

Uniqueness and Lipschitz stability for the identification of Lamé parameters from boundary measurements

Elena Beretta* Elisa Francini† Sergio Vessella‡

1 Introduction

A relevant inverse problem arising in nondestructive testing of materials is the one of determining, within an isotropic, linearly elastic three dimensional body Ω , the elastic properties of the body from measurements of traction and displacement taken on the exterior boundary of the domain Ω .

This leads mathematically to the formulation of the following boundary value problem for the system of linearized elasticity

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}u) &= 0 \text{ in } \Omega \subset \mathbb{R}^3, \\ u &= \psi \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is an open and bounded domain, $\widehat{\nabla}u$ denotes the strain tensor $\widehat{\nabla}u := \frac{1}{2}(\nabla u + (\nabla u)^T)$, $\psi \in H^{1/2}(\partial\Omega)$ is the boundary displacement field, and $\mathbb{C} \in L^\infty(\Omega)$ denotes the isotropic elasticity tensor with Lamé coefficients λ, μ :

$$\mathbb{C} = \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{Sym}, \text{ a.e. in } \Omega$$

where I_3 is 3×3 identity matrix and \mathbb{I}_{Sym} is the fourth order tensor such that $\mathbb{I}_{Sym}A = \hat{A}$, The strong convexity condition is assumed

$$\mu \geq \alpha_0 > 0, \quad 2\mu + 3\lambda \geq \beta_0 > 0 \text{ a.e. in } \Omega.$$

*Dipartimento di Matematica “G. Castelnuovo” Università di Roma “La Sapienza” (beretta@mat.uniroma1.it).

†Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze (francini@math.unifi.it)

‡Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze (sergio.vessella@dmd.unifi.it)

Under the above assumptions problem (1) has a unique weak solution $u \in H^1(\Omega)$ and the Dirichlet-to-Neumann linear map (DN map), $\Lambda_{\mathbb{C}}$, is well defined

$$\Lambda_{\mathbb{C}} : \psi \in H^{1/2}(\Omega) \rightarrow (\mathbb{C}\widehat{\nabla}u)\nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$$

where ν is the exterior unit normal to $\partial\Omega$.

The inverse problem consists in determining \mathbb{C} , i.e. λ and μ , from knowledge of the DN map $\Lambda_{\mathbb{C}}$.

This problem is closely related to the conductivity inverse problem arising in the modelling of EIT (Electrical Impedance Tomography). For the mathematical treatment of this problem we refer to the fundamental papers [A11],[AP], [Na] and [SU].

Unfortunately, the mathematical approach used to investigate the conductivity inverse problem fails partly when dealing in the elasticity framework. This is due to the fact that we have to deal with an elliptic system instead that with a scalar equation and we have to recover two parameters λ and μ instead of the sole conductivity parameter. As a consequence of these difficulties only partial results to this inverse problem are known and mainly concern the uniqueness issue. More precisely, the study of the problem was initiated in the 90's by Ikeata in [Ik] who considered a linearized version of it. In two dimensions Akamatsu, Nakamura and Steinberg in [ANS] and for dimension $n \geq 3$ Nakamura and Uhlmann in [NU] showed that one can determine uniquely and in a stable way $C^\infty(\overline{\Omega})$ Lamé parameters λ and μ and their derivatives on the boundary of a smooth domain Ω from the DN map. Local uniqueness has been proved in dimensions two by Nakamura and Uhlmann in [NU1] for $C^\infty(\overline{\Omega})$ Lamé parameters assuming that they are both close to positive constants. In three dimensions and higher Nakamura and Uhlmann in [NU2] and Eskin and Ralston in [ER] proved local uniqueness for smooth Lamé parameters whenever μ is close to a constant.

To our knowledge no result concerning stability estimates is known. Based on the results obtained by Alessandrini in [A11] who proved, *logarithmic* stability estimates for the conductivity inverse problem in the case of smooth conductivities and the example of Mandache in [M] who proved the optimality of this estimate also for the inverse elasticity problem in the case of smooth Lamé parameters *logarithmic* stability estimates or even worse ones are expected.

These considerations lead in recent years to look for different a priori assumptions on the unknown parameters which take into account the applied context from which the problem arises and give rise to better stability estimates ([V], [ABV], [BF], [BFV], [ABF],[BdHQ]). An attempt in this direction has been done by Alessandrini and Vessella in [AV] with for the conductivity inverse problem for unknown conductivities depending only on a finite number of parameters. In [AV] they proved *Lipschitz* continuous dependence from the DN map for conductivities that are constant on known subdomains, assuming ellipticity and $C^{1,\alpha}$ regularity at the interfaces joining contiguous domains.

In this paper we propose to consider $L^\infty(\Omega)$ elasticity tensors of the form

$$\mathbb{C}(x) = \sum_{j=1}^N (\lambda_j I_3 \otimes I_3 + 2\mu_j \mathbb{I}_{Sym}) \chi_{D_j}(x)$$

where the D_j 's, $j = 1, \dots, N$, are known disjoint Lipschitz domains representing a partition of Ω and $\lambda_j, \mu_j, j = 1, \dots, N$, are unknown constants.

We will prove that if \mathbb{C}^1 and \mathbb{C}^2 are of this form, assuming that the boundaries of D_j 's and of Ω contain flat portions, we have

$$\|\mathbb{C}^1 - \mathbb{C}^2\|_\infty \leq C \|\Lambda_{\mathbb{C}^1} - \Lambda_{\mathbb{C}^2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}.$$

where the stability constant C appearing in the estimate depends on various parameters like α_0, β_0 , on the regularity bounds on Ω and on the D_j 's and on their number N . In particular, the constructive character of the proof allows us to establish an estimate from above of the constant C .

We want now to emphasize that several significant examples fit in our analysis. Polyhedral partitions of Ω , appearing in any finite-element scheme used for effective reconstruction of the Lamé parameters (see for example [BJK]; and a layered configuration of the sets D_j 's arising in the study of composite laminates [Mi] and in geophysical prospection, [BC].

Our approach is based on the use of the following key ingredients: existence of singular solutions and study of their behaviour close to the flat discontinuity interfaces of the D_j 's, regularity estimates and quantitative estimates of unique continuation of solutions to system (1).

As already pointed out in [BF] a relevant difference with respect to the scalar case treated in [AV] is the issue of existence of singular solutions. In fact, in the case of strongly elliptic systems with L^∞ coefficients in dimension $n \geq 3$, existence of the fundamental solution and of the Green's function cannot be inferred without additional assumptions. In [HK] Hofmann and Kim prove their existence under the additional information that weak solutions of the system satisfy De Giorgi-Nash type local Hölder estimates. It is clear that in the case of a polyhedral partition of Ω solutions might not enjoy Hölder regularity at edges. On the other hand, in order to obtain our result, it is enough to construct singular solutions and analyze their behavior in a Lipschitz subset $\tilde{\mathcal{K}}$ at a given positive distance from edges. Nevertheless, while for the scalar (even for the complex valued treated in [BF]) conductivity equation is fairly easy to get a solution of the equation close to a flat interface by using the fundamental solution for the Laplace equation and suitable reflection arguments, this seems not to be possible for the Lamé system. In order to construct singular solutions, we make use of special fundamental solutions constructed by Rongved in [R] for isotropic biphasic laminates. Furthermore, looking at solutions of the elasticity system in $\tilde{\mathcal{K}}$ we can use the results of [CKVC] and of [LN] deriving regularity estimates for the solutions which allow us to obtain Hölder estimates of unique continuation in $\tilde{\mathcal{K}}$.

We would like to point out that our proof is based on the use of solutions having

boundary displacement fields supported in the flat portion Σ_1 of $\partial\Omega$. Hence, our stability result also holds replacing the full DN map with the *local* DN map that we will define in Section 2. Moreover, we expect to derive similar stability estimates also the case of domains D_j 's with $C^{1,\alpha}$ portions of interfaces and this analysis will be object of a forthcoming publication.

Finally, we would like to emphasize that *Lipschitz* stability estimates have become crucial also for the effective reconstruction of the unknown coefficients. In fact, recently, in [dHQS1] and [dHQS2], de Hoop, Qiu and Scherzer have shown that Lipschitz type stability estimates imply local convergence of iterative reconstruction algorithms and the radius of convergence of the iterates depends on the stability constant and hence an explicit determination of the dependence of such constant from the a-priori parameters, in particular from the partition number N , is crucial.

The plan of the paper is the following: section 2 contains the description of the main result. In paragraph 2.1 we introduce the notation and main definitions. In paragraph 2.2 we state the main a priori assumptions and the main result. We also reformulate the inverse elasticity problem in terms of the non-linear forward map F acting on a finite-dimensional subset of \mathbb{R}^{2N} and recall a general result (Proposition 2.5) that will let us show that F has a Lipschitz continuous inverse.

Section 3 contains some auxiliary results. In particular, in paragraph 3.2 we collect some properties concerning the fundamental solution in biphase elastic isotropic materials introduced by Rongved in [R] and we prove existence of singular solutions for our Lamé system. In paragraph 3.3 we state some regularity results and estimates of unique continuation concerning solutions of piecewise constant Lamé systems.

In section 4 we give the proof of our main result by verifying that the forward map F corresponding to our inverse problem, satisfies all the assumptions of Proposition 2.5 thus concluding the proof.

Finally, in the Appendix we recall some known results concerning solutions to Lamé systems with constant coefficients and the proof of the estimate of unique continuation stated in section 3.

Acknowledgements

We want to thank Antonino Morassi for pointing out to us the paper by Rongved on biphase fundamental solution and for stimulating discussions. This work has been supported by MIUR within the project PRIN 20089PWTPS003.

2 Main result

2.1 Notation and main definitions

For every $x \in \mathbb{R}^3$ we set $x = (x', x_3)$ where $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. For every $x \in \mathbb{R}^3$, r and L positive real numbers we will denote by $B_r(x)$, $B'_r(x')$ and

$Q_{r,L}(x)$ the open ball in \mathbb{R}^3 centered at x of radius r , the open ball in \mathbb{R}^2 centered at x' of radius r and the cylinder $B'_r(x') \times (x_3 - Lr, x_3 + Lr)$, respectively; in the sequel $B_r(0)$, $B'_r(0)$ and $Q_{r,L}(0)$ will be denoted by B_r , B'_r and $Q_{r,L}$, respectively. We will also denote by $\mathbb{R}_+^3 = \{(x', x_3) \in \mathbb{R}^3 : x_3 > 0\}$, $\mathbb{R}_-^3 = \{(x', x_3) \in \mathbb{R}^3 : x_3 < 0\}$, $B_r^+ = B_r \cap \mathbb{R}_+^3$, and $B_r^- = B_r \cap \mathbb{R}_-^3$.

For any subset D of \mathbb{R}^3 and any $h > 0$, we denote by

$$(D)_h = \{x \in D \mid \text{dist}(x, \mathbb{R}^3 \setminus D) > h\}.$$

Definition 2.1. Let Ω be a bounded domain in \mathbb{R}^3 . We shall say that a portion $\Sigma \subset \partial\Omega$ is of Lipschitz class with constants $r_0 > 0$, $L \geq 1$ if for any point $P \in \Sigma$, there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega \cap Q_{r_0,L} = \{(x', x_3) \in Q_{r_0,L} \mid x_3 > \psi(x')\}.$$

where ψ is a Lipschitz continuous function in B'_{r_0} such that

$$\psi(0) = 0 \text{ and } \|\psi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0.$$

We say that Ω is of Lipschitz class with constants r_0 and L if $\partial\Omega$ is of Lipschitz class with the same constants.

Remark 2.2. We use the convention of normalizing all norms in such a way that all their terms are dimensionally homogeneous. For example:

$$\|\psi\|_{C^{0,1}(B'_{r_0})} = \|\psi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla\psi\|_{L^\infty(B'_{r_0})}.$$

Similarly, denoting by $D^i u$ the vector which components are the derivatives of order i of the function u ,

$$\|u\|_{L^2(\Omega)} = \left(r_0^{-3} \int_{\Omega} u^2 \right)^{\frac{1}{2}}, \quad \|u\|_{C^k(\Omega)} = \sum_{i=0}^k r_0^i \|D^i u\|_{L^\infty(\Omega)},$$

$$\|u\|_{H^m(\Omega)} = r_0^{-3/2} \left(\sum_{i=0}^m r_0^{2i} \int_{\Omega} |D^i u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}$, $\|\cdot\|_{H^{-\frac{1}{2}}(\partial\Omega)}$, where Ω is a bounded subset of \mathbb{R}^3 whose boundary is smooth enough.

We will also make use of the following notations for matrices and tensors: for any 3×3 matrices A and B we set $A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}$ and $\hat{A} = \frac{1}{2}(A + A^T)$. By I_3 we denote the 3×3 identity matrix and by \mathbb{I}_{Sym} we denote the fourth order tensor such that $\mathbb{I}_{Sym} A = \hat{A}$.

In the whole paper we are going to consider isotropic elastic materials, hence the elasticity tensor \mathbb{C} is a fourth order tensor given by

$$\mathbb{C}(x) = \lambda(x)I_3 \otimes I_3 + 2\mu(x)\mathbb{I}_{Sym} \quad \text{for a.e. } x \text{ in } \Omega, \quad (2)$$

where Ω is a bounded domain in \mathbb{R}^3 of Lipschitz class, and the real valued functions $\lambda = \lambda(x)$ and $\mu = \mu(x) \in L^\infty(\Omega)$ are the Lamé moduli. We will also use Poisson's ratio $\nu(x) = \frac{\lambda(x)}{2(\lambda(x)+\mu(x))}$.

An elasticity tensor \mathbb{C} is *strongly convex* if there is a positive number ξ_0 such that, for almost every x in Ω ,

$$\mathbb{C}(x)A : A \geq \xi_0 |A|^2 \quad \text{for every } 3 \times 3 \text{ symmetric matrix } A. \quad (3)$$

In the isotropic case (2), the strong convexity condition takes the form

$$\mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geq \beta_0 \quad \text{for a.e. } x \text{ in } \Omega. \quad (4)$$

In these case, the Poisson's ratio has values in an compact subset of $(-1, \frac{1}{2})$. More precisely we can estimate

$$-1 + \frac{\beta_0 \alpha_0}{4} \leq \nu(x) \leq \frac{1}{2} - \frac{\alpha_0^2}{4} \quad \text{for a.e. } x \text{ in } \Omega. \quad (5)$$

In the sequel we will make use of the following norm in the linear space of isotropic tensors:

$$\|\mathbb{C}\|_\infty = \max \{ \|\lambda\|_{L^\infty(\Omega)}, \|\mu\|_{L^\infty(\Omega)} \}.$$

This norm is equivalent to the usual L^∞ norm for tensors in the space of isotropic tensors.

Our boundary measurements are represented by the Dirichlet to Neumann map. As a matter of fact, since we will restrict our measurements to boundary data that have support on some subset of the boundary, we will make use of a local Dirichlet to Neumann map.

Definition 2.3 (The Local Dirichlet to Neumann map). *Let Ω be a bounded domain of Lipschitz class and let Σ be an open portion of $\partial\Omega$. We denote by $H_{co}^{1/2}(\Sigma)$ the function space*

$$H_{co}^{1/2}(\Sigma) := \left\{ \phi \in H^{1/2}(\partial\Omega) : \text{supp } \phi \subset \Sigma \right\}$$

and by $H_{co}^{-1/2}(\Sigma)$ the topological dual of $H_{co}^{1/2}(\Sigma)$. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H_{co}^{1/2}(\Sigma)$ and $H_{co}^{-1/2}(\Sigma)$ based on the $L^2(\Sigma)$ scalar product. Then, given $\psi \in H_{co}^{1/2}(\Sigma)$, there exists a unique vector valued function $u \in H^1(\Omega)$ weak solution to the Dirichlet problem

$$\begin{cases} \text{div}(\mathbb{C}\widehat{\nabla}u) &= 0 \text{ in } \Omega, \\ u &= \psi \text{ on } \partial\Omega. \end{cases} \quad (6)$$

We define the local Dirichlet to Neumann linear map $\Lambda_{\mathbb{C}}^\Sigma$ as follows:

$$\Lambda_{\mathbb{C}}^\Sigma : \psi \in H_{co}^{1/2}(\Sigma) \rightarrow (\mathbb{C}\widehat{\nabla}u)n|_\Sigma \in H_{co}^{-1/2}(\Sigma).$$

Note that for $\Sigma = \partial\Omega$ we get the usual Dirichlet to Neumann map. For this reason we will set $\Lambda_{\mathbb{C}} := \Lambda_{\mathbb{C}}^{\partial\Omega}$.

