$\tilde{l}_h = l_h - \sum_{j \in \mathscr{S}_h} \left\{ \max_{\chi \in \Delta_j^\chi} H_h^{[j]} \begin{bmatrix} C_j & 0 \end{bmatrix} \chi + \max_{z \in \mathscr{Z}_j} H_h^{[j]z} z \right\} \tag{26}$$

and, for all $t \geq 0$

$$k_{h,t+N-1} = \sum_{j \in \mathscr{S}_h} H_h^{[j]} \tilde{y}_{t+N-1}^{[j]}. \tag{27}$$

Then, it is possible to show (see the Appendix) that

$$\begin{aligned} H_h^{[i]} \bar{y}_{t+N}^{[i]} \leq \tilde{l}_h - &\sum_{j \in \mathscr{S}_h \setminus \{i\}} H_h^{[j]} \tilde{y}_{t+N-1}^{[j]} \\ &- \frac{(n_h - 1)}{n_h}(\tilde{l}_h - k_{h,t+N-1}) \end{aligned} \tag{28}$$

guarantees that (3) holds and recursive feasibility. Therefore, (28) can be used in place of (3) in the following optimization problem associated with the reference output trajectory layer:

$$\min_{\bar{y}_{t+N}^{[i]}} V_i^y(\bar{y}_{t+N}^{[i]}, t) \tag{29}$$

subject to

$$\bar{y}_{t+N}^{[i]} - \tilde{y}_{t+N-1}^{[i]} \in \mathscr{B}_{q,\varepsilon}^{(\rho_i)}(0) \tag{30}$$

$$\bar{y}_{t+N}^{[i]} \in \mathbb{Y}_i \tag{31}$$

and (28), where

$$V_i^y(\bar{y}_{t+N}^{[i]}) = \gamma \|\bar{y}_{t+N}^{[i]} - \tilde{y}_{t+N-1}^{[i]}\|^2 + \|\bar{y}_{t+N}^{[i]} - y_{\text{set-point}}^{[i]}\|_{T_i}^2.$$

The weight T_i must verify the inequality

$$T_i > \gamma\, I_{p_i} \tag{32}$$

while γ is an arbitrarily small positive constant.

2) Computation of the Control Variables: The i-DPC problem solved by the ith robust MPC layer unit is defined as follows:

$$\min_{\hat{x}_t^{[i]},\hat{u}_{[t:t+N-1]}^{[i]}} V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) \tag{33}$$

where

$$V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) = \sum_{k=t}^{t+N-1} \|\hat{x}_k^{[i]} - \tilde{x}_k^{[i]}\|_{Q_i}^2 + \|\hat{u}_k^{[i]} - \tilde{u}_k^{[i]}\|_{R_i}^2 + \|\hat{x}_{t+N}^{[i]} - \tilde{x}_{t+N}^{[i]}\|_{P_i}^2 \tag{34}$$

subject to (15) and, for $k = t, \ldots, t+N-1$

$$x_t^{[i]} - \hat{x}_t^{[i]} \in \mathscr{E}_i \tag{35a}$$

$$\hat{z}_k^{[i]} - \tilde{z}_k^{[i]} \in \Delta_i^z \tag{35b}$$

$$\hat{x}_k^{[i]} \in \hat{\mathbb{X}}_i \tag{35c}$$

$$\hat{u}_k^{[i]} \in \hat{\mathbb{U}}_i \tag{35d}$$

and to the terminal constraint

$$\hat{x}_{t+N}^{[i]} - \tilde{x}_{t+N}^{[i]} \in \Sigma_i. \tag{36}$$

The weights Q_i and R_i in (34) must be taken as positive definite matrices while, in order to prove the convergence properties of the proposed approach, select the matrices P_i as the solutions of the (fully independent) Lyapunov equations

$$F_{ii}^T P_i F_{ii} - P_i = -(Q_i + K_i^T R_i K_i). \tag{37}$$

At time t, $(\hat{x}_{t|t}^{[i]}, \hat{u}_{[t:t+N-1]|t}^{[i]}, \bar{y}_{t+N|t}^{[i]})$ is the solution to the i-DPC problem and $\hat{u}_{t|t}^{[i]}$ is the input to the nominal system (15).

