American option valuation in a stochastic volatility model with transaction costs

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1. Introduction

The intrinsic limitations of the Black–Scholes model in describing real financial markets behaviour are very well known. Among the main assumptions underlying this model, the most relevant assumptions are probably constant volatility and no transaction costs. In this paper we consider the valuation problem of American options in a model in which both proportional transaction costs are taken into account and the volatility is assumed to evolve according to a stochastic process of the Cox–Ingersoll–Ross (CIR) type, as in the Heston model, see [11].

The pricing of a European option in presence of transaction costs was considered in several papers, we recall the asymptotic result by Whalley and Wilmott [16], where a correction term to the Black–Scholes pricing formula was derived. The pricing of American options in models with transaction costs has been considered by some authors. In particular, we cite the fundamental paper by Davis and Zariphopoulou [8], where the valuation problem is attacked via a utility indifference price approach: the related optimal control problem is formulated and the existence and uniqueness of the viscosity solution for the corresponding Hamilton–Jacobi–Bellman (HJB) equation is proved. We also cite the paper by Zakamouline [17], where there is a different formulation of the stochastic optimal control problem in terms of a quasi-variational inequality, which results to be more suitable for numerical applications. This formulation is proved to be equivalent to the one presented in [8] and a numerical procedure is illustrated in order to explicitly compute the solution.

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A large literature is available on the valuation problem of American options in a stochastic volatility framework, as well. We just recall the papers by Chiarella et al. [2,3], Chung et al. [4], and Clarke and Parrot [5].

On the contrary, the literature regarding option pricing in models including both transaction costs and stochastic volatility features is, to our knowledge, not so extensive. We cite the recent paper by Mariani et al. [12], where the authors proposed a numerical approximation scheme for European option prices in stochastic volatility models including transaction costs based on a finite-difference method.

In this paper our aim is to study the American option pricing problem in a modelling framework taking into account both stochastic volatility and transaction costs, based on the approach pioneered by Davis and Zariphopoulou [8].

The plan of the paper is as follows: In Section 2 we fix the notations and we introduce the model. In Section 3 we formulate the singular control problem related to the American option valuation problem and we obtain heuristically the associated HJB partial differential equation (PDE). In Section 4 we prove the existence of the viscosity solution for the HJB equation. In Section 5, we provide the comparison principle for our HJB equation, from which we deduce the uniqueness of the viscosity solution for the same problem. In Section 6 we reduce the problem dimensionality by a suitable choice of the utility function involved: we show how the choice of the exponential utility allows to formulate in a slightly simpler way the optimal control problem under investigation, we propose a discretization method for the associated variational inequality, and in Section 7 we discuss the numerical results obtained. Finally, in Section 8 we present some concluding remarks and discuss possible extensions of the present investigation.

2. A stochastic volatility model with transaction costs

In this section we introduce the American option valuation problem in a financial market with proportional transaction costs, written on a risky asset which evolves according to the Heston model, see [10,11]. In the formulation of the model we keep the notations introduced in [8]. We suppose to have the following multidimensional stochastic process:

$$\mathrm{d}z(t) = rz(t)\mathrm{d}t - (1+\lambda)S(t)\mathrm{d}L(t) + (1-\mu)S(t)\mathrm{d}M(t), \tag{1}$$

$$\mathrm{d}y(t) = \mathrm{d}L(t) - \mathrm{d}M(t), \tag{2}$$

$$\mathrm{d}S(t) = \alpha S(t)\mathrm{d}t + \sqrt{\nu(t)}S(t)\mathrm{d}W(t), \tag{3}$$

$$\mathrm{d}\nu(t) = \xi(\eta - \nu(t))\mathrm{d}t + \vartheta\sqrt{\nu(t)}\mathrm{d}Z(t). \tag{4}$$

In the above equations z represents the amount invested in the risk-free asset (the 'Bond'), S is the risky asset (the 'Stock'), r is the risk-free interest rate, α is the drift rate of the stock, λ and μ are the (proportional) costs of buying and selling a stock and $\sqrt{\nu}$ is the volatility function, which we shall suppose to be driven by the Wiener process Z according to a CIR-type dynamics. Parameters ξ, η and ϑ are assumed to be constant. To avoid a zero volatility we assume that $\xi\eta > \vartheta^2/2$ (the strict inequality is required in the proof of the comparison Theorem 5.1). The Wiener processes W and Z are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, with $T > 0$ the final horizon, is the natural filtration generated by the two Wiener processes and satisfies the usual conditions. W and Z are assumed to be correlated with coefficient ρ. $L(t)$ and $M(t)$ are the cumulative number of shares bought or sold, respectively, up to time

$t \in [0, T]$. Finally, Equations (1) and (2) imply that the trading strategies are self-financing.

The cash value of a number of shares $y \in \mathbb{R}$ of the stock when its price is $S \in (0, +\infty)$ is not simply yS, but is given by

$$c(y, S) = \begin{cases} (1+\lambda)yS, & \text{if } y < 0, \\ (1-\mu)yS, & \text{if } y \geq 0, \end{cases}$$

where λyS and μyS are the amounts that the investor has to pay, due to the presence of the transaction costs.

We shall formulate the American option valuation problem as a utility maximization problem in strict analogy with the approach pioneered by Davis and Zariphopoulou [8]. Let $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ be the buyer utility function, assumed to be concave, increasing and such that $\mathcal{U}(0) = 0$. Let us suppose that the buyer of the option owns an initial wealth x in cash. At time $t = 0$ he/she splits his/her wealth into the amounts x_1 and $x_2 = x - x_1$. He/she uses the quantity x_1 to buy x_1/p shares of American options written on the risky asset S, where p is the American option price we are going to define. The amount x_2, instead, is used to construct a portfolio π composed of the bond and the stock, in order to maximize the expected utility of his/her terminal wealth:

$$V(t, z, y, S, \nu) = \sup_{\mathcal{A}_{t,T}} \mathbb{E}[\mathcal{U}(z(T) + c(y(T), S(T)))|(z(t^-), y(t^-), S(t^-), \nu(t^-)) = (z, y, S, \nu)].$$

Here $0 \leq t \leq T$, $(z, y, S, \nu) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)$ is the state at time t^- and $\mathcal{A}_{t,T}$ is the set of admissible trading strategies (L, M) which we now define.

Definition 2.1. The set of admissible trading strategies $\mathcal{A}_{t,T}$, for every $0 \leq t \leq T$, is the set of two-dimensional right-continuous, measurable, $\mathbb{F}$-adapted and increasing stochastic processes $(L, M) = (L(u), M(u))_{t \leq u \leq T}$, with $L(t^-) = M(t^-) = 0$. Furthermore, (L, M) are such that the corresponding processes $(z(u), y(u), S(u))_{t \leq u \leq T}$ satisfy

$$(z(u), y(u), S(u)) \in \mathcal{S}_{\bar{K}}, \quad t \leq u \leq T, \tag{5}$$

where $\bar{K}$ is a positive constant and

$$\mathcal{S}_{\bar{K}} = \{(z, y, S) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : z + c(y, S) > -\bar{K}\}.$$

Remark 1. Note that the set of admissible trading strategies $\mathcal{A}_{t,T}$ depends also on the initial state (z, y, S, ν) at time t^-. Constraint (5) is required in the proof of the comparison Theorem 5.1 and it only rules out strategies which are clearly non-optimal, as the objective is the maximization of the utility of final wealth. Moreover, we note that $L(t)$ and $M(t)$ may be positive, i.e. there can be a jump at the initial time t.

