SPECTRAL TRIPLES FOR THE SIERPINSKI GASKET

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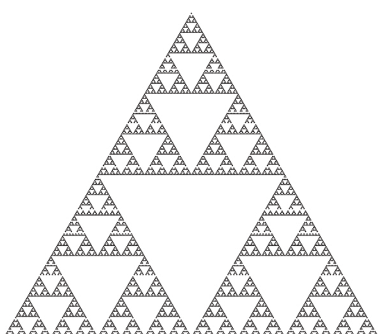
1. Introduction

The advent of Noncommutative Geometry allowed to consider from a geometrical and analytical point of view spaces which appear to be singular when analysed by using the classical tools of Differential Calculus and Riemannian Geometry.



In the present paper we approach from a NCG point of view the study of a compact subset K of the plane which is a central example among fractal sets, namely the Sierpiński Gasket. We associate to the gasket a family of spectral triples. For values of the parameters in a suitable range, the triple reconstructs the main known features of the gasket, namely its similarity dimension, Hausdorff measure, a distance which is bi-Lipschitz equivalent w.r.t. the Euclidean one, and the standard Dirichlet form, with the appearance of an *energy dimension*. Moreover, these triples pair non trivially with the K-theory of the gasket.

The fundamental topological property of K is its self-similarity, by which K can be reconstructed as a whole from the knowledge of any arbitrary small part of it. More precisely, considering the three similitudes w_1, w_2, w_3 of scaling parameter $1/2$ fixing respectively the vertices p_1, p_2, p_3 of an equilateral triangle, one may characterize K as the only compact set in $\mathbb{R}^2$ such that

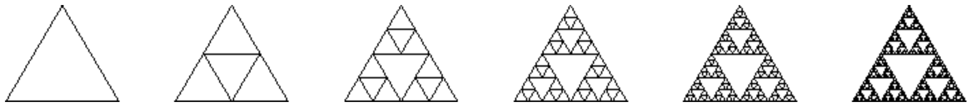
$$K = w_1(K) \cup w_2(K) \cup w_3(K)\,,$$

namely K is the fixed point of the map $K \mapsto w_1(K) \cup w_2(K) \cup w_3(K)$ which is a contraction with respect to the Hausdorff distance on compact subsets of the plane. This allows various approximations of K as, for example, the one given by finite graphs.

1991 *Mathematics Subject Classification.* 58B34, 28A80, 47D07, 46LXX.

Key words and phrases. Self-similar fractals, Noncommutative geometry, Dirichlet forms.

This work has been partially supported by GNAMPA, MIUR, the European Networks "Quantum Spaces - Noncommutative Geometry" HPRN-CT-2002-00280, and "Quantum Probability and Applications to Physics, Information and Biology", GDRE GREFI GENCO, and the ERC Advanced Grant 227458 OACFT "Operator Algebras and Conformal Field Theory".



The gasket K was introduced by Sierpiński for purely topological motivations [54]. Successively, using measure theory, it was noticed that it is a space with a non integer Hausdorff dimension $d_H = \frac{\ln 3}{\ln 2}$ [38], which later attracted the attention of Probabilists, who constructed a stochastic process X_t with continuous sample paths on K [46]. The process is symmetric w.r.t. the Hausdorff measure μ_H and has a self-adjoint generator Δ in $L^2(K, \mu_H)$, whose discrete spectrum was carefully studied by Fukushima and Shima in [29]. Finally, Kigami [44] introduced on the gasket (and other fractals) the notion of harmonic structure, the most symmetric choice of which produces on K the so called standard Dirichlet form, whose associated self-adjoint operator in $L^2(K, \mu_H)$ coincides with Δ.

We recall that the main object for the spectral description of the metric aspects of a geometry, introduced by Connes [22], is the so called spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It consists of an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ and a self-adjoint unbounded operator D, the so called Dirac operator. Main requests are the boundedness of the commutators $[D, a]$ of the elements $a \in \mathcal{A}$ with D, and the fact that D has discrete spectrum. Such triples are meant to generalize the role that, on a compact Riemannian manifold, is played by the algebra of smooth functions and by the Dirac operator acting on the Hilbert space of square integrable sections of the Clifford bundle. The integral $\int a\, dvol$ w.r.t. the Riemannian volume form is replaced by the functional

$$\mathcal{A} \ni a \mapsto \mathrm{tr}_\omega(a|D|^{-d_D})\,, \tag{1.1}$$

tr_ω being the Dixmier logarithmic trace on the algebra of compact operators on a separable Hilbert space. There is a unique exponent d_D (if any), called metric dimension (cf. [32, 23]), depending on the asymptotic distribution of the eigenvalues of D, which gives rise to a non-trivial functional on $\mathcal{A}$ via the formula above. In great generality, this functional is a positive trace on the algebra $\mathcal{A}$ (see [18]). In this way, on a manifold, one recovers the dimension and the integral. The differential 1-form da of a smooth function a on the manifold, is replaced by the commutator $[D, a]$, so that the Lipschitz seminorm $\|da\|_\infty$ is then replaced by the norm $\|[D, a]\|$ of the commutator. The Riemannian metric is generalized by the Connes distance between states on $\mathcal{A}$, defined through the formula

$$\rho_D(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : \|[D, a]\| \le 1\}\,,$$

that can be thought as generalized noncommutative Monge-Kantorovitch distance.
A simple but fruitful idea is to generalize the Dirichlet energy integral of a manifold

$$\mathcal{E}[a] = \int |da|^2$$

to the NCG setting of a spectral triple, by the ansatz:

$$\mathcal{E}[a] = \mathrm{tr}_\omega(|[D, a]|^2 |D|^{-\delta_D}). \tag{1.2}$$

Similar formulas have been recently considered in [9, 42]. Finally, we recall that spectral triples produce topological invariants. Indeed, a spectral triple gives rise to a class in the K-homology of the algebra $\mathcal{A}$, hence it pairs with K-theory. Such pairing may be expressed in terms of the Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ associated with the spectral triple, F being the phase of D. In particular, if the triple is odd, and u is an invertible element of $\mathcal{A}$, the pairing is given by the index map $u \to \mathrm{Ind}(P_+ u P_+)$, where P_+ is the projection on the positive part of the spectrum of F, and $P_+ u P_+$ is a Fredholm operator acting on the space $P_+\mathcal{H}$.

As mentioned above, one of the features of Noncommutative Geometry is that it non only applies to noncommutative manifolds, such as noncommutative tori, quantum groups [16] or the various quantum spheres, but it is also able to describe some classical (e.g. dealing with points of a topological space) but singular geometries. The first example of this fact was given by Connes in [22], where a spectral triple was assigned to the Cantor set. Such a triple could reconstruct dimension, measure and distance of the fractal set, as well as the pairing with K-theory. The peculiar aspects of the construction are the following: the triple is a direct sum of triples associated with elementary building blocks; the building blocks are the lacunas of the fractal, namely the boundaries of the removed intervals in the Cantor set. The first idea can easily be adapted to self-similar fractals, by choosing as building blocks the images of a suitable subset via compositions of similarities. This has been exploited in [15] for the Sierpinski gasket (as anticipated in [14]), where the building blocks are the lacunas of the gasket, meant as the boundaries of the removed triangles in K. Again, dimension, measure, distance and pairing with K-theory are reconstructed in spectral terms.

Let us mention here that, as far as the volume measure is concerned, a Connes'-like formula for P.C.F. self-similar fractals was obtained already in the paper of Kigami and Lapidus, [43].

Our aim here is to add to this list the standard Dirichlet form on the gasket by making use of formula (1.2). While for regular geometries the energy form may be considered as a derived object, given the other analytic and geometric tools, on fractals the point of view is reversed. The fundamental observation in this regard is that, for a large class of self-similar fractals, energy measures, representing the distribution of energy, are singular w.r.t. the self-similar measures, representing the distribution of the volume (cf. [7, 47, 46, 35, 37]). In our noncommutative picture, this singularity is reflected in the difference between the abscissas of convergence d_D and δ_D in the formulas (1.1), (1.2), quantities which would coincide with the dimension on Riemannian manifolds and on other "regular" spaces.

For parameters in a suitable range, δ_D takes the value $d_E = \frac{\log(12/5)}{\log 2} \approx 1.26$, which plays the role of an *energy dimension*. Such dimension is thus smaller than the Hausdorff dimension $d_H := \frac{\log 3}{\log 2} \approx 1.58$, and has no apparent counterpart in the classical analysis on fractals.

Let us try to explain why we call dimension such a number d_E. Hausdorff dimension is, as it is said, a rarefaction index, which separates two opposite, extreme behaviors. For values above, the associated external measures vanish; for values below they are always infinite; the Hausdorff dimension is the only value (if any) which can produce a non trivial measure. In noncommutative geometry, such a role is played by the abscissa of convergence of appropriate functionals, cf. Theorem 2.7 in [32]. In the case of the volume functional $s \to \mathrm{tr}(a|D|^{-s})$, the abscissa d_D separates the values for which the logarithmic singular trace vanishes identically from those for which such trace is infinite, the value d_D being the only value (if any) which can produce a non trivial trace functional. The same happens for our energy dimension: the abscissa of convergence of the energy functional $s \to \mathrm{tr}(|[D,a]|^2|D|^{-s})$ is the only value (if any) which can reproduce the standard Dirichlet form. This happens exactly when $\delta_D = d_E$.

It is interesting to notice that, in analogy to what has been discovered for the volume functional on Riemannian manifolds in [50], formula (1.2) above, even in its symmetrized version

$$\mathcal{E}[a] = \mathrm{tr}_\omega(|D|^{-\delta_D/2}|[D,a]|^2|D|^{-\delta_D/2})\,, \tag{1.3}$$

provides the value of the standard Dirichlet form, up to a multiplicative constant, on a suitable form core only, and not on the whole Dirichlet space of finite energy elements. On the other hand, we will show that the residue

$$\mathcal{E}[a] = \mathrm{Res}_{s=1}\,\mathrm{tr}(|D|^{-s\delta_D/2}|[D,a]|^2|D|^{-s\delta_D/2}) \tag{1.4}$$

does coincide, up to a multiplicative constant, with the standard Dirichlet form for all finite energy elements.

Let us mention here that the spectral triple for the gasket proposed in [33] Remark 2.14, whose building blocks were spectral triples associated with the boundary points of the edges of the gasket, could indeed produce the standard Dirichlet form exactly as above, with the same energy dimension. However, being based on a discrete approximation of K, it could not give rise to any pairing with K-theory.

We remark that noncommutative geometry provides also a replacement for the de Rham cohomology in terms of cyclic cohomology, however we do not pursue this direction here. Indeed differential forms have no classical analogue on fractals, and their study in this case is essentially based on the differential calculus developed in [19]. There, the authors associate a bimodule-valued derivation to a regular Dirichlet form and define differential 1-forms as the elements of the bimodule. In [20] this first order differential calculus on p.c.f. fractals has been developed further to a pseudo differential calculus by the construction of Fredholm modules associated to the Dirichlet form. Recent developments in this direction are also contained in [40], while in a recent paper of ours [17] concerning the gasket, we give a more concrete description of the 1-forms of [19] in such a way as to define their integrals on paths and their generalized potentials on suitable coverings and develop a Hodge-de Rham theory on the gasket.

We now come to a more detailed description of our family of spectral triples. As in [15], our building blocks are associated with the lacunas (boundaries of removed triangles) of the gasket, canonically identified with circles. The starting point is the observation that formula (1.4) may recover the standard Dirichlet form on K if and only if the standard triple on the circle is deformed (cf. Lemma 4.11 and Theorem 4.12). From the technical point of view, this is based on a single but fundamental result by A. Jonsson [41], which states that the restriction to lacunas of finite energy functions on K with respect to the standard energy form belongs to a suitable fractional Sobolev space. Accordingly, we deform the classical spectral triple for the circle $\mathbb{T}$, by replacing the standard Laplacian Δ with its roots Δ^α for suitable α's in $(0,1]$. Notice that, by the theory of Dirichlet forms (see [28]), the quadratic form $\mathcal{E}_\alpha$ on $L^2(\mathbb{T})$ determined by the self-adjoint operator Δ^α is a Dirichlet form. As proposed in [19], we construct a bimodule-valued derivation ∂_α, characterized by

$$\mathcal{E}_\alpha[a] = \|\partial_\alpha a\|^2\,, \qquad \Delta^\alpha = \partial_\alpha^* \circ \partial_\alpha\,,$$

and define a Dirac operator on $\mathbb{T}$ as

$$D := \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix}\,.$$

While this deformation does not quantize the algebra, which remains $C(\mathbb{T})$, a zest of noncommutativity is nonetheless present, since the left and right actions of functions on the Hilbert space do not coincide (functions do not commute with forms). This is related to the fact that, while for $\alpha = 1$ the distributional kernel giving rise to the energy on $\mathbb{T}$ is supported on the diagonal, this is no-longer true for $\alpha < 1$. In probabilistic terms, the stochastic process on

$\mathbb{T}$ generated by Δ^α is a diffusion (i.e. has continuous paths), when $\alpha = 1$, while it is purely jumping, when $\alpha < 1$.

A second deformation parameter β may be introduced, by replacing the scaling parameter $1/2$ for the gasket with $2^{-\beta}$. Even though also this deformation produces interesting spectral triples, as explained below, such deformation is not really needed, since the value $\beta = 1$ is not only acceptable, but is the only one which corresponds to the metric gasket embedded in $\mathbb{R}^2$. Therefore, apart form the comments here, the results concerning $\beta \neq 1$ are confined in Section 5.

An unexpected outcome of the construction is that the two parameters have a quite different role for the gasket as a whole. Indeed, α only plays the role of a threshold parameter. The condition $\alpha \leq \alpha_0 = \frac{\log(10/3)}{\log 4}$ is a necessary condition for formula (1.3) to be finite for finite energy functions, and to reproduce the standard Dirichlet form. If, furthermore, $\alpha > \left(2 - \frac{\log(5/3)}{\beta \log 2}\right)^{-1}$, one gets a full-fledged spectral triple, whose features only depend on β, which assumes the role of a deformation parameter.
In fact, for $\alpha_0 < \beta \leq 1$, the Connes metric $\rho_{D,\beta}$ is bi-Lipschitz w.r.t. the Euclidean distance raised to the power β or, equivalently, bi-Lipschitz w.r.t the geodesic metric, induced on K by the Euclidean metric, raised to the power β. Consequently, the metric dimension is given by $d_{D,\beta} = d_{D,1} \cdot \beta^{-1}$, and, as expected, the volume measure $\mu_{D,\beta}$ coincides (up to a multiplicative constant) with the Hausdorff measure for the dimension $d_{D,\beta}$, which in turn coincides with the Hausdorff measure for the dimension d_H. The energy dimension is given by $\delta_{D,\beta} = 2 - \frac{\log(5/3)}{\log 2}\beta^{-1}$, and the corresponding energy form do not even depend on β, apart from a multiplicative constant.

Neither α nor β affect the pairing with K-theory. However we had to tackle another difficulty concerning the Fredholm module associated with the spectral triple. In fact, in order to implement the deformation associated with the parameter α, we had to choose the Hilbert space as the module of differential forms, making the triple (and the Fredholm module) an even one. To recover the pairing with odd K-theory, we have to add a further grading, obtaining a 1-graded Fredholm module, which then has the correct pairing with odd K-theory.

We now discuss the relation with previous constructions. If we denote by $D_{\alpha,\beta}$ the Dirac operator on the gasket associated with the parameters α and β, one of the Dirac operators considered in [15] is essentially equivalent to $D_{1,1}$, and a simple relation holds: $|D_{1,1}|^\alpha = |D_{\alpha,\alpha}|$. But the only deformation that is needed is that of the Dirac on the circle, namely of the first parameter, and the choice $\alpha = \beta$ is not compatible with the reconstruction of the energy, cf. Section 5. Moreover, we needed to construct a suitable derivation ∂_α on the circle in such a way that $\partial_\alpha^* \partial_\alpha = \Delta^\alpha$, Δ being the Laplacian on the circle, and then to define the deformed Dirac operator on the circle as the anti-diagonal matrix in formula (3.14). This guarantees not only the correct spectral behavior, but also suitable commutation relations of the global Dirac $D_{\alpha,1}$ on the gasket with the elements of the algebra $\mathcal{A}$.