Remark 2: Note that the problems (29) and (33) are independent of each other. In fact, (29) does not depend on $\hat{x}_t^{[i]}$ and $\hat{u}_{[t:t+N-1]}^{[i]}$. Moreover, both the cost function V_i^N and the constraints (35) are independent of $\bar{y}_{t+N|t}^{[i]}$. □

According to (17), the input to the system (1a) is

$$u_t^{[i]} = \hat{u}_{t|t}^{[i]} + K_i(x_t^{[i]} - \hat{x}_{t|t}^{[i]}). \tag{38}$$

Moreover, set $\tilde{y}_{t+N}^{[i]} = \bar{y}_{t+N|t}^{[i]}$ and compute the references $\tilde{e}_{t+N}^{[i]}$ and $\tilde{x}_{t+N+1}^{[i]}$ with (7b) and (7a), respectively. Finally, set $\tilde{u}_{t+N}^{[i]} = K_i^x \tilde{x}_{t+N}^{[i]} + K_i^e \tilde{e}_{t+N}^{[i]}$ from (7f).

Denoting by $\hat{x}_{k|t}^{[i]}$ the state trajectory of system (15) stemming from $\hat{x}_{t|t}^{[i]}$ and $\hat{u}_{[t:t+N-1]|t}^{[i]}$, at time t it is also possible to compute $\hat{x}_{t+N|t}^{[i]}$. The properties of the proposed distributed MPC algorithm for tracking can now be summarized in the following result.

Theorem 1: Let Assumptions 1–4 be verified and the tuning parameters be selected as previously described. If at time $t = 0$ a feasible solution to the constrained problems (29), (33) exists and $k_{h,N-1} \leq \tilde{l}_h$ [see (26) and (27)] for all $h = 1, \ldots, n_c$ then, for all $i = 1, \ldots, M$

1) Feasible solutions to (29), (33) exist for all $t \geq 0$, i.e., constraints (28), (30), (31), (35), and (36), respectively, are verified. Furthermore, the constraints $(x_t^{[i]}, u_t^{[i]}) \in \mathbb{X}_i \times \mathbb{U}_i$ and for all $i = 1, \ldots, M$, and (3) for all $h = 1, \ldots, n_h$, are fulfilled for all $t \geq 0$.
2) If coupling constraints (3) are absent, then the resulting MPC controller asymptotically steers the ith system to the admissible set-point $y_{\text{feas.set-point}}^{[i]}$, where $y_{\text{feas.set-point}}^{[i]}$ is the solution to

$$y_{\text{feas.set-point}}^{[i]} = \operatorname*{argmin}_{y^{[i]} \in \mathbb{Y}_i} \|y^{[i]} - y_{\text{set-point}}^{[i]}\|_{T_i}^2. \tag{39}$$

When coupling static constraints are present, the convergence to the nearest feasible solution to the prescribed set-point may be prevented for some initial conditions. These situations are denoted deadlock solutions in [19]. Future work will be specifically devoted to this issue.

V. Control of Unicycle Robots

In this section, the proposed algorithm is applied to the problem of positioning a number of mobile robots in specified positions, while guaranteeing collision avoidance.

The dynamics of a single robot is described by a modified version of the first-order kinematic model [15]

$$\dot{x} = \upsilon \cos\phi \tag{40a}$$

$$\dot{y} = \upsilon \sin\phi \tag{40b}$$

$$\dot{\phi} = \omega \tag{40c}$$

$$\dot{\upsilon} = a \tag{40d}$$

where (x, y) is the Cartesian position of the robot, ϕ is its orientation angle, and υ is its linear velocity. The linear acceleration a and the angular velocity ω are inputs.

By resorting to a feedback linearization procedure (see [15]) a linear model of the robots can be used to describe the system's dynamics. Namely, define $\eta_1 = x$, $\eta_2 = \dot{x}$, $\eta_3 = y$, $\eta_4 = \dot{y}$, and the dynamics resulting from (40) is

$$\dot{\eta}_1 = \eta_2 \tag{41a}$$

$$\dot{\eta}_2 = a\cos\phi - \upsilon\omega\sin\phi \tag{41b}$$

$$\dot{\eta}_3 = \eta_4 \tag{41c}$$

$$\dot{\eta}_4 = a\sin\phi + \upsilon\omega\cos\phi. \tag{41d}$$

Now define two new fictitious input variables $a_x = a\cos\phi - \upsilon\omega\sin\phi$ and $a_y = a\sin\phi + \upsilon\omega\cos\phi$. From (41), the model (40) is transformed in a set of two decoupled double integrators with inputs a_x and a_y.