The map $\Lambda_{\mathbb{C}}^{\Sigma}$ can be identified with the bilinear form on $H_{co}^{1/2}(\Sigma) \times H_{co}^{1/2}(\Sigma)$ by

$$\tilde{\Lambda}_{\mathbb{C}}^{\Sigma}(\psi, \phi) := \langle \Lambda_{\mathbb{C}}^{\Sigma} \psi, \phi \rangle = \int_{\Omega} \mathbb{C} \widehat{\nabla} u : \widehat{\nabla} v \quad (7)$$

for all $\psi, \phi \in H_{co}^{1/2}(\Sigma)$ and where u solves (6) and v is any $H^1(\Omega)$ function such that $v = \phi$ on $\partial\Omega$.

We shall denote by $\|\cdot\|_{\star}$ the norm in $\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))$ defined by

$$\|T\|_{\star} = \sup\{\langle T\psi, \phi \rangle \mid \psi, \phi \in H_{co}^{1/2}(\Sigma), \|\psi\|_{H_{co}^{1/2}(\Sigma)} = \|\phi\|_{H_{co}^{-1/2}(\Sigma)} = 1\}$$

for every $T \in \mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))$.

We finally recall an extension to systems of Alessandrini's identity [Al1], [Is]. Let u_1 and u_2 be the solutions to

$$\operatorname{div}(\mathbb{C}^k \widehat{\nabla} u_k) = 0 \text{ in } \Omega$$

for $k = 1, 2$ respectively and with \mathbb{C}^k , $k = 1, 2$, satisfying (3). Then

$$\int_{\Omega} (\mathbb{C}^1 - \mathbb{C}^2) \widehat{\nabla} u_1 : \widehat{\nabla} u_2 = \langle (\Lambda_{\mathbb{C}^1} - \Lambda_{\mathbb{C}^2}) u_2, u_1 \rangle \quad (8)$$

where $\Lambda_{\mathbb{C}^1}$, $\Lambda_{\mathbb{C}^2}$ denote the Dirichlet to Neumann map corresponding to \mathbb{C}^1 , \mathbb{C}^2 respectively.

2.2 Main assumptions and statement of the main result

Let $A, L, \alpha_0, \beta_0, N$ be given positive numbers such that $N \in \mathbb{N}$, $\alpha_0 \in (0, 1)$, $\beta_0 \in (0, 2)$ and $L \geq 1$. We shall refer to them as the a priori data. Let r_0 be a positive number.

Our main assumptions are:

(A1) $\Omega \subset \mathbb{R}^3$ is an open bounded domain such that

$$|\Omega| \leq A r_0^3,$$

and we assume that

$$\overline{\Omega} = \cup_{j=1}^N \overline{D}_j,$$

where D_j , $j = 1, \dots, N$ are connected and pairwise non overlapping open domains of Lipschitz class with constants r_0, L .

We also assume that there exists one region, say D_1 such that $\partial D_1 \cap \partial \Omega$ contains an open flat portion Σ and that for every $j \in \{2, \dots, N\}$ there exist $j_1, \dots, j_M \in \{1, \dots, N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j,$$

and, for every $k = 2, \dots, M$

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a flat portion Σ_k such that

$$\Sigma_k \subset \Omega, \quad \forall k = 2, \dots, M.$$

Furthermore, for $k = 1, \dots, M$, we assume there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates such that $P_k = 0$ and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3, L} &= \{x \in Q_{r_0/3, L} : x_3 = 0\}, \\ D_{j_k} \cap Q_{r_0/3, L} &= \{x \in Q_{r_0/3, L} : x_3 < 0\}, \\ D_{j_{k-1}} \cap Q_{r_0/3, L} &= \{x \in Q_{r_0/3, L} : x_3 > 0\}; \end{aligned}$$

where we set $\Sigma_1 := \Sigma$.

For simplicity we will call D_{j_1}, \dots, D_{j_M} a *chain of domains connecting D_1 to D_j* . For any $k \in \{1, \dots, M\}$ we will denote by n_k the exterior unit vector to ∂D_k in P_k .

(A2) We assume the tensor \mathbb{C} to be isotropic piecewise constant of the form

$$\mathbb{C} = \sum_{j=1}^N \mathbb{C}_j \chi_{D_j}(x) \quad (9)$$

where

$$\mathbb{C}_j = \lambda_j I_3 \otimes I_3 + 2\mu_j \mathbb{I}_{Sym}$$

with constant Lamé coefficients λ_i and μ_i that satisfy

$$\alpha_0 \leq \mu_j \leq \alpha_0^{-1}, \quad \lambda_j \leq \alpha_0^{-1}, \quad 2\mu_j + 3\lambda_j \geq \beta_0, \quad j = 1, \dots, N. \quad (10)$$

For $j = 1, \dots, N$, we denote the Poisson's ratio by $\nu_j = \frac{\lambda_j}{2(\lambda_j + \mu_j)}$. Note that each ν_j satisfies (5).

In the sequel we will introduce a number of constants that we will always denote by C . The values of these constants might differ from one line to the other.

Theorem 2.4. *Let Ω and Σ satisfy assumption (A1). Then there exists a positive constant C depending on $L, A, N, \alpha_0, \beta_0$ only such that, for any \mathbb{C}^k , $k = 1, 2$ satisfying assumption (A2), we have*

$$\|\mathbb{C}^1 - \mathbb{C}^2\|_\infty \leq C \|\Lambda_{\mathbb{C}^1}^\Sigma - \Lambda_{\mathbb{C}^2}^\Sigma\|_\star. \quad (11)$$

A better evaluation of constant C is given in Remark 4.7.

In order to prove Theorem 2.4 we will first state it in terms of the forward map that maps Lamé parameters to the corresponding Dirichlet to Neumann map. Then, we will apply to the forward map the following general result:

Proposition 2.5. *Let M_1 and M_2 be positive numbers and $d \in \mathbb{N}$. Let \mathcal{A} and \mathbf{K} be an open subset and a compact subset of \mathbb{R}^d respectively. Assume that $\mathbf{K} \subset \mathcal{A}$,*

$$\text{dist}(\mathbf{K}, \mathbb{R}^d \setminus \mathcal{A}) \geq M_1, \text{ and } \mathbf{K} \subset B_{M_2}(0).$$

Let \mathcal{B} be a Banach space and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be such that:

- (i) *F is Frechét differentiable;*
- (ii) *the Frechét derivative $F' : \mathcal{A} \rightarrow \mathcal{L}(\mathbb{R}^d, \mathcal{B})$ is uniformly continuous with a modulus of continuity $\sigma_1(\cdot)$;*
- (iii) *$F|_{\mathbf{K}}$ is injective;*
- (iv) *$(F|_{\mathbf{K}})^{-1} : F(\mathbf{K}) \rightarrow \mathbf{K}$ is uniformly continuous with a modulus of continuity $\sigma_2(\cdot)$;*
- (v) *F' is injective in \mathbf{K} , namely there is a positive number q_0 such that*

$$\min_{x \in \mathbf{K}, |h|=1} \|F'(x)[h]\|_{\mathcal{B}} \geq q_0;$$

then we have

$$\|x_1 - x_2\|_{\mathbb{R}^d} \leq C \|F(x_1) - F(x_2)\|_{\mathcal{B}} \quad \text{for every } x_1, x_2 \in \mathbf{K},$$

where $C = \max\{\frac{2M_1}{\sigma_2^{-1}(\delta_1)}, \frac{2}{q_0}\}$, for $\delta_1 = \frac{1}{2} \min\{\delta_0, M_2\}$ with $\delta_0 = \sigma_1^{-1}(\frac{q_0}{2})$.

This proposition holds also in infinite dimensional spaces. For a proof of Proposition 2.5 in finite dimensional space we refer to [BaV, Prop.5].

Let us now introduce the forward map corresponding to our problem. In order to represent the set of Lamé parameters, we will use the following notation: let $\underline{L} := (\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N)$ denote a vector in \mathbb{R}^{2N} and denote by \mathcal{A} the open subset of \mathbb{R}^{2N} defined by

$$\mathcal{A} := \{\underline{L} \in \mathbb{R}^{2N} : \mu_j > 0, 2\mu_j + 3\lambda_j > 0, j = 1, \dots, N\}. \quad (12)$$

For each vector $\underline{L} \in \mathcal{A}$ we can define a piecewise constant isotropic elastic tensor $\mathbb{C}_{\underline{L}}$ (as in (9)) with Lamé parameters λ_j and μ_j for $j = 1, \dots, N$.

In this case, $\|\mathbb{C}_{\underline{L}}\|_{\infty}$ is equal to the norm in \mathbb{R}^{2N} given by

$$\|\underline{L}\|_{\infty} = \max_{j=1, \dots, N} \{\max\{|\lambda_j|, |\mu_j|\}\}.$$

The (nonlinear) forward map can be defined as follows:

Definition 2.6. Let Ω and Σ satisfy assumption **(A1)**.
Let us define $F : \mathcal{A} \rightarrow \mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))$ by

$$F(\underline{L}) = \Lambda_{\mathbb{C}\underline{L}}^{\Sigma}.$$

We can identify F with $\tilde{F} : \mathcal{A} \rightarrow \mathcal{B}$ such that $\tilde{F}(\underline{L}) = \tilde{\Lambda}_{\mathbb{C}\underline{L}}^{\Sigma}$ (defined in (7)), where \mathcal{B} is the Banach space of bilinear form on $H_{co}^{1/2}(\Sigma) \times H_{co}^{1/2}(\Sigma)$ with the standard norm.

Let ψ and $\phi \in H_{co}^{1/2}(\Sigma)$ and let $u_{\underline{L}}$ be the solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}_{\underline{L}} \widehat{\nabla} u) = 0 & \text{in } \Omega \\ u = \psi & \text{on } \partial\Omega, \end{cases}$$

and $v_{\underline{L}}$ solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}_{\underline{L}} \widehat{\nabla} v) = 0 & \text{in } \Omega \\ v = \phi & \text{on } \partial\Omega, \end{cases}$$

then

$$\tilde{F}(\underline{L})(\psi, \phi) = \langle F(\underline{L})(\psi), \phi \rangle = \int_{\Omega} \mathbb{C}_{\underline{L}} \widehat{\nabla} u_{\underline{L}} : \widehat{\nabla} v_{\underline{L}}.$$

In the sequel, we will write F and $\Lambda_{\mathbb{C}\underline{L}}^{\Sigma}$ instead of \tilde{F} and $\tilde{\Lambda}_{\mathbb{C}\underline{L}}^{\Sigma}$.

With the above notation, Theorem 2.4, can be stated as follows:

Theorem 2.7. Let Ω and Σ satisfy assumption **(A1)** and let $\mathbf{K} \subset \mathcal{A}$ be the compact subset

$$\mathbf{K} := \{\underline{L} \in \mathcal{A} : \alpha_0 \leq \mu_j \leq \alpha_0^{-1}, \lambda_j \leq \alpha_0^{-1}, 2\mu_j + 3\lambda_j \geq \beta_0, j = 1, \dots, N\}. \quad (13)$$

Then, there exists a positive constant C , depending on $L, A, N, \alpha_0, \beta_0$ only such that

$$\|\underline{L}^1 - \underline{L}^2\|_{\infty} \leq C \|F(\underline{L}^1) - F(\underline{L}^2)\|_{\star}$$

for every $\underline{L}^1, \underline{L}^2$ in \mathbf{K} .

Notice that Theorem 2.7 means that F is invertible on \mathbf{K} and its inverse is Lipschitz continuous.

In Section 4 we will show that the forward map of definition 2.6 satisfies all the assumptions of Proposition 2.5 for \mathcal{A} and \mathbf{K} defined as in (12) and (13) respectively and \mathcal{B} is the space of bilinear form on $H_{co}^{1/2}(\Sigma) \times H_{co}^{1/2}(\Sigma)$. Then, Theorem 2.7 is a consequence of Proposition 2.5.

3 Preliminary results

3.1 Further notation and definitions

In order to prove the main theorem we need to introduce some further notation and definitions.

Construction of an augmented domain and extension of \mathbb{C} .

First we extend the domain Ω to a new domain Ω_0 such that $\partial\Omega_0$ is of Lipschitz class and $B_{r_0/C}(P_1) \cap \Sigma \subset \Omega_0$, for some suitable constant $C \geq 1$ depending only on L . We proceed as in [Al-Ro-R-Ve, Sect. 6]. Set

$$\rho_1 = r_0/C_L, \text{ where } C_L = \frac{3\sqrt{1+L^2}}{L}, \quad (14)$$

and define, for every $x' \in B'_{\frac{r_0}{3}}$

$$\psi^+(x') = \begin{cases} \frac{\rho_1}{2} & \text{for } |x'| \leq \frac{\rho_1}{4L}, \\ \rho_1 - 2L|x'| & \text{for } \frac{\rho_1}{4L} < |x'| \leq \frac{\rho_1}{2L}, \\ 0 & \text{for } |x'| > \frac{\rho_1}{2L}. \end{cases}$$

Observe that, for every $x' \in B'_{\frac{r_0}{3}}$, $|\psi^+(x')| \leq \frac{\rho_1}{2}$ and $|\nabla_{x'}\psi^+(x')| \leq 2L$. Next, we denote by

$$D_0 = \{x = (x', x_3) \in Q_{r_0/3, L} \mid 0 \leq x_3 < \psi^+(x')\}, \\ \Omega_0 = \Omega \cup D_0.$$

It is straightforward to verify that

- i) Ω_0 has Lipschitz boundary with constants $\frac{r_0}{3}, 3L$.
- ii)

$$\Omega_0 \supset Q_{r_0/4LC_L, L}.$$

Let \mathbb{C} be an isotropic tensor that satisfies assumption **(A2)**. We still denote by \mathbb{C} its extension to Ω_0 such that $\mathbb{C}|_{D_0}A = 2\hat{A}$ for every 3×3 matrix A . This extended tensor is still an isotropic elasticity tensor of the form

$$\mathbb{C} = \sum_{j=0}^N \mathbb{C}_j \chi_{D_j}(x) \quad (15)$$

where each \mathbb{C}_j for $j = 0, \dots, N$ has Lamé parameters satisfying (10).

Construction of a walkway

Let us fix $j \in \{1, \dots, N\}$ and let D_{j_1}, \dots, D_{j_M} be a chain of domains connecting D_1 to D_j . For the sake of brevity set $D_k = D_{j_k}$, $k = 1, \dots, M$.

By [Al-Ro-R-Ve, Prop. 5.5] there exists $C'_L \geq 1$ depending on L only, such that $(D_k)_h$ is connected for every $k \in \{1, \dots, M\}$ and every $h \in (0, r_0/C'_L)$. Denote by

$$h_0 = \min \left\{ \frac{r_0}{6}, \frac{r_0}{C'_L}, \frac{\rho_1}{8\sqrt{1+4L^2}} \right\}, \quad (16)$$

where ρ_1 is as in (14).

Let us introduce the following sets:

i) $Q_{(k)}$, $k = 1, \dots, M$, is the cylinder centered at P_k such that by a rigid transformation of coordinates under which $P_k = 0$ and Σ_k belongs to the plane $\{(x', 0)\}$, is given by $Q_{(k)} = Q_{\rho_1/4L, L}$. Moreover we denote $Q_{(M)}^- = Q_{(M)} \cap D_{M-1}$;

ii) \mathcal{K} is the interior part of the set $\bigcup_{i=0}^{M-1} \overline{D}_i$;

iii) $\mathcal{K}_h = \bigcup_{i=0}^{M-1} (D_i)_h$, for every $h \in (0, h_0)$;

iv)

$$\tilde{\mathcal{K}}_h = \mathcal{K}_h \cup Q_{(M)}^- \cup \bigcup_{k=1}^{M-1} Q_{(k)}; \quad (17)$$

v)

$$K_0 = \left\{ x \in D_0 \mid \text{dist}(x, \partial\Omega) > \frac{\rho_1}{8} \right\}.$$

It is straightforward to verify that \mathcal{K}_h is connected and of Lipschitz class for every $h \in (0, h_0)$ and that (in a suitable coordinate system)

$$K_0 \supset B'_{\rho_1/4L}(P_1) \times \left(\frac{\rho_1}{8}, \frac{\rho_1}{4} \right). \quad (18)$$

3.2 Existence of singular solutions

3.2.1 Fundamental solution in the biphasic laminate

In our proof of Lipschitz stability estimates, as in the approach used by Alessandrini and Vessella for the conductivity equation [AV], a crucial role is played by singular solutions for the Lamé system. As a matter of fact we are not only interested in the existence of such singular solutions, but also in their asymptotic behavior close to the interfaces.

In the scalar case, this tool is granted by the existence of Green functions and by explicit expressions for solutions in the presence of an interface.