At time τ, the buyer could decide to exercise the option and to transfer the money to the portfolio, i.e. he/she receives the cash amount Kx_1/p and pays to the option writer the amount $x_1 S(\tau)/p$, which is the price of x_1/p shares of the underlying security. If $[y(\tau), z(\tau)]$ is the investor's portfolio composition at exercise time τ, after the money transfers are performed the new portfolio composition is given by

$$\left[y(\tau) - \frac{x_1}{p}, z(\tau) + \frac{Kx_1}{p}\right].$$

Therefore, let us define

$$V_1(t, z, y, S, \nu; x_1) = V\left(t, z + \frac{Kx_1}{p}, y - \frac{x_1}{p}, S, \nu\right). \tag{6}$$

Then it is relevant to introduce the following value function:

$$\begin{aligned} U(t, z, y, S, \nu; x_1) = & \sup_{\mathcal{A}_{t,T,\tau}} \mathbb{E}[V_1(\tau, z(\tau), y(\tau), S(\tau), \nu(\tau); x_1) | (z(t^-), y(t^-), S(t^-), \nu(t^-)) \\ & = (z, y, S, \nu)], \end{aligned} \tag{7}$$

where $\tau \in \mathcal{T}_{t,T}$, the set of $\mathbb{F}$-stopping times with values in $[t, T]$. Now we define the auxiliary functions:

$$\alpha(x_1, x_2, S, p) = U(0, S, 0, x_2; x_1),$$

$$\beta(x, S, p) = \sup_{x_1 + x_2 = x} \alpha(x_1, x_2, S, p),$$

$$X^*(p, S, x) = \arg\max \alpha(x_1, x - x_1, S, p).$$

We can finally define the writing price of the American option as follows:

$$p^*(S) = \sup_x \{p : X^*(p, S, x) > 0\}.$$

Hence, the fair price of the American option is defined to be the maximum price at which a positive investment is made in the option at time $t = 0$.

In the following sections, in order to avoid cumbersome notations, we shall drop the explicit dependence of $V_1(\tau, z(\tau), y(\tau), S(\tau), \nu(\tau); x_1)$ on all the variables in some of the formulas presented.

We conclude this section by noting that in [8], therefore, without stochastic volatility, the authors proved that the above definition of the price of the American option reduces to the classical Black–Scholes price when transaction costs vanish.

3. The singular control problem

In this section, in strict analogy with [8] and also [7], we derive heuristically the singular control problem associated with the American option valuation for a market model including both stochastic volatility and transaction costs.

We begin by restricting temporarily our interest to trading strategies which are absolutely continuous with respect to time, i.e. to those that can be written as

$$L(t) = \int_0^t \ell(s)\mathrm{d}s, \quad M(t) = \int_0^t m(s)\mathrm{d}s,$$

where $\ell(s)$ and $m(s)$ are non-negative functions uniformly bounded by a fixed constant $k < \infty$. In this particular case, Equations (1)–(4) become a vector stochastic differential equation with controlled drift and the value function of the approximate problem, denoted by V_k, satisfies the following HJB equation:

$$\max_{0\leq \ell, m\leq k}\left\{\left(\frac{\partial V_k}{\partial y}-(1+\lambda)S\frac{\partial V_k}{\partial z}\right)\ell-\left(\frac{\partial V_k}{\partial y}-(1-\mu)S\frac{\partial V_k}{\partial z}\right)m\right\}+\frac{\partial V_k}{\partial t}+\mathcal{L}V_k=0,$$

with terminal condition $V_k(T,z,y,S,\nu)=\mathcal{U}(z+c(y,S))$ for $(z,y,S,\nu)\in\mathbb{R}\times\mathbb{R}\times(0,+\infty)\times(0,+\infty)$. Here the differential operator $\mathcal{L}$ is given by

$$\mathcal{L}W=rz\frac{\partial W}{\partial z}+\alpha S\frac{\partial W}{\partial S}+\frac{1}{2}\nu S^2\frac{\partial^2 W}{\partial S^2}+\xi(\eta-\nu)\frac{\partial W}{\partial \nu}+\frac{1}{2}\vartheta^2\nu\frac{\partial^2 W}{\partial \nu^2}+\rho\vartheta\nu S\frac{\partial^2 W}{\partial S\partial \nu}.$$

The optimal trading strategy can be described by considering the following three possible cases:

(1)
$$\frac{\partial V_k}{\partial y}-(1+\lambda)S\frac{\partial V_k}{\partial z}\geq 0,$$

where the maximum is achieved by taking $m=0$ and buying at the maximum possible rate $\ell=k$;

(2)
$$\frac{\partial V_k}{\partial y}-(1-\mu)S\frac{\partial V_k}{\partial z}\leq 0,$$

where the maximum is achieved by taking $\ell=0$ and selling at the maximum possible rate $m=k$;

(3)
$$(1-\mu)S\frac{\partial V_k}{\partial z}\leq\frac{\partial V_k}{\partial y}\leq(1+\lambda)S\frac{\partial V_k}{\partial z},$$

where the maximum is achieved with $\ell=0$ and $m=0$, i.e. by neither buying nor selling.

These remarks suggest that the optimization problem turns out to be a free boundary problem, where, once the value function is known in the five-dimensional space, defined by the state of the investor (t,z,y,S,ν), the optimal trading strategy is determined by the previous inequalities. Moreover, the state space is divided into three regions called the Buy, Sell and No-Transaction regions, characterized by the same previous inequalities. In the limit $k\longrightarrow\infty$ the class of admissible trading strategies becomes the class defined before, see Definition 2.1.