To recover the real inputs, (ω, a) from (a_x, a_y) compute

$$\begin{bmatrix}\omega \\ a\end{bmatrix} = \frac{1}{\upsilon}\begin{bmatrix}-\sin\phi & \cos\phi \\ \upsilon\cos\phi & \upsilon\sin\phi\end{bmatrix}\begin{bmatrix}a_x \\ a_y\end{bmatrix}. \tag{42}$$

Note that, for obtaining (42), it is assumed that $\upsilon \neq 0$. This singularity point must be accounted for when designing control laws on the equivalent linear model [15]. In discrete time, from (41) and with sampling time $\tau = 5$ s, we obtain

$$A_{ii} = A = \begin{bmatrix}1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 1\end{bmatrix}, \quad B_{ii} = B = \begin{bmatrix}\frac{\tau^2}{2} & 0 \\ \tau & 0 \\ 0 & \frac{\tau^2}{2} \\ 0 & \tau\end{bmatrix}.$$

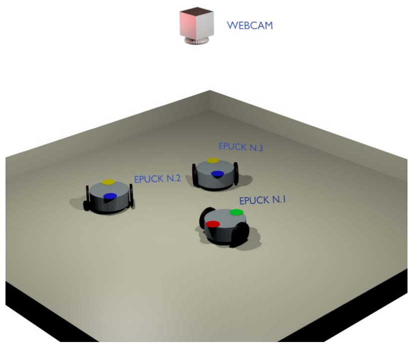

Fig. 3. Sketch of the experimental set-up.

The measured variables are x and y, i.e., η_1 and η_3 in (41). Note that, this case study is characterized by: 1) no dynamically coupling terms, i.e., $E_i = 0$ and $L_{ij} = 0$ for all $i, j = 1, \dots, M$ and 2) static coupling constraints on the position variables guaranteeing collision avoidance. Therefore, set $C_{zi} = C_i$ and $D_{zi} = 0$.

The experimental setup consists of three e-puck mobile robots [14]. To simplify the application of the algorithm, the control law is designed on a portable computer communicating with the e-puck robots through wireless connection. The measurement system consists of a camera, installed on the top of the 130×80 cm^2 working area. Position and orientation of each robot are detected using two colored circles, placed on the top of each agent, see Fig. 3.

The fact that no coupling terms are present (i.e., $E_i = 0$ and $\mathscr{F}_{ij} = 0$ for all i, j) greatly simplifies the design phase. More specifically, with reference to the off-line design steps outlined in Section IV

1) Assumptions 1 and 3 correspond to solve centralized small-scale (e.g., eigenvalue assignment) problems.
2) similarly to the previous step, Δ_i^{χ} can be computed as the RPI set for $\chi_{t+1}^{[i]} - \chi_{t+1}^{[i]ss} = \mathscr{F}_{ii}(\chi_t^{[i]} - \chi_t^{[i]ss}) + \tilde{w}_t^{[i]}$, with $\tilde{w}_t^{[i]} = -(I_6 - \mathscr{F}_i)^{-1}\mathscr{F}_i\mathscr{G}_i\mathscr{B}^{(2)}_{q(0),\varepsilon}$ with $\varepsilon = 5$ cm.
3) Since $E_i = 0$, $w_t^{[i]} = 0$ in (18), $\mathscr{E}_i$ is an arbitrarily small positively invariant set, and Δ_i^z can be chosen arbitrarily.
4) Since, in (25b), $\Gamma_i(I_{n+p} - \mathscr{F})^{-1}\mathscr{G}\prod_{i=1}^M \mathbb{Y}_j = (I_6 - \mathscr{F}_i)^{-1}\mathscr{G}_i\mathbb{Y}_i$, this step can be verified in a decentralized fashion. □

Collision avoidance constraints are in principle nonconvex and described using nonlinear inequalities. To circumvent this problem, suitable linear constraints are defined to replace non-convex ones and are obtained by tracing a line stemming from the center of each robot and corresponding to a tangent line to the circumference of the neighboring ones. For all $i = 1, 2, 3$, in the cost functions V_i^y and V_i^N we set $\gamma = 1$, $T_i = 4I_2$ and $Q_i = I_4$, $R_i = 0.01I_2$, respectively.

In the reported real experiment, the three robots are initially placed (at time $t = 1s$) at positions $(28, 52)$, $(39, 16)$, and $(90, 39)$-all coordinates are in cm. Fig. 4 shows the evolution of their motion in reaching the goal positions-i.e., $(86, 13)$, $(77, 55)$, and $(20, 39)$-at time $t = 45$ s while fulfilling collision avoidance constraints.