Unfortunately, the existence of the Green matrix cannot be inferred for elliptic systems with bounded coefficients in dimension 3 or higher.

Moreover, whereas for the scalar (even complex valued) conductivity equation is fairly easy to get a solution of the equation close to a flat interface by using the fundamental solution for the Laplace equation and suitable reflection arguments, this seems not to be possible for the Lamé system.

In order to construct singular solutions, we make use of special fundamental solutions constructed by Rongved in [R] for isotropic biphasic laminates.

Consider the isotropic tensor

$$\mathbb{C}_b = \mathbb{C}^+ \chi_{\mathbb{R}_+^3} + \mathbb{C}^- \chi_{\mathbb{R}_-^3}$$

where \mathbb{C}^+ and \mathbb{C}^- are constant isotropic elastic tensors given by

$$\mathbb{C}^+ = \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{sym}, \quad \mathbb{C}^- = \lambda' I_3 \otimes I_3 + 2\mu' \mathbb{I}_{sym},$$

with λ and μ and λ' and μ' satisfy (10). Denote Poisson's parameters by ν and ν' .

In [R] an explicit formula for a fundamental solution $\Gamma : \{(x, y) : x \in \mathbb{R}^3, y \in \mathbb{R}^3, x \neq y\} \rightarrow \mathbb{R}^{3 \times 3}$ of

$$\operatorname{div} \left(\mathbb{C}_b \widehat{\nabla} \Gamma(\cdot, y) \right) = \delta_y I_3,$$

is given. Here δ_y is the Dirac distribution concentrated at y .

This explicit formula is quite involved and some alternative formulations in more convenient tensor form have been proposed, for example in [MeRe].

For the purpose of the present work we need to point out some properties of biphasic fundamental solution. First of all, it is a fundamental solution, in the sense that $\Gamma(x, y)$, is continuous in $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \neq y\}$ and $\Gamma(x, \cdot)$ is locally integrable in \mathbb{R}^3 , for all $x \in \mathbb{R}^3$, and, for every vector valued function $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \mathbb{C}_b \widehat{\nabla} \Gamma(\cdot, y) : \widehat{\nabla} \phi = \phi(y).$$

Furthermore, for every $x, y \in \mathbb{R}^3$, $x \neq y$, we have

$$\Gamma(x, y) = \Gamma(y, x)^T, \quad (19)$$

$$|\Gamma(x, y)| \leq \frac{C}{|x - y|},$$

and, for any $r > 0$,

$$\|\nabla \Gamma(\cdot, y)\|_{L^2(\mathbb{R}^3 \setminus B_r(y))} \leq \frac{C}{(rr_0^3)^{1/2}}, \quad (20)$$

where C depends on α_0, β_0 only.

We will also need to use the explicit representation of the some components of biphasic fundamental solution. In particular, we will make use of explicit expression of the third column of Γ . For $x = (x_1, x_2, x_3)$ with $x_3 > 0$ and $y = (0, 0, c)$ with $c > 0$, from [R] we have,

$$\Gamma(x, y) \cdot e_3 = \frac{3 - 4\nu}{4(1 - \nu)} \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} - \frac{1}{4(1 - \nu)} \left[x_3 \begin{pmatrix} B_{x_1} \\ B_{x_2} \\ B_{x_3} \end{pmatrix} + \begin{pmatrix} \beta_{x_1} \\ \beta_{x_2} \\ \beta_{x_3} \end{pmatrix} \right], \quad (21)$$

where

$$B = \frac{1}{4\pi\mu} \left\{ \frac{1}{R_1} + \alpha \left[\frac{3 - 4\nu}{R_2} + \frac{2c(x_3 + c)}{R_2^3} \right] \right\},$$

$$\beta = -\frac{1}{4\pi\mu} \left\{ \frac{c}{R_1} + \alpha \left[\frac{c(3 - 4\nu)}{R_2} - \gamma \log(R_2 + x_3 + c) \right] \right\},$$

with

$$\alpha = F_1(\mu, \mu', \nu) := \frac{\mu - \mu'}{\mu + (3 - 4\nu)\mu'}, \quad (22)$$

$$\gamma = F_2(\mu, \mu', \nu, \nu') := \frac{4(1-\nu)\mu \left[(1-2\nu)(3-4\nu') - 2\frac{\nu-\nu'}{\mu-\mu'}\mu' \right]}{\mu' + (3-4\nu')\nu}, \quad (23)$$

$$R_1 = (x_1^2 + x_2^2 + (x_3 - c)^2)^{1/2}, \text{ and } R_2 = (x_1^2 + x_2^2 + (x_3 + c)^2)^{1/2}.$$

It is worth noticing that, for $\mathbb{C}^+ = \mathbb{C}^-$, the matrix Γ coincides with the free space fundamental matrix for constant isotropic elasticity tensor.

3.2.2 Singular solutions

With the aid of the biphasic fundamental solution Γ , let us construct singular solutions with singularities in some subset of the domain. Since Γ is defined only in the case of a flat interface, we will need to keep away from interfaces that are not flat.

For this reason we need to set the following notation: let \mathcal{F} be the union of the flat parts of $\cup_{j=0}^N \partial D_j$. By flat parts we intend that they can be represented as the graphs of a constant function in at least a ball of radius $\frac{r_0}{3}$ (as Σ_k in assumption **(A1)**). Let $\mathcal{D} = \cup_{j=0}^N \partial D_j \setminus \mathcal{F}$. The set \mathcal{D} contains the non flat parts of the interfaces.

Let $\mathbb{C} = \sum_{j=0}^N \mathbb{C}_j \chi_{D_j}$ with tensors \mathbb{C}_j satisfying **(A2)** for all j . Let $y \in \Omega_0 \setminus \mathcal{D}$ and let $r = \min(r_0/4, \text{dist}(y, \mathcal{D} \cup \partial\Omega_0))$. Then, in the sphere $B_r(y)$ either \mathbb{C} is constant, $\mathbb{C} = \mathbb{C}_j$ or, by a suitable choice of the coordinate system, $\mathbb{C} = \mathbb{C}_j + (\mathbb{C}_{j+1} - \mathbb{C}_j) \chi_{\{x_3 > a\}}$ for some $j = 0, 1, \dots, N$ and some a with $|a| < r$. Let

$$\mathbb{C}_y = \begin{cases} \mathbb{C}_j & \text{if } \mathbb{C} = \mathbb{C}_j \text{ in } B_r(y), \\ \mathbb{C}_j + (\mathbb{C}_{j+1} - \mathbb{C}_j) \chi_{\{x_3 > a\}} & \text{otherwise,} \end{cases}$$

and consider the biphasic fundamental solution to

$$\text{div}(\mathbb{C}_y \widehat{\nabla} \Gamma(\cdot, y)) = \delta_y I_3 \text{ in } \mathbb{R}^3.$$

Proposition 3.1. *Let Ω_0 and \mathbb{C} satisfy assumptions **(A1)** and **(A2)**. Then, for $y \in \Omega_0 \setminus \mathcal{D}$, there exists a unique function $G(\cdot, y)$, continuous in $\Omega \setminus \{y\}$ such that*

$$\int_{\Omega_0} \mathbb{C} \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \phi = \phi(y), \text{ for every } \phi \in C_0^\infty(\Omega_0).$$

and such that

$$G(\cdot, y) = 0 \quad \text{on } \partial\Omega_0.$$

Furthermore, if $\text{dist}(y, \mathcal{D} \cup \partial\Omega_0) \geq r_0/c_1$ for some $c_1 > 1$,

$$\|G(\cdot, y) - \Gamma(\cdot, y)\|_{H^1(\Omega_0)} \leq C r_0^{-1} \quad (24)$$

$$\|G(\cdot, y)\|_{H^1(\Omega_0 \setminus B_r(y))} \leq C (r r_0)^{-1/2}, \quad (25)$$

where C depends α_0, β_0, A, L and, increasingly, on c_1 . Furthermore,

$$G(x, y) = G(y, x)^T \quad \text{for every } x, y \in \Omega_0 \setminus \mathcal{D}. \quad (26)$$

Proof. Let us set

$$G(x, y) := \Gamma(x, y) + w(x, y),$$

where w is solution to

$$\begin{cases} \operatorname{div} \left(\mathbb{C} \widehat{\nabla}_x w(\cdot, y) \right) &= \operatorname{div}((\mathbb{C} - \mathbb{C}_y) \widehat{\nabla} \Gamma(\cdot, y)) \quad \text{in } \Omega_0 \\ w(\cdot, y) &= -\Gamma(\cdot, y) \quad \text{on } \partial\Omega_0. \end{cases} \quad (27)$$

Since $\mathbb{C} - \mathbb{C}_y = 0$ in $B_r(y)$ and $\Gamma(\cdot, y)$ is smooth and bounded in $\Omega_0 \setminus B_r(y)$, then $f = (\mathbb{C} - \mathbb{C}_y) \widehat{\nabla} \Gamma(\cdot, y) \in L^2(\Omega_0)$, $\operatorname{div} f \in H^{-1}(\Omega_0)$ and $\Gamma(\cdot, y)|_{\partial\Omega_0} \in H^{1/2}(\partial\Omega_0)$, hence this problem has a unique solution $w \in H^1(\Omega_0)$.

Furthermore, if $\operatorname{dist}(y, \mathcal{D} \cup \partial\Omega_0) \geq r_0/c_1$ for some $c_1 > 1$,

$$\|w(\cdot, y)\|_{H^1(\Omega_0)} \leq C(\|f(\cdot, y)\|_{H^{-1}(\Omega_0)} + \|\Gamma(\cdot, y)\|_{H^{1/2}(\partial\Omega_0)}) \leq Cr_0^{-1}$$

where C depends on α_0, β_0, A and c_1 . This proves (24). Estimate (25) follows from (24) and from (20).

The symmetry of function G follows by standard arguments of potential theory [E]. \square

3.2.3 Two useful properties of the biphasic fundamental solution.

Let $\mathbb{C}_b, \overline{\mathbb{C}}_b$ be given by

$$\mathbb{C}_b = \mathbb{C}^+ \chi_{\mathbb{R}_+^3} + \mathbb{C}^- \chi_{\mathbb{R}_-^3}, \quad \overline{\mathbb{C}}_b = \mathbb{C}^+ \chi_{\mathbb{R}_+^3} + \overline{\mathbb{C}}^- \chi_{\mathbb{R}_-^3},$$

where $\mathbb{C}^+, \mathbb{C}^-$ and $\overline{\mathbb{C}}^-$ are constant and strongly convex isotropic tensors whose Lamé coefficients satisfy (4)

Let $\Gamma_{\mathbb{C}_b}$ and $\Gamma_{\overline{\mathbb{C}}_b}$ be the biphasic fundamental solutions relative to operators $\operatorname{div}(\mathbb{C}_b \widehat{\nabla} \cdot)$ and $\operatorname{div}(\overline{\mathbb{C}}_b \widehat{\nabla} \cdot)$, respectively.

Proposition 3.2. *For every $l, m \in \mathbb{R}^3$ and every $y, z \in \mathbb{R}_+^3, y \neq z$ we have*

$$\int_{\mathbb{R}_-^3} (\mathbb{C}_b - \overline{\mathbb{C}}_b) \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, y) l : \widehat{\nabla} \Gamma_{\overline{\mathbb{C}}_b}(\cdot, z) m = (\Gamma_{\mathbb{C}_b}(y, z) - \Gamma_{\overline{\mathbb{C}}_b}(y, z)) m \cdot l.$$

Proof. Let us fix $l, m \in \mathbb{R}^3$ and $y, z \in \mathbb{R}_+^3, y \neq z$ and denote by

$$u = \Gamma_{\mathbb{C}_b}(\cdot, y) l, \quad v = \Gamma_{\overline{\mathbb{C}}_b}(\cdot, z) m.$$

Let $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(y), B_{\varepsilon_0}(z) \subset \mathbb{R}_+^3$ and $B_{\varepsilon_0}(y) \cap B_{\varepsilon_0}(z) = \emptyset$. Since $\mathbb{C}_b = \overline{\mathbb{C}}_b$ in \mathbb{R}_+^3 , we get trivially

$$\int_{\mathbb{R}_+^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))} (\mathbb{C}_b - \overline{\mathbb{C}}_b) \widehat{\nabla} u : \widehat{\nabla} v = 0, \quad \text{for every } \varepsilon \in (0, \varepsilon_0).$$

Hence

$$\int_{\mathbb{R}_-^3} (\mathbb{C}_b - \overline{\mathbb{C}}_b) \widehat{\nabla} u : \widehat{\nabla} v = \int_{\mathbb{R}^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))} (\mathbb{C}_b - \overline{\mathbb{C}}_b) \widehat{\nabla} u : \widehat{\nabla} v, \quad \text{for every } \varepsilon \in (0, \varepsilon_0). \quad (28)$$

Integration by parts in $\mathbb{R}^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))$ yields, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathbb{R}^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))} \mathbb{C}_b \widehat{\nabla} u : \widehat{\nabla} v = - \int_{\partial B_\varepsilon(y)} (\mathbb{C}_b \widehat{\nabla} u) n \cdot v - \int_{\partial B_\varepsilon(z)} (\mathbb{C}_b \widehat{\nabla} u) n \cdot v, \quad (29)$$

where n is the outward unit normal to $\partial(B_\varepsilon(y) \cup B_\varepsilon(z))$. Now by definition of fundamental solution we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(y)} (\mathbb{C}_b \widehat{\nabla} u) n \cdot v = v(y) \cdot l \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(z)} (\mathbb{C}_b \widehat{\nabla} u) n \cdot v = 0. \quad (30)$$

By (29) and (30) we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))} \mathbb{C}_b \widehat{\nabla} u : \widehat{\nabla} v = -v(y) \cdot l. \quad (31)$$

In a similar way, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3 \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))} \overline{\mathbb{C}}_b \widehat{\nabla} u : \widehat{\nabla} v = -u(z) \cdot m. \quad (32)$$

By (28), (31), and (32) we have

$$\int_{\mathbb{R}_-^3} (\mathbb{C}_b - \overline{\mathbb{C}}_b) \widehat{\nabla} u : \widehat{\nabla} v = u(z) \cdot m - v(y) \cdot l = \Gamma_{\mathbb{C}}(z, y) l \cdot m - \Gamma_{\overline{\mathbb{C}}}(y, z) m \cdot l.$$

By (19) the thesis follows. \square

Proposition 3.3. *Let h, k be real numbers and let \mathbb{H} be the fourth order tensor*

$$\mathbb{H}(x) = (h I_3 \otimes I_3 + 2k \mathbb{I}_{sym}) \chi_{\mathbb{R}_-^3}(x).$$

For every $l, m \in \mathbb{R}^3$ and every $y, z \in \mathbb{R}_+^3$, $y \neq z$, we have

$$\int_{\mathbb{R}_-^3} \mathbb{H} \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, y) l : \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, z) m = \left(\frac{d}{dt} \Gamma_{\mathbb{C}_b + t\mathbb{H}}(y, z) m \cdot l \right)_{|_{t=0}}.$$

Proof. Let us fix $l, m \in \mathbb{R}^3$ and $y, z \in \mathbb{R}_+^3$, $y \neq z$. Let t_0 be a positive number such that for every $t \in (-t_0, t_0)$ the tensor $\mathbb{C}_b + t\mathbb{H}$ is strongly convex.