Then we conjecture that the state space remains divided into a Buy, a Sell and a No-Transaction region, where the value function satisfies the following set of equations:

(i) In the Buy region we have

$$V(s,z,y,S,\nu)=V(s,z-(1+\lambda)S\delta y_b,y+\delta y_b,S,\nu),$$

where δy_b, the number of shares bought by the investor, can take any positive value up to the number required to reach the boundary of the Buy region; when $\delta y_b\longrightarrow 0$ the previous equation becomes

$$\frac{\partial V}{\partial y} - (1+\lambda)S\frac{\partial V}{\partial z} = 0.$$

(ii) In the Sell region the value function must satisfy the following equation:

$$V(s,z,y,S,\nu) = V(s, z + (1-\mu)S\delta y_s, y - \delta y_s, S, \nu),$$

where δy_s, the number of shares sold by the investor, can take any positive value up to the number required to reach the boundary of the Sell region. In the limit $\delta y_s \longrightarrow 0$ the previous equation becomes

$$\frac{\partial V}{\partial y} + (1-\mu)S\frac{\partial V}{\partial z} = 0.$$

(iii) In the No-Transaction region the value function is the solution of the following equation:

$$-\frac{\partial V}{\partial t} - \mathcal{L}V = 0 \tag{8}$$

and the following inequalities must hold:

$$(1-\mu)S\frac{\partial V}{\partial z} \leq \frac{\partial V}{\partial y} \leq (1+\lambda)S\frac{\partial V}{\partial z}.$$

A direct inspection of the sign of the left-hand side of Equation (8) suggests that this is positive in both the Buy and the Sell regions, in such a way that the set of equations provided above can be condensed in the following fully nonlinear PDE:

$$\min\left\{-\frac{\partial V}{\partial y} + (1+\lambda)S\frac{\partial V}{\partial z}, \frac{\partial V}{\partial y} - (1-\mu)S\frac{\partial V}{\partial z}, -\frac{\partial V}{\partial t} - \mathcal{L}V\right\} = 0,$$

for $(s,z,y,S,\nu) \in [0,T) \times \mathbb{R} \times \mathbb{R} \times (0,+\infty) \times (0,+\infty)$.

With regard to U, we remark that it follows from the definition of U that

$$U(t,z,y,S,\nu) \geq V_1(t,z,y,S,\nu), \quad \text{on } [0,T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0,+\infty).$$

Therefore, using for U the same arguments just used for V and taking into account the last inequality, we obtain, at least formally, the variational inequality that U must satisfy:

$$\min\left\{U - V_1, -\frac{\partial U}{\partial y} + (1+\lambda)S\frac{\partial U}{\partial z}, \frac{\partial U}{\partial y} - (1-\mu)S\frac{\partial U}{\partial z}, -\frac{\partial U}{\partial t} - \mathcal{L}U\right\} = 0.$$

4. Viscosity properties of the value functions

In the present section we characterize the two value functions V and U as the unique constrained viscosity solutions to the corresponding HJB equations. To this end, we consider a general HJB equation of the form:

$$F\left(t, X, W, \frac{\partial W}{\partial t}, D_X W, D_X^2 W\right) = 0, \quad \text{in } [0,T) \times \mathcal{S}, \tag{9}$$

where $\mathcal{S}$ is an open subset of $\mathbb{R}^n$, moreover $D_X W$ and $D_X^2 W$ are the gradient and the Hessian matrix of W with respect to X, respectively. We write the state vector as $X = (X_1, X_2) \in \mathcal{S}_1 \times \mathcal{S}_2 = \mathcal{S}$, where X_1 includes all the state variables on which some constraint is imposed, while X_2 is the set of state variables which is not subject to any constraints. In our model state X corresponds to (z, y, S, ν), with $X_1 = (z, y, S)$ and $X_2 = \nu$. Moreover, the set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is given by $\mathcal{S}_1 = \mathcal{S}_{\bar{K}}$ and $\mathcal{S}_2 = (0, +\infty)$. We assume that the function $F : [0,T] \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ is continuous and degenerate elliptic, i.e. for all $n \times n$ symmetric matrices $M, \hat{M}$ we have

$$F(t, X, r, q, p, M) \geq F(t, X, r, q, p, \hat{M}), \quad \text{if } M \leq \hat{M}.$$

Now we provide the definition of constrained viscosity solution to (9). For a general overview of the theory of viscosity solutions we refer to the User's Guide by Crandall et al. [6], and to the books by Fleming and Soner [9] and by Pham [13].

Definition 4.1. A continuous function $W : [0,T] \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2 \to \mathbb{R}$ is a constrained viscosity solution of (9) on $[0,T) \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2$ if:

(i) W is a viscosity subsolution of (9) on $[0,T) \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2$, that is for all $(t_0, X_0) \in [0,T) \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2$ and for all $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}}_1 \times \mathcal{S}_2)$ such that (t_0, X_0) is a local maximum point of $W - \varphi$ we have

$$F\left(t_0, X_0, W(t_0, X_0), \frac{\partial \varphi}{\partial t}(t_0, X_0), D_X \varphi(t_0, X_0), D_X^2 \varphi(t_0, X_0)\right) \leq 0.$$

(ii) W is a viscosity supersolution of (9) on $[0,T) \times \mathcal{S}_1 \times \mathcal{S}_2$, that is for all $(t_0, X_0) \in [0,T) \times \mathcal{S}_1 \times \mathcal{S}_2$ and for all $\varphi \in C^{1,2}([0,T) \times \mathcal{S}_1 \times \mathcal{S}_2)$ such that (t_0, X_0) is a local minimum point $W - \varphi$ we have

$$F\left(t_0, X_0, W(t_0, X_0), \frac{\partial \varphi}{\partial t}(t_0, X_0), D_X \varphi(t_0, X_0), D_X^2 \varphi(t_0, X_0)\right) \geq 0.$$

Theorem 4.2. The value function U is a constrained viscosity solution of

$$\min\left\{W - V_1, -\frac{\partial W}{\partial y} + (1+\lambda) S \frac{\partial W}{\partial z}, \frac{\partial W}{\partial y} - (1-\mu) S \frac{\partial W}{\partial z}, -\frac{\partial W}{\partial t} - \mathcal{L} W\right\} = 0, \tag{10}$$

on $[0,T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$.

Proof. We separate the proof into two steps.

(i) *U is a viscosity subsolution of (10).* Let $(t_0, X_0) \in [0,T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$, with $X_0 := (z_0, y_0, S_0, \nu_0)$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty))$ such that (t_0, X_0) is a local maximum point of $U - \varphi$. Without loss of generality, we can assume that $U(t_0, X_0) = \varphi(t_0, X_0)$ and $U \leq \varphi$ on $[0,T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$. We have to prove that

$$\min\left\{\varphi(t_0,X_0)-V_1(t_0,X_0),-\frac{\partial\varphi}{\partial y}(t_0,X_0)+(1+\lambda)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0),\frac{\partial\varphi}{\partial y}(t_0,X_0)\right.$$
$$\left.-(1-\mu)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0),-\frac{\partial\varphi}{\partial t}(t_0,X_0)-\mathcal{L}\varphi(t_0,X_0)\right\}\leq 0.$$

This amounts to say that at least one argument in the minimum operator must be non-positive.