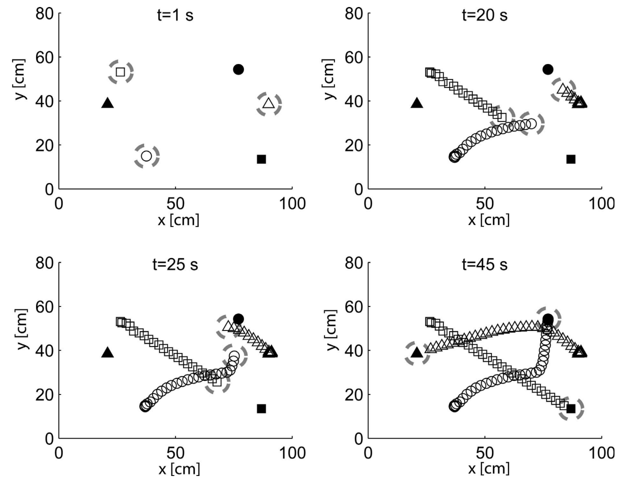

Fig. 4. Plots of the robot trajectories. Robot 1: □. Robot 2: ○. Robot 3: △. Symbols with white surface denote the position of the robots, while symbols with black surface denote the goal positions. Large circles with gray dashed line denote the area occupied by the robots.

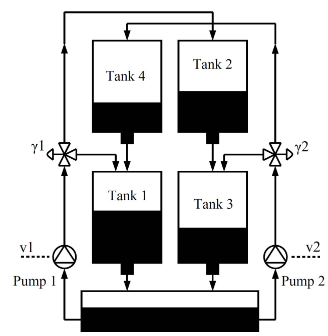

Fig. 5. Schematic representation of the four-tank system.

VI. Control of Dynamically Coupled Subsystems

Consider now the four-tank system (see Fig. 5) described in [9] and used as a benchmark to test several control algorithms, both centralized and distributed, e.g., in [1], [4], [8], and [13]. The goal is to control the water levels h_1 and h_3 of tanks 1 and 3 using the pump command voltages v_1 and v_2. The variables $x^{[i]}$, $i = 1, \dots, 4$ and $u^{[j]}$, $j = 1, 2$ are the variations of the levels h_i and of the voltages v_j with respect to the corresponding nominal working points. The obtained linearized and discretized system with sampling time $\tau = 0.5$ s has the form (5) with $n = 4$, $p = m = 2$ where

$$\mathbf{A} = \begin{bmatrix} 0.9921 & 0 & 0 & 0.0206 \\ 0 & 0.9835 & 0 & 0 \\ 0 & 0.0165 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9793 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 0.0417 & 2.47 \cdot 10^{-4} \\ 0.0156 & 0 \\ 1.30 \cdot 10^{-4} & 0.0311 \\ 0 & 0.0235 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The system is decomposed into two subsystems: the first one is composed by tanks 1 and 2, and the second one by tanks

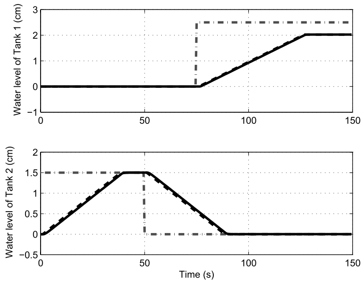

Fig. 6. Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) (black solid lines) and reference outputs $\tilde{y}^{[1]}$ (above) and $\tilde{y}^{[2]}$ (below) (black dashed lines). Gray dash-dotted lines: desired set-points $y^{[1,2]}_{\text{set-point}}$.

3 and 4. We set $x^{[1]} = (x_1, x_2)$, $u^{[1]} = u_1$, $y^{[1]} = y_1$, $x^{[2]} = (x_3, x_4)$, $u^{[2]} = u_2$, and $y^{[2]} = y_2$. The constraints on the system's variables are

$$\begin{aligned} x^{[1]}_{\min} &= \left[-12.4\ -1.4\right]^T, \quad x^{[1]}_{\max} = \left[27.6\ 38.6\right]^T \\ x^{[2]}_{\min} &= \left[-12.7\ -1.8\right]^T, \quad x^{[2]}_{\max} = \left[27.3\ 38.2\right]^T \\ u^{[1]}_{\min} &= -3, u^{[1]}_{\max} = 3, \qquad u^{[2]}_{\min} = -3, u^{[2]}_{\max} = 3. \end{aligned}$$

The matrices K_i and $\mathcal{K}_i$ fulfilling Assumptions 1 and 2 have been computed as described in Remark 1.