Since $\mathbb{H}(x) = 0$ for every $x \in \mathbb{R}_+^3$ we have trivially

$$\psi^{(l, m)}(y, z) := \int_{\mathbb{R}_-^3} \mathbb{H} \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, y) l : \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, z) m = \int_{\mathbb{R}_-^3} \mathbb{H} \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, y) l : \widehat{\nabla} \Gamma_{\mathbb{C}_b}(\cdot, z) m.$$

Hence, for every $t \in (-t_0, t_0) \setminus \{0\}$,

$$\begin{aligned} \psi^{(l,m)}(y, z) &= \frac{1}{t} \int_{\mathbb{R}^3} ((\mathbb{C}_b + t\mathbb{H}) - \mathbb{C}_b) \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, y) l : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, z) m = \\ &= \frac{1}{t} \int_{\mathbb{R}^3} ((\mathbb{C}_b + t\mathbb{H}) - \mathbb{C}_b) \widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(\cdot, y) l : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, z) m - \\ &\quad - \int_{\mathbb{R}_-^3} \mathbb{H} \left(\widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(\cdot, y) l - \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, y) l \right) : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, z) m. \end{aligned} \quad (33)$$

By (33) and by Proposition 3.2 we get

$$\begin{aligned} \psi^{(l,m)}(y, z) &= \frac{1}{t} ((\Gamma_{\mathbb{C}_b+t\mathbb{H}}(y, z)) - \Gamma_{\mathbb{C}_b}(y, z)) m \cdot l \\ &\quad - \int_{\mathbb{R}_-^3} \mathbb{H} \left(\widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(\cdot, y) l - \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, y) l \right) : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, z) m. \end{aligned} \quad (34)$$

Now, by straightforward calculation on the biphasic fundamental solution given in [R], we have, for every $x \in \mathbb{R}_-^3$,

$$\lim_{t \rightarrow 0} \left(\widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(x, y) l - \widehat{\nabla}\Gamma_{\mathbb{C}_b}(x, y) l \right) = 0, \quad (35)$$

and, for every $t \in (-t_0, t_0)$,

$$\left| \mathbb{H} \left(\widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(x, y) l - \widehat{\nabla}\Gamma_{\mathbb{C}_b}(x, y) l \right) : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(x, z) m \right| \leq C \min\{|x|^{-4}, 1\}, \quad (36)$$

where C depends on $\alpha_0, \beta_0, y, z, l$ and m only. By (35), (36), and applying the dominated convergence theorem, we get

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}_-^3} \mathbb{H} \left(\widehat{\nabla}\Gamma_{\mathbb{C}_b+t\mathbb{H}}(\cdot, y) l - \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, y) l \right) : \widehat{\nabla}\Gamma_{\mathbb{C}_b}(\cdot, z) m = 0. \quad (37)$$

Finally, by (37) and (34) the thesis follows. \square

3.3 Some estimates for solutions to the Lamé system

In this section we collect some properties concerning solutions to the linearized elasticity system with piecewise constant elasticity tensor that will be crucial to prove our main result.

First we state a regularity result for solutions of elliptic systems in composite materials from [LN] and [CKVC]. Afterwards we use this result in order to obtain Proposition 3.5, which is then used to prove a quantitative estimate of unique continuation for solutions of systems satisfying assumptions **(A1)** and **(A2)**.

Proposition 3.4. *Let \mathbb{C}^+ and \mathbb{C}^- be two isotropic elasticity tensors with constant Lamé coefficients λ, μ and λ', μ' respectively, satisfying assumption **(A2)**, let $R > 0$ and $\mathbb{C}_b = \mathbb{C}^+ \chi_{\mathbb{R}_+^3} + \mathbb{C}^- \chi_{\mathbb{R}_-^3}$. Let $v \in H^1(B_R)$ be a weak solution to*

$$\operatorname{div}(\mathbb{C}_b \widehat{\nabla} v) = 0 \text{ in } B_R.$$

Then, for every multiindex β' , $D_{x'}^{\beta'} v \in C^0(B_R)$ and $v \in C^\infty(\overline{B_R^\pm})$. Moreover for any $\delta > 0$ and $k \geq 0$

$$R^k \|D^k v\|_{L^\infty(\overline{B_{(1-\delta)R}^\pm})} \leq CR^{-3/2} \left[\int_{B_R} |v|^2 dx \right]^{1/2} \quad (38)$$

where $C = C(\delta, k, \alpha_0, \beta_0)$.

Note that, by $v \in C^\infty(\overline{B_R^\pm})$ we intend that $v|_{B_R^-}$ has a C^∞ extension to $\overline{B_R^-}$ and $v|_{B_R^+}$ has a C^∞ extension to $\overline{B_R^+}$.

Observe that, in particular, we have

$$\|v\|_{L^\infty(\overline{B_{(1-\delta)R})}} \leq CR^{-3/2} \left[\int_{B_R} |v|^2 dx \right]^{1/2}. \quad (39)$$

Beside this regularity result, the principal ingredients for proving estimates of unique continuation are the three sphere inequality, some stability estimates for the Cauchy problem and a smallness propagation estimate in a cone. All these results hold for constant elasticity tensors and are precisely described in the Appendix.

The flatness assumption on interfaces allows us to get a better estimate for the Lipschitz constant. The reason is the fact that solutions to the system with piecewise constant elasticity tensor have analytic extension beyond the flat interface. This property is stated in the following Proposition.

Here we use the notation of section 3.2.1 and assumptions **(A1)** and **(A2)**.

Proposition 3.5. *Let $v \in H_{loc}^1(\mathcal{K})$ be a solution to*

$$\operatorname{div}(\mathbb{C} \widehat{\nabla} v) = 0 \text{ in } \mathcal{K}. \quad (40)$$

Let us fix $k \in \{0, \dots, M\}$. Then there exist two positive constants C_1 and C , depending only on α_0, β_0 and L , such that $v|_{D_k}$ can be extended by a function \tilde{v} in the set $D_k \cup \Xi_{k+1}^{C_1}$, where

$$\Xi_{k+1}^{C_1} = \left\{ x \in D_k \cup D_{k+1} \cup \Sigma_k : \operatorname{dist}(x, B_{r_0/6}(P_k) \cap \Sigma_k) < \frac{r_0}{4C_1} \right\} \quad (41)$$

and

$$\|\tilde{v}\|_{L^\infty(\Xi_{k+1}^{C_1})} \leq C \|v\|_{L^2(\Xi_{k+1}^{C_1/2})}. \quad (42)$$

Proof of Proposition 3.5

It is not restrictive to assume that $P_k = 0$, Σ_k belongs to the plane $\{x_3 = 0\}$ and $B_{r_0/3}^+ \subset D_k$, $B_{r_0/3}^- \subset D_{k+1}$. By Proposition 3.4 we know that $v|_{D_k} \in C^\infty(D_k \cup \Sigma_k)$ and $v|_{D_{k+1}} \in C^\infty(D_{k+1} \cup \Sigma_k)$. Since \mathbb{C} is a constant tensor in each domain, for every $\beta' \in (\mathbb{N} \cup \{0\})^2$, $D_{x'}^{\beta'} v$ is a solution to (40).

Denote by $R = \frac{r_0}{6}$ and $v_+ := v|_{\overline{B_R^+}}$. For any x_0 in $\Sigma_k \cap B_R$, $D_{x'}^{\beta'} v \in C^0(B_R(x_0))$ and $v|_{\overline{B_R^+}(x_0)} \in C^\infty(\overline{B_R^+}(x_0))$.

Let us recall the following Caccioppoli inequality ([BBFM, pag.20]): let u be a solution to (40) then, for every $x_0 \in \Sigma_k \cap B_R$

$$\int_{B_{\rho_2}(x_0)} |\nabla u|^2 \leq \frac{C}{(\rho_2 - \rho_1)^2} \int_{B_{\rho_1}(x_0)} |u|^2 \quad (43)$$

for $0 < \rho_2 < \rho_1 < R$ where C depends on α_0 and β_0 only.

Denote by

$$\phi(x') = v_+(x', 0) \quad \text{and} \quad \psi(x') = \frac{\partial v_+}{\partial x_3}(x', 0). \quad (44)$$

We now prove that ϕ and ψ are analytic in $\Sigma_k \cap B_R$ and we estimate the derivatives $D_{x'}^{\beta'} \phi$ and $D_{x'}^{\beta'} \psi$ from above on $B_{R/4}(x_0)$ for every $\beta' \in (\mathbb{N} \cup \{0\})^2$ and for every $x_0 \in \Sigma_k \cap B_R$.

Starting from (43) and using the same iterative procedure followed to prove inequality (39) in [BF] we obtain

$$\sum_{|\beta|=N_0} \int_{B_{\frac{R}{2}}(x_0)} |D_{x'}^{\beta'} v|^2 \leq \left(C \left(\frac{2N_0}{R} \right)^2 \right)^{N_0} \int_{B_R(x_0)} |v|^2,$$

for every $N_0 \in \mathbb{N}$, where C depends on α_0 and β_0 only.

On the other side, by Proposition 3.4 we have

$$\|D_{x'}^{\beta'} v\|_{L^\infty(B_{\frac{R}{4}}(x_0))} \leq C \|D_{x'}^{\beta'} v\|_{L^2(B_{\frac{R}{2}}(x_0))}.$$

Hence, proceeding as in [BF, (41)] and applying Proposition 3.4 to $D_{x'}^{\beta'} v$, we get

$$\|D_{x'}^{\beta'} \phi\|_{L^\infty(B'_{\frac{R}{4}}(x_0))} \leq \beta'! \left(\frac{C}{R} \right)^{|\beta'|} \|v\|_{L^2(B_R(x_0))}, \quad (45)$$

and

$$\|D_{x'}^{\beta'} \psi\|_{L^\infty(B'_{\frac{R}{4}}(x_0))} \leq \beta'! \left(\frac{C}{R} \right)^{|\beta'|+1} \|v\|_{L^2(B_R(x_0))}. \quad (46)$$

where C depends on α_0 and β_0 only. By (45) and (46) and by the Cauchy-Kowalewski theorem we have that the solution \tilde{v} to the Cauchy problem

$$\begin{cases} \operatorname{div}(\mathbb{C} \widehat{\nabla} \tilde{v}) &= 0 \\ \tilde{v}(\cdot, 0) &= \phi \quad \text{on } \Sigma_k \cap B_R, \\ \frac{\partial \tilde{v}}{\partial x_3}(\cdot, 0) &= \psi \quad \text{on } \Sigma_k \cap B_R \end{cases}$$

is analytic in the neighborhood $\Xi_{k+1}^{C_1}$ of $\Sigma_k \cap B_R$. Therefore, taking into account (44), \tilde{v} is the analytic extension of v_+ in $\Xi_{k+1}^{C_1}$ and estimate (42) follows. \square

Finally we state a quantitative estimate of unique continuation.

Proposition 3.6. *Let ε_1, E_1 and h be positive numbers, $h < h_0$, where h_0 is defined in (16). Let $v \in H_{loc}^1(\mathcal{K})$ be a solution to*

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}v) = 0 \quad \text{in } \mathcal{K},$$

such that

$$\|v\|_{L^\infty(K_0)} \leq \varepsilon_1,$$

and

$$|v(x)| \leq E_1 \left(\frac{\operatorname{dist}(x, \Sigma_M)}{r_0} \right)^{-\frac{1}{2}} \quad \text{for every } x \in \mathcal{K}_{h/2}. \quad (47)$$

Then

$$|v(\tilde{x})| \leq C \left(\frac{r_0}{r} \right)^2 \varepsilon_1^{\overline{\theta}^m \tau_r} (E_1 + \varepsilon_1)^{1 - \overline{\theta}^m \tau_r},$$

where $r \in (0, \frac{r_0}{C})$, $\tilde{x} = P_M + rn_M$,

$$\tau_r = \tilde{\theta} \left(\frac{r}{r_0} \right)^\delta$$

and $\overline{m}, C, \delta, \tilde{\theta}, 0 < \tilde{\theta} < 1$, depend on A, L, α_0, β_0 and N .

The proof of the above Proposition is given in the Appendix.

4 Proof of the main result

This section contains the proof of the main result that consists in showing that the forward map introduced in definition 2.6 satisfies all the assumptions of Proposition 2.5.

4.1 Differentiability of F

Proposition 4.1. *The map*

$$F : \mathcal{A} \rightarrow \mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))$$

defined in (2.6) is Frechét differentiable in \mathcal{A} and

$$\langle F'(\underline{L})[\underline{H}]\psi, \phi \rangle = \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\mathbb{C}} : \widehat{\nabla}v_{\mathbb{C}} \quad (48)$$

where $\mathbb{H} = \mathbb{C}_{\underline{H}}$.

Moreover, $F' : \mathcal{A} \rightarrow \mathcal{L}(\mathbb{R}^{2N}, \mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma)))$ is Lipschitz continuous with Lipschitz constant $C_{F'}$ depending on A, L, α_0, β_0 only.

Proof. Fix $\underline{L} \in \mathcal{A}$ and let $\underline{H} \in \mathbb{R}^{2N}$ such that $\|\underline{H}\|_\infty$ is sufficiently small. By (8) we have

$$\langle (F(\underline{L} + \underline{H}) - F(\underline{L}))\psi, \phi \rangle = \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}+\underline{H}} : \widehat{\nabla}v_{\underline{L}}.$$

Hence, by setting,

$$\eta := \langle (F(\underline{L} + \underline{H}) - F(\underline{L}))\psi, \phi \rangle - \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}} : \widehat{\nabla}v_{\underline{L}} = \int_{\Omega} \mathbb{H}\widehat{\nabla}(u_{\underline{L}+\underline{H}} - u_{\underline{L}}) : \widehat{\nabla}v_{\underline{L}},$$

we have

$$|\eta| \leq Cr_0^2 \|\underline{H}\|_\infty \|\nabla(u_{\underline{L}+\underline{H}} - u_{\underline{L}})\|_{L^2(\Omega)} \|\phi\|_{H_{co}^{1/2}(\Sigma)}, \quad (49)$$

where C depends on A, L, α_0, β_0 only. Let us estimate $\|\nabla(u_{\underline{L}+\underline{H}} - u_{\underline{L}})\|_{L^2(\Omega)}$. For, observe that $w := u_{\underline{L}+\underline{H}} - u_{\underline{L}}$ is solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}_{\underline{L}}\widehat{\nabla}w) &= \operatorname{div}(\mathbb{H}\widehat{\nabla}u_{\underline{L}+\underline{H}}) \text{ in } \Omega, \\ w &= 0 \text{ on } \partial\Omega, \end{cases} \quad (50)$$

hence

$$\int_{\Omega} \mathbb{C}_{\underline{L}}\widehat{\nabla}w : \widehat{\nabla}w = \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}+\underline{H}} : \widehat{\nabla}w.$$

By Lax-Milgram Theorem and Korn inequality we get

$$\|\nabla w\|_{L^2(\Omega)} \leq C \|\underline{H}\|_\infty \|\nabla u_{\underline{L}+\underline{H}}\|_{L^2(\Omega)} \leq Cr_0^{-1} \|\underline{H}\|_\infty \|\psi\|_{H_{co}^{1/2}(\Sigma)}, \quad (51)$$

where C depends on A, L, α_0, β_0 only.

By inserting (51) into (49) we get

$$|\eta| \leq Cr_0 \|\underline{H}\|_\infty^2 \|\psi\|_{H_{co}^{1/2}(\Sigma)} \|\phi\|_{H_{co}^{1/2}(\Sigma)}, \quad (52)$$

where C on depends on A, L, α_0, β_0 only, that yields (48).

Let us now prove the Lipschitz continuity of F' . Let $\underline{L}^1, \underline{L}^2 \in \mathcal{A}$ and set

$$\begin{aligned} \omega &:= \langle (F'(\underline{L}^2) - F'(\underline{L}^1))[\underline{H}]\psi, \phi \rangle = \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}^2} : \widehat{\nabla}v_{\underline{L}^2} - \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}^1} : \widehat{\nabla}v_{\underline{L}^1} \\ &= \int_{\Omega} \mathbb{H}(\widehat{\nabla}u_{\underline{L}^2} - \widehat{\nabla}u_{\underline{L}^1}) : \widehat{\nabla}v_{\underline{L}^2} + \int_{\Omega} \mathbb{H}\widehat{\nabla}u_{\underline{L}^1} : (\widehat{\nabla}v_{\underline{L}^2} - \widehat{\nabla}v_{\underline{L}^1}). \end{aligned}$$

By reasoning as we did to derive (52) we obtain

$$|\omega| \leq C_{F'} r_0 \|\underline{H}\|_\infty \|\underline{L}^2 - \underline{L}^1\|_\infty \|\psi\|_{H_{co}^{1/2}(\Sigma)} \|\phi\|_{H_{co}^{1/2}(\Sigma)},$$

where $C_{F'}$ depends only on A, L, α_0 , and β_0 . \square

4.2 Injectivity of $F|_{\mathbf{K}}$ and uniform continuity of $(F|_{\mathbf{K}})^{-1}$

In the present subsection we will prove Theorem 4.2 whose statement is given below.

Let

$$\sigma(t) = \begin{cases} |\log t|^{-\frac{1}{\delta}} & \text{for } 0 < t < \frac{1}{e}, \\ t - \frac{1}{e} + 1 & \text{for } t \geq \frac{1}{e}, \end{cases} \quad (53)$$

where $\delta \in (0, 1)$ is as in Proposition 3.6. The function σ is strictly increasing, concave, and $\lim_{t \rightarrow 0} \sigma(t) = 0$. We have

Theorem 4.2. *For every $\underline{L}^1, \underline{L}^2 \in \mathbf{K}$ the following inequality holds true*

$$\|\underline{L}^1 - \underline{L}^2\|_{\infty} \leq C_* \sigma^N(\|F(\underline{L}^1) - F(\underline{L}^2)\|_*) \quad (54)$$

where $\sigma^N(\cdot)$ is the composition of the function $\sigma(\cdot)$ defined in (53) with itself N times and C_* is a constant depending on $A, L, \alpha_0, \beta_0, N$ only.