First we observe that $\varphi(t_0,X_0)\geq V_1(t_0,X_0)$, using the definition of U and the equality $U(t_0,X_0)=\varphi(t_0,X_0)$. If $\varphi(t_0,X_0)=V_1(t_0,X_0)$ we get the thesis. Hence, we suppose that

$$\varphi(t_0,X_0)-V_1(t_0,X_0)>0.$$

Now we argue by contradiction assuming that

$$\frac{\partial\varphi}{\partial y}(t_0,X_0)-(1+\lambda)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0)<0, \tag{11}$$

$$\frac{\partial\varphi}{\partial y}(t_0,X_0)-(1-\mu)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0)>0$$

and

$$\frac{\partial\varphi}{\partial t}(t_0,X_0)+\mathcal{L}\varphi(t_0,X_0)<0. \tag{12}$$

From the dynamic programming principle for U we have

$$U(t_0,X_0)=\max\left\{\sup_{\ell\in\mathbb{R}^+}U(t_0,z_0-(1+\lambda)S_0\ell,y_0+\ell,S_0,\nu_0),\right.$$
$$\left.\sup_{m\in\mathbb{R}^+}U(t_0,z_0+(1-\mu)S_0m,y_0-m,S_0,\nu_0)\right\}.$$

Suppose that there exists $\bar{\ell}>0$ such that

$$U(t_0,X_0)=\sup_{\ell\in[\bar{\ell},+\infty)}U(t_0,z_0-(1+\lambda)S_0\ell,y_0+\ell,S_0,\nu_0). \tag{13}$$

Then, using the dynamic programming principle, we deduce that

$$U(t_0,X_0)=U(t_0,z_0-(1+\lambda)S_0\ell,y_0+\ell,S_0,\nu_0),\quad 0\leq\ell\leq\bar{\ell}.$$

From this equality we obtain

$$\varphi(t_0,z_0-(1+\lambda)S_0\ell,y_0+\ell,S_0,\nu_0)-\varphi(t_0,X_0)\geq 0,\quad 0\leq\ell\leq\bar{\ell}.$$

As a consequence, dividing by ℓ and taking the limit as ℓ tends to 0, we get

$$\frac{\partial\varphi}{\partial y}(t_0,X_0)-(1+\lambda)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0)\geq 0,$$

which is in contradiction with (11). We conclude that there does not exist $\bar{\ell} > 0$ such that (13) holds, then

$$U(t_0, X_0) > U(t_0, z_0 - (1+\lambda)S_0\ell, y_0 + \ell, S_0, \nu_0), \quad \forall \ell > 0. \tag{14}$$

In an analogous way we can prove that

$$U(t_0, X_0) > U(t_0, z_0 + (1-\mu)S_0 m, y_0 - m, S_0, \nu_0), \quad \forall m > 0. \tag{15}$$

Now, it remains to show that if (12) holds, too, then we get a contradiction. Note that using the continuity of V_1 and the smoothness of φ we deduce the existence of $\delta > 0$ such that

$$\frac{\partial \varphi}{\partial y}(t,X) - (1+\lambda)S\frac{\partial \varphi}{\partial z}(t,X) < 0$$

and

$$\frac{\partial \varphi}{\partial y}(t,X) - (1-\mu)S\frac{\partial \varphi}{\partial z}(t,X) > 0,$$

for every

$$(t,X) \in \mathcal{B}(t_0,X_0) := (t_0 - \delta, t_0 + \delta) \times B_\delta(X_0) \cap [0,T) \times \bar{S}_{\bar{K}} \times (0,+\infty),$$

where $B_\delta(X_0)$ is the open ball of radius δ centred at X_0. Now, let $\varepsilon > 0$, then using the dynamic programming principle we find two controls L_ε and M_ε and a stopping time τ_ε such that for every stopping time $\tilde{\tau}$ we have

$$\begin{aligned} U(t_0, X_0) \leq \mathbb{E}[&U(\tilde{\tau}, X^{L_\varepsilon,M_\varepsilon}(\tilde{\tau}))\mathbb{1}_{\{\tilde{\tau}\leq\tau_\varepsilon\}} \\ &+ V_1(\tau_\varepsilon, X^{L_\varepsilon,M_\varepsilon}(\tau_\varepsilon))\mathbb{1}_{\{\tilde{\tau}>\tau_\varepsilon\}} | X^{L_\varepsilon,M_\varepsilon}(t_0^-) = X_0] + \varepsilon, \end{aligned} \tag{16}$$

where $X^{L_\varepsilon,M_\varepsilon}$ is the state process corresponding to controls L_ε and M_ε. Let us introduce the following stopping time:

$$\bar{\tau} = \inf\left\{t \in [t_0, T] : (t, X^{L_\varepsilon,M_\varepsilon}(t)) \notin \mathcal{B}(t_0, X_0)\right\}.$$

We have that $\mathbb{P}(\bar{\tau} > t_0) = 1$. Indeed, thanks to (14) and (15), we can choose L_ε and M_ε with no jumps at time t_0. Since $U(t_0, X_0) > V_1(t_0, X_0)$, we note that there exists an event $A \in \mathcal{F}_{t_0}$, with $\mathbb{P}(A) > 0$, such that $\tau_\varepsilon(\omega) > t_0$ for every $\omega \in A$.

Let τ be a stopping time such that $\tau \leq \bar{\tau} \wedge \tau_\varepsilon$. Applying Ito's formula to $\varphi(t, X^{L_\varepsilon,M_\varepsilon}(t))$ we get

$$\mathbb{E}\left[\varphi(\tau, X^{L_\varepsilon,M_\varepsilon}(\tau))|X^{L_\varepsilon,M_\varepsilon}(t_0)=X_0\right] = \varphi(t_0,X_0) + \mathbb{E}\left[\int_{t_0}^{\tau}\left(\frac{\partial\varphi}{\partial y}\left(t, X^{L_\varepsilon,M_\varepsilon}(t)\right)\right.\right.$$

$$- (1+\lambda)S^{L_\varepsilon,M_\varepsilon}(t)\frac{\partial\varphi}{\partial z}(t, X^{L_\varepsilon,M_\varepsilon}(t))\Big)\mathrm{d}L_\varepsilon(t) + \int_{t_0}^{\tau}\left((1-\mu)S^{L_\varepsilon,M_\varepsilon}(t)\frac{\partial\varphi}{\partial z}\left(t, X^{L_\varepsilon,M_\varepsilon}(t)\right)\right.$$

$$\left.\left. -\frac{\partial\varphi}{\partial y}(t, X^{L_\varepsilon,M_\varepsilon}(t))\right)\mathrm{d}M_\varepsilon(t) + \int_{t_0}^{\tau}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t, X^{L_\varepsilon,M_\varepsilon}(t))\mathrm{d}t|X^{L_\varepsilon,M_\varepsilon}(t_0)=X_0\right]$$

$$\leq \varphi(t_0,X_0) + \mathbb{E}\left[\int_{t_0}^{\tau}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t, X^{L_\varepsilon,M_\varepsilon}(t))\mathrm{d}t|X^{L_\varepsilon,M_\varepsilon}(t_0)=X_0\right].$$

Using the fact that $U \leq \varphi$, $U(t_0,X_0)=\varphi(t_0,X_0)$ and inequality (16) with $\tilde{\tau}=\tau$, we find

$$\varphi(t_0,X_0) - \varepsilon \leq \varphi(t_0,X_0) + \mathbb{E}\left[\int_{t_0}^{\tau}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t, X^{L_\varepsilon,M_\varepsilon}(t))\mathrm{d}t|X^{L_\varepsilon,M_\varepsilon}(t_0)=X_0\right].$$

Hence

$$\mathbb{E}\left[\int_{t_0}^{\tau}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t, X^{L_\varepsilon,M_\varepsilon}(t))\mathrm{d}t|X^{L_\varepsilon,M_\varepsilon}(t_0)=X_0\right] \geq -\varepsilon.$$

Let $\varepsilon' > 0$ and define the following stopping time:

$$\tau' = \inf\left\{t \in [t_0,T] : \left|\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t, X^{L_\varepsilon,M_\varepsilon}(t)) - \left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t_0,X_0)\right| > \varepsilon'\right\} \wedge \bar{\tau} \wedge \tau_\varepsilon.$$

Then $\tau'(\omega) > t_0$ for every $\omega \in A$, therefore, choosing $\varepsilon = \varepsilon'\mathbb{E}[\tau' - t_0]$, we find

$$\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(t_0,X_0) \geq -2\varepsilon.$$

From the arbitrariness of ε' we find a contradiction with (12).