The weights used are $Q_1 = Q_2 = I_2$, $R_1 = R_2 = 1$, $T_1 = T_2 = 1$, $\gamma = 10^{-6}$, $N = 3$. In the simulations, the reference trajectories $y^{[i]}_{\text{set-point}}$, $i = 1, 2$ are piecewise constant, see Fig. 6. The results achieved are shown in Fig. 6. Notably, the set-point $y^{[1]}_{\text{set-point}} = 2.5$ results infeasible to our algorithm, and hence the system output $y^{[1]}_t$ converges to the nearest feasible value.

VII. Conclusion

In this brief paper, a novel distributed scheme for tracking reference signals is proposed. As in all existing noncooperative schemes, a degree of conservativity is brought about when addressing couplings among the subsystems and for obtaining simple problems at the reference generator level. On the other hand, the main advantages are: scalability of the online implementation, limited transmission and computational load (also in view of the fact that the reference generator layer is independent from the robust MPC layer, and hence computations can be performed in a parallelized fashion), and simplicity of implementation. The algorithm has also proven to be very flexible, i.e., in the unicycle robot application, where it has already been successfully used for the solution of leader-following and/or formation control problems. Further generalizations will require the development of a distributed real time optimization scheme for the generation of the ideal set-point output trajectories and the definition of coupling constraints guaranteeing convergence to the optimal feasible solution.

Appendix

A. *Proof of Recursive Feasibility of Problem* (28)–(31)

Assume that, at step t, a solution $\bar{y}^{[i]}_{t+N|t}$ to (29) exists for all $i = 1, \dots, M$ and that $k_{h,t+N-1} \le \tilde{l}_h$ for all $h = 1 \dots, n_c$. First note that, since $k_{h,t+N-1} \le \tilde{l}_h$, $\sum_{j\in\mathcal{S}_h} H^{[j]}_h \tilde{y}^{[j]}_{t+N-1} \le \tilde{l}_h$. Recall that, in view of the relationship between the matrices $H^{[i]}_h$, $H^{[i]z}_h$, and C_i used in (23), for all $i = 1, \dots, M$

$$H^{[i]}_h y^{[i]}_{t+N-1} = H^{[i]z}_h z^{[i]}_{t+N-1} \in H^{[i]z}_h \tilde{z}^{[i]}_{t+N-1} \oplus H^{[i]z}_h \mathscr{Z}_i.$$

Since, in view of (4), $H^{[i]z}_h \tilde{z}^{[i]}_{t+N-1} = H^{[i]}_h C_i \tilde{x}^{[i]}_{t+N-1}$, we obtain

$$H^{[i]}_h y^{[i]}_{t+N-1} \in H^{[i]}_h C_i \tilde{x}^{[i]}_{t+N-1} \oplus H^{[i]z}_h \mathscr{Z}_i.$$

Recall that $C_i \tilde{x}^{[i]}_{t+N-1} = [C_i\ 0]\chi^{[i]}_{t+N-1}$ and that, from (13), $[C_i\ 0]\chi^{[i]}_{t+N-1} \in [C_i\ 0](\chi^{[i]ss}_{t+N-1} \oplus \Delta^{\chi}_i)$. Since $[C_i\ 0]\chi^{[i]ss}_{t+N-1} = \tilde{y}^{[i]}_{t+N-1}$, it follows that

$$H^{[i]}_h y^{[i]}_{t+N-1} \in H^{[i]}_h \tilde{y}^{[i]}_{t+N-1} \oplus H^{[i]}_h \left[C_i\ 0\right] \Delta^{\chi}_i \oplus H^{[i]z}_h \mathscr{Z}_i.$$