Remark 4.3. Observe that Theorem 4.2 provides the injectivity of $F|_{\mathbf{K}}$ and an estimate of the modulus of continuity of $(F|_{\mathbf{K}})^{-1}$.

In order to prove Theorem 4.2 we need to prove first some preliminary results. Let $j \in \{1, \dots, N\}$ be such that

$$\|\mathbb{C}_{\underline{L}^1} - \mathbb{C}_{\underline{L}^2}\|_{L^\infty(D_j)} = \|\mathbb{C}_{\underline{L}^1} - \mathbb{C}_{\underline{L}^2}\|_{L^\infty(\Omega_0)}$$

and let D_{j_1}, \dots, D_{j_M} be a chain of domains connecting D_1 to D_j . For the sake of brevity set $D_k = D_{j_k}$. Consider \mathcal{K}, K_0 and \mathcal{K}_h defined as in Section 3. Let $\mathcal{W}_k = \text{Int}(\cup_{j=0}^k \overline{D}_j)$, $\mathcal{U}_k = \Omega_0 \setminus \mathcal{W}_k$, for $k = 1, \dots, M-1$. The tensors $\mathbb{C}_{\underline{L}^1}$ and $\mathbb{C}_{\underline{L}^2}$ are extended as in (15) in all of Ω_0 . To simplify the notation we will set $\mathbb{C} := \mathbb{C}_{\underline{L}^1}$ and $\bar{\mathbb{C}} := \mathbb{C}_{\underline{L}^2}$. Finally let $\tilde{\mathcal{K}}_k = \mathcal{K}_h \cap \mathcal{W}_k$ and for $y, z \in \tilde{\mathcal{K}}_k$ define the matrix-valued function

$$\mathcal{S}_k(y, z) := \int_{\mathcal{U}_k} (\mathbb{C} - \bar{\mathbb{C}})(\cdot) \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z),$$

whose entries are given by

$$\mathcal{S}_k^{(p,q)}(y, z) := \int_{\mathcal{U}_k} (\mathbb{C} - \bar{\mathbb{C}})(\cdot) \widehat{\nabla} G^{(p)}(\cdot, y) : \widehat{\nabla} \bar{G}^{(q)}(\cdot, z) \quad p, q = 1, 2, 3$$

and where $G^{(p)}(\cdot, y)$ and $\bar{G}^{(q)}(\cdot, z)$ denote respectively the p -th column and the q -th column of the singular solutions of Proposition 3.1 corresponding to the tensors \mathbb{C} and $\bar{\mathbb{C}}$ respectively. From (25) we have that

$$|\mathcal{S}_k^{(p,q)}(y, z)| \leq C(d(y)d(z))^{-1/2}, \quad y, z \in \tilde{\mathcal{K}}_k,$$

where the constant C depends on the a priori parameters only and $d(y) = d(y, \mathcal{U}_k)$, $d(z) = d(z, \mathcal{U}_k)$. For any fixed $q = 1, 2, 3$ let us denote by $\mathcal{S}_k^{(\cdot, q)}(\cdot, z)$ the vector valued function whose elements are $\mathcal{S}_k^{(p,q)}(y, z)$, $p = 1, 2, 3$; analogously we define $\mathcal{S}_k^{(p, \cdot)}(y, \cdot)$, for any fixed $p = 1, 2, 3$.

First we prove

Proposition 4.4. For all $y, z \in \tilde{\mathcal{K}}_k$ we have that $\mathcal{S}_k^{(\cdot, q)}(\cdot, z)$ and $\mathcal{S}_k^{(p, \cdot)}(y, \cdot)$ belong to $H_{loc}^1(\tilde{\mathcal{K}}_k)$ and for any fixed $q \in \{1, 2, 3\}$

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\mathcal{S}_k^{(\cdot, q)}(\cdot, z)) = 0 \quad \text{in } \tilde{\mathcal{K}}_k, \quad (55)$$

and for any fixed $p \in \{1, 2, 3\}$

$$\operatorname{div}(\bar{\mathbb{C}}\widehat{\nabla}\mathcal{S}_k^{(p, \cdot)}(\cdot, z)) = 0 \quad \text{in } \tilde{\mathcal{K}}_k. \quad (56)$$

Proof. For seek of simplicity, in the proof we will omit the index k . Let us fix $q \in \{1, 2, 3\}$ and let us first show that the vector valued function $\mathcal{S}^{(\cdot, q)}(\cdot, z) \in H_{loc}^1(\tilde{\mathcal{K}})$ for fixed $z \in \tilde{\mathcal{K}}$. Let $\phi \in C_0^\infty(B_r(y_0))$ where $y_0 \in \tilde{\mathcal{K}}$ and $B_r(y_0) \subset \tilde{\mathcal{K}}$. Consider, for fixed $p, q \in \{1, 2, 3\}$

$$\begin{aligned} \int_{\tilde{\mathcal{K}}} \mathcal{S}^{(p, q)}(y, z) \partial_j \phi(y) dy &= \\ &= \int_{\tilde{\mathcal{K}}} \left[\int_{\mathcal{U}} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x G^{(p)}(x, y) : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) \partial_j \phi(y) dx \right] dy. \end{aligned}$$

Observe now that by (25) and by the fact that $B_r(y_0) \subset \tilde{\mathcal{K}}$ we have

$$\int_{B_r(y_0)} \int_{\mathcal{U}} |\nabla_x G^{(p)}(x, y)|^2 dx dy < +\infty, \quad \int_{B_r(y_0)} \int_{\mathcal{U}} |\nabla_x \bar{G}^{(q)}(x, z)|^2 dx dy < +\infty.$$

Hence, by Schwarz inequality, we have that, for fixed $z \in \tilde{\mathcal{K}}$,

$$(\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x G^{(p)}(x, y) : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) \partial_j \phi(y) \in L^1(\mathcal{U} \times B_r(y_0)),$$

so that we can interchange the order of integration and get

$$\begin{aligned} \int_{\tilde{\mathcal{K}}} \mathcal{S}^{(p, q)}(y, z) \partial_j \phi(y) dy &= \\ &= \int_{\mathcal{U}} \left[(\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x \int_{B_r(y_0)} G^{(p)}(x, y) \partial_j \phi(y) dy : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) \right] dx \end{aligned}$$

and using the symmetry of G almost everywhere in \mathcal{U} , (26), we get

$$\begin{aligned} \int_{B_r(y_0)} G^{(p)}(x, y) \partial_j \phi(y) dy &= \int_{B_r(y_0)} G^{(p)}(y, x) \partial_j \phi(y) dy \\ &= - \int_{B_r(y_0)} \partial_{y_j} G^{(p)}(y, x) \phi(y) dy, \end{aligned}$$

so that

$$\begin{aligned} \int_{\tilde{\mathcal{K}}} \mathcal{S}^{(p, q)}(y, z) \partial_j \phi(y) dy &= \\ &= - \int_{\mathcal{U}} \left[(\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x \int_{B_r(y_0)} \partial_{y_j} G^{(p)}(y, x) \phi(y) dy : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) \right] dx. \end{aligned}$$

Now recalling that $G(y, x) = w(y, x) + \Gamma(y, x)$, by the properties of Γ and by the boundary value problem satisfied by $\partial_{x_j} w(y, x)$ for any $j = 1, 2, 3$ it is straightforward to see that $\nabla_y \partial_{x_j} w(y, x) \in L^2(\mathcal{U} \times B_r(y_0))$ and hence

$$\partial_{y_j} \mathcal{S}^{(p,q)}(y, z) = \int_{\mathcal{U}} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x \partial_{y_j} G^{(p)}(y, x) : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) dx.$$

Now, arguing as in the first part of the proof and considering now a vector-valued test function $\Phi \in C_0^\infty(B_r(y_0))$ and by (26) we have

$$\begin{aligned} & \int_{B_r(y_0)} \mathbb{C}(y) \widehat{\nabla}_y \mathcal{S}^{(\cdot,q)}(y, z) : \widehat{\nabla}_y \Phi(y) dy = \\ & = \int_{\mathcal{U}} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla}_x \left[\int_{B_r(y_0)} \mathbb{C}(y) \widehat{\nabla}_y G(y, x) : \widehat{\nabla}_y \Phi(y) dy \right] : \widehat{\nabla}_x \bar{G}^{(q)}(x, z) dx \end{aligned}$$

and since

$$\int_{B_r(y_0)} \mathbb{C}(y) \widehat{\nabla}_y G^{(p)}(y, x) : \widehat{\nabla}_y \Phi(y) dy = 0, \text{ a.e. in } \mathcal{U}$$

for all $p = 1, 2, 3$ we finally have

$$\int_{B_r(y_0)} \mathbb{C}(y) \widehat{\nabla}_y \mathcal{S}^{(\cdot,q)}(y, z) : \widehat{\nabla}_y \Phi(y) dy = 0,$$

for all $\Phi \in C_0^\infty(B_r(y_0))$ and since y_0 is arbitrary (55) follows. Analogously we get (56). \square

Proposition 4.5. *If for a positive ε_0 and for some $k \in \{1, \dots, M-1\}$*

$$|\mathcal{S}_k(y, z)| \leq \varepsilon_0 r_0^{-1} \text{ for every } (y, z) \in K_0 \times K_0 \quad (57)$$

then

$$|\mathcal{S}_k(y_r, z_{\bar{r}})| \leq C r_0^{-1} \left(\frac{r_0}{r}\right)^{5/2} \left(\frac{r_0}{\bar{r}}\right)^2 \left(\frac{\varepsilon_0}{C_1 + \varepsilon_0}\right)^{(\bar{\theta}^{\bar{n}k})^2 \tau_r \tau_{\bar{r}}} \quad (58)$$

where $y_r = P_{k+1} + r n_{k+1}$, $z_{\bar{r}} = P_{k+1} + \bar{r} n_{k+1}$, $P_{k+1} \in \Sigma_{k+1}$, $\bar{r}, r \in (0, r_0/C)$, $\tau_r = \bar{\theta} \left(\frac{r}{r_0}\right)^\delta$, $\tau_{\bar{r}} = \bar{\theta} \left(\frac{\bar{r}}{r_0}\right)^\delta$ and $\bar{n}, C, C_1, \delta, \bar{\theta} \in (0, 1)$ depend on A, L, α_0, β_0 only.

Proof. Fix $z \in K_0$ and consider the function $v(y) := \mathcal{S}_k^{(\cdot,q)}(y, z)$, for fixed q . By Proposition 4.4 we know that v is solution to

$$\operatorname{div}(\mathbb{C} \widehat{\nabla} v(\cdot)) = 0 \quad \text{in } \tilde{K}_k.$$

Moreover, from Proposition 3.1, we get

$$|v(y)| \leq C_1 r_0^{-1}, \quad y \in \tilde{K}_k,$$

where C_1 depends on A, L, α_0, β_0 only and from (57) we have

$$|v(y)| \leq r_0^{-1} \varepsilon_0, \quad y \in K_0.$$

Then, applying Proposition 3.6 for $\varepsilon_1 = r_0^{-1} \varepsilon_0$ and $E_1 = C_1 r_0^{-1}$, we have

$$|v(y_r)| = |\mathcal{S}_k^{(\cdot, q)}(y_r, z)| \leq C r_0^{-1} \left(\frac{r_0}{r}\right)^2 \left(\frac{\varepsilon_0}{C_1 + \varepsilon_0}\right)^{\tilde{\theta}^{\tilde{n}_k} \tau_r}.$$

Now let us consider, for fixed p ,

$$\bar{v}(z) := \mathcal{S}_k^{(p, \cdot)}(y_r, z)$$

which is solution to

$$\operatorname{div}(\mathbb{C} \widehat{\nabla} \bar{v}(\cdot)) = 0 \quad \text{in } \tilde{\mathcal{K}}_k$$

and which satisfies

$$|\bar{v}(z)| \leq r_0^{-1} C \left(\frac{r_0}{r}\right)^2 \left(\frac{\varepsilon_0}{C_1 + \varepsilon_0}\right)^{\tilde{\theta}^{\tilde{n}_k} \tau_r}, \quad z \in K_0.$$

By Proposition 3.1 we have

$$|\bar{v}(z)| \leq C r_0^{-1} \left(\frac{r}{r_0}\right)^{-1/2} \left(\frac{d(z)}{r_0}\right)^{-1/2}, \quad z \in \tilde{\mathcal{K}}_k,$$

where C depends on A, L, α_0, β_0 only. Hence, applying again Proposition 3.6 to $\bar{v}(z)$, we get

$$|\bar{v}(z_{\bar{r}})| \leq C r_0^{-1} \left(\frac{r_0}{r}\right)^{5/2} \left(\frac{r_0}{\bar{r}}\right)^2 \left(\frac{\varepsilon_0}{C_1 + \varepsilon_0}\right)^{(\tilde{\theta}^{\tilde{n}_k})^2 \tau_r \tau_{\bar{r}}}$$

which proves (58). \square

Proof of Theorem 4.2

Observe first that $\|F(\underline{\mathbf{L}}^1) - F(\underline{\mathbf{L}}^2)\|_* = \|\Lambda_{\mathbb{C}} - \Lambda_{\bar{\mathbb{C}}}\|_*$. Denote by

$$\varepsilon := \|F(\underline{\mathbf{L}}^1) - F(\underline{\mathbf{L}}^2)\|_*.$$

Then from identity (8), we derive that for every $y, z \in K_0$ and for $|l| = |m| = 1$.

$$\left| \int_{\Omega_0} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla} G(x, y) l : \widehat{\nabla} \bar{G}(x, z) m dx \right| \leq C r_0^{-1} \varepsilon, \quad (59)$$

where C depends on α_0, β_0, A, L only.

Let

$$\delta_k := \max_{0 \leq j \leq k} \{|\lambda_j - \bar{\lambda}_j|, |\mu_j - \bar{\mu}_j|\},$$

where $k \in \{0, 1, \dots, M\}$. Recalling that by construction

$$\mathbb{C}|_{D_0} = \bar{\mathbb{C}}|_{D_0}$$

we have that $\delta_0 = 0$. In order to obtain (54) we use a recursive argument. More precisely, we prove that for a suitable increasing sequence $\{\omega_k(\varepsilon)\}_{0 \leq k \leq M}$ satisfying $\varepsilon \leq \omega_k(\varepsilon)$ for every $k = 0, \dots, M$ we have

$$\delta_k \leq \omega_k(\varepsilon) \implies \delta_{k+1} \leq \omega_{k+1}(\varepsilon), \text{ for every } k = 0, \dots, M-1.$$

Without loss of generality we can choose $\omega_0(\varepsilon) = \varepsilon$. Suppose now that for some $k \in \{1, \dots, M-1\}$ we have

$$\delta_k \leq \omega_k(\varepsilon). \quad (60)$$

Consider

$$\mathcal{S}_k(y, z) := \int_{\mathcal{U}_k} (\mathbb{C} - \bar{\mathbb{C}})(\cdot) \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z)$$

and fix $z \in K_0$. From Proposition 3.1 and from (59) we get, for $y, z \in K_0$

$$|\mathcal{S}_k(y, z)| \leq \frac{C}{r_0} (\varepsilon + \omega_k(\varepsilon)),$$

where C depends on A, L, α_0, β_0 only. By (58) and choosing $\bar{r} = cr$ with $c \in [1/4, 1/2]$ we easily get that there are constants $C_0, \delta \in (0, 1)$ and θ_* depending only on A, L, α_0, β_0 and, increasingly, on M , such that for fixed $l, m \in \mathbb{R}^3$ such that $|l| = |m| = 1$,

$$|\mathcal{S}_k(y_r, z_{\bar{r}}) m \cdot l| \leq Cr_0^{-1} \left(\frac{r_0}{r}\right)^{9/2} \varsigma \left(\omega_k(\varepsilon), \frac{r}{r_0}\right), \quad (61)$$

where

$$\varsigma(t, s) = \left(\frac{t}{1+t}\right)^{\theta_* s^{2\delta}} \quad (62)$$

