Nonclassical diffusion with memory

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We consider the nonclassical diffusion equation with hereditary memory

$$\boldsymbol{u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)\,\mathrm{d}s + \varphi(u) = f}$$

on a bounded three-dimensional domain. Setting the problem in the past history framework, and with very general assumptions on the memory kernel κ, we prove that the related solution semigroup possesses a global attractor of optimal regularity.

Keywords: nonclassical diffusion; hereditary memory; global attractors of optimal regularity

1. Introduction

1.1. The equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider for $t > 0$ the equation in the unknown variable $u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$

$$u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)\,\mathrm{d}s + \varphi(u) = f, \tag{1.1}$$

subject to Dirichlet boundary conditions

$$u_{|\partial\Omega} = 0. \tag{1.2}$$

The problem is supplemented with the initial conditions

$$u(0) = u_0 \qquad \text{and} \qquad u(-s)_{|s>0} = g_0(s), \tag{1.3}$$

where $u_0 : \Omega \to \mathbb{R}$ and $g_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are given functions, the latter accounting for the initial past history of u.

1.2. Assumptions on the constitutive terms

Let the (time-independent) external force f belong to the space $L^2(\Omega)$, and let the nonlinearity $\varphi \in \mathcal{C}(\mathbb{R})$ fulfill $\varphi(0) = 0$ along with the growth restriction

$$|\varphi(u) - \varphi(v)| \leq c|u-v|\left(1 + |u|^4 + |v|^4\right) \tag{1.4}$$

and the dissipation condition

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1, \tag{1.5}$$

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where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet operator $-\Delta$. The convolution (or memory) kernel κ is a nonnegative summable function having the explicit form

$$\kappa(s) = \int_s^\infty \mu(\sigma)\,\mathrm{d}\sigma,$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing (hence nonnegative) piecewise absolutely continuous function. In particular, μ is allowed to exhibit (infinitely many) jumps. Moreover, we require that

$$\kappa(s) \le \Theta\mu(s), \tag{1.6}$$

for some $\Theta > 0$ and every $s > 0$. As shown in [1], this is completely equivalent to the requirement that

$$\mu(\sigma + s) \le M\mathrm{e}^{-\delta\sigma}\mu(s), \tag{1.7}$$

for some $M \ge 1$, $\delta > 0$, every $\sigma \ge 0$ and almost every $s > 0$. Our analysis, however, will not exclude the degenerate case $\kappa \equiv 0$.

1.3. *A brief overview of the problem*

Equation (1.1) arises in the classical diffusion theory when assuming that the diffusing species behaves as a linearly viscous fluid, which leads to include its velocity gradient in the constitutive laws [2]. Besides, the convolution term takes into account the influence of the past history of u on its future evolution, providing a more accurate description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers (see, e.g., [3]). This model, with special regard to the case $\kappa \equiv 0$ corresponding to the nonclassical diffusion equation

$$u_t - \Delta u_t - \Delta u + \varphi(u) = f, \tag{1.8}$$

has deserved some attention in the literature during the last years (see, e.g., [4–9]). In all the papers dealing with the memory relaxation (1.1) of (1.8), the kernel μ is assumed to satisfy the inequality

$$\mu'(s) + \delta\mu(s) \le 0, \tag{1.9}$$

introduced in the seminal article [10] and commonly adopted thereafter. However, as shown in [11], the condition (1.7) of the present paper turns out to be the most general one, for its failure prevents the uniform decay of solutions to systems with memory, no matter how are the equations involved. It is also evident that (1.7) with $M = 1$ boils down to (1.9), whereas the same condition for $M > 1$ is much more general. Indeed, any compactly supported decreasing μ fulfills (1.7) for some $M > 1$, but it cannot comply with (1.9) if, for instance, exhibits flat zones. Accordingly, (1.7) allows to treat a greater variety of models with memory, including equations with delay of the form

$$u_t - \Delta u_t - \Delta u - \int_0^T \Delta u(t-s)\,\mathrm{d}s + \varphi(u) = f.$$

In the recent work [8], rephrasing the problem in the past history framework of Dafermos [10], and assuming (1.9), a dissipative solution semigroup $S(t)$ is obtained, acting on a suitable phase space accounting for the past values of the variable u. The semigroup is shown to possess the global attractor. Actually, the authors of [8] take $f \in H^{-1}(\Omega)$ and use a simple approximation argument with a sequence $f_n \in L^2(\Omega)$ such that $f_n \to f$. However, no regularity of the attractor is given, not even for $f \in L^2(\Omega)$. In fact, such a regularity when $f \in L^2(\Omega)$ is implicitly proved in [9], which deals with the nonautonomous case.

In this note, our aim is twofold. Firstly, we prove the existence of the attractor within the more general (in fact, the most general) condition (1.6) on the kernel. Secondly, we establish the optimal regularity of the attractor (also for the case $\kappa \equiv 0$) in a rather direct way, simplifying quite drastically the argument used in [9], which is imported from the analogous paper without memory [5].