From the definition of $\tilde{l}_h$, it is easy to see that the tightened constraint $k_{h,t+N-1} \le \tilde{l}_h$ implies (3). Furthermore, the fulfillment of (28) at time t implies that

$$\begin{aligned} k_{h,t+N} &= \sum_{i\in\mathcal{S}_h} H^{[i]}_h \tilde{y}^{[i]}_{t+N} \\ &\le -(n_h-1)k_{h,t+N-1} + n_h\tilde{l}_h - (n_h-1)(\tilde{l}_h - k_{h,t+N-1}) \\ &\le \tilde{l}_h \end{aligned} \tag{43}$$

which, in turn, implies that (3) will be verified also at time $t + N$. Finally, we prove that a solution to (29) exists at step $t + 1$ for all $i = 1, \dots, M$. In fact taking $\bar{y}^{[i]}_{t+N+1} = \bar{y}^{[i]}_{t+N|t} = \tilde{y}^{[i]}_{t+N}$ one has $\bar{y}^{[i]}_{t+N|t} - \tilde{y}^{[i]}_{t+N} = 0 \in \mathscr{B}^{(p_i)}_{q,\varepsilon}(0)$ and $\bar{y}^{[i]}_{t+N|t} \in \mathbb{Y}_i$, hence verifying (30) and (31), respectively. Furthermore

$$\begin{aligned} H^{[i]}_h \tilde{y}^{[i]}_{t+N+1} &= H^{[i]}_h \tilde{y}^{[i]}_{t+N} \\ &\le \tilde{l}_h - \sum_{j\in\mathcal{S}_h\setminus\{i\}} H^{[j]}_h \tilde{y}^{[j]}_{t+N} - \frac{(n_h-1)}{n_h}(\tilde{l}_h - k_{h,t+N}). \end{aligned}$$

In fact, $k_{h,t+N} \le \tilde{l}_h - \frac{(n_h-1)}{n_h}(\tilde{l}_h - k_{h,t+N})$ in view of the fact that $k_{h,t+N} \le \tilde{l}_h$, as it is proved in (43).

B. *Proof of Convergence for the Reference Management Layer*

In the absence of coupling constraints (3), since at time $t + 1$, $\bar{y}^{[i]}_{t+N+1} = \tilde{y}^{[i]}_{t+N}$ is a feasible solution, in view of the optimality of the solution $\bar{y}^{[i]}_{t+N+1|t+1}$

$$\begin{aligned} V^y_i(\bar{y}^{[i]}_{t+N+1|t+1}, t+1) &\le V^y_i(\bar{y}^{[i]}_{t+N|t}, t+1) \\ &\le \|\bar{y}^{[i]}_{t+N|t} - y^{[i]}_{\text{set-point}}\|^2_{T_i}. \end{aligned} \tag{44}$$

In view of the fact that $\bar{y}^{[i]}_{t+N+1|t+1} = \tilde{y}^{[i]}_{t+N+1}$ for all t, write $V^y_i(\bar{y}^{[i]}_{t+N+1|t+1}, t+1) = \gamma\,\|\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N}\|^2 + \|\tilde{y}^{[i]}_{t+N+1} - y^{[i]}_{\text{set-point}}\|^2_{T_i}$, and rewrite (44) as $\|\tilde{y}^{[i]}_{t+N+1} - y^{[i]}_{\text{set-point}}\|^2_{T_i} \le \|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{\text{set-point}}\|^2_{T_i} - \gamma\,\|\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N}\|^2$. From this we infer that, as $t \to \infty$, $\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N} \to 0$ and

$$\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{\text{set-point}}\|^2_{T_i} \to \text{const.} \tag{45}$$

Assume, by contradiction, that $\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{\text{set-point}}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c^o_i$, where

$$c^o_i = \|y^{[i]}_{\text{feas.set-point}} - y^{[i]}_{\text{set-point}}\|^2_{T_i}. \tag{46}$$

Note that, this implies that $\tilde{y}^{[i]}_{t+N} \neq y^{[i]}_{\text{feas.set-point}}$.

Assume that, given $\bar{t}$, for all $t \geq \bar{t}$ the optimal solution to (29) is $\bar{y}^{[i]}_{t+N|t} = \bar{y}^{[i]}$, where $\|\bar{y}^{[i]} - y^{[i]}_{\text{set-point}}\|^2_{T_i} = \bar{c}_i$. It results that $V^y_i(\bar{y}^{[i]}_{t+N|t}, t) = \bar{c}_i$. On the other hand, an alternative solution is given by $\bar{\bar{y}}^{[i]}_{t+N}$, where $\bar{\bar{y}}^{[i]}_{t+N} = \lambda_i \bar{y}^{[i]} + (1-\lambda_i) y^{[i]}_{\text{feas.set-point}}$, with $\lambda_i \in [0,1)$. This solution is feasible provided that: 1) $\bar{\bar{y}}^{[i]}_{t+N} - \bar{y}^{[i]} \in \mathscr{B}^{p_i}_{q,\varepsilon}(0)$, which can be verified if $(1-\lambda_i)$ is sufficiently small and 2) $\bar{\bar{y}}^{[i]}_{t+N} \in \mathbb{Y}_i$, which is also satisfied if $(1-\lambda_i)$ is sufficiently small (since $\mathbb{Y}_i$ is convex and $\bar{y}^{[i]} \neq y^{[i]}_{\text{feas.set-point}}$).