Let us choose $l = m = e_3$ and split

$$\mathcal{S}_k(y_r, z_{\bar{r}}) e_3 \cdot e_3 = I_1 + I_2, \quad (63)$$

where

$$I_1 = \int_{B_{r_1} \cap D_{k+1}} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla} G(x, y_r) e_3 : \widehat{\nabla} \bar{G}(x, z_{\bar{r}}) e_3 dx \quad (64)$$

and

$$I_2 = \int_{\mathcal{U}_{k+1} \setminus (B_{r_1} \cap D_{k+1})} (\mathbb{C} - \bar{\mathbb{C}})(x) \widehat{\nabla} G(x, y_r) e_3 : \widehat{\nabla} \bar{G}(x, z_{\bar{r}}) e_3 dx$$

and where $r_1 = \frac{r_0}{4LC_L}$ for C_L as in (14). Then, from Proposition 3.1, we derive immediately that

$$|I_2| \leq \frac{C}{r_0}. \quad (65)$$

By (60) we have that

$$|\bar{\lambda}_k - \lambda_k| \leq \omega_k(\varepsilon), \quad |\bar{\mu}_k - \mu_k| \leq \omega_k(\varepsilon). \quad (66)$$

and hence, using (10),

$$|\bar{\nu}_k - \nu_k| \leq C\omega_k(\varepsilon), \quad (67)$$

where C depends on α_0 only. Estimates (66) and (67) together with (24) and (25) give

$$|I_1| \geq \left| \int_{B_{r_1} \cap D_{k+1}} (\mathbb{C}_b^{k+1} - \bar{\mathbb{C}}_b^{k+1})(\cdot) \widehat{\nabla} \Gamma_{k+1}(\cdot, y_r) e_3 : \widehat{\nabla} \bar{\Gamma}_{k+1}(\cdot, z_{\bar{r}}) e_3 \right| - \frac{C}{\sqrt{r_0 r}} - \frac{C\omega_k(\varepsilon)}{r}, \quad (68)$$

where Γ_{k+1} and $\bar{\Gamma}_{k+1}$ are the biphasse fundamental solutions introduced in Section 3.2 where the elastic phase correspond to elastic tensors \mathbb{C}_b^{k+1} and $\bar{\mathbb{C}}_b^{k+1}$ given by

$$\begin{aligned} \mathbb{C}_b^{k+1} &= \mathbb{C}_k \chi_{\mathbb{R}_+^3} + \mathbb{C}_{k+1} \chi_{\mathbb{R}_-^3} \\ \bar{\mathbb{C}}_b^{k+1} &= \mathbb{C}_k \chi_{\mathbb{R}_+^3} + \bar{\mathbb{C}}_{k+1} \chi_{\mathbb{R}_-^3} \end{aligned}$$

up to a rigid transformation that maps Σ_{k+1} into $x_3 = 0$. Furthermore by (61), (63) and (65) we obtain

$$|I_1| \leq Cr_0^{-1} \left(\left(\frac{r_0}{r} \right)^{9/2} \varsigma \left(\omega_k(\varepsilon), \frac{r}{r_0} \right) + 1 \right), \quad (69)$$

where C depends on A, L, α_0, β_0 only. Hence, by (69) and (68) and by performing the change of variables $x = rx'$ in the integral at the right hand side of (64), we have

$$\left| \int_{B_{\rho_0/r}^-} (\mathbb{C}_b^{k+1} - \bar{\mathbb{C}}_b^{k+1})(x') \widehat{\nabla} \Gamma_{k+1}(x', e_3) e_3 : \widehat{\nabla} \bar{\Gamma}_{k+1}(x', ce_3) e_3 dx' \right| \leq \delta_0 \left(\frac{r}{r_0} \right), \quad (70)$$

where

$$\delta_0 \left(\frac{r}{r_0} \right) = C \left[\left(\frac{r_0}{r} \right)^{7/2} \varsigma \left(\omega_k(\varepsilon), \frac{r}{r_0} \right) + \sqrt{\frac{r}{r_0}} \right].$$

Since we have

$$\left| \int_{\mathbb{R}_-^3 \setminus B_{\rho_0/r}^-} (\mathbb{C}_b^{k+1} - \bar{\mathbb{C}}_b^{k+1})(x') \widehat{\nabla} \Gamma_{k+1}(x', e_3) e_3 : \widehat{\nabla} \bar{\Gamma}_{k+1}(x', ce_3) e_3 dx' \right| \leq C \frac{r}{r_0},$$

where C depends on A, L, α_0, β_0 only, by (70) and by applying Proposition 3.2 we obtain

$$\left| (\Gamma_{k+1}(e_3, ce_3) - \bar{\Gamma}_{k+1}(e_3, ce_3)) e_3 \cdot e_3 \right| \leq \delta_0 \left(\frac{r}{r_0} \right) + C \frac{r}{r_0}. \quad (71)$$

For seek of simplicity in what follows we will omit the indices k and $k + 1$ and write $\mu = \mu_k$, $\mu' = \mu_{k+1}$ and, in a similar way, define λ , λ' , ν , ν' . We will bar corresponding Lamé coefficients for $\bar{\mathbb{C}}$.

Using the explicit form (21) of the biphasse fundamental solution we have

$$\begin{aligned} \Gamma_{k+1}(e_3, ce_3)e_3 \cdot e_3 - \bar{\Gamma}_{k+1}(e_3, ce_3)e_3 \cdot e_3 &= \frac{1}{4\pi(1-c)} \left(\frac{1}{\mu} - \frac{1}{\bar{\mu}} \right) + \\ &+ \frac{1}{16\pi(1+c)} \left[\frac{\alpha[(3-4\nu)^2 - \gamma + 3 - 4\nu]}{\mu(1-\nu)} - \frac{\bar{\alpha}[(3-4\bar{\nu})^2 - \bar{\gamma} + 3 - 4\bar{\nu}]}{\bar{\mu}(1-\bar{\nu})} \right] + \\ &+ \frac{c}{4\pi(c+1)^3} \left[\frac{\alpha}{\mu(1-\nu)} - \frac{\bar{\alpha}}{\bar{\mu}(1-\bar{\nu})} \right], \end{aligned}$$

where

$$\begin{aligned} \alpha &= F_1(\mu, \mu', \nu), & \bar{\alpha} &= F_1(\bar{\mu}, \bar{\mu}', \bar{\nu}), \\ \gamma &= F_2(\mu, \mu', \nu, \nu'), & \bar{\gamma} &= F_2(\bar{\mu}, \bar{\mu}', \bar{\nu}, \bar{\nu}') \end{aligned}$$

and F_1 and F_2 have been defined in (22) and (23).

From (71), (66) and (67) we obtain, by elementary calculation, for every $c \in [1/4, 1/2]$

$$|p(c)| \leq C \left(\delta_0 \left(\frac{r}{r_0} \right) + \frac{r}{r_0} + \omega_k(\varepsilon) \right) \quad (72)$$

where

$$p(c) := [4(\alpha - \bar{\alpha})(3 - 4\nu)(1 - \nu) - (\alpha\gamma - \bar{\alpha}\bar{\gamma})](1 + c)^2 + 4c(\alpha - \bar{\alpha})$$

and C depends on α_0 only. Now, if $\omega_k(\varepsilon) < 1/e$ then we choose $r = r_\varepsilon$ where

$$r_\varepsilon = \frac{r_0}{C} |\log \omega_k(\varepsilon)|^{-\frac{1}{4\delta}},$$

where C depends on α_0, β_0, A, L and $\delta \in (0, 1)$ is as in Proposition 4.5 and by (72) we get

$$|p(c)| \leq C' |\log \omega_k(\varepsilon)|^{-\frac{1}{8\delta}}, \quad (73)$$

for every $c \in [1/4, 1/2]$, where C depends on α_0, β_0, A, L and $\delta \in (0, 1)$.

Otherwise, if $\omega_k(\varepsilon) \geq 1/e$, since $p(c)$ is bounded, we can trivially write

$$|p(c)| \leq C e \omega_k(\varepsilon). \quad (74)$$

Estimates (73) and (74) yields

$$|p(c)| \leq C \sigma(\omega_k(\varepsilon)), \quad (75)$$

where C depends on α_0, β_0, A, L only.

From (75) we easily get

$$|\alpha - \bar{\alpha}| \leq C \sigma(\omega_k(\varepsilon)) \quad , \quad |\alpha\gamma - \bar{\alpha}\bar{\gamma}| \leq C \sigma(\omega_k(\varepsilon)). \quad (76)$$

where C depends on A, L, α_0 and β_0 .

Now, by simple but tedious calculations, from (76) and (5) we derive

$$|\nu' - \bar{\nu}'| \leq C\sigma(\omega_k(\varepsilon)), |\lambda' - \bar{\lambda}'| \leq C\sigma(\omega_k(\varepsilon)), |\mu' - \bar{\mu}'| \leq C\sigma(\omega_k\varepsilon)$$

Hence we have

$$\delta_{k+1} \leq \omega_{k+1}(\varepsilon) := C\sigma(\omega_k(\varepsilon)).$$

Finally, by iteration and recalling that $\omega_0(\varepsilon) = \varepsilon$ we get (54). \square

4.3 Injectivity of $F'(\underline{L})$ and estimate from below of $F'_{|\mathbf{K}}$

Proposition 4.6. *Let $F : \mathcal{A} \rightarrow \mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))$ be the map introduced in definition 2.6. Let us define*

$$q_0 := \min \{ \|F'(\underline{L})[\underline{H}]\|_* \mid \underline{L} \in \mathbf{K}, \underline{H} \in \mathbb{R}^{2N}, \|\underline{H}\|_\infty = 1 \}$$

we have

$$(\sigma^N)^{-1} (1/C_\star) \leq q_0, \quad (77)$$

where $\sigma(\cdot)$ is defined by (53) and $C_\star, C_\star > 1$, depends on A, L, α_0, β_0 and N only.

Proof. By the definition of q_0 we have that there exist $\underline{L}_0 \in \mathbf{K}$ and $\underline{H}_0 = (h_{0,1}, \dots, h_{0,N}, k_{0,1}, \dots, k_{0,N})$, $\|\underline{H}_0\|_\infty = 1$, such that

$$\|F'(\underline{L}_0)[\underline{H}_0]\|_* = q_0.$$

Therefore, by (48) and Proposition 4.1 we have

$$\left| \int_\Omega \mathbb{H}_0 \widehat{\nabla} G(\cdot, y) l : \widehat{\nabla} G(\cdot, z) m \right| \leq \frac{C}{r_0} q_0 \quad \text{for every } y, z \in \mathcal{K}_0, \quad (78)$$

for every l, m unit vectors of \mathbb{R}^3 , where C depends on α_0, β_0, L and A only, $\mathbb{H}_0 = \mathbb{C}_{\underline{H}_0}$, $G(\cdot, y)$ denotes the singular solution defined in Section 3.2.2.

From now on vector $(0, h_{0,1}, \dots, h_{0,N}, 0, k_{0,1}, \dots, k_{0,N})$ will still be denoted by \underline{H}_0 .

Let us fix $j \in \{1, \dots, N\}$ and let D_{j_1}, \dots, D_{j_M} be a chain of domains connecting D_1 to D_j , where

$$\max\{|h_{0,j}|, |k_{0,j}|\} = \|\underline{H}_0\|_\infty = 1$$

For the sake of brevity set $D_i = D_{j_i}$, $i = 1, \dots, M$ and order domains and entries in \underline{H}_0 accordingly. For every $i \in \{0, \dots, M-1\}$ denote by $\Gamma_{i+1}(\cdot, \cdot)$ the biphas fundamental solution introduced in Section 3.2 where the elastic phases correspond to the elastic tensor given by

$$\mathbb{C}_b^{(i+1)}(x) = \mathbb{C}_i \chi_{\mathbb{R}_+^3} + \mathbb{C}_{i+1} \chi_{\mathbb{R}_-^3}(x).$$

Here $\lambda_0 = 0$ and $\mu_0 = 1$. Now, for any $i \in \{0, \dots, M-1\}$ let us denote by

$$\eta_i := \max_{0 \leq p \leq i} \{|h_{0,p}|, |k_{0,p}|\}.$$

In order to obtain (77), we use a recursive argument. More precisely, we prove that for a suitable increasing sequence $\{\omega_i(q_0)\}_{0 \leq i \leq M}$ satisfying $q_0 \leq \omega_i(q_0)$ for every $i = 0, \dots, M$ we have

$$\eta_i \leq \omega_i(q_0) \implies \eta_{i+1} \leq \omega_{i+1}(q_0), \text{ for every } i = 0, \dots, M-1,$$

from which, taking into account that we can choose $\omega_0(q_0) = q_0$, we will obtain (77).

For any $i \in \{0, \dots, M-1\}$, let $\mathcal{W}_i = \text{Int}(\cup_{j=0}^i \overline{D}_j)$, $\mathcal{U}_i = \Omega \setminus \mathcal{W}_i$, $\tilde{\mathcal{K}} = \mathcal{K}_h \cap \mathcal{W}_i$ and, for $y, z \in \tilde{\mathcal{K}}$, let $\mathcal{T}_i(y, z) = \left\{ \mathcal{T}_i^{(p,q)}(y, z) \right\}_{1 \leq p, q \leq 3}$ be the matrix valued function whose elements are given by

$$\mathcal{T}_i^{(p,q)}(y, z) := \int_{\mathcal{U}_i} \mathbb{H}_0(\cdot) \widehat{\nabla} G^{(p)}(\cdot, y) : \widehat{\nabla} G^{(q)}(\cdot, z), \quad p, q = 1, 2, 3.$$

Moreover, for any fixed $q = 1, 2, 3$, let us denote by $\mathcal{T}_i^{(\cdot, q)}(\cdot, z)$ the vector valued function whose elements are $\mathcal{T}_i^{(p,q)}(\cdot, z)$, $p = 1, 2, 3$; analogously we define $\mathcal{T}_i^{(p, \cdot)}(y, \cdot)$, for any fixed $p = 1, 2, 3$.

Let us fix l, m unit vectors of \mathbb{R}^3 . By (78) we have, for every $y, z \in K_0$,

$$\begin{aligned} |\mathcal{T}_i(y, z) l \cdot m| &\leq \left| \int_{\Omega} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y) l : \widehat{\nabla} G(\cdot, z) m \right| \\ &+ \left| \int_{\Omega \cap \mathcal{W}_i} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y) l : \widehat{\nabla} G(\cdot, z) m \right| \leq \frac{C}{r_0} q_0 + \frac{C}{r_0} \eta_i \\ &\leq \frac{C}{r_0} (q_0 + \omega_i(q_0)). \end{aligned}$$

where C depends on α_0, β_0, L and A only.

Arguing similarly to the proof of Proposition 4.5, we have that there exist C_0, C_1 such that for every $r \in (0, r_0/C_0)$ the following inequality holds true (recall $r_1 = \frac{r_0}{4L}$ where ρ_1 is defined in (14))

$$|\mathcal{T}_i(y_r, z_{\bar{r}})| \leq C r_0^{-1} \left(\frac{r_0}{r} \right)^{9/2} \varsigma(\omega_i(q_0), r/r_0), \quad (79)$$

where ς is defined as in (62).

Now we have trivially

$$\begin{aligned} \int_{D_{i+1} \cap B_{r_1}} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y_r) l : \widehat{\nabla} G(\cdot, z_{\bar{r}}) m &= \mathcal{T}_i(y_r, z_{\bar{r}}) l \cdot m \\ &- \int_{\mathcal{U}_i \setminus B_{r_1}} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y_r) l : \widehat{\nabla} G(\cdot, z_{\bar{r}}) m. \end{aligned} \quad (80)$$

On the other side we have

$$\left| \int_{\mathcal{U}_i \setminus B_{r_1}} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y_r) l : \widehat{\nabla} G(\cdot, z_{\bar{r}}) m \right| \leq \frac{C}{r_0},$$

where C depend on A, L, α_0, β_0 and M only.

By the above inequality, (80) and (79) we have

$$\left| \int_{D_{i+1} \cap B_{r_1}} \mathbb{H}_0 \widehat{\nabla} G(\cdot, y_r) l : \widehat{\nabla} G(\cdot, z_{\bar{r}}) m \right| \leq \frac{C}{r_0} \left(1 + \left(\frac{r_0}{r} \right)^{9/2} \varsigma(\omega_i(q_0), r/r_0) \right)$$

where C depend on A, L, α_0, β_0 and M only.

Denote by \mathbb{H} the tensor given by

$$\widetilde{\mathbb{H}}(x) = (h_{0,i+1} I_3 \otimes I_3 + 2k_{0,i+1} \mathbb{I}_{sym}) \chi_{\mathbb{R}_-^3}(x).$$

From Proposition 3.1 we have, for every $c \in (1/4, 1/2)$ and choosing $l = m = e_3$,

$$\begin{aligned} \left| \int_{B_{r_1}^-} \widetilde{\mathbb{H}} \widehat{\nabla} \Gamma_{i+1}(x, re_3) e_3 : \widehat{\nabla} \Gamma_{i+1}(x, cre_3) e_3 dx \right| &\leq \\ &\leq \frac{C}{r_0} \left(\left(\frac{r_0}{r} \right)^{1/2} + \left(\frac{r_0}{r} \right)^{9/2} \varsigma(\omega_i(q_0), r/r_0) \right) \end{aligned} \quad (81)$$

where C depend on A, L, α_0, β_0 and M only.