We finally mention that the analysis of this paper can also be made within a different approach, namely, working in the so-called minimal state framework [12], rather than in the past history one. Quite interestingly, the results in the minimal state framework can be deduced from the corresponding ones in the past history setting proved here (see [13–15] for a comparison). This might be the object of a future note.

Remark 1.1

It would be also interesting to consider (1.1) with free parameters $\alpha, \beta \ge 0$, that is,

$$u_t - \alpha\Delta u_t - \beta\Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)\,\mathrm{d}s + \varphi(u) = f.$$

Clearly, the picture significantly changes, depending whether or not α and β vanish. Here, we limit ourselves to mention the degenerate case $\alpha = \beta = 0$, firstly considered in [16]. In that situation, as noted by the authors, there is not in general enough dissipation in the system. Nonetheless, stability results have been obtained in [17, 18] in space dimension one and two, provided that the memory kernel is sufficiently concentrated at zero (see also [19] for the linear case).

2. Functional setting and notation

2.1. General notation

Given a Banach space $\mathcal{X}$, we denote by $\mathrm{B}_{\mathcal{X}}(R)$ the closed ball in $\mathcal{X}$ of radius $R \geq 0$, that is,

$$\mathrm{B}_{\mathcal{X}}(R) = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq R\}.$$

Moreover, we denote by

- $\mathfrak{J}$ the space of continuous increasing functions $J : [0,\infty) \to [0,\infty)$,
- $\mathfrak{D}$ the space of continuous decreasing functions $d : [0,\infty) \to [0,\infty)$ with $d(\infty) < 1$.

Throughout the paper, $c \geq 0$ will stand for a generic constant. We will also use, often without explicit mention, the Sobolev embeddings, as well as the Young, Hölder, and Poincaré inequalities.

2.2. Geometric spaces

Introducing the strictly positive Dirichlet operator

$$A = -\Delta, \qquad \mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega),$$

we define the scale of compactly nested Hilbert spaces

$$\mathrm{H}^r = \mathrm{dom}\left(A^{r/2}\right), \quad r \in \mathbb{R}$$

with inner products and norms given by

$$\langle u, v\rangle_r = \left\langle A^{r/2}u, A^{r/2}v\right\rangle_{L^2(\Omega)} \qquad \text{and} \qquad \|u\|_r = \left\|A^{r/2}u\right\|_{L^2(\Omega)}.$$

We will always omit the index r whenever $r = 0$. The symbol $\langle\cdot,\cdot\rangle$ will also stand for the duality product between H^r and its dual space H^{-r}. We recall the relations

$$\mathrm{H} = L^2(\Omega), \quad \mathrm{H}^1 = H_0^1(\Omega), \quad \mathrm{H}^2 = H^2(\Omega) \cap H_0^1(\Omega),$$

along with the generalized Poincaré inequalities

$$\lambda_1 \|u\|_r^2 \leq \|u\|_{r+1}^2,$$

where $\lambda_1 > 0$ is the first eigenvalue of A.

2.3. History spaces

For $r \in \mathbb{R}$, we introduce the so-called history spaces (r is omitted if zero)

$$\mathcal{M}^r = L^2_\mu\left(\mathbb{R}^+; \mathrm{H}^{r+1}\right)$$

endowed with the weighted L^2-inner products

$$\langle \eta, \xi\rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s)\langle\eta(s), \xi(s)\rangle_{r+1}\,\mathrm{d}s.$$

We will also consider the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, namely, the linear operator

$$T\eta = -\eta', \qquad \mathrm{dom}(T) = \left\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\right\},$$

the *prime* standing for weak derivative. The following inequality holds [11, 20]

$$\langle T\eta, \eta\rangle_{\mathcal{M}} \leq 0, \quad \forall \eta \in \mathrm{dom}(T). \tag{2.1}$$

We recall that if $u \in L^1_{\mathrm{loc}}(0,\infty;\mathrm{H}^1)$, then for every $\eta_0 \in \mathcal{M}$, the Cauchy problem on $[0,\infty)$

$$\begin{cases} \eta_t = T\eta + u \\ \eta^0 = \eta_0 \end{cases}$$

has a unique solution $\eta \in \mathcal{C}([0,\infty), \mathcal{M})$ given by

$$\eta^t(s) = \begin{cases} \int_0^s u(t-\sigma)\,\mathrm{d}\sigma & s \leq t, \\ \eta_0(s-t) + \int_0^t u(t-\sigma)\,\mathrm{d}\sigma & s > t. \end{cases} \tag{2.2}$$

Finally, we introduce the *extended history spaces* (again, r is omitted if zero).