We conclude this review about the dependence of our constructions on α and β by noticing that the requests concerning spectral triple properties reflect into independent bounds on the parameters, which finally give rise to a quite small fraction of the (β, α)-plane. The fact that this set is indeed non empty is not at all obvious, and only an analysis of a larger family of fractals and their Dirichlet forms may reveal the reasons of its non-triviality.

We note here that our triples indeed violate one of the requests of a spectral triple as defined in [22], since the kernel of the Dirac operator is infinite dimensional. However, this degeneracy of the kernel does not cause any harm in the construction, when taking the point of view of reading $|D|^{-s}$ as the functional calculus of D with the function $f(t) = 0$ for $t = 0$ and $f(t) = |t|^{-s}$ for $t \neq 0$.

The question is more subtle when the associated Fredholm module is concerned. Indeed, denoting by $P_\pm$ the projection on the positive, resp. negative, part of the spectrum of D, the two formulas for the pairing of the module with the K-theory class of an invertible element u given by $\mathrm{Ind}(P_+\pi(u)P_+)$ and $-\mathrm{Ind}(P_-\pi(u)P_-)$, which are equivalent when the dimension of the kernel of D is finite, may be expected to differ. We call a Fredholm module *tamely degenerate* when such equality holds, hence the kernel of D is irrelevant from the K-theoretical point of view, and check that this condition is satisfied for our triples.

We describe now some technical aspects of our construction. First, in order to construct a Dirac operator for the α-deformed triples on the circle, we had to define a differential square root of Δ^α, or, in other terms, a derivation implementing the corresponding Dirichlet form. This has been done by realizing the corresponding Dirichlet form in terms of an integral operator, whose distributional kernel is written in terms of a special function, the so called Clausen cosine function Ci_s. We show that $-\mathrm{Ci}_{-2\alpha} \geq 0$, for $0 < \alpha < 1$, and describe the Dirac operator in terms of the derivation ∂_α given by $\partial_\alpha f(x,y) = (-2\pi\,\mathrm{Ci}_{-2\alpha}(x-y))^{1/2}(f(x)-f(y))$. By means of some explicit estimates on Ci_α we can show the relation of the Connes' distance for the α-deformed circle and the α-power of the Riemannian distance. In this sense, our deformed circles may be considered as quasi-circles, since the α-power of the Riemannian distance clearly satisfies the so-called reverse triangle inequality [1]. As for the case of the gasket described above, the α-deformation rescales the Hausdorff dimension of the circle and leaves the volume invariant (up to a multiplicative constant).

Second, our study of the noncommutative formula for the standard Dirichlet form produces an interesting situation when Dixmier traces are concerned. Indeed, when used to describe the volume form in noncommutative geometry, the Dixmier trace is computed for elements which belong to the principal ideal of compact operators generated by $|D|^{-d}$, and the same happens for the computation of the energy form according to formula (1.2) when regular spaces are considered, namely when the metric dimension d_D and the energy dimension δ_D coincide. It is known that the theory of singular traces on principal ideals ([56, 2, 31] etc.) is in a sense simpler than the corresponding theory on symmetrically normed ideals. In the case of fractal spaces however, there is no principal ideal containing all elements of the form $|[D,f]|^2|D|^{-\delta}$, and Dixmier traces on symmetrically normed ideals and the analysis in [12, 13] play a key role.

As for the organization of the paper, it consists of this introduction, four sections, and an appendix. The first section is devoted to some results on (possibly degenerate) spectral triples and Fredholm modules, the second to the description of the α-deformed circles, the third to the construction and results of the triples on the gasket, and the fourth to the two-parameter triples. The appendix contains some estimates concerning the Clausen functions.

The results contained in this paper have been described in several conferences, such as Cardiff 2010, Cambridge 2010, Cornell 2011, Paris 2011, Messina 2011, Cortona 2012, Marseille 2012.

2. Spectral Triples

2.1. **Spectral Triples and their Fredholm Modules.** We generalize here the notion of Spectral Triple and of Fredholm module, by allowing the Dirac operator D to have an infinite dimensional kernel. This generalization, with respect to the situation usually considered in the literature, will be useful later on, when we will construct Spectral Triples on the circle and the Sierpinski Gasket, whose Dirac operators have an *infinite dimensional kernel*. Because of that, some extra work will be needed to construct an associated Fredholm module having nontrivial pairing with K-theory.

We recall that the notion of Fredholm module $(F, \pi, \mathcal{H})$ on a compact topological spaces K is a generalization of the theory of elliptic pseudodifferential operators on a compact manifold. In its *odd* form, one requires that the elements f of the algebra of continuous functions $C(K)$ are represented as bounded operators $\pi(f)$ on a Hilbert space $\mathcal{H}$ on which, moreover, a distinguished self-adjoint operator F of square 1 is considered, the *symmetry*, in such a way that the commutators $[F, \pi(f)]$ are compact operators.
In its simplest example, F is the Hilbert transform, a 0-order, pseudodifferential operator acting on the space of square integrable functions on the circle. When continuous functions are considered as multiplication operators acting on the same Hilbert space, the compactness of the commutators above results from the fundamental theorem of the theory of pseudo-differential operators, according to which commutators of 0-order operators have order -1 and are thus compact.
As a consequence of the theory, operators like FuF, constructed by (matrix ampliations) of F and continuous functions u with values in unitary matrices, are Fredholm operators and their indexes give a pairing between the Fredholm module $(F, \pi, \mathcal{H})$ and the odd K-theory group generated by continuous functions on K with values in unitary matrices. In the even case, indexes constructed similarly to those above give a pairing between the Fredholm module $(F, \pi, \mathcal{H})$ and the even K-theory group, generated by continuous functions on K with values in the space of matrix projections. By a famous theorem due to Serre and Swan, elements of the even K-theory group have a direct geometric interpretation in terms of equivalence classes of locally trivial vector bundles on K.
The notion of Fredholm module, introduced by Atiyah [3] on compact spaces and generalized to C*-algebras by Mishchenko [36], Brown-Douglas-Fillmore [11], and Kasparov [39], lies at the core of the Connes' noncommutative differential geometry [22], where the operator $df := i[F, \pi(f)]$ is the operator theoretical substitute for the differential of the function f.
On the other hand, the notion of spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is meant to encode the *metric* aspects of a generalized geometry. To remain in a commutative framework, where the relevant C*-algebra is $C(K)$ as above, a suitable dense subalgebra $\mathcal{A}$ of $C(K)$ is required to act on a Hilbert space $\mathcal{H}$ together with an unbounded, self-adjoint operator D, having discrete spectrum, in such a way that the commutators $[D, f]$ are bounded for all $a \in \mathcal{A}$.
The archetypical example of a spectral triple arises in Riemannian geometry where D is the Dirac operator on the Clifford bundle and the boundedness of the commutator above is equivalent to the Lipschitz continuity of the function involved.
The metric aspects of a space endowed with a spectral triple are recovered by regarding the compact operator $|D|^{-1}$ as the operator theoretic realization of the infinitesimal arc element ds, whereas its local features are recovered through the asymptotic behavior of the spectrum of D and related operators.

Definition 2.1. (Spectral Triple) A (possibly kernel-degenerate, compact) Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ consists of an involutive unital algebra $\mathcal{A}$, acting faithfully on a Hilbert space $\mathcal{H}$ trough a representation π, and a self-adjoint operator $(D, \mathrm{dom}(D))$ on it, subject to the conditions

(i) the commutators $[D,\pi(a)]$, initially defined on the domain $\mathrm{dom}(D)\subset\mathcal{H}$ through the sesquilinear forms

$$(\xi,[D,\pi(a)]\eta):=(D\xi,\pi(a)\eta)-(\pi(a)^*\xi,D\eta)\qquad \xi,\eta\in dom(D)\,,$$

extend to bounded operators on $\mathcal{H}$, for all $a\in\mathcal{A}$;

(ii) the operator D^{-1} is compact on $\ker(D)^\perp$.

The operator $(D,\mathrm{dom}(D))$ is referred to as the *Dirac operator* of the Spectral Triple.

Notice that if $\ker(D)$ is finite dimensional the condition in (ii) reduces to the compactness of the operators $(I+D^2)^{-1}$. We recover in this way the original definition of a Spectral Triple by Connes [22].

Definition 2.2. (Fredholm Module) A (possibly kernel-degenerate) Fredholm Module $(\mathcal{H},\pi,F)$ over a C*-algebra A consists of a Hilbert space $\mathcal{H}$, a representation $\pi:A\to\mathcal{B}(\mathcal{H})$, and a bounded operator $F\in\mathcal{B}(\mathcal{H})$ such that

(i) F^2-I is a compact operator on $\ker(F)^\perp$,

(ii) F^*-F is a compact operator,

(iii) the commutators $[F,\pi(a)]$ are compact operators, for all $a\in A$.

Remark 2.3. In the following, in many occasions, the representation π will be omitted, the $*$-algebra $\mathcal{A}$ and the C*-algebra A being identified with subalgebras of $\mathcal{B}(\mathcal{H})$.

The classical formulation of Atiyah is recovered when $\ker(F)$ is finite dimensional. Again, the above generalization is required to deal with the Fredholm Modules we will construct on quasicircles and on the Sierpinski Gasket.

A classical result by Baaj and Julg [4] shows that Spectral Triples give rise to Fredholm modules by taking $F=\mathrm{sgn}(D)$ (or any other function which is asymptotic to $\mathrm{sgn}(t)$ for $|t|\to\infty$), whenever $\dim\ker(D)<+\infty$. We need to generalize this result to allow $\dim\ker(D)=+\infty$. The key point is to show that the boundedness of the commutator $[D,a]$ implies the compactness of $[F,a]$ even if $\ker(D)$ is not finite dimensional.

Proposition 2.4. *Let $(\mathcal{A},\mathcal{H},D)$ be a (possibly kernel-degenerate) Spectral Triple over a unital $*$-algebra $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$. Then, setting $F:=\mathrm{sgn}(D)$, $(F,\mathcal{H})$ is a (possibly kernel-degenerate) Fredholm module over the C*-algebra A given by the norm closure $\overline{\mathcal{A}}$ of the $*$-algebra $\mathcal{A}$.*

Proof. Since F is self-adjoint and F^2-I vanishes on $\ker(F)^\perp=\ker(D)^\perp$ by construction, the first two requirements in the Definition 2.2 of a Fredholm module hold true.
To verify the third requirement, let us first observe that $\frac{1}{\sqrt{x}}=\frac{2}{\pi}\int_0^{+\infty}\frac{dt}{x+t^2}$ for any $x>0$, from which it follows that $F=\frac{2}{\pi}\int_0^{+\infty}\frac{D}{t^2+D^2}\,dt$, where the integral converges in norm since

$$\frac{2}{\pi}\int_k^{+\infty}\frac{D}{t^2+D^2}\,dt=\begin{cases}F\big(I-\frac{2}{\pi}\arctan\big(k|D|^{-1}\big)\big) & \text{on } (\ker D)^\perp\\ 0 & \text{on } \ker D\end{cases}$$

hence

$$\|\frac{2}{\pi}\int_k^{+\infty}\frac{D}{t^2+D^2}\,dt\|\le 1-\frac{2}{\pi}\arctan\big(\frac{k}{\lambda}\big)$$

where $\lambda>0$ is the first non-zero eigenvalue of $|D|$. Let now a belong to $\mathcal{A}$, so that $a\,\mathrm{dom}(D)\subset\mathrm{dom}(D)$ and $Da-aD$ is bounded on $\mathrm{dom}(D)$. Our aim is to prove first that the operator defined as

$$C:=\frac{2}{\pi}\int_0^{+\infty}\frac{t}{t^2+D^2}\,[D,a]\,\frac{t}{t^2+D^2}\,dt-\frac{2}{\pi}\int_0^{+\infty}\frac{D}{t^2+D^2}\,[D,a]\,\frac{D}{t^2+D^2}\,dt$$

is well defined and compact and then to show that it coincides with the commutator $[F, a]$. Observe that $t \in (0, +\infty) \mapsto D(t^2 + D^2)^{-1}$ is a $\mathcal{K}(\mathcal{H})$-valued continuous function that can be continuously extended to $[0, +\infty)$ by assigning to it the value $D/D^2 \in \mathcal{K}(\mathcal{H})$ at $t = 0$. Here we are denoting by D/D^2 the compact operator which is the inverse of D on $\ker(D)^\perp$ and vanishes on $\ker(D)$. Indeed, with λ as above, we have

$$\Big\| \frac{D}{t^2 + D^2} - \frac{D}{D^2} \Big\| = \Big\| \frac{t^2}{(t^2 + D^2)} \frac{D}{D^2} \Big\| = \frac{t^2}{(t^2 + \lambda^2)\lambda} \to 0 \qquad t \to 0\,.$$

Since $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$, and

$$\Big\| \frac{D}{t^2 + D^2} \, [D, a] \, \frac{D}{t^2 + D^2} \Big\| = \| [D, a] \| \cdot \Big\| \frac{D}{t^2 + D^2} \Big\|^2 \le \frac{1}{4} \| [D, a] \| \, t^{-2} \in L^1([1, \infty))\,,$$

we have that $\frac{2}{\pi} \int_0^{+\infty} \frac{D}{t^2+D^2} \, [D, a] \, \frac{D}{t^2+D^2} dt$ is a compact operator.

On the other hand, $t \in (0, +\infty) \mapsto t(t^2+D^2)^{-1} \, [D, a] \, t(t^2+D^2)^{-1}$ is a $\mathcal{K}(\mathcal{H})$-valued continuous function which can be extended continuously to $t = 0$. Indeed, when restricted to $\ker(D)^\perp$ it appears as a continuous function of $t \in [0, +\infty)$ of products of operators in which at least one factor is compact, and when restricted to $\ker(D)$ it reduces to

$$\frac{t}{t^2 + D^2} \, [D, a] \, \frac{t}{t^2 + D^2} = \frac{t}{t^2 + D^2} \, (Da - aD) \, \frac{t}{t^2 + D^2} = \frac{t}{t^2 + D^2} \, Da \, \frac{1}{t} = \frac{D}{t^2 + D^2} \, a$$

so that it converges in $\mathcal{B}(\mathcal{H})$ to the compact operator $(D/D^2)\, a$ as $t \to 0$. Since, again, $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$ and

$$\Big\| \frac{t}{t^2 + D^2} \, [D, a] \, \frac{t}{t^2 + D^2} \Big\| = \| [D, a] \| \cdot \Big\| \frac{t}{t^2 + D^2} \Big\|^2 \le \| [D, a] \| \, t^{-2} \in L^1([1, \infty))\,,$$

we have that $\frac{2}{\pi} \int_0^{+\infty} \frac{t}{t^2+D^2} \, [D, a] \, \frac{t}{t^2+D^2} \, dt$ is a compact operator.
By the formulas above we have

$$[F, a] = \frac{2}{\pi} \int_0^{+\infty} \left(\frac{D}{t^2 + D^2} \, a - a \, \frac{D}{t^2 + D^2} \right) dt$$

where the integral converges in norm. Therefore the identity $C = [F, a]$ follows if we prove that in the integral representations of C and $[F, a]$, the integrands coincide as quadratic forms. In fact, for fixed $\xi, \eta \in \mathcal{H}$, the vectors $\frac{t}{t^2+D^2}\xi$, $\frac{t}{t^2+D^2}\eta$ $\frac{D}{t^2+D^2}\xi$, $\frac{D}{t^2+D^2}\eta$ belong to $\mathrm{dom}(D)$, and

$$\begin{aligned}
&\left(\xi, \left(\frac{t}{t^2 + D^2} \, [D, a] \, \frac{t}{t^2 + D^2} - \frac{D}{t^2 + D^2} \, [D, a] \, \frac{D}{t^2 + D^2} \right) \eta \right) \\
&= \left(\frac{t}{t^2 + D^2} \xi, [D, a] \, \frac{t}{t^2 + D^2} \eta \right) - \left(\frac{D}{t^2 + D^2} \xi, [D, a] \, \frac{D}{t^2 + D^2} \eta \right) \\
&= \left(\frac{D}{t^2 + D^2} \xi, a \frac{t^2}{t^2 + D^2} \eta \right) - \left(\frac{D^2}{t^2 + D^2} \xi, a \frac{D}{t^2 + D^2} \eta \right) \\
&- \left(\frac{t^2}{t^2 + D^2} \xi, a \frac{D}{t^2 + D^2} \eta \right) + \left(\frac{D}{t^2 + D^2} \xi, a \frac{D^2}{t^2 + D^2} \eta \right) \\
&= \left(\frac{D}{t^2 + D^2} \xi, a\eta \right) - \left(\xi, a \frac{D}{t^2 + D^2} \eta \right) \\
&= \left(\xi, \left(\frac{D}{t^2 + D^2} \, a - a \, \frac{D}{t^2 + D^2} \right) \eta \right) .
\end{aligned}$$

$\square$

Recall that a *symmetry* of a Hilbert space $\mathcal{H}$ is a bounded operator $\gamma \in \mathcal{B}(\mathcal{H})$ such that

$$\gamma^* = \gamma\,, \qquad \gamma^2 = I\,.$$

Definition 2.5. (Even and odd Spectral Triples and Fredholm Modules)
A Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ is called *even* if there exists a symmetry γ such that

$$D\gamma + \gamma D = 0\,, \qquad a\gamma - \gamma a = 0 \qquad a \in \mathcal{A}\,.$$

A Fredholm Module $(F, \mathcal{H})$ is called *even* if there exists a symmetry γ such that

$$F\gamma + \gamma F = 0\,, \qquad a\gamma - \gamma a = 0 \qquad a \in \mathcal{A}\,.$$

In other words, the operator D (resp. F) of an even Spectral Triple (resp. Fredholm module) acts as an antidiagonal matrix with respect to the orthogonal decomposition of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ in eigenspaces $\mathcal{H}_\pm$ of the symmetry γ corresponding to its eigenvalues ± 1, while the elements of $\mathcal{A}$ act diagonally.