(ii) *U is a viscosity supersolution of (10).* Let $(t_0,X_0) \in [0,T)\times\mathcal{S}_{\bar{K}}\times(0,+\infty)$, with $X_0 := (z_0, y_0, S_0, \nu_0)$, and $\varphi \in C^{1,2}([0,T)\times\mathcal{S}_{\bar{K}}\times(0,+\infty))$ such that (t_0,X_0) is a local minimum point of $U-\varphi$. We assume that $U(t_0,X_0)=\varphi(t_0,X_0)$ and $U \geq \varphi$ on $[0,T)\times\mathcal{S}_{\bar{K}}\times(0,+\infty)$. We have to prove that

$$\min\left\{\varphi(t_0,X_0) - V_1(t_0,X_0), -\frac{\partial\varphi}{\partial y}(t_0,X_0) + (1+\lambda)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0), \frac{\partial\varphi}{\partial y}(t_0,X_0)\right.$$

$$\left. -(1-\mu)S_0\frac{\partial\varphi}{\partial z}(t_0,X_0), -\frac{\partial\varphi}{\partial t}(t_0,X_0) - \mathcal{L}\varphi(t_0,X_0)\right\} \geq 0.$$

Therefore we need to show that each argument of the minimum operator is non-negative. Clearly $\varphi(t_0,X_0) \geq V_1(t_0,X_0)$, using the definition of U and the fact that $U(t_0,X_0)=\varphi(t_0,X_0)$.

Now consider the trading strategy: $L(t)=\ell > 0$ and $M(t)=0$, $t_0 \leq t \leq T$. By the dynamic programming principle

$$U(t_0, z_0, y_0, S_0, \nu_0) \geq U(t_0, z_0 - (1+\lambda)S_0\ell, y_0 + \ell, S_0, \nu_0).$$

This inequality holds for φ as well, and, by taking the left-hand side to the right-hand side, dividing by ℓ and sending $\ell \to 0$, we get

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1+\lambda)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) \leq 0.$$

In an analogous way we can prove that

$$\frac{\partial \varphi}{\partial y}(t_0, X_0) - (1-\mu)S_0 \frac{\partial \varphi}{\partial z}(t_0, X_0) \geq 0.$$

Finally, to prove that the last argument inside the minimum operator is positive, consider the following trading strategy: $L(t) = 0$ and $M(t) = 0$, $t_0 \leq t \leq T$. Denote by $X^0(t)$ the corresponding dynamic of the state process. Thanks to the dynamic programming principle we have

$$U(t_0, X_0) \geq \mathbb{E}[U(t, X^0(t)) | X^0(t_0) = X_0], \quad t_0 \leq t \leq T.$$

This inequality also holds for φ. Applying Ito's formula to $\varphi(t, X^0(t))$ we get

$$\mathbb{E}\left[\int_{t_0}^{t} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(s, X^0(s))\mathrm{d}s | X^0(t_0) = X_0\right] \leq 0.$$

Therefore, dividing by $t - t_0$ and sending $t \downarrow t_0$ we deduce the thesis.

We also have the following theorem regarding the value function V, whose proof is not reported, since it is very similar to that of Theorem 4.2.

THEOREM 4.3. The value function V is a constrained viscosity solution of

$$\min\left\{-\frac{\partial W}{\partial y} + (1+\lambda)S\frac{\partial W}{\partial z}, \frac{\partial W}{\partial y} - (1-\mu)S\frac{\partial W}{\partial z}, -\frac{\partial W}{\partial t} - \mathcal{L}W\right\} = 0, \qquad (17)$$

on $[0, T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$.

5. Comparison theorem and uniqueness

In this section we prove a comparison theorem, Theorem 5.1, which allows us to show that the two value functions V and U are the unique constrained viscosity solutions to the corresponding HJB equations. We do this under the additional assumption, very useful also for numerical applications, that the utility function is of exponential type. More precisely, we assume that $\mathcal{U}$ satisfies the following inequality for every $(z, y, S, \nu) \in \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$:

$$\mathcal{U}(z + c(y, S)) \leq M - \mathrm{e}^{-\gamma(z + c(y,S))}, \qquad (18)$$

where M and γ are positive constants.

THEOREM 5.1. Suppose that assumption (18) holds true. Let u be a bounded upper semi-continuous viscosity subsolution of (10) on $[0,T)\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$ and v be a lower semi-continuous function which is bounded from below exhibits sublinear growth and is a viscosity supersolution of (10) on $[0,T)\times\mathcal{S}_{\bar{K}}\times(0,+\infty)$. Suppose that $u(T,X)\leq v(T,X)$ for every $X\in\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$. Then $u\leq v$ on $[0,T]\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$.

Proof. First we construct a positive strict supersolution of (10) on $[0,T]\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$ when $\mathcal{U}$ satisfies (18). Let $\beta>1/2$ be such that $\xi\eta>\beta\vartheta^2$. Then define $h:[0,T]\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)\to\mathbb{R}$ as follows:

$$h(t,z,y,S,\nu)=M-\mathrm{e}^{-\gamma(z+kyS)}+\frac{1}{\nu^{2\beta-1}}+C_1(T-t)+C_2,$$

where C_1 is a positive constant that will be fixed later, while constant k satisfies

$$1-\mu<k<1+\lambda.$$

Finally, C_2 is a positive constant that makes h strictly positive and $h\geq V_1+K'$ on $[0,T]\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$, for some constant $K'>0$ (we observe that V_1 is bounded on $[0,T]\times\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$, thanks to assumption (18)). Note that $h(T,z,y,S,\nu)>\mathcal{U}(z+c(y,S))$, for every $(z,y,S,\nu)\in\bar{\mathcal{S}}_{\bar{K}}\times(0,+\infty)$, taking C_2 large enough. It remains to prove the strict supersolution property. We have

$$\begin{aligned}
H\left(t,X,h,\frac{\partial h}{\partial t},D_Xh,D_X^2h\right)&=\min\left\{h-V_1,-\frac{\partial h}{\partial y}+(1+\lambda)S\frac{\partial h}{\partial z},\frac{\partial h}{\partial y}-(1-\mu)S\frac{\partial h}{\partial z},\right.\\
&\left.-\frac{\partial h}{\partial t}-\mathcal{L}h\right\}\geq\mathrm{e}^{-\gamma(z+kyS)}\min\left\{K'\mathrm{e}^{\gamma(z+kyS)},S(1+\lambda-k),S(k-(1-\mu)),\right.\\
&C_1\mathrm{e}^{\gamma(z+kyS)}-r\gamma(z+kyS)+\frac{1}{2}\nu\gamma^2k^2y^2S^2-(\alpha-r)\gamma kyS\\
&\left.+\xi(\eta-\nu)\frac{2\beta-1}{\nu^{2\beta}}\mathrm{e}^{\gamma(z+kyS)}-\vartheta^2\frac{(2\beta-1)\beta}{\nu^{2\beta}}\mathrm{e}^{\gamma(z+kyS)}\right\}.
\end{aligned}$$