$$\mathcal{H}^r = \mathrm{H}^{r+1} \times \mathcal{M}^r.$$

3. The dissipative semigroup

Following the approach of Dafermos [10], the original problem (1.1)–(1.2) can be translated into the system in the unknown variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} u_t + Au_t + Au + \displaystyle\int_0^\infty \mu(s) A\eta(s)\,\mathrm{d}s + \varphi(u) = f, \\ \eta_t(s) = T\eta(s) + u, \end{cases} \tag{3.1}$$

where the initial conditions (1.3) become

$$u(0) = u_0 \qquad \text{and} \qquad \eta^0(s) = \int_0^s g_0(\sigma)\,\mathrm{d}\sigma.$$

At a formal level, this is obtained by defining the auxiliary variable

$$\eta = \eta^t(x,s) : \Omega \times [0,\infty) \times \mathbb{R}^+ \to \mathbb{R},$$

accounting for the *integrated past history* of u, as

$$\eta^t(s) = \int_0^s u(t-\sigma)\,\mathrm{d}\sigma, \tag{3.2}$$

Actually, the correspondence between the original problem and its reformulation in the history framework is not only formal but can be rigourously justified with the proper functional setting (see [20] for more details).

The well-posedness result, stated in the next theorem, follows by a standard application of a Galerkin scheme (see, e.g., [8]).

Theorem 3.1
System (3.1) generates a strongly continuous semigroup $S(t) : \mathcal{H} \to \mathcal{H}$. Thus, for every $t \geq 0$ and every $z = (u_0, \eta_0) \in \mathcal{H}$,

$$S(t)z = \big(u(t), \eta^t\big)$$

is the unique solution at time t to (3.1) with initial datum z. In addition,

$$u_t \in L^2_{\mathrm{loc}}\left(0,\infty;\mathrm{H}^1\right),$$

while η fulfills the explicit representation formula (2.2).

In fact, the semigroup $S(t)$ turns out to map $\mathcal{H}^r$ into $\mathcal{H}^r$ for every $r \in [0,1]$. Accordingly, for any fixed initial datum $z \in \mathcal{H}^r$, we denote the corresponding r-energy at time t by (as usual, r is omitted whenever zero)

$$E_r(t) = \frac{1}{2}\left[\|u(t)\|^2_{r+1} + \|u(t)\|^2_r + \|\eta^t\|^2_{\mathcal{M}^r}\right].$$

This functional, clearly equivalent to the $\mathcal{H}^r$-norm of the solution, is what one obtains when performing the natural energy estimates.

The dissipative character of $S(t)$ is witnessed by the existence of a bounded absorbing set, capable to capture in finite time all trajectories originating from any given bounded set of initial data.

Theorem 3.2
There exist $c_0 \geq 0$ and $\varepsilon_0 > 0$ such that

$$E(t) \leq 2E(0)\mathrm{e}^{-\varepsilon_0 t} + c_0,$$

for all initial data in $\mathcal{H}$.

Proof
In light of (1.5), there is $\nu > 0$ (possibly very small) such that

$$\langle \varphi(u), u\rangle \geq -(1-\nu)\|u\|_1^2 - c.$$

Accordingly, the function

$$\gamma = f - \varphi(u)$$

is easily seen to satisfy the inequality

$$\langle \gamma, u\rangle \leq a\|u\|_1^2 + q,$$

for some $a \in [0,1)$ and $q \geq 0$. The claim follows by applying Lemma A.1 with $r = 0$, $h(t) \equiv 0$ and $Q(t) \equiv q$. □

As a straightforward consequence, there exists $R_0 > 0$ such that the ball $\mathbb{B}_0 = \mathrm{B}_{\mathcal{H}}(R_0)$ is an absorbing set for $S(t)$ on $\mathcal{H}$. This means that for any $R > 0$, there is an entering time $t_R \geq 0$ for which

$$S(t)\mathrm{B}_{\mathcal{H}}(R) \subset \mathbb{B}_0, \quad \forall t \geq t_R.$$

4. The global attractor

4.1. Statement of the result

The main theorem of this section is the following.

Theorem 4.1
The semigroup $S(t)$ possesses the global attractor $\mathfrak{A}$.

By definition, the global attractor of $S(t)$ is the unique compact set $\mathfrak{A} \subset \mathcal{H}$ which is at the same time fully invariant and attracting for the semigroup (see, e.g., [21–25]). Namely,

(i) $S(t)\mathfrak{A} = \mathfrak{A}$ for every $t \geq 0$ and
(ii) for every $R \geq 0$,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathrm{B}_{\mathcal{H}}(R), \mathfrak{A}) = 0,$$

where $\mathrm{dist}_{\mathcal{H}}$ denotes the standard Hausdorff semidistance in $\mathcal{H}$. We also recall that for an arbitrarily fixed $\tau \in \mathbb{R}$, the global attractor can be given the form [24]

$$\mathfrak{A} = \{Z(\tau) : Z \text{ CBT}\},$$

where a complete bounded trajectory (CBT) of the semigroup is a function $Z \in \mathcal{C}_{\mathrm{b}}(\mathbb{R}, \mathcal{H})$ satisfying

$$Z(\tau) = S(t)Z(\tau - t), \quad \forall t \geq 0,\ \forall \tau \in \mathbb{R}.$$