According to this alternative solution

$$V^y_i(\bar{\bar{y}}^{[i]}_{t+N}, t) = \gamma \|\bar{\bar{y}}^{[i]}_{t+N} - \bar{y}^{[i]}\|^2 + \|\bar{\bar{y}}^{[i]}_{t+N} - y^{[i]}_{\text{set-point}}\|^2_{T_i}.$$

Now, if (32) is verified, then $V^y_i(\bar{\bar{y}}^{[i]}_{t+N}, t) < V^y_i(\bar{y}^{[i]}, t)$. This contradicts the assumption that $\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{\text{set-point}}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c^o_i$. Therefore, the only asymptotic solution compatible with (29), is that corresponding with $\tilde{y}^{[i]}_{t+N} = y^{[i]}_{\text{feas.set-point}}$.

It is now proved that $\tilde{y}^{[i]}_t \to y^{[i]}_{\text{feas.set-point}}$ for $t \to \infty$. In view of Assumption 2, this implies that $\mathscr{C}_i \chi^{[i]}_t = C_i \tilde{x}^{[i]}_t \to y^{[i]}_{\text{feas.set-point}}$ for all $i = 1, \ldots, M$.

C. Proof of Recursive Feasibility of the i-DPC Problem

Assume that, at step t, a solution to (33) exists for all $i = 1, \ldots, M$, i.e., $(\hat{x}^{[i]}_{t|t}, \hat{u}^{[i]}_{[t:t+N-1]|t})$. Next, we prove that, at step $t+1$, a solution to (33) exists for all $i = 1, \ldots, M$. To do so, we prove that the tuple $(\hat{x}^{[i]}_{t+1|t}, \hat{u}^{[i]}_{[t+1:t+N]|t})$ satisfies the constraints (15), (35a)–(36) and is therefore a feasible (possibly suboptimal) solution to (33). Here, $\hat{u}^{[i]}_{[t+1:t+N]|t}$ is obtained with

$$\hat{u}^{[i]}_{t+N|t} = \tilde{u}^{[i]}_{t+N} + K_i(\hat{x}^{[i]}_{t+N|t} - \tilde{x}^{[i]}_{t+N}). \tag{47}$$

First, note that, in view of the robust positive invariance of sets $\mathscr{E}_i$ with respect to (18), $i = 1, \ldots, M$, $x^{[i]}_{t+1} - \hat{x}^{[i]}_{t+1|t} \in \mathscr{E}_i$, and therefore (35a) is verified. Furthermore, in view of the feasibility of (35b)–(35d) at step t, it follows that (35b)–(35d) are satisfied at step $t+1$ for $k = t+1, \ldots, t+N-1$ and, from (36) and (25a)

$$\begin{aligned}\hat{z}^{[i]}_{t+N|t} - \tilde{z}^{[i]}_{t+N} &= (C_{zi} + D_{zi}K_i)(\hat{x}^{[i]}_{t+N|t} - \tilde{x}^{[i]}_{t+N}) \\ &\in (C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta^z_i.\end{aligned}$$

Hence, constraint (35b) is verified for $k = t+N$. Furthermore

$$\begin{bmatrix}\hat{x}^{[i]}_{t+N|t}\\ \hat{u}^{[i]}_{t+N|t}\end{bmatrix} \in \begin{bmatrix}\tilde{x}^{[i]}_{t+N}\\ \tilde{u}^{[i]}_{t+N}\end{bmatrix} \oplus \begin{bmatrix}I_{n_i}\\ K_i\end{bmatrix}\Sigma_i$$

where from (13)

$$\begin{bmatrix}\tilde{x}^{[i]}_{t+N}\\ \tilde{u}^{[i]}_{t+N}\end{bmatrix} \in \begin{bmatrix}I_{n_i} & 0\\ K^x_i & K^e_i\end{bmatrix}\left(\chi^{[i]ss}_{t+N} \oplus \Delta^\chi_i\right). \tag{48}$$