Corollary 2.6. *Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be an even Spectral Triple. Then, setting $F := \operatorname{sgn}(D)$, $(F, \mathcal{H}, \gamma)$ is an even Fredholm module over the C^*-algebra $\overline{\mathcal{A}}$.*

Fredholm modules represent elements of the K-homology groups of the algebra A [22]. These can be paired with elements of the K-theory groups of A. In particular, odd Fredholm modules couple with elements of the group $K_1(A)$, whose elements are represented by invertible or unitary elements of A. Indeed, assume F to be selfadjoint. In this case, for any invertible element $u \in A$, the operator $P_+\pi(u)P_+$ is Fredholm on $P_+\mathcal{H}$, where P_+ is the projection on the positive part of the spectrum of F, and the pairing is given by

$$\langle F, u\rangle = \operatorname{Ind}(P_+\pi(u)P_+).$$

In the following, we allow F to have a infinite dimensional kernel. The following Proposition justifies in some cases the treatment of such kernel-degenerate Fredholm modules.

Proposition 2.7. *Let F be a self-adjoint operator whose spectrum is $\sigma(F) := \{-1, 0, 1\}$. Assume $[F, \pi(a)]$ to be compact for any $a \in A$, and denote by P_λ the spectral projection for the eigenvalue $\lambda \in \{-1, 0, 1\}$. Then*

(i) when u is invertible, $P_\lambda\pi(u)P_\lambda$ is Fredholm, for all $\lambda \in \{-1, 0, 1\}$, and

$$\sum_\lambda \operatorname{Ind}(P_\lambda\pi(u)P_\lambda) = 0\,,$$

(ii) $[P_\lambda, \pi(a)]$ is compact, for any $a \in A$, and $\lambda \in \{-1, 0, 1\}$.

Proof. (i) As $[F, \pi(u)] \in \mathcal{K}(\mathcal{H})$, for all $\lambda, \lambda' \in \{-1, 0, 1\}$, we have $P_\lambda[F, \pi(u)]P_{\lambda'} \in \mathcal{K}(\mathcal{H})$,

$$P_\lambda[F, \pi(u)]P_{\lambda'} = (\lambda - \lambda')P_\lambda\pi(u)P_{\lambda'}$$

and, in particular, $P_\lambda[F, \pi(u)]P_\lambda = 0$. Since

$$\pi(u) = \sum_\lambda P_\lambda\pi(u)P_\lambda + \sum_{\lambda \neq \lambda'} (\lambda - \lambda')^{-1} P_\lambda[F, \pi(u)]P_{\lambda'}$$

and u is invertible, then $\sum_\lambda P_\lambda\pi(u)P_\lambda$ and $P_\lambda\pi(u)P_\lambda$ are Fredholm operators, for all $\lambda \in \{-1, 0, 1\}$, and

$$\sum_\lambda \operatorname{Ind}(P_\lambda\pi(u)P_\lambda) = \operatorname{Ind}(\pi(u)) = 0\,.$$

(ii) Observe that, for all $a \in \mathcal{A}$ and $\{\lambda, \lambda', \lambda''\}$ a permutation of $\{-1, 0, 1\}$, we have

$$[P_\lambda, \pi(a)] = \sum_{\lambda,\mu\in\sigma(F)} P_\lambda[P_\lambda, \pi(a)]P_\mu = P_\lambda\pi(a)P_{\lambda'} + P_\lambda\pi(a)P_{\lambda''} - P_{\lambda'}\pi(a)P_\lambda - P_{\lambda''}\pi(a)P_\lambda.$$

Since all summands in the last expression are compact by (i), the thesis follows. $\square$

Corollary 2.8. *Let $(\mathcal{H}, \pi, F)$ be a Fredholm module over a C^*-algebra A, in the sense of Definition 2.2, with $F^* = F$, and $F^2 = I$ on $(\ker F)^\perp$, and assume that for all invertible $u \in A$ we have*

$$\mathrm{Ind}(P_0\pi(u)P_0) = 0\,.$$

Then there exists a Fredholm module $(\mathcal{H}, \pi, F')$ such that $F'^2 = I$ and

$$\mathrm{Ind}(P_\lambda\pi(u)P_\lambda) = \mathrm{Ind}(P'_\lambda\pi(u)P'_\lambda) \qquad \lambda = -1\,, +1\,.$$

Here P_λ (resp. P'_λ) denotes the projection of the operator F (resp. F') associated to the eigenvalue $\lambda \in \{-1, +1\}$.

Proof. Defining $F' := F + P_0$ we have $F'^* = F'$, $F'^2 = I$ and $\sigma(F') = \{-1, +1\}$. Since $[F', \pi(a)] = [F, \pi(a)] + [P_0, \pi(a)]$, by Proposition 2.7 (ii) we have $[F', \pi(u)] \in \mathcal{K}(\mathcal{H})$, so that $(\mathcal{A}, (\pi, \mathcal{H}), F')$ is a Fredholm module. Finally, since $P'_1 = P_1 + P_0$, and $P_1\pi(u)P_0), P_0\pi(u)P_1$ are compact, by the proof of Proposition 2.7 (i), and since, by assumption, $\mathrm{Ind}(P_0\pi(u)P_0) = 0$, we have

$$\mathrm{Ind}(P'_1\pi(u)P'_1) = \mathrm{Ind}((P_1{+}P_0)\pi(u)(P_1{+}P_0)) = \mathrm{Ind}(P_1\pi(u)P_1){+}\mathrm{Ind}(P_0\pi(u)P_0) = \mathrm{Ind}(P_1\pi(u)P_1)\,.$$

$\square$

Definition 2.9. A (possibly kernel-degenerate) odd Fredholm module $(\mathcal{H}, \pi, F)$ will be called tamely degenerate if

$$\mathrm{Ind}(P_0\pi(u)P_0) = 0\,, \tag{2.1}$$

for all invertible $u \in Mat_k(A)$, $k \in \mathbb{N}$, where P_0 denotes the projection onto $\ker F \otimes I_{\mathbb{C}^k}$.

Corollary 2.8 proves that a tamely degenerate Fredholm module is equivalent to a (non-kernel-degenerate) Fredholm module, as far as their indexes are concerned.

We now recall the definition of p-graded Fredholm module.

Definition 2.10. [[34], Defs 8.1.11 & A.3.1] Let $p \in \{-1, 0, , ...\}$. A p-graded Fredholm module over a C*-algebra A is given by the following data:

(a) a separable Hilbert space $\mathcal{H}$;
(b) $p+1$ unitary operators $\varepsilon_0, ..., \varepsilon_p$ such that $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0$ if $i \neq j$, $\varepsilon_i^2 = -1$, for $i \neq 0$, $\varepsilon_0^2 = 1$.
(c) a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $[\varepsilon_i, \pi(a)] = 0$ for any $i = 0, \dots, p$, any $a \in \mathcal{A}$
(d) an operator F on $\mathcal{H}$ such that $\varepsilon_i F - F\varepsilon_i = 0$, $i \neq 0$, $\varepsilon_0 F + F\varepsilon_0 = 0$, and, for all $a \in \mathcal{A}$, $(F^2 - 1)$, $F - F^*$, $[F, \pi(a)]$ are compact.

In particular, odd Fredholm modules are (-1)-graded, and even Fredholm modules are (0)-graded.

Endowed with the equivalence relation given by stable homotopy [34], the set of equivalence classes of p-graded Fredholm modules becomes an abelian group, with addition given by direct sum, which is denoted $K^{-p}(A) = KK^{-p}(A, \mathbb{C})$, and called $(-p)$-th K-homology group of A. Because of Bott periodicity (cf. Proposition 8.2.13 in [34]), $K^{-p}(A)$ and $K^{-p-2}(A)$ are naturally isomorphic, so there are really two K-homology groups of A, the odd one $K^1(A)$, and the even one $K^0(A)$. It turns out that (equivalence classes of) p-graded Fredholm modules pair with odd K-theory when p is odd, and with even K-theory, when p is even.

A particular instance of Bott periodicity, which we will need in the following sections, says that, given a 1-graded Fredholm module $\mathcal{F} = (\mathcal{H}, \pi, F, \gamma, \varepsilon)$, and setting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, $\pi = \pi^+ \oplus \pi^-$, $F = \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} I_{\mathcal{H}^+} & 0 \\ 0 & -I_{\mathcal{H}^-} \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 0 & -iV \\ -iV^* & 0 \end{pmatrix}$, $F^+ = VF_{21} = F_{12}V^*$, then $\mathcal{F}^* = (\mathcal{H}^+, \pi^+, F^+)$ is an odd Fredholm module on A, giving the same pairing with K-theory. Proposition 2.12 shows that weakening some of the conditions in the definition of 1-graded module does not alter the previous result.

Let us observe that, given an even Fredholm module $(\pi, \mathcal{H}, F, \gamma)$ on $\mathcal{A}$, we can make it a 1-graded Fredholm module $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ simply by adding a skew-adjoint unitary operator ε which commutes with F, anticommutes with γ, and commutes with $\pi(a)$ (possibly up to compact operators).

Definition 2.11. (1-graded Fredholm Module) A (possibly kernel-degenerate) 1-graded Fredholm Module $(\mathcal{H}, \pi, F, \gamma, \varepsilon)$ over a C*-algebra A, consists of an even (possibly kernel-degenerate) Fredholm Module $(\mathcal{H}, \pi, F, \gamma)$, and an operator $\varepsilon \in \mathcal{U}(\mathcal{H})$ such that

(*i*) $\varepsilon^2 + I = 0$ on $\ker(F)^\perp$,
(*ii*) $\varepsilon^* + \varepsilon = 0$,
(*iii*) the commutators $[\varepsilon, \pi(a)]$ are compact operators, for all $a \in A$.

Proposition 2.12. *Let $(\mathcal{H}, \pi, F, \gamma, \varepsilon)$ be a (possibly kernel-degenerate) 1-graded Fredholm module, with F self-adjoint. Then $\varepsilon_0 = P^+ - P^-$, where $P^\pm \in \mathrm{Proj}(\mathcal{H})$, $P^+ + P^- = I$. Setting $\mathcal{H}^\pm := P^\pm\mathcal{H}$, one gets $\pi = \pi^+ \oplus \pi^-$, $\varepsilon = \begin{pmatrix} 0 & -iV \\ -iV^* & 0 \end{pmatrix}$, where $V : \mathcal{H}^- \to \mathcal{H}^+$ is a partial isometry, and $F = \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}$. Setting $F^+ = VF_{21} = F_{12}V^*$, $F^- = V^*F_{12} = F_{21}V$, we have that the spectrum of $F^\pm$ is $\{-1, 0, 1\}$, and let $P^\pm$, $N^\pm$, $Z^\pm$ be the spectral projections on the positive, negative, and zero eigenvalue of $F^\pm$.*

Moreover, $(\mathcal{H}^+, \pi^+, F^+)$, $(\mathcal{H}^-, \pi^-, F^-)$ are (possibly kernel-degenerate) odd Fredholm modules, and, for all invertible $u \in Mat_k(A)$, it holds (with $P^\pm$ denoting $P^\pm \otimes I_{\mathbb{C}^k}$, and analogously for $N^\pm$, $Z^\pm$, and $\pi^\pm$ properly extended to $Mat_k(A)$)

$$\mathrm{Ind}(P^+\pi^+(u)P^+) = \mathrm{Ind}(P^-\pi^-(u)P^-),$$
$$\mathrm{Ind}(N^+\pi^+(u)N^+) = \mathrm{Ind}(N^-\pi^-(u)N^-),$$
$$\mathrm{Ind}(Z^+\pi^+(u)Z^+) = \mathrm{Ind}(Z^-\pi^-(u)Z^-).$$

Finally, if any of the modules $(\mathcal{H}, \pi, F, \gamma, \varepsilon)$, $(\mathcal{H}^+, \pi^+, F^+)$, $(\mathcal{H}^-, \pi^-, F^-)$ is tamely degenerate, then so are the other two.

Proof. In the course of this proof we set $A \approx B$ to mean equality modulo compact operators, *i.e.* $A - B$ is a compact operator. From the properties of γ, it follows that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm$ is the eigenspace relative to the eigenvalue ± 1 of γ. Moreover $\pi = \pi^+ \oplus \pi^-$, where $\pi^\pm A \to \mathcal{B}(\mathcal{H}^\pm)$ is a representation of A. From $\gamma\varepsilon + \varepsilon\gamma = 0$ and $\varepsilon + \varepsilon^* = 0$ it follows $\varepsilon = \begin{pmatrix} 0 & iV \\ iV^* & 0 \end{pmatrix}$, where $V : \mathcal{H}^- \to \mathcal{H}^+$. In addition, for $a \in A$, we have $0 \approx \varepsilon\pi(a) - \pi(a)\varepsilon = i \begin{pmatrix} 0 & V\pi^-(a) - \pi^+(a)V \\ V^*\pi^+(a) - \pi^-(a)V^* & 0 \end{pmatrix} \implies V\pi^-(a) \approx \pi^+(a)V$.

As for F, $0 = F\gamma + \gamma F \implies F = \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}$, whereas $0 = F\varepsilon - \varepsilon F \implies F_{12}V^* = VF_{21}$, $V^*F_{12} = F_{21}V$. Moreover, denoting by P_0 the projection onto $\ker F$, we have $\begin{pmatrix} VV^* & 0 \\ 0 & V^*V \end{pmatrix} = -\varepsilon^2 = I - P_0 = F^2 = \begin{pmatrix} F_{12}F_{21} & 0 \\ 0 & F_{21}F_{12} \end{pmatrix}$, so that $VV^* = F_{12}F_{21}$, and $V^*V = F_{21}F_{12}$ are projections, so that V is a partial isometry.

Let us set $F^+ = VF_{21}$, $F^- = V^*F_{12}$, so that $(F^+)^* = F_{21}^*V^* = F_{12}V^* = F^+$, $\begin{pmatrix} F^+ & 0 \\ 0 & F^- \end{pmatrix} = -i\varepsilon F \implies \begin{pmatrix} (F^+)^2 & 0 \\ 0 & (F^-)^2 \end{pmatrix} = -\varepsilon F \varepsilon F = -\varepsilon^2 F^2 = F^2 = I - P_0$, which implies that the spectrum of $F^\pm$ is $\{-1, 0, 1\}$, and let $P^\pm$, $N^\pm$, $Z^\pm$ be the spectral projections on the positive, negative, and zero eigenvalue of $F^\pm$. Therefore, $F^\pm = P^\pm - N^\pm$, and $P^+ + N^+ = I - Z^+ = VV^*$, $P^- + N^- = I - Z^- = V^*V$. Moreover, $F^+V = VF_{21}V = VF^-$, so that $(P^+ - N^+)V = V(P^- - N^-)$. Besides, $(P^+ + N^+)V = VV^*V = V(P^- + N^-)$, from which we conclude $P^+V = VP^-$, and $N^+V = VN^-$.