4.2. Proof of Theorem 4.1

Appealing to a general result of the theory of infinite dimensional dynamical systems (see, e.g., [21–25]), it is enough to show that the semigroup is asymptotically compact. This will be obtained in two steps, exploiting in a crucial way the technical Lemma A.1 in Appendix. In what follows, $\mathbb{B}_0$ is a bounded absorbing set, whose existence has been established in Theorem 3.2.

Proposition 4.2
For any $t \geq 0$, there exists a closed bounded set $\mathcal{B}(t) \subset \mathcal{H}^{1/3}$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathcal{B}(t)) \leq c_1 \mathrm{e}^{-\omega t},$$

for some constants $c_1 \geq 0$ and $\omega > 0$.

Proof
We prove the proposition by showing that for any $z \in \mathbb{B}_0$, we can split the solution $S(t)z = (u(t), \eta^t)$ into the sum

$$S(t)z = \left(\hat{v}(t), \hat{\xi}^t\right) + \left(\hat{w}(t), \hat{\zeta}^t\right)$$

whose terms satisfy the estimates

$$\left\| \left(\hat{v}(t), \hat{\xi}^t\right) \right\|_{\mathcal{H}} \leq c_1 \|z\|_{\mathcal{H}} \mathrm{e}^{-\omega t}, \tag{4.1}$$

and

$$\left\| \left(\hat{w}(t), \hat{\zeta}^t\right) \right\|_{\mathcal{H}^{1/3}} \leq J(t), \tag{4.2}$$

for some $c_1 \geq 0$, $\omega > 0$ and $J \in \mathfrak{J}$. In which case, the closed ball

$$\mathcal{B}(t) = \mathrm{B}_{\mathcal{H}^{1/3}}(J(t))$$

fulfills the claim.

The assumptions on the nonlinearity φ allow us to write (see, e.g., [26])

$$\varphi(u) = \varphi_0(u) + \varphi_1(u),$$

for some continuous functions φ_0, φ_1 satisfying

$$\varphi_0(u)u \geq 0, \qquad |\varphi_0(u) - \varphi_0(v)| \leq c|u - v|(|u| + |v|)^4,$$
$$\liminf_{|u|\to\infty} \frac{\varphi_1(u)}{u} > -\lambda_1, \qquad |\varphi_1(u)| \leq c(1 + |u|).$$

For $z \in \mathbb{B}_0$, let $\left(\hat{v}(t), \hat{\xi}^t\right)$ and $\left(\hat{w}(t), \hat{\zeta}^t\right)$ be the solutions to the problems

$$\begin{cases} \hat{v}_t + A\hat{v}_t + A\hat{v} + \displaystyle\int_0^\infty \mu(s) A\hat{\xi}(s)\,\mathrm{d}s + \varphi_0(\hat{v}) = 0, \\ \hat{\xi}_t = T\hat{\xi} + \hat{v}, \\ \left(\hat{v}(0), \hat{\xi}^0\right) = z, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \hat{w}_t + A\hat{w}_t + A\hat{w} + \displaystyle\int_0^\infty \mu(s)A\hat{\zeta}(s)\,\mathrm{d}s = \hat{g}, \\ \hat{\zeta}_t = T\hat{\zeta} + \hat{w}, \\ \left(\hat{w}(0), \hat{\zeta}^0\right) = 0, \end{cases} \tag{4.4}$$

where

$$\hat{g} = f - \varphi_0(u) + \varphi_0(\hat{v}) - \varphi_1(u).$$

The proof of (4.1) follows at once by Lemma A.1 for $r = 0$ and $\gamma = -\varphi_0(\hat{v})$. Indeed, because

$$\langle \gamma, \hat{v} \rangle = -\langle \varphi_0(\hat{v}), \hat{v} \rangle \leq 0,$$

the hypotheses of the lemma are verified with

$$a = 0 \qquad \text{and} \qquad h(t) = Q(t) \equiv 0.$$

In order to prove (4.2), note that

$$|\hat{g}| = |f - \varphi_0(u) + \varphi_0(\hat{v}) - \varphi_1(u)| \leq |f| + c|\hat{w}|(|u| + |\hat{v}|)^4 + c(1 + |u|).$$