In turn, in view of (9) and similarly to (10)

$$\chi^{[i]ss}_{t+N} \in \Gamma_i (I_{n+p} - \mathscr{F})^{-1}\mathscr{G}\prod_{i=1}^{M}\mathbb{Y}_j. \tag{49}$$

This eventually implies that, in view of (25b)

$$\begin{bmatrix}\hat{x}^{[i]}_{t+N|t}\\ \hat{u}^{[i]}_{t+N|t}\end{bmatrix} \in \begin{bmatrix}I_{n_i} & 0\\ K^x_i & K^e_i\end{bmatrix}\left(\Gamma_i(I_{n+p} - \mathscr{F})^{-1}\mathscr{G}\prod_{i=1}^{M}\mathbb{Y}_j \oplus \Delta^\chi_i\right) \oplus \begin{bmatrix}I_{n_i}\\ K_i\end{bmatrix}\Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i$$

which verifies constraints (35c) and (35d) for $k = t+N$. Note that, in view of (7a), (15), and (47)

$$\hat{x}^{[i]}_{t+N+1|t} - \tilde{x}^{[i]}_{t+N+1} = F_{ii}(\hat{x}^{[i]}_{t+N|t} - \tilde{x}^{[i]}_{t+N}) \tag{50}$$

and therefore $\hat{x}^{[i]}_{t+N+1|t} - \tilde{x}^{[i]}_{t+N+1} \in \Sigma_i$ in view of the definition of Σ_i as a positively invariant set for (24), hence verifying (36). Therefore, also the constraint (36) is verified at step $t+1$.

D. Proof of Convergence for the Robust MPC Layer

At time t the pair $(\hat{x}^{[i]}_{t|t}, \hat{u}^{[i]}_{[t:t+N-1]|t})$ is a solution to (33), leading to the optimal cost $V^{*N}_i(t)$. Since $(\hat{x}^{[i]}_{t+1|t}, \hat{u}^{[i]}_{[t+1:t+N]|t})$ is a feasible solution to (33) at time $t+1$, by optimality $V^{*N}_i(t+1) \leq V^N_i(\hat{x}^{[i]}_{t+1|t}, \hat{u}^{[i]}_{[t+1:t+N]|t})$ and, by applying standard arguments in MPC [17] $V^{*N}_i(t+1) \leq V^{*N}_i(t) - (\|\hat{x}^{[i]}_{t|t} - \tilde{x}^{[i]}_t\|^2_{Q_i} + \|\hat{u}^{[i]}_{t|t} - \tilde{u}^{[i]}_t\|^2_{R_i})$, so that, for all $i = 1, \ldots, M$, $\hat{x}^{[i]}_{t|t} \to \tilde{x}^{[i]}_t$ and $\hat{u}^{[i]}_{t|t} \to \tilde{u}^{[i]}_t$ as $t \to \infty$. Now, consider (5a) and, collectively, the model (7a) and equation (17). We have that

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{A}\,\mathbf{x}_t + \mathbf{B}\left(\hat{\mathbf{u}}_{t|t} + \mathbf{K}(\mathbf{x}_t - \hat{\mathbf{x}}_{t|t})\right)\\ \tilde{\mathbf{x}}_{t+1} &= \mathbf{A}\,\tilde{\mathbf{x}}_t + \mathbf{B}\tilde{\mathbf{u}}_t + \mathbf{B}\mathbf{K}(\tilde{\mathbf{x}}_t - \tilde{\mathbf{x}}_t)\end{aligned} \tag{51}$$

for all $t \geq 0$. Denote $\Delta\mathbf{x}_t = \mathbf{x}_t - \tilde{\mathbf{x}}_t$, $\Delta\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t} - \tilde{\mathbf{x}}_t$, and $\Delta\hat{\mathbf{u}}_t = \hat{\mathbf{u}}_{t|t} - \tilde{\mathbf{u}}_t$. From (51), $\Delta\mathbf{x}_{t+1} = \mathbf{F}\Delta\mathbf{x}_t + \mathbf{B}\left(\Delta\hat{\mathbf{u}}_t - \mathbf{K}\Delta\hat{\mathbf{x}}_t\right)$. Since $\mathbf{B}\left(\Delta\hat{\mathbf{u}}_t - \mathbf{K}\Delta\hat{\mathbf{x}}_t\right) \to 0$ as $t \to \infty$, in view of Assumption 1, it holds that $\Delta\mathbf{x}_t \to 0$ as $t \to \infty$, which implies that asymptotically $C_i x^{[i]}_t \to C_i \tilde{x}^{[i]}_t$.

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