In order to conclude that $(\mathcal{H}^\pm, \pi^\pm, F^\pm)$ are odd Fredholm modules, we only need to prove the properties of $F^\pm$. For all $a \in A$, we have

$$I - P_0 \approx (F^2 - I)\pi(a) = \begin{pmatrix} \big((F^+)^2 - I\big)\pi^+(a) & 0 \\ 0 & \big((F^-)^2 - I\big)\pi^-(a) \end{pmatrix}$$

$\implies \big((F^\pm)^2 - I\big)\pi^\pm(a) \approx I - Z^\pm$. $0 \approx F\pi(a) - \pi(a)F \implies$

$$\begin{aligned} 0 &\approx -i\varepsilon F\pi(a) + i\varepsilon\pi(a)F \approx -i\varepsilon F\pi(a) + \pi(a)i\varepsilon F \\ &= \begin{pmatrix} F^+\pi^+(a) - \pi^+(a)F^+ & 0 \\ 0 & F^-\pi^-(a) - \pi^-(a)F^- \end{pmatrix} \end{aligned}$$

$\implies F^\pm\pi^\pm(a) - \pi^\pm(a)F^\pm \approx 0$. $0 \approx (F^* - F)\pi(a) \implies$

$$\begin{aligned} 0 \approx &- i\varepsilon(F^* - F)\pi(a) = -(iF^*\varepsilon - i\varepsilon F)\pi(a) = (iF^*\varepsilon^* + i\varepsilon F)\pi(a) \\ =& \big((-i\varepsilon F)^* - (-i\varepsilon F)\big)\pi(a) = \begin{pmatrix} \big((F^+)^* - F^+\big)\pi^+(a) & 0 \\ 0 & \big((F^-)^* - F^-\big)\pi^-(a) \end{pmatrix} \end{aligned}$$

$\implies \big((F^\pm)^* - F^\pm\big)\pi^\pm(a) \approx 0$, that is, $(\mathcal{H}^\pm, \pi^\pm, F^\pm)$ are (possibly kernel-degenerate) odd Fredholm modules. Finally, for all invertible $u \in A$, we have

$$\begin{aligned} \mathrm{Ind}(P^+\pi^+(u)P^+) &= \mathrm{Ind}(P^+VV^*\pi^+(u)VV^*P^+) = \mathrm{Ind}(VP^-V^*V\pi^-(u)P^-V^*) \\ &= \mathrm{Ind}(VP^-\pi^-(u)P^-V^*) = \mathrm{Ind}(P^-\pi^-(u)P^-), \end{aligned}$$

where the second equality follows from the intertwining properties of V, and $V\pi^-(a) \approx \pi^+(a)V$, and the last equality follows from the fact that $V \in \mathcal{U}(P^-\mathcal{H}, P^+\mathcal{H})$. The equality $\mathrm{Ind}(N^+\pi^+(u)N^+) = \mathrm{Ind}(N^-\pi^-(u)N^-)$ is proved analogously, whereas $\mathrm{Ind}(Z^+\pi^+(u)Z^+) = \mathrm{Ind}(Z^-\pi^-(u)Z^-)$ follows from the above and Proposition 2.7 (i). An analogous argument proves the above equalities for any invertible $u \in Mat_k(A)$.

Therefore, if $(\mathcal{H}, \pi, F, \gamma, \varepsilon)$ is tamely degenerate, $0 = \mathrm{Ind}(P_0\pi(u)P_0) = \mathrm{Ind}(Z^+\pi^+(u)Z^+) + \mathrm{Ind}(Z^-\pi^-(u)Z^-)$, which implies $\mathrm{Ind}(Z^+\pi^+(u)Z^+) = \mathrm{Ind}(Z^-\pi^-(u)Z^-) = 0$, that is $(\mathcal{H}^\pm, \pi^\pm, F^\pm)$ are tamely degenerate. Viceversa, $(\mathcal{H}^+, \pi^+, F^+)$ is tamely degenerate $\iff$ $(\mathcal{H}^-, \pi^-, F^-)$ is, and in this case $(\mathcal{H}, \pi, F, \gamma, \varepsilon)$ is tamely degenerate as well. $\square$

2.2. **Spectral triples on self-similar fractals.** A self-similar fractal (in $\mathbb{R}^n$) is described by a finite set of similitudes $w_1, \dots w_k$, with scaling parameters $\lambda_1, \dots \lambda_k$, $\lambda_i < 1$, as the unique compact set X such that

$$\bigcup_{i=1}^{k} w_i(X) = X.$$

A standard way to construct spectral triples on such fractal is the following:

- Select a subset $S \subset X$ together with a triple $\mathcal{T} = (\pi, \mathcal{H}, D)$ on $\mathcal{C}(S)$.
- Set $\mathcal{T}_\emptyset = (\pi_\emptyset, \mathcal{H}_\emptyset, D_\emptyset)$ on $\mathcal{C}(X)$, where $\pi_\emptyset(f) = \pi(f|_S)$, $\mathcal{H}_\emptyset = \mathcal{H}$, $D_\emptyset = D$.
- Set $\mathcal{T}_\sigma := (\pi_\sigma, \mathcal{H}_\emptyset, D_\sigma)$ on $\mathcal{C}(X)$, with $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$, $D_\sigma = \lambda_\sigma^{-1} D_\emptyset$, $\lambda_\sigma = \prod_{i=1}^{|\sigma|} \lambda_{\sigma_i}$.
- Set $\mathcal{T} = \bigoplus_\sigma \mathcal{T}_\sigma$ on $\mathcal{C}(X)$ and consider the *-algebra $\mathcal{A} = \{f \in \mathcal{C}(X) : [D, f] \text{ is bdd}\}$.

This type of construction was used in [32, 33] to reproduce some of the features of the fractals. It was also the basis of the construction of the triples for the Sierpinski gasket K in [14, 15], by choosing S as the lacuna $\ell_\emptyset$ isometrically identified with the circle $\mathbb{T}$, with the standard triple given by $\mathcal{H} = L^2(\mathbb{T})$ and $D = -id/d\vartheta$.

As shown below (Lemma 4.11), this choice does not allow the recovery of the energy via residues. In order to do so, one has to deform the triple on the circle by replacing the standard Laplacian Δ with one of its fractional powers Δ^α, $\alpha < 1$. However, in order to obtain a Dirac operator D based on a suitable "differential" square root of Δ^α, we need to double the Hilbert space. So, the deformed triples on $\mathbb{T}$, which we describe below, are obtained by deforming the triple $(\mathcal{A}, \mathcal{K}, D)$, where $\mathcal{A} = Lip(\mathbb{T})$, $\mathcal{K} := L^2(\Omega^*(\mathbb{T})) = L^2(\Omega^1(\mathbb{T})) \oplus L^2(\Omega^0(\mathbb{T}))$, and $D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$. However, such a triple on $\mathbb{T}$ (and its associated Fredholm module) is even, with standard grading $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so that, to maintain the correct pairing with odd K-theory, we add a further grading ε,

$$\varepsilon = -i \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}, \text{ where } Ve^{int} = \frac{\operatorname{sgn}(n)}{n} de^{int} = ie^{int} dt \,, t \in \mathbb{T}. \tag{2.2}$$

It is not difficult to show that $(\mathcal{A}, \mathcal{K}, D, \gamma, \varepsilon)$ is a 1-graded spectral triple, that is to say, $(\mathcal{C}(\mathbb{T}), \mathcal{K}, F, \gamma, \varepsilon)$ is a 1-graded Fredholm module according to Definition 2.10 above, where F is the phase of D. Such 1-graded Fredholm module is equivalent to the original odd one by Bott periodicity, cf. Proposition 2.12. The triple $(\mathcal{A}, \mathcal{K}, D, \gamma, \varepsilon)$ allows the required α-deformation, which is described in the next Section.

3. Spectral triples on quasi-circles

In this section we build a family of spectral triples on the algebra $C(\mathbb{T})$ of continuous functions on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, depending on a parameter $\alpha \in (0, 1]$. For $\alpha = 1$ we consider the triple $(\mathcal{A}, \mathcal{K}, D, \gamma, \varepsilon)$ given at the end of the preceding section, which describes the circle with the standard differential structure. The rest of the Section is devoted to the construction of the triples for $\alpha < 1$, which may be considered as deformations of the original one, the circle being replaced by quasi-circles.

3.1. **Preliminaries about quadratic forms on $\mathbb{T}$.** We will use the following notation. For any $f \in C(\mathbb{T})$, the Fourier coefficients are $f_k := \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} \, dt$, $k \in \mathbb{Z}$, the convolution between f and $g \in C(\mathbb{T})$ is $f * g(t) := \frac{1}{2\pi} \int_{\mathbb{T}} f(t - \vartheta) g(\vartheta) \, d\vartheta$, and if Ψ is a distribution on $\mathbb{T}$ and $f \in C^\infty(\mathbb{T})$, the (sesquilinear) pairing $\langle \Psi, f \rangle$ is defined as the (weakly continuous) extension of the scalar product in $L^2(\mathcal{T})$. For any positive sequence $\{a_k\}$ of polynomial growth on $\mathbb{Z}$ we consider the quadratic form on functions in $\mathcal{C}^\infty(\mathbb{T})$ given by

$$Q[f] = \sum_{k\in\mathbb{Z}} a_k |f_k|^2,$$

and the distribution Φ given by the Fourier series $\sum_{k\in\mathbb{Z}} a_k e^{ikt}$, so that $\langle\Phi, f\rangle = \sum_{k\in\mathbb{Z}} a_k f_k$, and

$$Q[f] = \langle\Phi, f^* * f\rangle,$$

where $f^*(t) := \overline{f}(-t)$.

Definition 3.1. A sequence $\{a_k \in \mathbb{C} : k \in \mathbb{Z}\}$ is called *positive definite* if

$$\sum_{m,n\in\mathbb{Z}} a_{m-n}\overline{c}_m c_n \geq 0 \tag{3.1}$$

for any finitely supported sequence $\{c_k\}$. A sequence $\{a_k\}$ is called *conditionally positive definite* if

$$\sum_{m,n\in\mathbb{Z}} a_{m-n}(\partial\overline{c})_m(\partial c)_n \geq 0 \tag{3.2}$$

for any finitely supported sequence $\{c_k \in \mathbb{C} : k \in \mathbb{Z}\}$, where $(\partial c)_k = c_k - c_{k-1}$. A sequence is *(conditionally) negative definite* if it is the opposite of a (conditionally) positive definite one.

Theorem 3.2. *Let $\{a_k\}$ be a conditionally positive definite sequence. Then there exist a positive measure μ on $\mathbb{T}$ and a constant b such that*

$$\langle\Phi, f\rangle = \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\ d\mu + a_0 f(0) + \frac{1}{2i}(a_1 - a_{-1})f'(0) + bf''(0).$$

Proof. The proof is analogous to that of Thm 1, Chapter II of [30], but we give the details for the convenience of the reader.

Passing to Fourier series, eq. (3.2), which clearly holds also for fast decreasing sequences c_k, may be rephrased as

$$\langle |1 - e^{-it}|^2 \Phi, |f|^2\rangle \geq 0, \tag{3.3}$$

where $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}$. Since such sums describe all $\mathcal{C}^\infty$ functions, and $|1 - e^{-it}|^2 = 2(1 - \cos t)$, this is equivalent to $\langle(1 - \cos t)\Phi, g\rangle \geq 0$ for any positive function $g \in \mathcal{C}^\infty$, namely $(1 - \cos t)\Phi$ is a positive measure ν. Equivalently, $\langle\Phi, (1 - \cos t)g\rangle = \int g\, d\nu$ for any $g \in \mathcal{C}^\infty$. Since any function h with a zero of order 2 may be written as $h = (1 - \cos t)g$, we get $\langle\Phi, h\rangle = \int h(t)(1 - \cos t)^{-1}\, d\nu$ for any function h with a zero of order 2 in $t = 0$. We then separate the part of ν with support in 0, setting $\nu = b\delta_0 + (1 - \cos t)\mu$, thus getting

$$\langle\Phi, h\rangle = \int_{(0,2\pi)} h(t)\, d\mu + bh''(0)\,.$$

Then, since $f(t) - f(0) - f'(0)\sin t$ has a zero of order 2 for $t = 0$, we get

$$\begin{aligned}\langle\Phi, f\rangle &= \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\ d\mu + \langle\Phi, f(0) + f'(0)\sin t\rangle + bf''(0)\\ &= \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\ d\mu + a_0 f(0) + \frac{1}{2i}(a_1 - a_{-1})f'(0) + bf''(0).\end{aligned}$$

□

3.2. **Sobolev norms and Clausen functions.** Let $s \in \mathbb{C}$. Then the polylogarithm function of order s is defined as

$$\mathrm{Li}_s(z) := \sum_{k\in\mathbb{N}} \frac{z^k}{k^s}, \quad |z| < 1.$$

It has an analytic continuation on the whole complex plane with the line $[1, +\infty)$ removed, cf. the Appendix. The Clausen cosine function $\mathrm{Ci}_s(t)$ is defined as the sum of the Fourier series

$$\sum_{k\in\mathbb{N}} \frac{\cos kt}{k^s}, \quad \mathrm{Re}\, s > 1.$$

When $\mathrm{Re}\, s \le 1$ it can be defined as the real part of $\mathrm{Li}_s(e^{it})$, hence it is a smooth function for $t \neq 0$.

Some properties of the Clausen function are contained in Lemma A.1 and Proposition A.2.

Proposition 3.3. *Let $\alpha \in (0,1)$, $a_k = |k|^{2\alpha}$, $k \in \mathbb{Z}$, and Φ the associated distribution as above. Then*

(*i*) *the sequence a_k is conditionally negative definite,*
(*ii*) *for any $\mathcal{C}^\infty$ function f,*

$$\langle \Phi, f \rangle = \frac{1}{\pi} \int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)(f(t) - f(0))\, dt\,.$$

In particular, the Clausen function $\mathrm{Ci}_{-2\alpha}$ is negative.

Proof. (*i*) It is well known that k^2 is a conditionally negative definite sequence, therefore so is $k^{2\alpha}$, for $\alpha \in (0,1]$ ([8], page 78).

(*ii*) Assume $f(0) = 0$. Since Φ is even, the pairing with the odd part of f vanishes, while, by Proposition A.2, the pairing with the even part is given by the integral against $\frac{1}{\pi}\mathrm{Ci}_{-2\alpha}$. According to the results of Theorem 3.2, the measure $d\mu$ (which is now negative) should be replaced by $\frac{1}{\pi}\mathrm{Ci}_{-2\alpha}(t)\, dt$, showing in particular that $\mathrm{Ci}_{-2\alpha}$ is negative. For a general f, again using the parity of Φ, the pairing becomes

$$\langle \Phi, f \rangle = \frac{1}{\pi} \int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)(f(t) - f(0))\, dt + b f''(0)\,,$$

hence we get the result if we show that $b = 0$. By definition, for any continuous function g, $\langle \Phi, (1 - \cos t) g(t) \rangle = b\, g(0) + \int (1 - \cos t) g(t)\, d\mu$. In particular, if g has suitably small support, $b\, g(0) = \lim_{\varepsilon\to 0} \langle \Phi, (1 - \cos t) g(t/\varepsilon) \rangle$. Choosing $g(t) = \chi_{[-1,1]}(1 - |t|)$, a direct computation shows that $b = 0$. ☐

Corollary 3.4. *Let $\alpha \in (0,1]$, $a_k = |k|^{2\alpha}$, $k \in \mathbb{Z}$, and denote by $\mathcal{E}_\alpha$ the corresponding quadratic form. Then*

(*i*) *$\|f\|_2^2 + \mathcal{E}_\alpha[f]$ is the square of the norm for the Sobolev space $H^\alpha(\mathbb{T})$,*
(*ii*) *the quadratic form $\mathcal{E}_\alpha$ is given by*

$$\mathcal{E}_\alpha[f] = -\frac{1}{2\pi} \int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x - y)|f(x) - f(y)|^2\, dxdy - b\|f'\|_2^2,$$

where $b = 0$ when $\alpha < 1$, while $\mathrm{Ci}_{-2} = 0$ and $b = -1$ when $\alpha = 1$.