Now we show that we can choose C_1 large enough in such a way that the last argument in the minimum operator is strictly positive. Note that the function $D(\zeta)=\nu\gamma^2k^2\zeta^2/2-(\alpha-r)\gamma k\zeta$ has minimum value equal to $-(\alpha-r)^2/(2\nu)$. Consequently, the last argument inside the minimum operator is greater than or equal to the following expression:

$$\left(C_1-\frac{(\alpha-r)^2}{2\nu}-\frac{\xi(2\beta-1)}{\nu^{2\beta-1}}+\frac{(\xi\eta-\beta\vartheta^2)(2\beta-1)}{\nu^{2\beta}}\right)\mathrm{e}^{\gamma(z+kyS)}-r\gamma(z+kyS).$$

Since $\xi\eta>\beta\vartheta^2$, the function $G(\nu)=-((\alpha-r)^2)/(2\nu)-\xi(2\beta-1)/\nu^{2\beta-1}+(\xi\eta-\beta\vartheta^2)(2\beta-1)/\nu^{2\beta}$ is bounded from below by a constant: $G(\nu)\geq-A$ for every $\nu>0$, where A is a positive constant. Take $C_1=A+C_3$, where C_3 is a positive constant that will be fixed below. Then

$$\left(C_1 - \frac{(\alpha - r)^2}{2\nu} - \frac{\xi(2\beta - 1)}{\nu^{2\beta-1}} + \frac{(\xi\eta - \beta\vartheta^2)(2\beta - 1)}{\nu^{2\beta}}\right) e^{\gamma(z+kyS)} - r\gamma(z + kyS)$$
$$\geq C_3 e^{\gamma(z+kyS)} - r\gamma(z + kyS).$$

We can choose C_3 large enough so that the function $F(x) = C_3 e^x - rx$, with $x \geq -\gamma\bar{K}$, is bounded from below by a constant and, in particular, is strictly positive. In conclusion, we have proved that there exists a strictly positive constant δ such that

$$H\left(t, X, h, \frac{\partial h}{\partial t}, D_X h, D_X^2 h\right) \geq \delta,$$

on $[0, T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$.

To conclude the proof of the theorem, define the function $w^\varepsilon = (1 - \varepsilon)v + \varepsilon h$, with $0 < \varepsilon < 1$. Then $u(T, X) \leq w^\varepsilon(T, X)$ for every $X \in \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$. Moreover w^ε is a viscosity supersolution of $H - \varepsilon\delta = 0$ on $[0, T) \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$. Now we may apply Lemma 4.1 in [8] to u and w^ε and we deduce that $u \leq w^\varepsilon$ on $[0, T] \times \bar{\mathcal{S}}_{\bar{K}} \times (0, +\infty)$. Therefore, sending $\varepsilon \downarrow 0$, we get the thesis. □

COROLLARY 5.2. Under assumption (18), the value functions U and V are the unique constrained viscosity solutions to (10) and (17), respectively.

Proof. Thanks to assumption (18) we have that both U and V are bounded. Therefore we may apply the comparison Theorem 5 and we deduce the thesis. □

6. Negative exponential utility, dimensionality reduction and numerical discretization

In this section we assume a particular expression for the utility function describing the preferences of the investor. More precisely, we suppose that the utility function is a negative exponential utility of the following kind:

$$\mathcal{U}(x) = -\exp(-\gamma x).$$

Thanks to this assumption the dimensionality of the problem can be substantially reduced. Moreover, the solution of the optimization problem does not depend on the investor's initial wealth. This kind of utility function describes the preferences of an investor exhibiting constant risk-aversion, for this reason is sometimes called a constant absolute risk vversion utility function. This choice has been also adopted in [7,17] dealing, respectively, with European and American option pricing with transaction costs with constant volatility.

This specific choice seems restrictive to some extent, but [1] has shown that the dependence of option prices on the specific form of the utility function is very weak. Therefore we decided to stick on this particular choice, which greatly simplifies the computational procedure.

Now we introduce the following discount factor:

$$\delta(t, T) = \exp(-r(T - t)),$$

and the 'reduced utility functions':

$$U(t,z,y,S,\nu) = \exp\left(-\gamma\frac{z}{\delta(t,T)}\right)Q(t,y,S,\nu),$$
$$V(t,z,y,S,\nu) = \exp\left(-\gamma\frac{z}{\delta(t,T)}\right)Q_0(t,y,S,\nu),$$
$$V_1(t,z,y,S,\nu) = \exp\left(-\gamma\frac{z}{\delta(t,T)}\right)Q_1(t,y,S,\nu).$$

Remark 1. In order to slightly simplify the notation, and in analogy with [17], in the definition of U and V_1 provided by (7) and (6), respectively, we have assumed that a single option is purchased by the investor, in such a way that $x_1 = p$ and that, when the option exercise takes place, only the z argument in functions U and V_1 changes by the amount $g(S) := \max(K - S, 0)$, while the y argument is left unchanged. According to this assumption, the definitions of V_1 and U can be reformulated as follows:

$$V_1(t,z,y,S,\nu) = V\big(t, z + g(S), y, S, \nu\big),$$

which implies

$$Q_1(t,z,y,S,\nu) = \exp\left(-\gamma\frac{g(S)}{\delta(t,T)}\right)Q_0(t,y,S,\nu) \tag{19}$$

and

$$U(t,z,y,S,\nu) = \sup_{\mathcal{A}_{t,T},\tau}\mathbb{E}[V_1(\tau, z(\tau), y(\tau), S(\tau), \nu(\tau))\big|(z(t^-), y(t^-), S(t^-), \nu(t^-))$$
$$= (z,y,S,\nu)].$$

The purchase price of an American option simply becomes the value p such that

$$V(t,z,y,S,\nu) = U(t, z-p, y, S, \nu),$$

i.e.

$$p = \frac{\delta(t,T)}{\gamma}\log\left(\frac{Q_0(t,y,S,\nu)}{Q(t,y,S,\nu)}\right).$$