Thus, the Sobolev embeddings

$$\mathrm{H}^{2/3} \subset L^{18/5}(\Omega), \qquad \mathrm{H}^1 \subset L^6(\Omega), \qquad \mathrm{H}^{4/3} \subset L^{18}(\Omega),$$

jointly with Theorem 3.2 and (4.1), entail

$$\begin{aligned} \left\langle \hat{g}, A^{1/3}\hat{w} \right\rangle &\leq \|f\|\|\hat{w}\|_{2/3} + c(\|u\|_{L^6} + \|\hat{v}\|_{L^6})^4 \|\hat{w}\|_{L^{18}} \|A^{1/3}\hat{w}\|_{L^{18/5}} + c(1 + \|u\|)\|\hat{w}\|_{2/3} \\ &\leq c\|\hat{w}\|_{4/3} + c(\|u\|_1 + \|\hat{v}\|_1)^4 \|\hat{w}\|_{4/3}^2 + c\|u\|_1\|\hat{w}\|_{4/3} \\ &\leq \alpha\|\hat{w}\|_{4/3}^2 + \beta, \end{aligned} \tag{4.5}$$

for some positive constants α and β, depending on R_0. Hence, Lemma A.1 for $r = 1/3$ and $\gamma = \hat{g}$ applies with

$$a = 0, \qquad h(t) \equiv \alpha, \qquad Q(t) \equiv \beta.$$

Because $(\hat{w}(0), \hat{\zeta}^0) = 0$, that is, the initial energy vanishes, we obtain (4.2) with

$$J(t) = \beta \mathrm{e}^{2\alpha t},$$

up to a multiplicative constant. □

Proposition 4.3
For any $t \geq 0$, there exists a compact set $\mathcal{K}(t) \subset \mathcal{H}$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathcal{K}(t)) \leq c_2 \mathrm{e}^{-\omega t},$$

for some $c_2 \geq 0$, with $\omega > 0$ as in Proposition 4.2.

Proof
For any $z \in \mathbb{B}_0$, we consider the decomposition of the previous proof and we set, for every fixed t,

$$\Xi_t = \bigcup_{z \in \mathbb{B}_0} \hat{\zeta}^t.$$

Exploiting the representation formula (2.2) for $\hat{\zeta}^t$, taking into account that $\hat{\zeta}^0 = 0$ and $\hat{w} \in L^\infty(0, \infty; \mathrm{H}^1)$, we learn that $\Xi_t \subset \mathrm{dom}(T)$, and by elementary computations, we obtain

$$\sup_{z \in \mathbb{B}_0} \left\| T\hat{\zeta}^t \right\|_{\mathcal{M}} \leq c \qquad \text{and} \qquad \sup_{z \in \mathbb{B}_0} \left\| \hat{\zeta}^t(s) \right\|_1^2 \leq cs^2,$$

for some $c = c(R_0) > 0$. Besides, by (4.2),

$$\sup_{z \in \mathbb{B}_0} \left\| \hat{\zeta}^t \right\|_{\mathcal{M}^{1/3}} \leq J(t).$$

Because

$$s \mapsto cs^2\mu(s) \in L^1\left(\mathbb{R}^+\right),$$

we infer from Lemma A.2 for $r = 1/3$ that Ξ_t is precompact in $\mathcal{M}$. If we now define

$$\mathcal{K}(t) = \mathrm{B}_{\mathrm{H}^{4/3}}(J(t)) \times \overline{\Xi}_t,$$

the bar standing for closure in $\mathcal{M}$, then $\mathcal{K}(t)$ is compact in $\mathcal{H}$ and fulfills the claim, in light of (4.1)–(4.2). □

Proposition 4.3 says that $S(t)$ is asymptotically compact, hence possesses the global attractor $\mathfrak{A}$.

5. Regularity of the global attractor

5.1. Statement of the result

So far, we proved the existence of the global attractor $\mathfrak{A}$. However, as the estimate of Proposition 4.3 is not uniform-in-time, no regularity of $\mathfrak{A}$ can be deduced at this stage. To overcome this obstacle, we will exploit some ideas from [26], providing a tool apt to establish the regularity of the attractor without uniform-in-time estimates (Appendix A.2). Indeed, we have

Theorem 5.1
The *global attractor* $\mathfrak{A}$ of $S(t)$ is bounded in $\mathcal{H}^1$. Besides, for every CBT $Z = (u, \eta)$, the formal equality (3.2) actually holds true for all real times t. In particular, it follows that $\eta \in \mathrm{dom}(T)$.

A direct consequence of the last assertion of the theorem is the following corollary, whose proof is left to the reader.

Corollary 5.2
Given $u \in L^\infty(\mathbb{R}; \mathrm{H}^1)$ and defining $\eta = \eta^t(s)$ for all real times t by the formula (3.2), the function $Z = (u, \eta)$ is a CBT of $S(t)$ if and only if u solves the equation

$$u_t(t) + Au_t(t) + Au(t) + \int_0^\infty \kappa(s)Au(t-s)\,\mathrm{d}s + \varphi(u(t)) = f$$

for every $t \in \mathbb{R}$.

5.2. Proof of Theorem 5.1

We begin with an intermediate result.