Now, performing the change of variables $x = r\xi$ in the integral on the right hand side of (81) we obtain

$$\begin{aligned} \left| \int_{B_{r_1/r}^-} \widetilde{\mathbb{H}} \widehat{\nabla} \Gamma_{i+1}(\xi, e_3) e_3 : \widehat{\nabla} \Gamma_{i+1}(\xi, ce_3) e_3 d\xi \right| &\leq \\ &\leq C \left[\left(\frac{r}{r_0} \right)^{1/2} + \left(\frac{r_0}{r} \right)^{7/2} \varsigma \left(\omega_i(q_0), \frac{r}{r_0} \right) \right]. \end{aligned}$$

Therefore, for every $\varrho \in (0, 1/C_1)$, we have

$$\left| \int_{\mathbb{R}_-^3} \widetilde{\mathbb{H}} \widehat{\nabla} \Gamma_{i+1}(\cdot, e_3) e_3 : \widehat{\nabla} \Gamma_{i+1}(\cdot, ce_3) e_3 \right| \leq C \left(\varrho^{1/2} + \varrho^{-7/2} \varsigma(\omega_i(q_0), \varrho) \right), \quad (82)$$

where C depend on A, L, α_0, β_0 and M only. Now, if $\omega_i(q_0) < e^{-1/2}$ then we choose $\varrho = \frac{1}{2C_1} |\log \omega_i(q_0)|^{-\frac{1}{4\delta}}$, otherwise if $\omega_i(q_0) \geq e^{-1/2}$ then we estimate from above the right hand side of (82) trivially. Hence, by Proposition 3.3 we have

$$\left| \left(\frac{d}{dt} \Gamma_{\mathbb{C}_b^{(i+1)} + t\tilde{\mathbb{H}}}(e_3, ce_3) e_3 \cdot e_3 \right) \Big|_{t=0} \right| \leq C\sigma(\omega_i(q_0)), \quad (83)$$

where σ is defined by (53) and C depends on A, L, α_0, β_0 and M only.

By explicit calculation from (21), denoting by

$$\begin{aligned} \lambda(t) &= \lambda_{i+1} + th_{0,i+1}, & \mu(t) &= \mu_{i+1} + tk_{0,i+1}, \\ \nu(t) &= \frac{\lambda(t)}{2(\lambda(t) + \mu(t))}, \end{aligned}$$

and by

$$\alpha(t) = F_1(\mu_i, \mu(t), \nu_i), \quad \gamma(t) = F_2(\mu_i, \mu(t), \nu_i, \nu(t)),$$

for F_1 and F_2 as in (22) and (23), we get

$$\begin{aligned} & \frac{1}{16\pi\mu_i(1-\nu_i)} \left(\frac{d}{dt} \Gamma_{\mathbb{C}_{i+1} + t\tilde{\mathbb{H}}}(e_3, ce_3) m \cdot l \right) \Big|_{t=0} = \\ & = \frac{1}{(1+c)^3} \{ [4(1-\nu_i)(3-4\nu_i)\alpha'(0) + (\alpha\gamma)'(0)](1+c)^2 + 4c\alpha'(0) \}. \end{aligned}$$

Therefore, from (83) and (5) we find easily

$$\begin{cases} |\alpha'(0)| \leq C\sigma(\omega_i(q_0)), \\ |(\alpha\gamma)'(0)| \leq C\sigma(\omega_i(q_0)). \end{cases} \quad (84)$$

The first condition of (84) gives

$$\left| \frac{4(1-\nu_i)\mu_i}{(\mu_i + (3-4\nu_i)\mu_{i+1})^2} k_{0,i+1} \right| \leq C\sigma(\omega_i(q_0))$$

hence, by recalling (5) and (10), we have

$$|k_{0,i+1}| \leq C\sigma(\omega_i(q_0)). \quad (85)$$

Taking into account (85), the second equation of (84) implies

$$\left| \frac{8(1-\nu_i)\mu_i\mu_{i+1}^2}{2(\lambda_{i+1} + \mu_{i+1})^2(\mu_{i+1} + (3-4\nu_{i+1})\mu_i)^2} h_{0,i+1} \right| \leq C\sigma(\omega_i(q_0)),$$

hence, again by (5) and (10),

$$|h_{0,i+1}| \leq C\sigma(\omega_i(q_0)),$$

where C depends on A, L, α_0, β_0 and M only.

Therefore

$$\eta_{i+1} \leq \omega_{i+1}(q_0) := C\sigma(\omega_i(q_0)),$$

where C depends on A, L, α_0, β_0 and M only.

Finally, by iteration we get

$$1 = \eta_M \leq C\sigma^M(q_0) \leq C\sigma^N(q_0)$$

and the thesis follows. \square

Remark 4.7. Observe that the above proposition implies that the Frechét derivative $F'(\underline{L})$ is injective for every $\underline{L} \in \mathbf{K}$ and hence point (v) of Proposition 2.5 is completely proved. Therefore we have

$$\|\underline{L}^1 - \underline{L}^2\|_\infty \leq C\|F(\underline{L}^1) - F(\underline{L}^2)\|_* \quad \forall \underline{L}^1, \underline{L}^2 \in \mathbf{K},$$

where $C = \max\{\frac{2R_1}{(\sigma_2)^{-1}(\delta_1)}, \frac{2}{q_0}\}$, $M_1 = \min\{\frac{\beta_0}{\sqrt{13}}, \alpha_0\}$, $M_2 = \frac{\sqrt{2N}}{\alpha_0}$, $\sigma_2(\cdot) = C_*\sigma^N(\cdot)$, $q_0 = (\sigma^N)^{-1}(1/C_*)$, $\delta_1 = \frac{1}{2}\min\{\delta_0, M_2\}$ and $\delta_0 = \frac{q_0}{2C_{F'}}$ and we recall that C_* is the constant that occurs in Theorem 4.2, $C_{F'}$ is the Lipschitz constant of F' introduced in Proposition 4.1, and C_* has been introduced in Proposition 4.6.

5 Appendix

For the convenience of the reader, we recall here some quantitative estimates of unique continuation. Although such estimates have been proved in the general case where the elasticity tensor is of class $C^{1,1}$, here we give the statements in the special case we are interested in, namely we assume that

$$\mathbb{C} = \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{sym}, \quad (86)$$

where λ and μ are real numbers satisfying (4).

The following theorem is an immediate consequence of [Al-M, Theorem 5.1] and standard estimate of smallness propagation [Al-Ro-R-Ve, proof of Theorem 1.10]

Theorem 5.1 (Three sphere inequality). *Let u be a solution to the Lamé system*

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}u) = 0 \quad \text{in } B_{\bar{r}},$$

for some positive number \bar{r} . Then, for every r_1, r_2, r_3 , such that $0 < r_1 \leq r_2 < r_3 \leq \bar{r}$, we have

$$\int_{B_{r_2}} |u|^2 \leq C \left(\int_{B_{r_1}} |u|^2 \right)^{\theta_0} \left(\int_{B_{r_3}} |u|^2 \right)^{1-\theta_0}, \quad (87)$$

where C and θ_0 , $0 < \theta_0 < 1$, only depend on α_0 , β_0 , $\frac{r_2}{r_3}$ and, increasingly, on $\frac{r_1}{r_3}$.

The following theorem has been proved in [M-R].

Theorem 5.2 (Stability estimate for the Cauchy problem). *Let \mathbb{C} be as in (86). Let u be the solution to the Cauchy problem*

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}u) &= 0 \text{ in } B_{r_0/3}^+, \\ u(x', 0) &= h(x') \text{ on } B'_{r_0/3}, \\ \frac{\partial u}{\partial x_3}(x', 0) &= g(x') \text{ on } B'_{r_0/3}, \end{cases}$$

where $h \in H^{\frac{1}{2}}(B'_{r_0/3})$ and $g \in H^{-\frac{1}{2}}(B'_{r_0/3})$. We have

$$\begin{aligned} \|u\|_{L^\infty(B_{r_0/6}^+)} + r_0 \|\nabla u\|_{L^\infty(B_{r_0/6}^+)} &\leq \\ &\leq C \|u\|_{H^1(B_{r_0/6}^+)}^{1-\theta_1} \left(\|h\|_{L^2(B'_{r_0/3})} + \|g\|_{H^{-\frac{1}{2}}(B'_{r_0/3})} \right)^{\theta_1}, \end{aligned}$$

where C and θ_1 , $0 < \theta_1 < 1$, only depend on α_0, β_0 .

In order to state the following result of smallness propagation in a cone (Proposition 5.3) we introduce some notation. Given $z \in \mathbb{R}^3$, $\zeta \in \mathbb{R}^3$, $|\zeta| = 1$, $\gamma \in (0, \frac{\pi}{2})$, we denote by

$$C(z, \zeta, \gamma) = \left\{ x \in \mathbb{R}^3 \mid \frac{(x-z) \cdot \zeta}{|x-z|} > \cos \gamma \right\}$$

the open cone having vertex z , axis in the direction ζ and width 2γ and, for any $\rho > 0$, we denote by

$$C_\rho(\gamma) = C(0, -e_3, \gamma) \cap Q_{\rho, H_\gamma},$$

where $H_\gamma = \frac{1}{\tan \gamma}$.

Let $\gamma_1, \gamma_2, \gamma_3 \in (0, \frac{\pi}{2})$ be such that $\gamma_1 < \gamma_2 < \gamma_3$.

Denote by

$$\begin{aligned} t_0 &= \frac{H_{\gamma_3} \rho}{1 + \sin \gamma_3}, \\ \chi &= \frac{1 - \sin \gamma_2}{1 - \sin \gamma_1}, \end{aligned}$$

and, for any $t \in (0, t_0]$, denote by

$$s_k = \chi^{k-1} t, \quad w_k = -s_k e_3, \quad k \in \mathbb{N},$$

$$r_3^{(k)} = s_k \sin \gamma_3, \quad r_2^{(k)} = s_k \sin \gamma_2, \quad r_1^{(k)} = s_k \sin \gamma_1, \quad k \in \mathbb{N}. \quad (88)$$

Notice that we have

$$B_{r_1^{(k+1)}}(w_{k+1}) \subset B_{r_2^{(k)}}(w_k) \subset B_{r_3^{(k)}}(w_k) \subset C_\rho(\gamma_3), \quad \text{for every } k \in \mathbb{N}. \quad (89)$$

Let r be a given number such that $r \in (0, \chi t_0]$. Let k_0 be the smallest integer number such that $\chi^{k-1} \leq \frac{r}{t_0}$ and let

$$t := \chi^{-(k_0-1)} r.$$

Notice that

$$\chi t_0 \leq t \leq t_0$$

and

$$w_{k_0} = -re_3, r_1^{(k_0)} = r \sin \gamma_1. \quad (90)$$

Proposition 5.3. *Let \mathbb{C} be as in (86) and let u be a solution to the Lamé system*

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}u) = 0 \quad \text{in } C_\rho(\gamma_3).$$

Assume that

$$\int_{B_{t \sin \gamma_1}(w_1)} |u|^2 \leq \varepsilon^2, \quad \int_{C_\rho(\gamma_3)} |u|^2 \leq E^2, \quad (91)$$

where ε, E are given positive numbers such that $\varepsilon \leq E$. Then we have

$$|u(-re_3)| \leq \frac{C}{r^{3/2}} \varepsilon^{\eta_r} E^{1-\eta_r}, \quad (92)$$

where

$$\eta_r = \bar{\theta} \left(\frac{r}{t} \right)^{\frac{|\log \bar{\theta}|}{|\log \chi|}},$$

C and $\bar{\theta}, \bar{\theta} \in (0, 1)$ depend on $\alpha_0, \beta_0, \gamma_1, \gamma_2$ and γ_3 only.

Proof of Proposition 5.3

Let $r_i^{(k)}$ for $i = 1, 2, 3$ be as in (88). By Theorem 5.1 we have, for every $k \in \mathbb{N}$,

$$\int_{B_{r_2^{(k)}}(w_k)} |u|^2 \leq C \left(\int_{B_{r_1^{(k)}}(w_k)} |u|^2 \right)^{\bar{\theta}} \left(\int_{B_{r_3^{(k)}}(w_k)} |u|^2 \right)^{1-\bar{\theta}}, \quad (93)$$

where C and $\bar{\theta}, \bar{\theta} \in (0, 1)$, depend on $\alpha_0, \beta_0, \gamma_1, \gamma_2$ and γ_3 only.

Denote by

$$\sigma_k := E^{-2} \int_{B_{r_1^{(k)}}(w_k)} |u|^2.$$

Since, by (89), $B_{r_1^{(k)}} \subseteq B_{r_2^{(k-1)}}$, we have

$$\sigma_k \leq E^{-2} \int_{B_{r_2^{(k-1)}}(w_{k-1})} |u|^2, \quad \text{for every } k \geq 2. \quad (94)$$

By the second inequality in (91) and by (93) and (94) we get

$$\sigma_k \leq C\bar{\theta}^{k-1}, \text{ for every } k \geq 2.$$

Iterating the last inequality and taking into account the first inequality in (91) we get

$$\sigma_k \leq C^{\frac{1}{1-\bar{\theta}}} \left(\frac{\varepsilon^2}{E^2} \right)^{\bar{\theta}^{k-1}}, \text{ for every } k \geq 2.$$

Now, we choose $k = k_0$ in the above inequality and notice that

$$\bar{\theta}^{k_0-1} \geq \eta_r,$$

hence

$$\int_{B_{r_1}^{(k_0)}(w_{k_0})} |u|^2 \leq C\varepsilon^{2\eta_r} E^{2(1-\eta_r)},$$

where C depends on $\alpha_0, \beta_0, \gamma_1, \gamma_2$ and γ_3 only.

Finally, by (90) and by the estimate (39),

$$\|u\|_{L^\infty\left(B_{\frac{r \sin \gamma_1}{2}}(-re_3)\right)}^2 \leq \frac{C}{(r \sin \gamma_1)^3} \int_{B_{r \sin \gamma_1}(re_3)} |u|^2,$$

where C depends on α_0 and β_0 only and (92) follows. \square

We finally end this appendix by proving our main result on quantitative estimate of unique continuation for solutions of Lamé system with piecewise constant coefficients. In this proof the elasticity tensor \mathbb{C} is of the form (15).

Proof of Proposition 3.6

Denote by $C_2 = \max\left\{6, 4LC_L, 4C_1, \frac{2r_0}{h_0}\right\}$ and by $\rho_2 = \frac{r_0}{C_2}$, where C_L and C_1 are defined in (14) and in Proposition 3.5 respectively and h_0 is defined in (16). Notice that C_2 does not depend on r_0 and that $\rho_2 \leq \frac{\rho_1}{16}$. It is not restrictive to assume $n_1 = e_3$. Let us denote $x_0 = P_1 + n_1 \frac{3}{16} \rho_1$. We have by (18)

$$K_0 \supset B'_{\rho_2}(P_1) \times \left(\frac{\rho_1}{8}, \frac{\rho_1}{4} \right) \supset B_{\rho_2}(x_0).$$

Moreover, by Proposition 3.5 we have that the function $v|_{B'_{\rho_2}(P_1) \times (0, \frac{\rho_1}{4})}$ can be extended analytically to a function v_0 on $B'_{\rho_2}(P_1) \times \left(-\frac{r_0}{4C_1}, \frac{\rho_1}{4}\right)$ and

$$\|v_0\|_{L^\infty\left(B'_{\rho_2}(P_1) \times \left(-\frac{\rho_1}{4}, \frac{r_0}{4C_1}\right)\right)} \leq C(E_1 + \varepsilon_1), \quad (95)$$

where C depends on A, L, α_0, β_0 and N only.

Let us construct a chain of spheres of radius $\rho_2/4$ such that the first is $B_{\rho_2/4}(x_0)$, all the spheres are externally tangent and the last one is centered at $P_1 + \frac{\rho_2}{2} n_1$. We choose such a chain so that the spheres of radius ρ_2 concentric

with those of the chain are contained in $B'_{\rho_2}(P_1) \times \left(-\frac{r_0}{4C_1}, \frac{\rho_1}{4}\right)$. The number of spheres of the chain is certainly smaller than a constant m_1 depending on L, α_0 and β_0 only.