Proof. (*i*) Obvious.
(*ii*) We have

$$\mathcal{E}_\alpha[f] = \langle \Phi, f^* * f \rangle = \frac{1}{\pi} \int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)((f^* * f)(t) - (f^* * f)(0))\, dt + b(f^* * f)''(0).$$

Since $\mathrm{Ci}_{-2\alpha}(x-y) = \mathrm{Ci}_{-2\alpha}(y-x)$, we have

$$
\begin{aligned}
2\int_{\mathbb{T}} &\mathrm{Ci}_{-2\alpha}(t)((f^* * f)(t) - (f^* * f)(0))\, dt \\
&= 2\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)\big(\overline{f}(x-t)f(x) - \overline{f}(x)f(x)\big)\, dt\, dx \\
&= 2\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big(\overline{f}(y)f(x) - \overline{f}(x)f(x)\big)\, dy\, dx \\
&= \int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big(\overline{f}(y)f(x) - \overline{f}(x)f(x) + \overline{f}(x)f(y) - \overline{f}(y)f(y)\big)\, dy\, dx \\
&= -\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big|f(x)-f(y)\big|^2\, dy\, dx\,.
\end{aligned}
$$

As for the second summand,

$$(f^* * f)''(0) = -\big((f')^* * f'\big)(0) = -\|f'\|_2^2\,,$$

which proves the equation. Since the quadratic form gives rise to the Sobolev norm, the last summand should vanish when $\alpha < 1$. For $\alpha = 1$, the Clausen function vanishes by a direct computation, and $\mathcal{E}_\alpha[f] = \|f'\|_2^2$, giving $b = -1$. ☐

3.3. **The construction of the triple.** Let us consider, for each fixed $0 < \alpha \leq 1$, the Dirichlet form $\mathcal{E}_\alpha$ on $L^2(\mathbb{T})$, with domain $\mathcal{F}_\alpha := \{f \in L^2(\mathbb{T}) : \mathcal{E}_\alpha[f] < +\infty\}$.
As shown in Corollary 3.4, the Sobolev space $H^\alpha(\mathbb{T})$ coincides with $\mathcal{F}_\alpha$ and has norm

$$\|f\|_\alpha^2 = \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{E}_\alpha[f]\,.$$

We summarize below the main known properties of the Dirichlet spaces on the circle we are considering. Proofs may be found in [28].

Proposition 3.5. *The Dirichlet space $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ on $L^2(\mathbb{T})$ is regular in the sense that the Dirichlet algebra $\mathcal{F}_\alpha \cap C(\mathbb{T})$ is dense both in $C(\mathbb{T})$ with respect to the uniform norm and in $\mathcal{F}_\alpha$ with respect to the graph norm. In particular, the algebra $C^\gamma(\mathbb{T})$ of Hölder continuous functions of order $\gamma \in (\alpha, 1]$ is a form core contained in the Dirichlet algebra. We observe that $\mathcal{F}_\alpha \subset C(\mathbb{T})$, for $\alpha > \frac{1}{2}$.*

We now construct a Spectral Triple associated to the above Dirichlet space for each value of the parameter $0 < \alpha < 1$, the case $\alpha = 1$ having been described above. The construction is given in terms of a closable derivation with values in a suitable bimodule, underlying any regular Dirichlet form (see [19], [21]). By Corollary 3.4,

$$\mathcal{E}_\alpha[f] = \frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\, dzdw,$$

where we set $\varphi_\alpha = -2\pi\,\mathrm{Ci}_{-2\alpha}$.

The linear map defined as

$$\partial_\alpha : \mathcal{F}_\alpha \to L^2(\mathbb{T}\times\mathbb{T}) \qquad \partial_\alpha(f)(z,w) = \varphi_\alpha(z-w)^{1/2}(f(z)-f(w))$$

is a closed operator acting on $L^2(\mathbb{T})$, since $\mathcal{E}_\alpha[f] = \|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})}$.

Endowing the Hilbert space $L^2(\mathbb{T}\times\mathbb{T})$ with the $C(\mathbb{T})$-bimodule structure defined by the left and right actions of $C(\mathbb{T})$ given by

$$(f\xi)(z,w) := f(z)\xi(z,w)\,, \qquad (\xi g)(z,w) := \xi(z,w)g(w)\,, \qquad z,w \in \mathbb{T}\,,$$

and by the anti-linear involution

$$(\mathcal{J}\xi)(z,w) := \overline{\xi(w,z)}, \qquad z,w\in\mathbb{T}\,,$$

for $f,g\in C(\mathbb{T})$ and $\xi\in L^2(\mathbb{T}\times\mathbb{T})$, it is easy to see that the map ∂_α is a *derivation* on the Dirichlet algebra $\mathcal{F}_\alpha\cap C(\mathbb{T})$, since it is *symmetric*

$$\mathcal{J}(\partial_\alpha(f)) = \partial_\alpha(\overline{f}), \qquad f\in C^\gamma(\mathbb{T}),$$

and satisfies the *Leibniz rule*

$$\partial_\alpha(fg) = (\partial_\alpha f)g + f(\partial_\alpha g), \qquad f,g\in C^\gamma(\mathbb{T})\,.$$

Moreover, the map ∂_α is a *differential square root* of the self-adjoint operator Δ^α on $L^2(\mathbb{T})$ having $(\mathcal{E}_\alpha,\mathcal{F}_\alpha)$ as closed quadratic form, because of the identities

$$\mathcal{E}_\alpha[f] := \|\Delta^{\alpha/2}f\|^2_{L^2(\mathbb{T})} = \|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})}, \qquad f\in\mathcal{F}_\alpha\,.$$

We accommodate in the following Lemma some technical results which will be useful later.

Lemma 3.6. *Let us denote by $\{e_k : k\in\mathbb{Z}\}$ the orthonormal basis of eigenfunctions of the standard Laplacian Δ:*

$$e_k(t) := e^{ikt}, \qquad \Delta e_k = k^2 e_k.$$

(1) $\mathcal{E}_\alpha(e_k,e_j) = (e_k,\partial_\alpha^*\partial_\alpha e_j) = |k|^{2\alpha}\delta_{kj}$.
(2) Let $e'_n = |n|^{-\alpha}\partial_\alpha e_n$, $n\in\mathbb{Z}\setminus\{0\}$. Then, the family $\{e'_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an orthonormal basis for the range of ∂_α.
(3) The following equation holds:

$$\partial_\alpha^*((\partial_\alpha e_p)e_n) = \frac{1}{2}(|p|^{2\alpha}+|n+p|^{2\alpha}-|n|^{2\alpha})e_{n+p}. \tag{3.4}$$

(4) For any $k,p\in\mathbb{Z}$, $(|p|^{2\alpha}+|k|^{2\alpha}-|k-p|^{2\alpha})^2 \le 4|k|^{2\alpha}|p|^{2\alpha}$.
(5) Let us consider the map $S_f : C(\mathbb{T})\to L^2(\mathbb{T}\times\mathbb{T})$ defined, for a fixed $f\in C(\mathbb{T})$, as $S_f g := (\partial_\alpha f)g$, with $g\in C(\mathbb{T})$. Then, for $s>\alpha^{-1}$ and $f\in H^\alpha$, the operators $(\partial_\alpha\partial_\alpha^*)^{-s/4}S_fS_f^*(\partial_\alpha\partial_\alpha^*)^{-s/4}$ and $(\partial_\alpha^*\partial_\alpha)^{-s/4}S_f^*S_f(\partial_\alpha^*\partial_\alpha)^{-s/4}$ are trace class, and

$$\mathrm{tr}((\partial_\alpha\partial_\alpha^*)^{-s/4}S_fS_f^*(\partial_\alpha\partial_\alpha^*)^{-s/4}) \le 2\zeta(\alpha s)\mathcal{E}_\alpha[f] = \mathrm{tr}((\partial_\alpha^*\partial_\alpha)^{-s/4}S_f^*S_f(\partial_\alpha^*\partial_\alpha)^{-s/4}). \tag{3.5}$$

$$\mathrm{Res}_{s=1/\alpha}\,\mathrm{tr}((\partial_\alpha\partial_\alpha^*)^{-s/4}S_fS_f^*(\partial_\alpha\partial_\alpha^*)^{-s/4}) = \begin{cases} 0 & \text{if } \alpha<1, \\ 2\mathcal{E}_\alpha[f] & \text{if } \alpha=1. \end{cases} \tag{3.6}$$

$$\mathrm{Res}_{s=1/\alpha}\,\mathrm{tr}((\partial_\alpha^*\partial_\alpha)^{-s/4}S_f^*S_f(\partial_\alpha^*\partial_\alpha)^{-s/4}) = \frac{2}{\alpha}\mathcal{E}_\alpha[f]. \tag{3.7}$$

Proof. The equality $\partial_\alpha^*\partial_\alpha = \Delta^\alpha$ gives (1), while (2) follows from a direct computation, and (3) amounts to verify that $(\partial_\alpha e_k,(\partial_\alpha e_p)e_n) = \frac{1}{2}(p^{2\alpha}+(n+p)^{2\alpha}-n^{2\alpha})\delta_{k,n+p}$. We now show (4). We observe that it certainly holds for $p=0$ or $k=0$. When they do not vanish, we must prove that

$$-1 \le \frac{|p|^{2\alpha}+|k|^{2\alpha}-|k-p|^{2\alpha}}{2|k|^\alpha|p|^\alpha} \le 1, \tag{3.8}$$

or, setting $|p/k| = e^{2t}$, where we may assume $t\ge 0$,

$$-1 \le \frac{1}{2}\left(e^t \mp e^{-t}\right)^{2\alpha} - \cosh(2\alpha t) \le 1, \tag{3.9}$$

the $\pm$ sign being the sign of pk. Taking the worst cases, we get

$$-1 \le \frac{1}{2}\left(e^t - e^{-t}\right)^{2\alpha} - \cosh(2\alpha t), \quad \frac{1}{2}\left(e^t + e^{-t}\right)^{2\alpha} - \cosh(2\alpha t) \le 1, \tag{3.10}$$

or, equivalently, $2\sinh(\alpha t) \le (2\sinh t)^\alpha$ and $(2\cosh t)^\alpha \le 2\cosh(\alpha t)$. Passing to the logarithms, it is enough to prove that $f_\alpha(t) := \log(2\sinh(\alpha t)) - \alpha\log(2\sinh t) \le 0$ and $g_\alpha(t) := \alpha\log(2\cosh t) - \log(2\cosh(\alpha t)) \le 0$. This follows because both functions tend to 0 for $t \to +\infty$, and $f'_\alpha(t) = \alpha(\coth(\alpha t) - \coth t) \ge 0$ for $\alpha \in [0,1]$ since coth is decreasing, and $g'_\alpha(t) = \alpha(\tanh t - \tanh(\alpha t)) \ge 0$ for $\alpha \in [0,1]$ since tanh is increasing.
As for the inequality in (3.5), we have

$$\begin{aligned} \|S_f^*\partial_\alpha e_k\|^2 &= \sum_n |(e_n, S_f^*\partial_\alpha e_k)|^2 = \sum_n |(\partial_\alpha f, ((\partial_\alpha e_k)e_{-n})|^2 \\ &= \frac{1}{4}\sum_n (|k|^{2\alpha} + |n-k|^{2\alpha} - |n|^{2\alpha})^2 |(f, e_{n+k})|^2 \\ &= \frac{1}{4}\sum_p (|k|^{2\alpha} + |p|^{2\alpha} - |p-k|^{2\alpha})^2 |(f, e_p)|^2 \\ &\le |k|^{2\alpha}\sum_p |p|^{2\alpha}|(f,e_p)|^2 \ = |k|^{2\alpha}\mathcal{E}_\alpha[f]\,, \end{aligned} \tag{3.11}$$

where the inequality in the last row follows by (4). Then

$$\begin{aligned} \mathrm{tr}((\partial_\alpha\partial_\alpha^*)^{-s/4} S_f S_f^* (\partial_\alpha\partial_\alpha^*)^{-s/4}) &= \sum_k ((\partial_\alpha\partial_\alpha^*)^{-s/4} e'_k, S_f S_f^* (\partial_\alpha\partial_\alpha^*)^{-s/4} e'_k) \\ = \sum_k |k|^{-(s+2)\alpha}\|S_f^*\partial_\alpha e_k\|^2 &\le \sum_k |k|^{-s\alpha}\|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})} = 2\zeta(\alpha s)\mathcal{E}_\alpha[f]\,. \end{aligned} \tag{3.12}$$

Concerning the equality in (3.5) we have

$$\begin{aligned} \mathrm{tr}((\partial_\alpha^*\partial_\alpha)^{-s/4} S_f^* S_f (\partial_\alpha^*\partial_\alpha)^{-s/4}) &= \sum_k ((\partial_\alpha^*\partial_\alpha)^{-s/4} e_k, S_f^* S_f (\partial_\alpha^*\partial_\alpha)^{-s/4} e_k) \\ &= \sum_k |k|^{-s\alpha}\|S_f e_k\|^2 = \sum_k |k|^{-s\alpha}\|(\partial_\alpha f)e_k)\|^2 = 2\zeta(\alpha s)\mathcal{E}_\alpha[f]\,. \end{aligned} \tag{3.13}$$

Eq. (3.6) for $\alpha = 1$ is straightforward, we now prove it for $\alpha < 1$. By (3.11),

$$|k|^{-2\alpha}\|S_f^*\partial_\alpha e_k\|^2 = \sum_p \left(\frac{|k|^{2\alpha} + |p|^{2\alpha} - |p-k|^{2\alpha}}{2|k|^\alpha|p|^\alpha}\right)^2 (|p|^\alpha|(f,e_p)|)^2,$$

where the first factor in the series is bounded by 1, and the second is in $\ell^1(\mathbb{Z})$. By dominated convergence,

$$\lim_{|k|\to\infty} |k|^{-2\alpha}\|S_f^*\partial_\alpha e_k\|^2 = \sum_p \lim_{|k|\to\infty}\left(\frac{|k|^{2\alpha} + |p|^{2\alpha} - |p-k|^{2\alpha}}{2|k|^\alpha|p|^\alpha}\right)^2 (|p|^\alpha|(f,e_p)|)^2 = 0.$$

Formula (3.12) and the asymptotic character of the residue prove the thesis. Finally, (3.7) follows directly by (3.13). □

We now come to the promised family of spectral triples. As mentioned above, we consider a deformation of the standard L^2 De Rham complex $(L^2(\Omega^*(\mathbb{T})), \partial)$ on $\mathbb{T}$, where $L^2(\Omega^0(\mathbb{T}))$ resp. $L^2(\Omega^1(\mathbb{T}))$ denotes the L^2 functions, resp. L^2 1-forms on $\mathbb{T}$, and ∂ is the (densely defined) external derivation. Our deformation will be the L^2 complex $(L^2(\Omega^*_\alpha(\mathbb{T})), \partial_\alpha)$ on

$\mathbb{T}$, where $L^2(\Omega^0_\alpha(\mathbb{T})) := L^2(\Omega^0(\mathbb{T}))$, $L^2(\Omega^1(\mathbb{T})) := L^2(\mathbb{T}\times\mathbb{T})$, and the deformed external derivation ∂_α has been defined above. The triple $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$ is then given by the Hilbert space $\mathcal{K}_\alpha := L^2(\Omega^*_\alpha(\mathbb{T})) = L^2(\Omega^1_\alpha(\mathbb{T})) \oplus L^2(\Omega^0(\mathbb{T}))$, the Dirac operator $(D_\alpha, \mathrm{dom}(D_\alpha))$ on $\mathcal{K}_\alpha$ is defined as

$$D_\alpha := \begin{pmatrix} 0 & \partial_\alpha \\ \partial^*_\alpha & 0 \end{pmatrix}, \quad \text{so that} \quad D_\alpha \begin{pmatrix} \xi \\ f \end{pmatrix} = \begin{pmatrix} \partial_\alpha f \\ \partial^*_\alpha \xi \end{pmatrix}, \tag{3.14}$$

on the domain $\mathrm{dom}(D_\alpha) := \mathrm{dom}(\partial^*_\alpha) \oplus \mathcal{F}_\alpha$, and the $*$-algebra $\mathcal{A}_\alpha$ is defined as $\mathcal{A}_\alpha = \{f \in C(\mathbb{T}) : \|[D, L_f]\| < \infty\}$, where, if $f \in C(\mathbb{T})$, L_f denotes its left action on $\mathcal{K}_\alpha$ resulting from the direct sum of those on $L^2(\mathbb{T}\times\mathbb{T})$ and on $L^2(\mathbb{T})$. We also consider the seminorm p_α given by

$$p_\alpha(f)^2 = \frac{1}{2\pi} \sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy. \tag{3.15}$$