Lemma 5.3
The *global attractor* $\mathfrak{A}$ is bounded in $\mathcal{H}^{1/3}$.

Proof
Because $\mathfrak{A}$ is fully invariant, it is contained in every closed attracting set. We reach the claim by proving the existence of $R_\star > 0$ such that

$$\mathbb{B}_\star = \mathrm{B}_{\mathcal{H}^{1/3}}(R_\star)$$

attracts (in fact, exponentially) the bounded absorbing set $\mathbb{B}_0$. This amounts to finding $c_\star \geq 0$ and $\omega_\star > 0$ for which

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathbb{B}_\star) \leq c_\star \mathrm{e}^{-\omega_\star t}.$$

To this end, we apply Lemma A.3 with $r = 1/3$. In order to verify the assumptions therein, let $z \in \mathbb{B}_0$ be fixed. For any $y \in \mathbb{B}_0$ and $x \in \mathcal{H}^{1/3}$ such that $y + x = z$, we define the operators

$$V_z(t)y = \left(v(t), \xi^t\right) \qquad \text{and} \qquad U_z(t)y = \left(w(t), \zeta^t\right),$$

where $(v(t), \xi^t)$ and $(w(t), \zeta^t)$ solve the systems (4.3) and (4.4) without the *hats*, respectively, with initial data

$$\left(v(0), \xi^0\right) = y \qquad \text{and} \qquad \left(w(0), \zeta^0\right) = x,$$

but where g is now defined as

$$g = f - \varphi_0(u) + \varphi_0(v) - \varphi_1(u).$$

Condition (i) of Lemma A.3 holds by construction, while (ii) follows by the exponential decay (4.1), which now reads

$$\left\|\left(v(t), \xi^t\right)\right\|_{\mathcal{H}} = \|V_z(t)y\|_{\mathcal{H}} \leq c_1\|y\|_{\mathcal{H}}\mathrm{e}^{-\omega t}. \tag{5.1}$$

In order to prove (iii), we estimate the nonlinearity g as follows. We write

$$\begin{aligned} |g| &\leq |f| + c|w|(|u| + |v|)^4 + c(1 + |u|) \\ &\leq |f| + c|w|(|v| + |\hat{v}|)^4 + c(|u| + |v|)|\hat{w}|^4 + c(1 + |u|) \end{aligned}$$

and with analogous computations as in (4.5), we obtain

$$\begin{aligned}\langle g, A^{1/3}w\rangle &\le \|f\|\|w\|_{2/3} + c(\|v\|_{L^6} + \|\hat{v}\|_{L^6})^4\|w\|_{L^{18}}\|A^{1/3}w\|_{L^{18/5}}\\ &\quad + c(\|u\|_{L^6} + \|v\|_{L^6})\|\hat{w}\|^4_{L^{18}}\|A^{1/3}w\|_{L^{18/5}} + c(1+\|u\|)\|w\|_{2/3}\\ &\le c\|w\|_{4/3} + c(\|v\|_1 + \|\hat{v}\|_1)^4\|w\|^2_{4/3}\\ &\quad + c(\|u\|_1 + \|v\|_1)\|\hat{w}\|^4_{4/3}\|w\|_{4/3} + c\|u\|_1\|w\|_{4/3}.\end{aligned}$$

Exploiting the decay estimates (4.1) and (5.1) together with (4.2), we arrive at

$$\langle g(t), w(t)\rangle_{1/3} \le \left(\frac{1}{2} + c\mathrm{e}^{-4\omega t}\right)\|w(t)\|^2_{4/3} + Q(t),$$

for some $Q \in \mathfrak{I}$. Both c and Q depend only on $\mathbb{B}_0$. Hence, we can apply Lemma A.1 for $r = 1/3$ and $\gamma = g$, where

$$a = \frac{1}{2} \qquad \text{and} \qquad h(t) = c\mathrm{e}^{-4\omega t}.$$

The conclusion is

$$\|U_z(t)x\|_{\mathcal{H}^{1/3}} \le c\|x\|_{\mathcal{H}^{1/3}}\mathrm{e}^{-\varepsilon t} + J(t),$$

for some $\varepsilon > 0$ and $J \in \mathfrak{I}$. This establishes (iii) and finishes the proof. □

Conclusion of the proof of Theorem 5.1
Knowing that $\mathfrak{A}$ is fully invariant and bounded in $\mathcal{H}^{1/3}$, we split the solution $S(t)z$ with $z \in \mathfrak{A}$ into the sum

$$S(t)z = L(t)z + K(t)z,$$

where $L(t)z = (v(t), \xi^t)$ and $K(t)z = (w(t), \zeta^t)$ solve the systems

$$\begin{cases} v_t + Av_t + Av + \displaystyle\int_0^\infty \mu(s)A\xi(s)\,\mathrm{d}s = 0,\\ \xi_t = T\xi + v,\\ (v(0), \xi^0) = z\end{cases}$$

and

$$\begin{cases} w_t + Aw_t + Aw + \displaystyle\int_0^\infty \mu(s)A\zeta(s)\,\mathrm{d}s + \varphi(u) = f,\\ \zeta_t = T\zeta + w,\\ (w(0), \zeta^0) = 0.\end{cases}$$