By an iterated application of three sphere inequality (87) with $r_1 = \frac{\rho_2}{4}, r_2 = \frac{3\rho_2}{4}, r_3 = \rho_2$ and by (95) we have

$$\|v_0\|_{L^2(B_{3\rho_2/4}(P_1 + \frac{\rho_2}{2}n_1))} \leq C\varepsilon_1^{\theta_0^{m_1}} (E_1 + \varepsilon_1)^{1-\theta_0^{m_1}}, \quad (96)$$

where $\theta_0, 0 < \theta_0 < 1$, depends on α_0 and β_0 only and C depends on A, L, α_0 and β_0 only. Since $B_{3\rho_2/4}(P_1 + \frac{\rho_2}{2}n_1) \supset B_{\rho_2}(P_1)$, by (96) we have trivially

$$\|v_0\|_{L^2(B_{\rho_2/4}(P_1))} \leq C\varepsilon_1^{\theta_0^{m_1}} (E_1 + \varepsilon_1)^{1-\theta_0^{m_1}}. \quad (97)$$

By the above inequality and by Caccioppoli inequality, [BBFM, pag.20], we have

$$\|\nabla v_0\|_{L^2(B_{\rho_2/8}(P_1))} \leq \frac{C}{r_0} \varepsilon_1^{\theta_0^{m_1}} (E_1 + \varepsilon_1)^{1-\theta_0^{m_1}}, \quad (98)$$

where C depends on A, L, α_0 and β_0 only.

By (97) and (98) we get the following trace inequality

$$\|v_0\|_{L^2(B'_{\rho_2/16}(P_1))} + r_0 \|(\mathbb{C}\widehat{\nabla}v_0)n_1\|_{H^{-\frac{1}{2}}(B'_{\rho_2/16}(P_1))} \leq C\varepsilon_1^{\theta_0^{m_1}} (E_1 + \varepsilon_1)^{1-\theta_0^{m_1}}. \quad (99)$$

Now, let us recall the following transmission conditions

$$v|_{D_0} = v|_{D_1} \text{ on } \Sigma_1, \quad \left(\mathbb{C}\widehat{\nabla}v\right)|_{D_0} n_1 = \left(\mathbb{C}\widehat{\nabla}v\right)|_{D_1} n_1 \text{ on } \Sigma_1. \quad (100)$$

Let us denote by $B_{\rho_2/32}^-(P_1) = D_1 \cap B_{\rho_2/32}(P_1)$. By Theorem 5.2, (100), (99), (47) and the Caccioppoli inequality we have

$$\|v|_{D_1}\|_{L^\infty(B_{\rho_2/32}^-(P_1))} + r_0 \|\nabla v|_{D_1}\|_{L^\infty(B_{\rho_2/32}^-(P_1))} \leq C\varepsilon_1^{\theta_1\theta_0^{m_1}} (E_1 + \varepsilon_1)^{1-\theta_1\theta_0^{m_1}}, \quad (101)$$

where $\theta_0 \in (0, 1)$ and $\theta_1 \in (0, 1)$ depend on α_0 and β_0 only and C depends on A, L, α_0 and β_0 only.

Now, we prove by induction what follows: given $m_2 = \frac{3A(64)^{3(N-2)}C_2^3}{4\pi}$ and $\rho_k = (64)^{-k+2}$, there exist a constant C depending on A, L, α_0, β_0 and (increasingly) on M , such that for every $k \in \{1, 2, \dots, M-1\}$ the following inequality holds true

$$\begin{aligned} \|v|_{D_k}\|_{L^\infty(B_{\rho_{k+1}/32}^-(P_k))} + r_0 \|\nabla v|_{D_k}\|_{L^\infty(B_{\rho_{k+1}/32}^-(P_k))} &\leq \\ &\leq C\varepsilon_1^{\theta_1^k\theta_0^{m_1+(k-1)m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^k\theta_0^{m_1+(k-1)m_2}}. \end{aligned} \quad (102)$$

If $k = 1$ then (102) is proved in (101). Now, assume that (102) holds true for $k, 1 \leq k \leq M-2$. Let us denote by v_k the analytic extension of $v|_{\tilde{\mathcal{K}}_{h_0/2} \cap D_k}$ to

$(\tilde{\mathcal{K}}_{h_0/2} \cap D_{k+1}) \cup (\Xi_{k+1}^{C_1} \cap \bar{D}_k)$ where h_0 , $\tilde{\mathcal{K}}_{h_0/2}$ and $\Xi_{k+1}^{C_1}$ are defined by (16), (17) and (41) respectively. Let us denote $x_k = P_k - \frac{\rho_k}{32}n_k$. By the induction hypothesis we have trivially

$$\|v_k\|_{L^2(B_{\rho_{k+1}/64}(x_k))} \leq C\varepsilon_1^{\theta_1^k \theta_0^{m_1+(k-1)m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^k \theta_0^{m_1+(k-1)m_2}}, \quad (103)$$

where C depends on A, L, α_0, β_0 and (increasingly) k only.

Let us construct a chain of spheres of radius $\rho_{k+1}/4 \cdot 64$ such that the first is $B_{\rho_{k+1}/4 \cdot 64}(x_k)$, all the spheres are externally tangent and the last one is centered at $P_{k+1} + \frac{\rho_{k+1}}{4 \cdot 64}n_{k+1}$. We choose such a chain so that the spheres of radius $\rho_{k+1}/64$ concentric with those of the chain are contained in $\tilde{\mathcal{K}}_{h_0/2} \cap D_k$. The number of spheres of the chain is certainly smaller than a constant m_2 .

By an iterated application of three sphere inequality (87) with $r_1 = \frac{\rho_{k+1}}{4}$, $r_2 = \frac{3\rho_{k+1}}{4}$, $r_3 = \rho_{k+1}$ and by (103) we have

$$\begin{aligned} \|v_k\|_{L^2(B_{\rho_{k+1}/4}(P_{k+1}))} &\leq \|v_k\|_{L^2(B_{3\rho_{k+1}/4}(P_{k+1} + \frac{\rho_{k+1}}{4}n_{k+1}))} \leq \\ &\leq C\varepsilon_1^{\theta_1^k \theta_0^{m_1+k m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^k \theta_0^{m_1+k m_2}}, \end{aligned}$$

where C depends on A, L, α_0, β_0 and (increasingly) N only.

Now, proceeding in exactly the same way followed to prove (101) we have

$$\begin{aligned} \|v|_{D_{k+1}}\|_{L^\infty(B_{\rho_{k+1}/32}^-(P_{k+1}))} + r_0 \|\nabla v|_{D_{k+1}}\|_{L^\infty(B_{\rho_{k+1}/32}^-(P_{k+1}))} &\leq \\ &\leq C\varepsilon_1^{\theta_1^{k+1} \theta_0^{m_1+k m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^{k+1} \theta_0^{m_1+k m_2}}, \end{aligned}$$

where C depends on A, L, α_0, β_0 and N only. Therefore, inequality (102) holds true for every $k \in \{1, 2, \dots, M-1\}$.

In particular by (102) we have

$$\begin{aligned} \|v|_{D_{M-1}}\|_{L^\infty(B_{\rho_M/32}^-(P_M))} + r_0 \|\nabla v|_{D_{M-1}}\|_{L^\infty(B_{\rho_M/32}^-(P_M))} &\leq \\ &\leq C\varepsilon_1^{\theta_1^{M-1} \theta_0^{m_1+(M-2)m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^{M-1} \theta_0^{m_1+(M-2)m_2}}. \quad (104) \end{aligned}$$

Now by a rigid transformation of coordinate under which we have $P_M = 0$ and $n_M = e_3$ we have $Q_{(M)}^- = B'_{\rho_1/4L} \times (0, \frac{\rho_1}{4})$. In what follows, first we derive an error smallness estimate in a ball contained in the cylinder $B'_{\rho_1/8L} \times (0, \frac{\rho_1}{4})$ and then we apply Proposition 5.3 in the cone $C_{\rho_1/8L}(\gamma_3)$, where $\gamma_3 = \arctan \frac{1}{2L}$. In order to make precise such a step we adopt all the notation introduced before such a proposition. Therefore we denote $H_{\gamma_3} = 2L$, $t_0 = \frac{\rho_1}{8L} \frac{H_{\gamma_3}}{1+\sin \gamma_3}$. Since $\sin \gamma_3 = \frac{1}{\sqrt{4L^2+1}}$ we have

$$t_0 = \frac{\rho_1}{4} \frac{\sqrt{4L^2+1}}{1+\sqrt{4L^2+1}}.$$

Let γ_1, γ_2 be such that

$$\sin \gamma_1 = \frac{1}{4} \sin \gamma_3, \quad \sin \gamma_2 = \frac{3}{4} \sin \gamma_3$$

and set

$$\chi = \frac{1 - \sin \gamma_2}{1 - \sin \gamma_1} = \frac{4\sqrt{4L^2 + 1} - 3}{4\sqrt{4L^2 + 1} - 1}.$$

Let $r \in (0, \chi t_0)$ be fixed and

$$t := \chi^{-(k_0-1)} r$$

where k_0 is the smallest integer number such that $\chi^{k-1} \leq \frac{t_0}{r}$. Moreover let $\bar{P} = P_M + tn_M$. Proceeding as before, by (104) we get

$$\|v\|_{L^2(B_{\rho_M/4 \cdot 64}(\bar{P}))} \leq \varepsilon_2 := C \varepsilon_1^{\theta_1^{M-1} \theta_0^{m_1+(M-1)m_2}} (E_1 + \varepsilon_1)^{1-\theta_1^{M-1} \theta_0^{m_1+(M-1)m_2}},$$

where C depends on A, L, α_0, β_0 and M only.

Now denote by $\rho = \min\{\rho_M/4 \cdot 64, t \sin \gamma_1\}$. By applying Theorem 5.1 with $r_1 = \rho, r_2 = t \sin \gamma_1$ and $r_3 = t \sin \gamma_3$ we get

$$\|v\|_{L^2(B_{t \sin \gamma_1}(\bar{P}))} \leq \varepsilon_2 := C \varepsilon_2^{\theta_2} (E_1 + \varepsilon_2)^{1-\theta_2},$$

where $C, \theta_2, 0 < \theta_2 < 1$ depend on L, α_0 and β_0 only. Now we use Proposition 5.3 and we get

$$|v(\tilde{x})| \leq C \left(\frac{r_0}{r}\right)^{2-\frac{\eta_r}{2}} \varepsilon_2^{\eta_r} (E_1 + \varepsilon_2)^{1-\eta_r}, \quad (105)$$

where $\tilde{x} = P_M + rn_M, C$ depends on A, L, α_0, β_0 and (increasingly) N only and

$$\eta_r = \theta_2 \left(\frac{r}{t}\right)^{\frac{|\log \theta_2|}{|\log \chi|}}. \quad (106)$$

Finally, denoting by $\tilde{\theta} = \min\{\theta_1, \theta_2, \theta_3\}$ and $\bar{m} = \max\{m_1 + 1, m_2 + 1\}$ we get

$$|v(\tilde{x})| \leq C \left(\frac{r_0}{r}\right)^{2-\frac{\eta_r}{2}} \varepsilon_1^{\bar{\theta}^{\bar{m}M} \eta_r} (E_1 + \varepsilon_1)^{1-\bar{\theta}^{\bar{m}M} \eta_r}$$

and by $t \geq \chi t_0$ the thesis follows. \square

References

- [ANS] M. Akamatsu, G. Nakamura, S. Steinberg, Identification of Lamé coefficients from boundary observations *Inverse Problems*, 7, 335–354 (1991).
- [Al1] G. Alessandrini, Stable determination of conductivity by boundary measurements, *Applicable Analysis* 27, 153–172 (1988).

- [ABV] G. Alessandrini, E. Beretta, S. Vessella, Determining cracks by boundary measurements-Lipschitz Stability, *SIAM J. Math. Anal.* 27, 361-375 (1996).
- [Al-M] G. Alessandrini, A. Morassi, Strong unique continuation for the Lamé system of elasticity, *Comm. PDE*, 26, 1787–1810 (2001).
- [Al-Ro-R-Ve] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, The stability for the Cauchy problem for elliptic equations, *Inverse Problems* 25, 1–47 (2009).
- [AV] G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, *Adv. in Appl. Math.*, 35, 207–241 (2005).
- [ABF] H. Ammari, E. Beretta, E. Francini, Reconstruction of thin conducting inhomogeneities from electrostatic measurements, II. The case of multiple segments, *Applicable Analysis*, Vol. 85, 67-85 (2006).
- [AP] K. Astala, L. Paivarinta, Calderón’s inverse conductivity problem in the plane, *Ann. of Math.*, 163, 265–299 (2006).
- [BaV] V. Bacchelli, S. Vessella, Lipschitz stability for a stationary 2D inverse problem with unknown polygonal boundary, *Inverse Problems* 22, 16271658 (2006).
- [BBFM] E. Beretta, E. Bonnetier, E. Francini, A. Mazzucato, An asymptotic formula for the displacement field in the presence of small anisotropic elastic inclusions, *Inverse Problems and Imaging* 6, 1-23 (2012).
- [BdHQ] E. Beretta, M. V. de Hoop, L. Qiu, Lipschitz stability of an inverse boundary value problem for a Schroedinger type equation, arXiv:1203.1650, (2012).
- [BF] E. Beretta, E. Francini, Lipschitz stability for the impedance tomography problem. The complex case, *Comm. PDE*, 36, 1723–1749 (2011).
- [BFV] E. Beretta, E. Francini and S. Vessella, Determination of a linear crack in an elastic body from boundary measurements. Lipschitz stability, *SIAM J. Math. Anal.* 40, 984–1002 (2008).
- [BC] M. Bonnet, A. Constantinescu, Inverse problems in elasticity, *Inverse problems*, 2, R1-R50 (2005).
- [BJK] B. M. Brown, M. Jais, I. W. Knowles, A variational approach to an elastic inverse problem, *Inverse Problems*, 21, 1953–1973 (2005)
- [C] S. Campanato, Sistemi ellittici in forma divergenza. Regolarità all’interno. *Quaderni della Scuola Normale Superiore di Pisa*, 1980.
- [CKVC] M. Chipot, D. Kinderlehrer, G. Vergara-Caffarelli, Smoothness of linear laminates. *Arch. Rational Mech. Anal.*, 96, 81–96 (1986).

- [dHQS1] M. V. de Hoop, L. Qiu, O. Scherzer, Local Analysis of Inverse Problems: Höelder Stability and Iterative Reconstruction, *Inverse Problems*, 28 (2012).
- [dHQS2] M. V. de Hoop, L. Qiu, O. Scherzer, A convergence analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints, arXiv:1206.3706, (2012).
- [ER] G. Eskin, J. Ralston, On the inverse boundary value problem for linear isotropic elasticity, *Inverse Problems*, 18, 907–922 (2002).
- [E] L. Evans, Partial differential equations, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.
- [Fr] A. Friedman, Generalized functions and partial differential equations, Dover publication, New York, 2005.
- [HK] S. Hofmann, S. Kim, The Green function estimates for strongly elliptic systems of second order, *Manuscripta Math.*, 124, 139–172 (2007).
- [Ik] M. Ikehata, Inversion formulas for the linearized problem for an inverse boundary value problem in elastic prospection, *SIAM J. Appl. Math.* 50, 1635–1644 (1990).
- [Is] V. Isakov, Inverse problems for partial differential equations, 2nd edn, Springer, 2006.
- [LN] Y. Y. Li, L. Nirenberg, Estimates for elliptic systems from composite materials, *Comm. Pure Appl. Math.*, 56, 892–925 (2003).
- [M] N. Mandache, Exponential instability in an inverse problem for the Schrödinger equation, *Inverse Problems*, 17, 1435–1444 (2001).
- [Mi] G. Milton, The Theory of Composites, Cambridge University Press, 2002.
- [MeRe] C. Mengcheng, T. Renji, An explicit tensor expression for the fundamental solutions of a bimaterial space problem, *Applied Mathematics and Mechanics*, 18, 331–340 (1997).
- [M-R] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, *Inverse Problems* 20, 453–480(2004).
- [Na] G. Nakamura, Inverse problems for elasticity, Selected papers on analysis and differential equations, 71–85, Amer. Math. Soc. Transl. Ser. 2, 211, Amer. Math. Soc., Providence, RI, 2003.
- [NU1] G. Nakamura, G. Uhlmann, Identification of Lamé Parameters by Boundary Measurements, *American Journal of Mathematics*, 115, 1161–1187 (1993).

- [NU2] G. Nakamura, G. Uhlmann, Erratum: Global uniqueness for an inverse boundary value problem arising in elasticity, *Invent. Math.* 152, 205–207 (2003)(Erratum to *Invent. Math.* 118 (3) (1994) 457–474).
- [NU] G. Nakamura, G. Uhlmann, Inverse boundary problems at the boundary for an elastic system, *SIAM J. Math. Anal.*, 26, 263–79 (1995).
- [R] L. Rongved, Force interior to one of two joined semi-infinite solids, Proc. 2nd Midwestern Conf. Solid Mech, 1–13, 1955.
- [SU] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.*, 125, 153–169 (1987).
- [V] S. Vessella, Locations and strengths of point sources: stability estimates, *Inverse Problems* 8, 911-917 (1992).