Proposition 3.7. *For $\alpha \in (0,1)$ the algebra $\mathcal{A}_\alpha$ defined above coincides with $\{f \in C(\mathbb{T}) : p_\alpha(f) < \infty\}$, and $C^{0,\alpha+\varepsilon}(\mathbb{T}) \subset \mathcal{A}_\alpha$, hence it is a uniformly dense subalgebra of $C(\mathbb{T})$. Moreover, for $\alpha \geq \frac{1}{2}$, $\mathcal{A}_\alpha \subset C^{0,\alpha}(\mathbb{T})$. Analogous results hold true upon replacing the left module structure of $\mathcal{K}_\alpha$ by the right one.*

Proof. Let us consider first the map $S_f : C(\mathbb{T}) \to L^2(\mathbb{T}\times\mathbb{T})$ defined as $S_f g = (\partial_\alpha f)g$ for a fixed $f \in C(\mathbb{T})$. This map extends to a bounded map on $L^2(\mathbb{T})$, provided $f \in \mathcal{A}_\alpha$, because

$$\begin{aligned} \|S_f g\|^2_{L^2(\mathbb{T}\times\mathbb{T})} &= \frac{1}{4\pi^2}\int_{\mathbb{T}\times\mathbb{T}} |(\partial_\alpha f)(z,w)g(w)|^2\, dzdw \\ &= \frac{1}{4\pi^2}\int_{\mathbb{T}} |g(w)|^2 \int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\, dzdw \\ &\leq \frac{1}{2\pi}\|g\|^2_{L^2(\mathbb{T})} \sup_{w\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\, dz, \qquad g \in L^2(\mathbb{T}), \end{aligned}$$

so that

$$\|S_f\|^2 = \frac{1}{2\pi}\sup_{w\in\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\, dz = p_\alpha(f)^2\,. \tag{3.16}$$

Let us compute now, using the Leibniz rule for the derivation ∂_α, the quadratic form of the commutator $[D_\alpha, L_f]$, defined on the domain $\mathrm{dom}(D_\alpha) := \mathrm{dom}(\partial^*_\alpha) \oplus \mathcal{F}_\alpha$:

$$\begin{aligned} \big(\xi' \oplus g' | [D_\alpha, L_f]\xi \oplus g\big) &= \big(D_\alpha(\xi'\oplus g')|L_f(\xi\oplus g)\big) - \big(L_{f^*}(\xi'\oplus g')|D_\alpha(\xi\oplus g)\big) \\ &= \big(\partial_\alpha g' \oplus \partial^*_\alpha \xi' | f\xi \oplus fg\big) - \big(f^*\xi' \oplus f^* g' | \partial_\alpha g \oplus \partial^*_\alpha \xi\big) \\ &= (\partial_\alpha g'|f\xi) + (\partial^*_\alpha\xi'|fg) - (\xi'|f\partial_\alpha g) - (f^*g'|\partial^*_\alpha\xi) \\ &= (f^*\partial_\alpha g'|\xi) + (\xi'|\partial_\alpha(fg)) - (\xi'|f\partial_\alpha g) - (\partial_\alpha(f^*g')|\xi) \\ &= (\xi'|(\partial_\alpha f)g) - ((\partial_\alpha f^*)g'|\xi)\,. \\ &= (\xi'|S_f g) - (S_{f^*}g'|\xi)\,. \end{aligned}$$

Hence

$$[D_\alpha, L_f] = \begin{pmatrix} 0 & S_f \\ -S^*_{f^*} & 0 \end{pmatrix}, \qquad a \in \mathcal{A}_\alpha, \tag{3.17}$$

therefore $[D_\alpha, L_f]$ extends to a bounded operator on $\mathcal{K}_\alpha$ if and only if $p_\alpha(f) < \infty$, and $\|[D_\alpha, L_f]\| = \|S_f\| = p_\alpha(f)$. The relations w.r.t. the spaces of Hölder continuous functions follow by Proposition A.3. □

Theorem 3.8. *Let $\alpha \in (0,1]$. The triple $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$ described above is a densely defined Spectral Triple on the algebra $C(\mathbb{T})$, in the sense of Connes. In particular,*

(i) D_α^{-1} has compact resolvent, and the function $\zeta_D(s) = \mathrm{tr}(|D_\alpha|^{-s}) = 4\zeta(\alpha s)$, where by $|D_\alpha|^{-s}$ we mean the functional calculus with the function $\begin{cases} t^{-s} & t > 0, \\ 0 & t = 0. \end{cases}$

(ii) The dimension of the triple is α^{-1}, and $\mathrm{Res}_{s=1/\alpha} \mathrm{tr}(f|D_\alpha|^{-s}) = \frac{4}{\alpha} \int f(t)\, dt$.

(iii) The distance d_D induced on $\mathbb{T}$ by the spectral triple satifies, for any $\varepsilon > 0$, $d_D(x,y) \geq \frac{1}{c_\varepsilon}|x-y|^{\alpha+\varepsilon}$, $x, y \in \mathbb{T}$. Moreover, if $\alpha \geq \frac{1}{2}$, $d_D(x,y) \leq \frac{1}{\tilde{c}_\alpha}|x-y|^\alpha$, $x,y \in \mathbb{T}$. Here, c_ε and $\tilde{c}_\alpha$ are as in Proposition A.3.

(iv) The Dirichlet form $\mathcal{E}_\alpha$ can be recovered, for any $f \in H^\alpha(\mathbb{T})$, via the formulas

$$\mathcal{E}_\alpha[f] = \frac{2}{\alpha} \lim_{s \to 1} (s-1)\, \mathrm{tr}(|D|^{s/2}|[D,f]|^2|D|^{s/2})$$

Proof. (i) Notice first that, since the self-adjoint operators $\partial_\alpha^*\partial_\alpha$ and $\partial_\alpha\partial_\alpha^*$ are unitarily equivalent on the orthogonal complement of their kernels, it suffices to prove that $\partial_\alpha^*\partial_\alpha$ has discrete spectrum on $L^2(\mathbb{T})$. Indeed, Lemma 3.6 (1) shows that the spectrum of the self-adjoint operator $\partial_\alpha^*\partial_\alpha$ is discrete and coincides with $\{k^{2\alpha} : k \in \mathbb{N}\}$. Since any non-zero eigenvalue of D_α has multiplicity 4, we get the formula for ζ_D.

(ii) Follows by (i) and a straightforward computation.

(iii) Observe that, using the notation of Proposition A.3,

$$\begin{aligned} d_D(x,y) &= \sup\{|f(x) - f(y)| : \|[D_\alpha, f]\| \leq 1\} = \sup\{|f(x)-f(y)| : p_\alpha(f) \leq 1\} \\ &\geq \frac{1}{c_\varepsilon} \sup\left\{|f(x) - f(y)| : \|f\|_{C^{0,\alpha+\varepsilon}(\mathbb{T})} \leq 1\right\} = \frac{1}{c_\varepsilon}|x-y|^{\alpha+\varepsilon}, \end{aligned}$$

and, if $\alpha \geq \frac{1}{2}$,

$$\begin{aligned} d_D(x,y) &= \sup\{|f(x)-f(y)| : p_\alpha(f) \leq 1\} \\ &\leq \frac{1}{\tilde{c}_\alpha} \sup\left\{|f(x)-f(y)| : \|f\|_{C^{0,\alpha}(\mathbb{T})} \leq 1\right\} = \frac{1}{\tilde{c}_\alpha}|x-y|^\alpha. \end{aligned}$$

(iv) Follows by (3.17) and Lemma 3.6 (5).

□

As explained above, the pairing with odd K-theory is recovered by a 1-graded Fredholm module $(\mathcal{K}_\alpha, F_\alpha, \gamma, \varepsilon_\alpha)$, where F_α is the phase of D_α, and the further grading ε_α is a simple deformation of the grading ε described in formula (2.2), namely $\varepsilon_\alpha = -i \begin{pmatrix} 0 & V_\alpha \\ V_\alpha^* & 0 \end{pmatrix}$, where V_α is given by $V_\alpha e_j = \mathrm{sgn}(j) e_j' = \mathrm{sgn}(j)|j|^{-\alpha}\partial_\alpha e_j$, $j \neq 0$, $V_\alpha e_0 = 0$.

Proposition 3.9. *Let $\alpha \in (0,1]$. The quadruple $\mathcal{F}_\alpha = (\mathcal{K}_\alpha, F_\alpha, \gamma, \varepsilon_\alpha)$ is a tamely degenerate 1-graded Fredholm module on $C(\mathbb{T})$. The module $\mathcal{F}_\alpha^+$ constructed as in Proposition 2.12, is an odd Fredholm module on $C(\mathbb{T})$, and the pairing with K-Theory gives $\langle \mathcal{F}_\alpha^+, e_k \rangle = k$.*

Proof. For notational simplicity, we drop the index α. Let us observe that $F = \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}$, where W is the partial isometry given by $We_i = e_i'$, $i \neq 0$, $We_0 = 0$. As a consequence, setting $Se_j = \mathrm{sgn}(j)e_j$, we get $V = WS$, hence $i\varepsilon = (I \oplus S)F(I \oplus S)$. A direct computation shows $P_0 = [\ker(F)] = [\ker(F^2)] = 1 - F^2 = Q_0 \oplus (1 - S^2)$, where Q_0 is the projection on

$\ker(\partial_\alpha^*)$. Then the support of $(I \oplus S)F(I \oplus S)$ coincides with the support of F, which is $1 - P_0$. Therefore $-\varepsilon^*\varepsilon = \big((I \oplus S)F(I \oplus S)\big)^2 = F^2 = 1 - P_0$. We then compute

$$i\varepsilon F = \begin{pmatrix} WSW^* & 0 \\ 0 & S \end{pmatrix} = iF\varepsilon\,, \tag{3.18}$$

hence $[\varepsilon, F] = 0$. Moreover, ε is clearly skew-adjoint. We now prove the compactness of $[\varepsilon, \pi(f)]$. Indeed,

$$\begin{aligned}[i\varepsilon, \pi(f)] =& [(I \oplus S)F(I \oplus S), \pi(f)] \\ =& [(I \oplus S), \pi(f)]F(I \oplus S) + (I \oplus S)[F, \pi(f)](I \oplus S) + (I \oplus S)F[(I \oplus S), \pi(f)]\,.\end{aligned}$$

The compactness of $[F, \pi(f)]$ follows by the spectral triple properties (cf. Proposition 2.4), and the compactness of $[(I \oplus S), \pi(f)] = 0 \oplus [S, f]$ follows by the Toeplitz theory. Therefore $\mathcal{F}$ is a kernel-degenerate 1-graded Fredholm module, and so, by Proposition 2.12, $\mathcal{F}^+$ is a kernel-degenerate ungraded Fredholm module. Moreover, $\mathcal{F}^+$ is indeed non-degenerate, because $F^+ = V^*W = S$ has a one-dimensional kernel, hence $(F^+)^2 - I$ is compact. ☐

4. Spectral triples on the Sierpinski gasket

4.1. **Sierpinski Gasket and its Dirichlet form.** We denote by K the Sierpiński gasket. Introduced in [54] as a curve with a dense set of ramification points, it has been the object of various investigations in Analysis [44], Probability [47, 5] and Theoretical Physics [51]. Let $p_0, p_1, p_2 \in \mathbb{R}^2$ be the vertices of an equilateral triangle of unit length and consider the contractions $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane: $w_i(x) := p_i + \frac{1}{2}(x - p_i) \in \mathbb{R}^2$. Then K is the unique fixed-point w.r.t. the contraction map $E \mapsto w_0(E) \cup w_1(E) \cup w_2(E)$ in the set of all compact subsets of $\mathbb{R}^2$, endowed with the Hausdorff metric. Two ways of approximating K are shown in Figures 1 and 2.

Let us denote by $\Sigma_m := \{0, 1, 2\}^m$ the set of words of length $m \geq 0$ composed by m letters chosen in the alphabet of three letters $\{0, 1, 2\}$ and by $\Sigma := \bigcup_{m\geq 0} \Sigma_m$ the whole vocabulary (by definition $\Sigma_0 := \{\emptyset\}$). A word $\sigma \in \Sigma_m$ has, by definition, length m and this is denoted by $|\sigma| := m$. For $\sigma = \sigma_1\sigma_2 \ldots \sigma_m \in \Sigma_m$ let us denote by w_σ the contraction $w_\sigma := w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_m}$.

Let $V_0 := \{p_0, p_1, p_2\}$ be the set of vertices of the equilateral triangle and $E_0 := \{e_0, e_1, e_2\}$ the set of its edges, with e_i opposite to p_i. Then, for any $m \geq 1$, $V_m := \bigcup_{|\sigma|=m} w_\sigma(V_0)$ is the set of vertices of a finite graph (*i.e.* a one-dimensional simplex) denoted by (V_m, E_m) whose edges are given by $E_m := \bigcup_{|\sigma|=m} w_\sigma(E_0)$ (see Figure 2). The self-similar set K can be reconstructed also as an Hausdorff limit either of the increasing sequence V_m of vertices or of the increasing sequence E_m of edges, of the above finite graphs. Set $V_* := \cup_{m=0}^\infty V_m$, and $E_* := \cup_{m=0}^\infty E_m$.

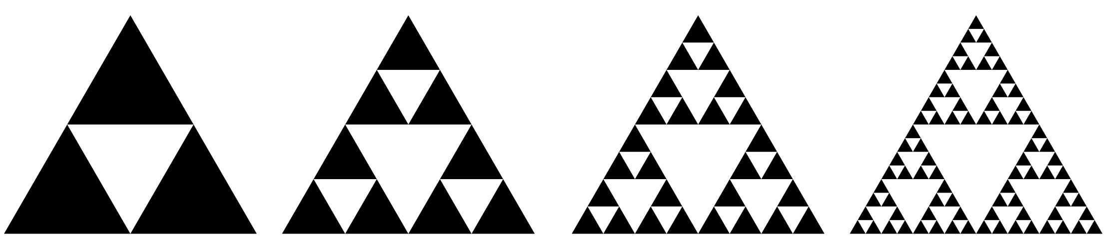


Figure 1. Approximations from above of the Sierpinski gasket.

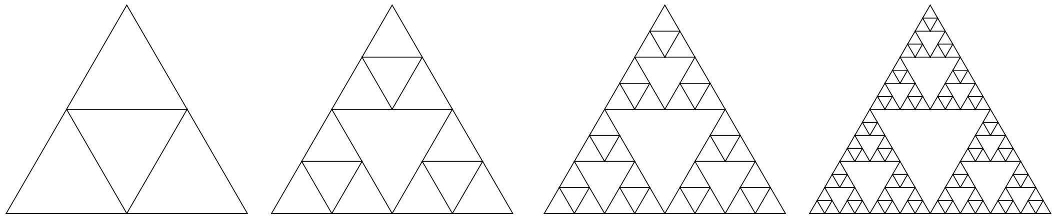


FIGURE 2. Approximations from below of the Sierpinski gasket.

In the present work a central role is played by the quadratic form $\mathcal{E} : C(K) \to [0, +\infty]$ given by

$$\mathcal{E}[f] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e\in E_m} |f(e_+) - f(e_-)|^2,$$

where each edge e has been arbitrarily oriented, and e_-, e_+ denote its source and target. It is a regular Dirichlet form since it is lower semicontinuous, densely defined on the subspace $\mathcal{F} := \{f \in C(K) : \mathcal{E}[f] < \infty\}$ and satisfies the *Markovianity property*

$$\mathcal{E}[f \wedge 1] \le \mathcal{E}[f] \qquad f \in C(K; \mathbb{R}). \tag{4.1}$$

The existence of the limit above and the mentioned properties are consequences of the theory of *harmonic structures* on self-similar sets developed by Kigami [44].