The same argument (in fact, easier) as the one used for (4.3) shows that the linear semigroup $L(t)$ decays exponentially in $\mathcal{H}$, that is,

$$\|L(t)z\|_{\mathcal{H}} \le c\mathrm{e}^{-\omega t}. \tag{5.2}$$

Next, calling

$$\gamma = f - \varphi(u),$$

owing to (1.4), the Sobolev embedding $\mathrm{H}^{6/5} \subset L^{10}(\Omega)$ and the $\mathcal{H}^{1/3}$-regularity of the attractor, we have

$$\langle \gamma, u\rangle_1 \le \|f\|\|u\|_2 + c\left(1 + \|u\|^5_{6/5}\right)\|u\|_2 \le \frac{1}{2}\|u\|^2_2 + q,$$

for some $q = q(\mathfrak{A}) > 0$. Hence, we can apply Lemma A.1 for $r = 1$, where

$$a = \frac{1}{2}, \qquad h(t) \equiv 0, \qquad \mathcal{Q}(t) \equiv q.$$

Because $K(0)z = 0$, we conclude that

$$\|K(t)z\|_{\mathcal{H}^1} \le \rho, \tag{5.3}$$

for some $\rho = \rho(\mathfrak{A}) > 0$. Collecting (5.2)–(5.3), and setting $\mathbb{B}_1 = \mathrm{B}_{\mathcal{H}^1}(\rho)$, we obtain

$$\mathrm{dist}_{\mathcal{H}}(\mathfrak{A}, \mathbb{B}_1) = \mathrm{dist}_{\mathcal{H}}(S(t)\mathfrak{A}, \mathbb{B}_1) \le c\mathrm{e}^{-\omega t} \to 0$$

as $t \to \infty$, yielding the inclusion $\mathfrak{A} \subset \mathbb{B}_1$.

We are left to show that for every CBT $Z = (u, \eta)$, the formal equality (3.2) holds for all $t \in \mathbb{R}$. Let then $Z(t) = (u(t), \eta^t)$ be a CBT, that is, a solution lying on $\mathfrak{A}$. Fixed an arbitrary $k > 0$, let us consider the solution at time $\tau > 0$ with initial data $Z(t-k)$

$$S(\tau)Z(t-k) = (v(\tau), \xi^\tau).$$

Observing that

$$v(\tau) = u(t-k+\tau) \qquad \text{and} \qquad \xi^\tau = \eta^{t-k+\tau},$$

the representation formula (2.2) applied to ξ^τ yields

$$\eta^{t-k+\tau}(s) = \xi^\tau(s) = \int_0^s v(\tau-\sigma)\, \mathrm{d}\sigma = \int_0^s u(t-k+\tau-\sigma)\, \mathrm{d}\sigma,$$

for every $s \le \tau$. Letting now $k = \tau$, we obtain (3.2) for all $s \le \tau$, and from the arbitrariness of $\tau > 0$ the claim follows. □

Appendix

A.1. An auxiliary problem

Let $r \in [0, 1]$ be fixed. For a sufficiently regular function $\gamma = \gamma(t)$ on $[0, \infty)$, let us consider the Cauchy problem in the unknown $Z(t) = (u(t), \eta^t)$

$$\begin{cases} u_t + A\left[u_t + u + \displaystyle\int_0^\infty \mu(s)\eta(s)\, \mathrm{d}s\right] = \gamma, \\ \eta_t = T\eta + u, \\ Z(0) = z \in \mathcal{H}^r. \end{cases}$$

Given a nonnegative constant $a < 1$, a nonnegative locally summable function h on $\mathbb{R}^+$ and a function $Q \in \mathfrak{I}$, the following lemma holds.

Lemma A.1
Assume that

$$\langle \gamma(t), u(t)\rangle_r \le (a + h(t))\|u(t)\|_{r+1}^2 + Q(t).$$

Then, there exists $\varepsilon > 0$ small such that the r-energy of the system fulfills the estimate

$$E_r(t) \le 2I(t)\mathrm{e}^{-\varepsilon t}E_r(0) + \frac{1}{\varepsilon}I(t)Q(t),$$

where

$$I(t) = \exp\left[\int_0^t 2h(\tau)\, \mathrm{d}\tau\right].$$

Proof
Multiplying the first equation by u in H^r and the second one by η in $\mathcal{M}^r$, on account of (2.1), we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}E_r + \|u\|_{r+1}^2 = \langle \gamma, A^r u\rangle + \left\langle TA^{r/2}\eta, A^{r/2}\eta\right\rangle_{\mathcal{M}} \le (a+h)\|u\|_{r+1}^2 + Q.$$