As a consequence, we may express the variational inequalities for U and V in terms of Q, Q_0 and Q_1, obtaining

$$\min\left\{Q - Q_1, -\frac{\partial Q}{\partial y} - \gamma(1+\lambda)S\frac{Q}{\delta(t,T)}, \frac{\partial Q}{\partial y} + (1-\mu)S\frac{Q}{\delta(t,T)}, -\frac{\partial Q}{\partial t} - \mathcal{D}W\right\} = 0 \tag{20}$$

and

$$\min\left\{-\frac{\partial Q_0}{\partial y} - \gamma(1+\lambda)S\frac{Q_0}{\delta(t,T)}, \frac{\partial Q_0}{\partial y} + (1-\mu)S\frac{Q_0}{\delta(t,T)}, -\frac{\partial Q_0}{\partial t} - \mathcal{D}W\right\} = 0, \tag{21}$$

where operator $\mathcal{D}$ is defined as follows:

$$\mathcal{D}Q = \alpha S\frac{\partial Q}{\partial S} + \frac{1}{2}\nu S^2\frac{\partial^2 Q}{\partial S^2} + \xi(\eta - \nu)\frac{\partial Q}{\partial \nu} + \frac{1}{2}\vartheta^2\nu\frac{\partial^2 Q}{\partial \nu^2} + \rho\vartheta\nu S\frac{\partial^2 Q}{\partial S\partial \nu},$$

and Q_1 is given by (19). Now, functions $Q(t, y, S, \nu)$ and $Q_0(t, y, S, \nu)$ are defined on a four-dimensional space $[0, T] \times \mathbb{R} \times (0, +\infty) \times (0, +\infty)$. The terminal conditions are given by

$$Q_0(T, y, S, \nu) = -\mathrm{e}^{-\gamma c(y,S)} \quad \text{and} \quad Q(T, y, S, \nu) = -\mathrm{e}^{-\gamma\left(g(S)+c(y,S)\right)}.$$

6.1. *Discretization and solution of the problem*

As in [7,17], we can couple the variational HJB inequalities (20) and (21) with a Markov chain approximation. More precisely, in both [7,17], the authors deal with the classical lognormal model, solving the pricing problem with a binomial model. Moreover, in [17] the author provides an alternative characterization of the value function which is based on a global maximum, and that is well suited for the application of the Markov chain approximation technique.

Since in this article we deal with stochastic volatility, first of all we have to introduce a tree-based method to price American options (without transaction costs) in the Heston model. Then we can easily couple this method with the variational HJB inequalities (20) and (21) (or their alternative characterization [17]) exploiting the Markov chain approximation as in [7,17].

We consider the tree-based model presented in [15]: the pricing approach is based on a modification of a combined tree for stock prices and volatilities, where the value of the derivative is computed on a two-dimensional grid (in stock and volatility) at each time step, exploiting interpolating techniques. In all our experiments we deal with the bilinear interpolation technique suggested in [15]. This pricing procedure allows to circumvent the problem of dealing with non-recombining tree, which often happens when dealing with lattice methods for the Heston model.

Coupling the model presented in [15] with the Markov chain approximation applied, among the others, in [7,17], we obtain the following procedure to compute Q and Q_0. Let us consider a discrete time grid $\{0, \delta t, 2\delta t, \ldots, N\delta t\}$ with $N = T/\delta t$, T being the maturity of the American derivative. The Markov chain for the discrete stock price $S(t)$ and volatility $\nu(t)$ processes are modelled according to [15], i.e.

$$\nu((i+1)\delta t) = \max\Big(\nu(i\delta t) + \xi\big(\eta - \nu(i\delta t)\big)\delta t + Y^1\vartheta\sqrt{\nu(i\delta t)\delta t}, 0\Big),$$

$$S((i+1)\delta t) = S(i\delta t)\mathrm{e}^{(\alpha-(1/2)\nu(i\delta t))\delta t + Y^2\sqrt{\nu(i\delta t)\delta t}},$$

where Y^1, Y^2 have values in $\{-1, 1\}$. We refer to [15] for the distribution of the two-dimensional random variable (Y^1, Y^2) and for the construction of the two-dimensional binomial tree, avoiding the problems related to the fact that the considered tree is not recombining. Moreover, the discrete time equation for the amount invested in the risk-free asset is

$$z((i+1)\delta t) = z(i\delta t)\mathrm{e}^{r\delta t}.$$

Following [7], after defining a grid for the number of shares, i.e. $y = y_j = j\delta y, j = -J, \ldots, J$, the discretization scheme for the HJB equation (21) is

$$
\begin{aligned}
Q_0(i\delta t, y_j, S(i\delta t), \nu(i\delta_t)) = \max(&\mathbb{E}[Q_0((i+1)\delta t, y_j, S((i+1)\delta t), \nu((i+1)\delta_t))], \\
&F_b(i\delta t, \delta y, S(i\delta t))Q_0(i\delta t, y_{j+1}, S(i\delta t), \nu(i\delta_t)), \\
&F_s(i\delta t, \delta y, S(i\delta t))Q_0(i\delta t, y_{j-1}, S(i\delta t), \nu(i\delta_t))),
\end{aligned}
\tag{22}
$$

with

$$
F_b(t, \delta y, S) = \mathrm{e}^{\gamma((1+\lambda)\delta y S/\delta(t,T))}, \quad \text{and} \quad F_s(t, \delta y, S) = \mathrm{e}^{-\gamma((1-\mu)\delta y S/\delta(t,T))};
$$

and

$$
Q_0(i\delta t, y_{j\pm 1}, S(i\delta t), \nu(i\delta_t)) = \mathbb{E}\left[Q_0((i+1)\delta t, y_{j\pm 1}, S((i+1)\delta t), \nu((i+1)\delta_t))\right],
$$

where the expected values are computed exploiting the two-dimensional binomial tree [15]. Notice that the first line in (22) corresponds to do nothing, while the second (third) one corresponds to buy (sell) δy shares of the stock. Similarly, we have

$$
\begin{aligned}
Q(i\delta t, y_j, S(i\delta t), \nu(i\delta_t)) = \max\Big(&\mathbb{E}\left[Q((i+1)\delta t, y_j, S((i+1)\delta t), \nu((i+1)\delta_t))\right], \\
&F_b(i\delta t, \delta y, S(i\delta t))Q(i\delta t, y_{j+1}, S(i\delta t), \nu(i\delta_t)), \\
&F_s(i\delta t, \delta y, S(i\delta t))Q(i\delta t, y_{j-1}, S(i\delta t), \nu(i\delta_t)), \\
&Q_1(i\delta t, y_j, S(i\delta t), \nu(i\delta_t))\Big),
\end{aligned}
$$

where the last line corresponds to the early exercise of the American contract, and can be evaluated exploiting (19) and function Q_0 computed according to (22). The above discretization procedure works also for the alternative characterization of the value function based on a global maximum [17]: the main difference is that, to compute Q_0 (and similarly to compute Q), (22) is replaced by

$$
\begin{aligned}
&Q_0(i\delta t, y_j, S(i\delta t), \nu(i\delta_t)) = \max(\mathbb{E}[Q_0((i+1)\delta t, y_j, S((i+1)\delta t), \nu((i+1)\delta_t))], \\
&\max_{l=1,\dots,J-j} F_b(i\delta t, l\delta y, S(i\delta t))Q_0(i\delta t, y_j + l\delta y, S(i\delta t), \nu(i\delta_t)), \\
&\max_{l=-J-j,\dots,-1} F_s(i\delta t, -l\delta y, S(i\delta t))Q_0(i\delta t, y_j + l\delta y, S(i\delta t), \nu(i\delta_t))),
\end{aligned}
$$

i.e. selling or buying all the possible number of shares of the stock (remaining on the grid $y = y_j = j\delta y, j = -J, \dots, J$) and not only δy shares (therefore the name 'global maximum' [17]). From a financial point of view, this last approach is more reasonable, since in computing the numerical solution we consider all the possible strategies the agent can implement, i.e. doing nothing, buy or sell any number of shares, or early exercise the option.