As a result of the theory of Dirichlet forms [10, 28], the domain $\mathcal{F}$ is an involutive subalgebra of $C(K)$ and, for any fixed $f, g \in \mathcal{F}$, the functional

$$\mathcal{F} \ni h \mapsto \Gamma(f, g)(h) := \frac{1}{2}\big(\mathcal{E}(f, hg) - \mathcal{E}(fg, h) + \mathcal{E}(g, fh)\big) \in \mathbb{R} \tag{4.2}$$

defines a finite Radon measure called the *energy measure* (or *carré du champ*) *of f and g*. In particular, for $f \in \mathcal{F}$, the measure $\Gamma(f, f)$ is nonnegative and one has the representation

$$\mathcal{E}[f] = \int_K 1\, d\Gamma(f, f) = \Gamma(f, f)(K) \qquad f \in \mathcal{F}\,.$$

In applications, f may represent a configuration of a system, $\mathcal{E}[f]$ its corresponding total energy and $\Gamma(f, f)$ represents its distribution. In homological terms, Γ is (up to the constant $1/2$) the Hochschild co-boundary of the 1-cocycle $\phi(f_0, f_1) := \mathcal{E}(f_0, f_1)$ on the algebra $\mathcal{F}$.

The Dirichlet or energy form $\mathcal{E}$ should be considered as a Dirichlet integral on the gasket. It is closable with respect to any Borel regular probability measure on K which is positive on open sets and vanishes on finite sets (see [44] Theorem 3.4.6 and [45] Theorem 2.6). Once the measure m has been chosen, $\mathcal{E}$ is the quadratic form of a positive, self-adjoint operator on $L^2(K, m)$, which may be thought of as a Laplace-Beltrami operator on K. A function $f \in \mathcal{F}$ is said to be *harmonic* in a open set $A \subset K$ if, for any $g \in \mathcal{F}$ vanishing on the complementary set A^c, one has

$$\mathcal{E}(f, g) = 0\,.$$

As a consequence of the Markovianity property (4.1), a Maximum Principle holds true for harmonic functions on the gasket [44]. In particular, one calls 0-*harmonic* a function u on K which is harmonic in V_0^c. Equivalently, for given boundary values on V_0, u is the unique function in $\mathcal{F}$ such that $\mathcal{E}[u] = \min\{\mathcal{E}[v] : v \in \mathcal{F}, v|_{V_0} = u\}$. More generally, one may call m-*harmonic* a function that, given its values on V_m, minimizes the energy among all functions in $\mathcal{F}$. For such functions we have

$$\mathcal{E}[u] = \left(\frac{5}{3}\right)^m \sum_{e \in E_m} |u(e_+) - u(e_-)|^2 .$$

Definition 4.1. (Cells, lacunas) For any word $\sigma \in \Sigma_m$, define a corresponding *cell* in K as follows

$$C_\sigma := w_\sigma(K) .$$

We will also define the *lacuna* $\ell_\emptyset$, see Fig. 3, as the boundary of the first removed triangle according to the approximation in Fig. 1. For any $\sigma \in \Sigma$, the lacuna ℓ_σ is defined as $\ell_\sigma := w_\sigma(\ell_\emptyset)$. We shall use the notation $\mathcal{E}_{C_\sigma}[u] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e\in E_m, e\subset C_\sigma} |u(e_+) - u(e_-)|^2$.

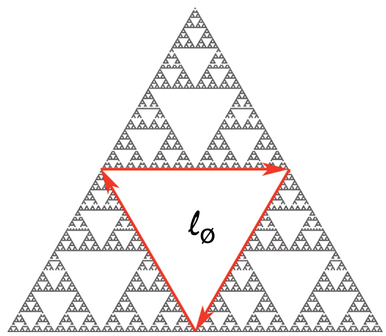


FIGURE 3. The lacuna $\ell_\emptyset$

4.2. **The dimensional spectrum and volume.** We now choose $\alpha \in (0, 1]$ and construct a triple on K according to the prescriptions given in Section 2.2. Let S be the main lacuna $\ell_\emptyset$ of the gasket, identified isometrically with $\mathbb{T}$, and consider the triple $\mathcal{T} = (\pi, \mathcal{H}, D_\alpha)$ constructed in Section 3.3. Then $\mathcal{T}_\emptyset = (\pi_\emptyset, \mathcal{H}_\emptyset, D_\emptyset)$ is given by $\pi_\emptyset(f) = \pi(f|_{\ell_\emptyset})$, $\mathcal{H}_\emptyset = \mathcal{H}$, $D_\emptyset = D_\alpha$, and, for any $\sigma \in \cup_n\{0, 1, 2\}^n$, consider the triple $(\pi_\sigma, \mathcal{H}_\sigma, D_\sigma)$ on $\mathcal{C}(K)$, where $\mathcal{H}_\sigma = \mathcal{H}_\emptyset$, $D_\sigma = 2^{|\sigma|} D_\emptyset$, and $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$.

Definition 4.2. Let us consider the following triple: $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H} = \oplus_{\sigma\in\Sigma}\mathcal{H}_\sigma$, $D = \oplus_{\sigma\in\Sigma} D_\sigma$, and $\mathcal{A}$ is the subalgebra of $\mathcal{C}(K)$ consisting of functions with bounded commutator with D, acting on $\mathcal{H}$ via the representation $\pi = \oplus_{\sigma\in\Sigma}\pi_\sigma$. According to the prescriptions of noncommutative geometry, we set $\oint f := \mathrm{tr}_\omega(f|D|^{-d})$, where tr_ω is the (logarithmic) Dixmier trace, and d is the abscissa of convergence of the zeta function $s \to \mathrm{tr}(|D|^{-s})$.

Theorem 4.3. *Let $\alpha \in (0, 1]$. The zeta function $\mathcal{Z}_D$ of $(\mathcal{A}, \mathcal{H}, D)$, i.e. the meromorphic extension of the function $s \in \mathbb{C} \mapsto \mathrm{tr}(|D|^{-s})$, is given by*

$$\mathcal{Z}_D(s) = \frac{4\zeta(\alpha s)}{1 - 3 \cdot 2^{-s}} ,$$

where ζ denotes the Riemann zeta function. Therefore, the dimensional spectrum of the spectral triple is

$$\mathcal{S}_{dim} = \{\alpha^{-1}\} \cup \left\{ \frac{\log 3}{\log 2}\left(1 + \frac{2\pi i}{\log 3} k\right) : k \in \mathbb{Z} \right\} \subset \mathbb{C} .$$

As a consequence, the metric dimension d_D of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, namely the abscissa of convergence of its zeta function, is $\max\{\alpha^{-1}, d_H\}$, $d_H = \frac{\log 3}{\log 2}$ *being the Hausdorff dimension.*

When $\alpha > \frac{\log 2}{\log 3}$, i.e. $d_D = d_H$, $\mathcal{Z}_D$ has a simple pole in d_D, and the measure associated via Riesz theorem with the functional $f \to \oint f$ coincides with a multiple of the Hausdorff measure H_{d_H}:

$$\mathrm{vol}(f) \equiv \int_K f \, d\,\mathrm{vol} := tr_\omega(f|D|^{-d_H}) = \frac{4d_H}{\log 3}\frac{\zeta(d_H)}{(2\pi)^{d_H}} \int_K f \, dH_{d_H} \qquad f \in C(K).$$

Proof. The non vanishing eigenvalues of $|D_\sigma|$ are exactly $\{2^{|\sigma|}(2\pi k^\alpha)\}$, each one with multiplicity 4.

Hence $\mathrm{tr}(|D_\sigma|^{-s}) = 4 \cdot 2^{-s|\sigma|}(2\pi)^{-s}\sum_{k>0}(k^\alpha)^{-s} = 4(2\pi)^{-s}2^{-s|\sigma|}\zeta(\alpha s)$ and for $\mathrm{Re}\, s > d_H$ we have

$$\begin{aligned}\mathrm{tr}(|D|^{-s}) &= \sum_\sigma \mathrm{tr}(|D_\sigma|^{-s}) = 4(2\pi)^{-s}\zeta(\alpha s)\sum_\sigma 2^{-s|\sigma|} \\ &= 4(2\pi)^{-s}\zeta(\alpha s)\sum_{n\geq 0} 2^{-sn}\sum_{|\sigma|=n} 1 \\ &= 4(2\pi)^{-s}\zeta(\alpha s)\sum_{n\geq 0}\left(3\cdot 2^{-s}\right)^n = 4(2\pi)^{-s}\zeta(\alpha s)(1-3\cdot 2^{-s})^{-1}\,.\end{aligned}$$

As the Riemann zeta function has just one pole at $s=1$ we have $\mathcal{S}_{dim} = \{\alpha^{-1}\} \cup \{d_H(1+\frac{2\pi i}{\log 3}k) : k \in \mathbb{Z}\} \subset \mathbb{C}$. Now we assume that $\alpha > \frac{\log 2}{\log 3}$, i.e. $d_D = \frac{\log 3}{\log 2}$, and prove that the volume measure is a multiple of the Hausdorff measure H_{d_H}.

Clearly, the functional $\mathrm{vol}(f) = tr_\omega(f|D|^{-d_H})$ makes sense also for bounded Borel functions on K, and we recall that the logarithmic Dixmier trace may be calculated as a residue (cf. [22]): $tr_\omega(f|D|^{-d_H}) = \mathrm{Res}_{s=d_H}\ \mathrm{tr}(f|D|^{-s})$, when the latter exists. Then, for any word τ,

$$\begin{aligned}tr_\omega(\chi_{C_\tau}|D|^{-d_H}) &= \mathrm{Res}_{s=d_H}\ \mathrm{tr}(\chi_{C_\tau}|D|^{-s}) \\ &= \lim_{s\to d_H^+}(s-d_H)\ \mathrm{tr}(\chi_{C_\tau}|D|^{-s}) \\ &= \lim_{s\to d_H^+}(s-d_H)\sum_\sigma \mathrm{tr}(\chi_{C_\tau}\circ w_\sigma|D_\sigma|^{-s}),\end{aligned}$$

and we note that $\chi_{C_\tau}\circ w_\sigma$ is not zero either when $\sigma < \tau$ or when $\sigma \geq \tau$. In the latter case, $\chi_{C_\tau}\circ w_\sigma = 1$. Since $d_H > 1$, $\mathrm{tr}(\chi_{C_\tau}|D_\sigma|^{-s}) \leq \mathrm{tr}(|D_\sigma|^{-s}) = 4(2\pi)^{-s}2^{-s|\sigma|}\zeta(\alpha s) \to 4(2\pi)^{-d_H}3^{-|\sigma|}\zeta(\alpha d_H)$ when $s \to d_H^+$, hence $\lim_{s\to d_H^+}(s-d_H)\,\mathrm{tr}(\chi_{C_\tau}|D_\sigma|^{-s}) = 0$. Therefore we may forget about the finitely many $\sigma < \tau$, and get

$$\begin{aligned}tr_\omega(\chi_{C_\tau}|D|^{-d_H}) &= \lim_{s\to d_H^+}(s-d_H)\sum_{\sigma\geq\tau}\mathrm{tr}(|D_\sigma|^{-s}) \\ &= \lim_{s\to d_H^+}(s-d_H)4(2\pi)^{-s}\zeta(\alpha s)\sum_\sigma 2^{-s(|\sigma|+|\tau|)} \\ &= 4\frac{\zeta(\alpha d_H)}{(2\pi)^{d_H}}2^{-d_H|\tau|}\lim_{s\to d_H^+}\frac{s-d_H}{1-3\cdot 2^{-s}} \\ &= \frac{4d_H}{\log 3}\frac{\zeta(\alpha d_H)}{(2\pi)^{d_H}}\left(\frac{1}{3}\right)^{|\tau|} = \frac{4d_H}{\log 3}\frac{\zeta(\alpha d_H)}{(2\pi)^{d_H}}H_{d_H}(C_\tau)\,.\end{aligned}$$

This implies that for any $f \in \mathcal{C}(K)$ for which $f \leq \chi_{C_\tau}$, $\mathrm{vol}(f) \leq \frac{4d_H}{\log 3}\frac{\zeta(\alpha d_H)}{(2\pi)^{d_H}}\left(\frac{1}{3}\right)^{|\tau|}$, therefore points have zero volume, and $\mathrm{vol}(\chi_{\dot{C}_\tau}) = \mathrm{vol}(\chi_{C_\tau})$, where $\dot{C}_\tau$ denotes the interior of C_τ. As a consequence, for the simple functions given by finite linear combinations of characteristic functions of cells or vertices, $\mathrm{vol}(\varphi) = \dfrac{4d_H}{\log 3}\dfrac{\zeta(\alpha d_H)}{(2\pi)^{d_H}}\displaystyle\int \varphi\, dH_{d_H}$. Since continuous function are Riemann integrable w.r.t. such simple functions, the thesis follows. □

Remark 4.4. In this case the functional $f \to \oint f$ does not reproduce the Hausdorff measure outside the algebra of continuous functions. Indeed such functional only depends on the behavior of f on the union of all lacunas, a set which is negligible w.r.t. the Hausdorff measure.

4.3. **The commutator condition and Connes metric.** In this section we will show that for $\alpha \in (0, 1]$ the triple $(\mathcal{A}, \mathcal{H}, D)$ considered above is a spectral triple in the sense of Connes [22], up to the infinite dimensionality of $\ker(D)$. Moreover, the commutator $\|[D, f]\|$ gives a Lip-norm in the sense of Rieffel [52]. Such condition for spectral triples has been recently considered in [6], where these triples are called spectral metric spaces.

Definition 4.5. We shall consider the following seminorms on functions defined on lacunas ℓ_σ:

$$L_{\sigma,\eta}(f) = \|f\|_{C^{0,\eta}(\ell_\sigma)} 2^{|\sigma|(1-\eta)}$$

Proposition 4.6. *Let $\alpha \in (0, 1]$. If $f \in C^{0,1}(K)$, p_α is defined in (3.15), and c_ε is given in Proposition A.3, then*

$$\|[D, f]\| = \sup_{\sigma\in\Sigma} 2^{|\sigma|} p_\alpha(f \circ w_\sigma) \le c_{1-\alpha} \sup_{\sigma\in\Sigma} L_{\sigma,1}(f) \le c_{1-\alpha}\|f\|_{C^{0,1}(K)}.$$

Proof. By equation (3.16) and Proposition A.3

$$\begin{aligned}\|[D, f]\| &= \|\bigoplus_{\sigma\in\Sigma}[D_\sigma, \pi_\sigma(f)]\| = \sup_{\sigma\in\Sigma} \|[D_\sigma, \pi_\sigma(f)]\| \\ &= \sup_{\sigma\in\Sigma} 2^{|\sigma|}\|[D_\alpha, \pi_\emptyset(f \circ w_\sigma)]\| = \sup_{\sigma\in\Sigma} 2^{|\sigma|}\|S_{f\circ w_\sigma}\| = \sup_{\sigma\in\Sigma} 2^{|\sigma|}p_\alpha(f \circ w_\sigma) \\ &\le c_{1-\alpha} \sup_{\sigma\in\Sigma} 2^{|\sigma|}\|f \circ w_\sigma\|_{C^{0,1}(\ell_\emptyset)} = c_{1-\alpha} \sup_{\sigma\in\Sigma} \|f\|_{C^{0,1}(\ell_\sigma)}.\end{aligned}$$

□

The previous Proposition gives an estimate from above of the norm of the commutator. However, by making use of Lemma A.1, we may get an estimate from below.