Following [27], we introduce a further functional in order to reconstruct E_r, namely,

$$\Psi(t) = \int_0^\infty \kappa(s)\left\|\eta^t(s)\right\|_{r+1}^2\, \mathrm{d}s.$$

In light of (1.6), Ψ satisfies an upperbound

$$\Psi(t) \le \Theta\left\|\eta^t\right\|_{\mathcal{M}^r}^2 \le 2\Theta E_r(t).$$

Besides, from the second equation of the system, and exploiting again (1.6), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi = -\|\eta\|_{\mathcal{M}^r}^2 + 2\int_0^\infty \kappa(s)\langle \eta(s), u\rangle_{r+1}\, \mathrm{d}s \le -\frac{1}{2}\|\eta\|_{\mathcal{M}^r}^2 + 2\Theta^2\kappa(0)\|u\|_{r+1}^2.$$

Therefore, for $\varepsilon > 0$ to be fixed, we define the functional

$$\Phi(t) = E_r(t) + 2\varepsilon\Psi(t),$$

which satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \left(1 - a - 4\varepsilon\Theta^2\kappa(0)\right)\|u\|_{r+1}^2 + \varepsilon\|\eta\|_{\mathcal{M}^r}^2 \leq h\|u\|_{r+1}^2 + Q \leq 2h\Phi + Q.$$

Choosing ε small enough such that (here we use the Poincaré inequality)

$$\varepsilon\left(1 + \frac{1}{\lambda_1}\right) \leq 1 - a - 4\varepsilon\Theta^2\kappa(0),$$

we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 2\varepsilon E_r \leq 2h\Phi + Q.$$

Up to further reducing ε, we also have

$$E_r(t) \leq \Phi(t) \leq 2E_r(t).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \leq (2h - \varepsilon)\Phi + Q,$$

and a direct application of the Gronwall lemma gives (recall that Q is increasing)

$$\begin{aligned} E_r(t) &\leq I(t)\mathrm{e}^{-\varepsilon t}\Phi(0) + I(t)Q(t)\mathrm{e}^{-\varepsilon t}\int_0^t \mathrm{e}^{\varepsilon\tau}\,\mathrm{d}\tau \\ &\leq 2I(t)\mathrm{e}^{-\varepsilon t}E_r(0) + \frac{1}{\varepsilon}I(t)Q(t), \end{aligned}$$

with $I(t)$ as in the statement of the lemma. □

A.2. *Two technical lemmas*

We finally recall two results needed in the course of the investigation. The first is a compactness lemma in the space $\mathcal{M}$ proved in [28] (see Lemma 5.5 therein). The second one is Theorem 3.1 from [26], written here in a suitable form for our scopes.

Lemma A.2
Let Ξ be a subset of $\mathrm{dom}(T)$, and let $r > 0$. If

$$\sup_{\eta\in\Xi}\left[\|\eta\|_{\mathcal{M}^r} + \|T\eta\|_{\mathcal{M}}\right] < \infty$$

and the map

$$s \mapsto \sup_{\eta\in\Xi}\mu(s)\|\eta(s)\|_1^2$$

belongs to $L^1(\mathbb{R}^+)$, then Ξ is precompact in $\mathcal{M}$.

Lemma A.3
Let $\mathbb{B}_0 \subset \mathcal{H}$ be an absorbing set for $S(t)$, and let $r > 0$. For every $z \in \mathbb{B}_0$, assume there exist two operators $V_z(t)$ and $U_z(t)$ acting on $\mathcal{H}$ and $\mathcal{H}^r$, respectively, with the following properties:

(i) given any $y \in \mathbb{B}_0$ and any $x \in \mathcal{H}^r$ satisfying the relation $y + x = z$,

$$S(t)z = V_z(t)y + U_z(t)x;$$

(ii) there exists $d_1 \in \mathfrak{D}$ such that for any $y \in \mathbb{B}_0$,

$$\sup_{z\in\mathbb{B}_0}\|V_z(t)y\|_{\mathcal{H}} \leq d_1(t)\|y\|_{\mathcal{H}};$$

(iii) there exist $d_2 \in \mathfrak{D}$ and $J \in \mathfrak{J}$ such that for any $x \in \mathcal{H}^r$,

$$\sup_{z\in\mathbb{B}_0}\|U_z(t)x\|_{\mathcal{H}^r} \leq d_2(t)\|x\|_{\mathcal{H}^r} + J(t).$$

Then, $\mathbb{B}_0$ is exponentially attracted by a closed ball of $\mathcal{H}^r$ centered at zero; namely, there exist (strictly) positive constants $R, K, \varkappa$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathrm{B}_{\mathcal{H}^r}(R)) \leq K\mathrm{e}^{-\varkappa t}.$$

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