7. Numerical results

The algorithm developed in the previous section was implemented, computing the price of an American put contract. In our numerical experiments we deal with the parameters considered in [15], i.e. $r = \alpha = 0.1$, and $S(0) = 9$, $\nu(0) = 0.0625$, $\eta = 0.16$, $\vartheta = 0.9$, $\xi = 5$ and $\rho = 0.1$. Moreover, the American put has strike $K = 10$ and maturity $T = 0.25$. The discretization parameters of the Markov Chain are $\delta t = 0.007$, $\delta y = 0.2$ and $J = 50$.

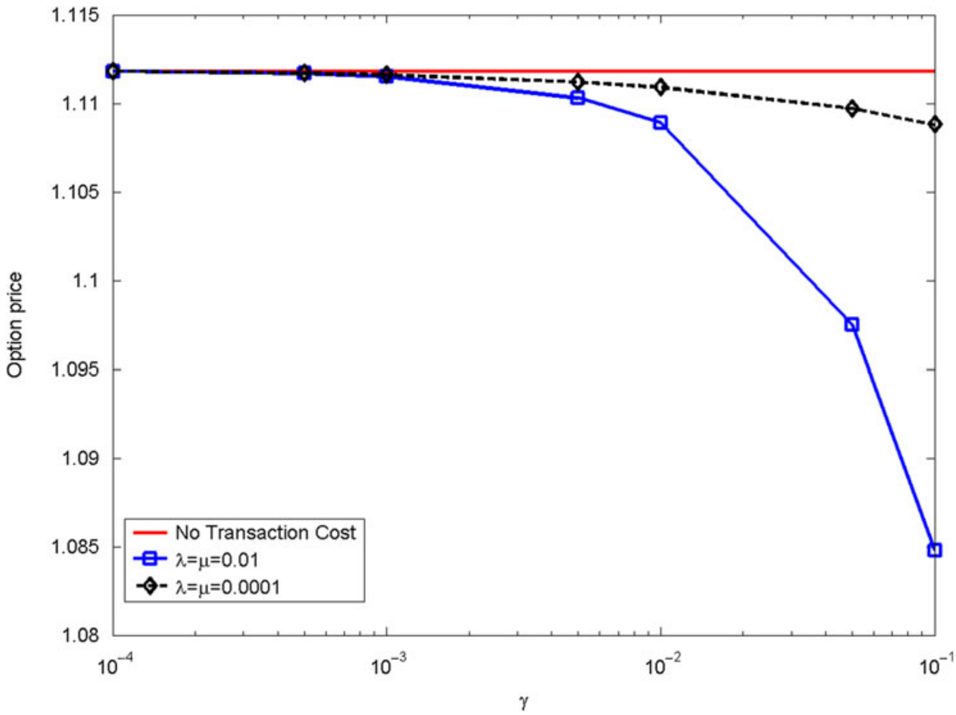

Figure 1. Price of an American put option plotted against the agent absolute risk-aversion.

In Figure 1 we plot the American put price for different values of parameter γ and considering two different sets of proportional costs: $\lambda = \mu = 1\%$ and $\lambda = \mu = 0.01\%$, while in Table 1 we deal with the influence of proportional transaction costs on the option price setting $\gamma = 0.1$.

It is clear that the option price decreases when both the proportional transaction costs and the absolute risk-aversion γ increase. These results are in line with what is presented in [17] when a classical lognormal model is considered. Therefore, as expected, moving from the lognormal model to the Heston stochastic volatility model does not change the behaviour of the derivative price with respect to γ and the proportional costs' parameters λ and μ. Moreover, decreasing λ, the option price approaches to its value when no transaction costs are considered.

To conclude, in Figure 2 we show the early exercise boundary at time $(T/2)$ in the space (S, ν), i.e. underlying asset and variance, dealing with the same American put as above and considering different values of the risk-aversion parameter γ and the proportional transaction costs $\lambda = \mu$. As expected, the early exercise boundary moves up

Table 1. Price of an American put option, $\gamma = 0.1$.

	$\lambda = \mu$				
	0.0001	0.0005	0.001	0.005	0.01
Price	1.1088	1.1033	1.0988	1.0848	1.0848

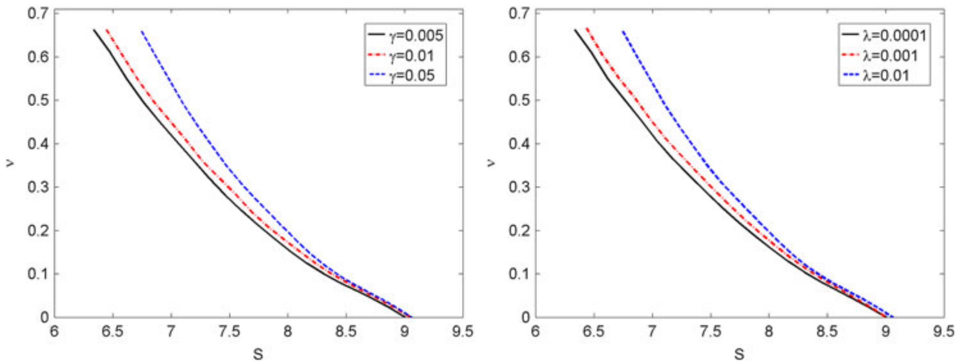

Figure 2. Early exercise boundary at time $T/2$, setting $\lambda = \mu = 0.01$ (left) and $\gamma = 0.05$ (right).

as both $\lambda = \mu$ and γ increase, i.e. as both the proportional transaction costs and the risk-aversion increase.

8. Concluding remarks

In this paper we investigated the American option pricing valuation problem in a continuous-time financial model in which transaction costs are considered and the volatility is assumed to be described by a stochastic process of a CIR type, as in the Heston model. We provided a formulation of this problem as a singular control problem for which we proved existence and uniqueness of a viscosity solution. By assuming a specific assumption on the utility function describing the investor's preferences, and after reformulating our singular control problem through a variational inequality, we also presented a discretization method and some numerical results. The results achieved in this work may be extended in several directions. First, different transaction costs models can be considered, as an example fixed transaction costs, and different stochastic dynamics for the volatility can be assumed, as an example the Stein–Stein model, see [14]. Moreover, a more systematic numerical investigation could be performed in order to provide results also in different modelling frameworks like those just mentioned. All these issues will be the topics of our future investigation.

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