Lemma 4.7. *Let $\alpha \in [\frac{1}{2}, 1]$, and $\tilde{c}_\alpha$ be as in Proposition A.3. Then*

$$\|[D, f]\| \ge \tilde{c}_\alpha \sup_{\sigma\in\Sigma} L_{\sigma,\alpha}(f).$$

Proof.

$$\|[D, f]\| = \sup_{\sigma\in\Sigma} 2^{|\sigma|}p_\alpha(f \circ w_\sigma) \ge \tilde{c}_\alpha \sup_{\sigma\in\Sigma} 2^{|\sigma|}\|f \circ w_\sigma\|_{C^{0,\alpha}(\ell_\emptyset)} = \tilde{c}_\alpha \sup_{\sigma\in\Sigma} \|f\|_{C^{0,\alpha}(\ell_\sigma)} 2^{|\sigma|(1-\alpha)}.$$

□

Proposition 4.8. *Let $\alpha \in [\frac{1}{2}, 1]$. There exists a constant $k(\alpha)$ such that*

$$\|f\|_{C^{0,1}(K)} \le k(\alpha)\|[D, f]\|. \tag{4.3}$$

Proof. Our aim is to estimate $|f(x)-f(y)|$ for a continuous function f for which $\|[D,f]\|<\infty$.
1^{st} step. Let C_σ be a cell of level m, x a point in C_σ. We now construct inductively a sequence of cells $C_{\sigma(j,x)}$, $j\geq 1$, such that $x\in C_{\sigma(j,x)}$, $C_{\sigma(1,x)}:=C_\sigma$, $C_{\sigma(j+1,x)}\subset C_{\sigma(j,x)}$, $|\sigma(j,x)|=m+j-1$ (if x is not a vertex such sequence is uniquely determined). We then construct a sequence $\{x_j\}_{j\geq 1}$ of points as follows: x_1 is a vertex of ℓ_σ contained in $C_{\sigma(2,x)}$, x_j is the unique point in $\ell_{\sigma(j-1,x)}\cap\ell_{\sigma(j,x)}$, $j>1$. By construction, $x_j\to x$ and the points x_j,x_{j+1} belong to the lacuna $\ell_{\sigma(j,x)}$.

We now observe that, by Lemma 4.7,

$$\begin{aligned}|f(x_{j+1})-f(x_j)| &\leq \|f\|_{C^{0,\alpha}(\ell_{\sigma(j,x)})}d(x_{j+1},x_j)^\alpha \leq L_{\sigma(j,x),\alpha}(f)2^{-(m+j-1)(1-\alpha)}(\mathrm{diam}(\ell_{\sigma(j,x)}))^\alpha\\ &\leq \tilde{c}_\alpha^{-1}2^{-\alpha}\|[D,f]\|2^{-(m+j-1)}.\end{aligned}$$

As a consequence,

$$|f(x_1)-f(x)|\leq\sum_{j\geq 1}|f(x_{j+1})-f(x_j)|\leq\tilde{c}_\alpha^{-1}2^{1-\alpha}\|[D,f]\|2^{-m}.$$

2^{nd} step. If x_0 is a vertex of level $n\neq 0$, and $m\geq n$, the butterfly shaped neighborhood $W(x_0,m)$ is the union of the two cells of level m containing x. For $x,y\in K$, let $W(x_0,m)$ be a minimal butterfly shaped neighborhood containing them. Observe that, by minimality, at least one of the points, say x, does not belong to $W(x_0,m+1)$, hence $\rho_{geo}(x,y)\geq\rho_{geo}(x,x_0)\geq 2^{-(m+1)}$.

Let us now choose $W(x_1,m+1)$ contained in one of the wings of $W(x_0,m)$ and containing both x and x_0. Reasoning as in the first step,

$$|f(x_0)-f(x)|\leq|f(x_0)-f(x_1)|+|f(x_1)-f(x)|\leq 2\tilde{c}_\alpha^{-1}2^{1-\alpha}\|[D,f]\|2^{-m},$$

hence,

$$\begin{aligned}|f(y)-f(x)| &\leq |f(y)-f(x_0)|+|f(x_0)-f(x)|\leq 4\tilde{c}_\alpha^{-1}2^{1-\alpha}\|[D,f]\|2^{-m}\\ &\leq 8\tilde{c}_\alpha^{-1}2^{1-\alpha}\|[D,f]\|\,\rho_{geo}(x,y).\end{aligned}$$

The thesis follows. □

Theorem 4.9. *For any $\alpha\in(0,1]$, the algebra $\mathcal{A}$ contains $\mathcal{C}^{0,1}(K)$ and the triple $(\mathcal{A},\mathcal{H},D)$ is a spectral triple. Moreover, for $\alpha\in[\frac{1}{2},1]$, $\mathcal{A}$ coincides with $\mathcal{C}^{0,1}(K)$, and the seminorm $f\to\|[D,f]\|$ is a Lip-norm according to Rieffel* [52]. *In particular, the metric*

$$\rho_D(x,y)=\sup_{f\in\mathcal{A}}\frac{|f(x)-f(y)|}{\|[D,f]\|}$$

is bi-Lipschitz w.r.t. the Euclidean geodesic metric ρ_{geo} on K.

Proof. It follows from Proposition 4.6 that $\mathcal{A}$ is dense in $C(K)$, which, together with the results in the previous Sections, give the spectral triple property. The Lip-norm property follows by Proposition 4.8, cf. [52]. Indeed, it implies that functions for which $\|[D,f]\|\leq 1$ are equicontinuous, which gives the compactness property of the set of elements for which $\|[D,f]\|\leq 1$ and $\|f\|\leq 1$, and it also implies that $\|[D,f]\|$ vanishes only on constant functions. The equivalence of the seminorms follows from Propositions 4.6 and 4.8. The other results easily follow. □

4.4. **The gasket in K-homology.** Let $\alpha \in (0,1]$, and denote by $(\mathcal{A}, \mathcal{H}, D)$ the spectral triple for the gasket considered above. Let F be the phase of D, and, for any $\sigma \in \Sigma$, denote by $\gamma_\sigma, \varepsilon_\sigma, S_\sigma$, a copy of the operators γ, ε, S associated, by Proposition 3.9, to the lacuna ℓ_σ, identified with $\mathbb{T}$. Finally, let $\gamma = \oplus_{\sigma\in\Sigma}\gamma_\sigma$, $\varepsilon = \oplus_{\sigma\in\Sigma}\varepsilon_\sigma$, $S = \oplus_{\sigma\in\Sigma}S_\sigma$.

Theorem 4.10. *Let $\alpha \in (0,1]$. The quintuple $\mathcal{F} = (\mathcal{H}, \pi, F, \gamma, \varepsilon)$ is a tamely degenerate 1-graded Fredholm module on $\mathcal{A}$, as in Definition 2.9. The ungraded Fredholm module $\mathcal{F}^+ = (\mathcal{H}^+, \pi^+, F^+)$ associated to it by Proposition 2.12 is a tamely degenerate module on $\mathcal{A}$, and $F^+ = S$. The module $\mathcal{F}^+$ is non trivial in K-homology. In particular, it pairs non trivially with the generators of the (odd) K-theory of the gasket associated with the lacunas.*

Proof. First step. We check the compactness of $[\varepsilon, \pi(f)] = \oplus_\sigma[\varepsilon_\sigma, \pi_\emptyset(f \circ w_\sigma)]$. As in the proof of Proposition 3.9, this amounts to prove that $\oplus_\sigma[S_\sigma, \pi^0_\emptyset(f \circ w_\sigma)]$ is compact, where $\pi^i_\emptyset$ denotes the action of $C(\ell_\emptyset)$ on $L^2(\Omega^i(\ell_\emptyset))$, $i = 0, 1$. Even though each summand is compact, the compactness of the direct sum is not obvious.

We first consider an affine function f in the plane, restricted to the gasket, and observe that consequently $f \circ w_\sigma|_{\ell_\emptyset} = const + 2^{-|\sigma|}f|_{\ell_\emptyset}$. Let us denote by $\{s_n\}$ the sequence of the singular values with multiplicity, arranged in a non increasing order, of $[S_\emptyset, \pi^0_\emptyset(f)]$. Then, for any given σ,

$$[S_\sigma, \pi^0_\emptyset(f \circ w_\sigma)] = 2^{-|\sigma|}[S_\emptyset, \pi^0_\emptyset(f)],$$

namely the sequence of singular values for $\oplus_\sigma[S_\sigma, \pi_\emptyset(f \circ w_\sigma)]$ is $\{2^{-|\sigma|}s_n : \sigma \in \Sigma, n \in \mathbb{N}\}$, showing that $[\varepsilon, \pi(f)]$ is compact.

Now, for any given $n \in \mathbb{N}$, consider the piece-wise affine functions $\mathrm{Aff}_n(K)$ on the gasket, which are affine when restricted to cells of level n. Reasoning as above, we obtain that, for $|\sigma| = n$, the operator $\oplus_{\tau\geq\sigma}[\varepsilon_\sigma, \pi_\emptyset(f \circ w_\tau)]$ is compact, from which the compactness of $[\varepsilon, \pi(f)]$ follows again. Since $\cup_n \mathrm{Aff}_n(K)$ is dense in $\mathcal{A}$, the thesis is proved. The other properties being obvious, we have proved that $\mathcal{F}$ is a kernel-degenerate 1-graded Fredholm module. Therefore, by Proposition 2.12, $\mathcal{F}$ is a kernel-degenerate ungraded Fredholm module.

Second step. According to Proposition 2.12, it is sufficient to prove the tame degeneracy of the ungraded Fredholm module. Since $K^1(K)$ is a direct sum of countably many copies of $\mathbb{Z}$ it is sufficient to verify the equation (2.1) only for the generators, namely for the unitaries u_σ having winding number 1 around ℓ_σ and winding number 0 around all other lacunas. However, for the unitary u_σ, the global index in (2.1) is equal to the index on the lacuna ℓ_σ, which is clearly trivial. Tameness follows. □

4.5. **The Dirichlet form.** Let us recall that the integral $\oint a$ of an element $a \in \mathcal{A}$ in noncommutative geometry is defined as the Dixmier trace $\mathrm{tr}_\omega(a|D|^d)$, where d is the metric dimension of the triple. Such integral may be computed, for a positive bounded a, in two equivalent ways:

$$\lim_{s\to1}(s-1)\,\mathrm{tr}\left((|D|^{-d/2}a|D|^{-d/2})^s\right); \tag{4.4}$$

$$\lim_{s\to1}(s-1)\,\mathrm{tr}(a|D|^{-sd}) = d^{-1}\lim_{t\to d}(t-d)\,\mathrm{tr}(a|D|^{-t}); \tag{4.5}$$

when such limits exist, cf. [22] Proposition 4 p.306, and [13] Corollary 3.7 (in this case the noncommutative integral is independent of the choice of the ultrafilter ω on $\mathbb{N}$).

However, things change when we consider unbounded a's. First of all, we replace $a|D|^{-sd}$ with $|D|^{-sd/2}a|D|^{-sd/2}$ in such a way that the trace is well defined (possibly infinite). Moreover, while the boundedness of $(s-1)\,\mathrm{tr}\left((|D|^{-d/2}a|D|^{-d/2})^s\right)$ for $s > 1$ is equivalent to

$|D|^{-d/2}a|D|^{-d/2} \in \mathcal{L}^{1,\infty}$ (cf. [12] Thm. 4.5), the boundedness of $(s-1)\,\mathrm{tr}(|D|^{-sd/2}a|D|^{-sd/2})$ for $s > 1$ is in general a weaker condition (cf. Lemma 4.20). Indeed, when classical d-manifolds M are concerned, with $|D| = \Delta^{1/2}$, [50] shows that the residue at 1 of $(s-1)\,\mathrm{tr}(|D|^{-sd/2}f|D|^{-sd/2})$ is finite and gives the integral of f on M w.r.t. the volume form (up to a multiplicative constant) for any function $f \in L^1(M)$, that the same is true for the residue at 1 of $(s-1)\,\mathrm{tr}\left((|D|^{-d/2}a|D|^{-d/2})^s\right)$ only when $f \in L^{1+\varepsilon}(M)$, and examples are given of $f \in L^1(M)$ such that $|D|^{-d/2}a|D|^{-d/2}$ does not belong to $\mathcal{L}^{1,\infty}$.

Our aim here is to use the NCG translation table to define a Dirichlet energy on spectral triples, cf. also [9, 42]. Starting with the classical Dirichlet integral $f \mapsto \int_M |\nabla f|^2\, d\mathrm{vol}$ on a Riemannian manifold M, we replace the gradient ∇f with the commutator $[D, f]$, the integral as explained above, and get the nonnegative quadratic functional

$$a \mapsto \mathcal{E}_D[a] := \mathrm{tr}_\omega\left(|D|^{-\delta/2}|[D,a]|^2|D|^{-\delta/2}\right), \tag{4.6}$$

for a suitable δ. As above, we may hope to compute the energy also as

$$\lim_{s\to 1}(s-1)\,\mathrm{tr}(|D|^{s\delta/2}|[D,a]|^2|D|^{s\delta/2}) = \delta^{-1}\lim_{t\to\delta}(t-\delta)\,\mathrm{tr}(|D|^{t/2}|[D,a]|^2|D|^{t/2}). \tag{4.7}$$

For classical Riemannian d-manifolds M, the energy is finite for functions in the Sobolev space $f \in H^1(M)$, namely $|\nabla f|^2 \in L^1(M)$, therefore the analysis in [50] (for $\delta = d$) shows that formula (4.7) is finite and recovers a multiple of the Dirichlet energy form for all $f \in H^1(M)$, while formula (4.6) recovers a multiple of the Dirichlet energy form only for functions in a proper subset.

This will be the case also in our present setting (even though we could not produce a counterexample) where the manifold is replaced by the Sierpinski gasket and the Dirichlet integral by the standard Dirichlet form. In particular, we prove that formula (4.7) is finite and recovers a multiple of the standard Dirichlet form for all finite energy functions, while we are able to prove that formula (4.6) recovers a multiple of the standard Dirichlet form only on a form core.

Moreover, as a counterpart of the results of Kusuoka [47], Ben-Bassat-Strichartz-Teplyaev [7] (see also [35, 37]) showing that self-similar measures and energy measures are singular on the gasket, the exponent δ in (4.6), (4.7), which we call *energy dimension*, is smaller than the volume dimension d. As a consequence, we cannot exclude that the multiplicative constant relating $\mathcal{E}_D[a]$ in formula (4.6) with the standard Dirichlet form may even depend on the generalized limit ω.

Let us remark here that we prove that the forms described in (4.6) and (4.7) coincide (up to a constant) with the standard energy form on the gasket on suitable domains. The question whether such formulas directly give a Dirichlet form, either for general spectral triples or for the case of fractals, is not treated here, and will be the subject of future research.

Also, the residue and the Dixmier trace formulas, which are essentially equivalent for the case of the volume, are proved here in a quite independent way, the second requiring rather technical results on singular traces; this is why the domains of their validity are different, and the two constants giving the relation with the standard energy are unrelated, only estimates being available.

Now we prove that formula (4.7) reproduces a multiple of the standard Dirichlet form on the gasket. In the following Theorem, a result of Jonsson [41] on the regularity of the trace of a finite energy function on an edge of the gasket, will imply that the standard Dirichlet form on the gasket can be recovered via the spectral triple only if α is not too close to 1. In this section, when f is a continuous function on the gasket, $\mathcal{E}[f]$ denotes the values of the (possibly infinite) standard Dirichlet form on f. Let us first observe that

$$\begin{aligned}(4.8)\qquad \operatorname{tr}(|D|^{-s/2}|[D,f]|^2|D|^{-s/2}) &= \sum_\sigma \operatorname{tr}(|D_\sigma|^{-s/2}|[D_\sigma,\pi_\sigma(f)]|^2|D_\sigma|^{-s/2}) \\ &= \sum_\sigma 2^{(2-s)|\sigma|}\operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma)]|^2|D_\emptyset|^{-s/2}).\end{aligned}$$

However, the following holds.

Lemma 4.11. *Let $s > \frac{1}{\alpha}$, $\alpha_0 = \frac{\log(10/3)}{\log 4} \approx 0.87$. Then:*

- *(i)* $\operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,g]|^2|D_\emptyset|^{-s/2})$ *is finite if and only if $g \in H^\alpha(\ell_\emptyset)$.*
- *(ii)* $\operatorname{tr}(|D|^{-s/2}|[D,f]|^2|D|^{-s/2}) < \infty$, $\forall f$ *with finite energy on $K \Rightarrow \alpha \le \alpha_0$.*

Proof. (i) By (3.17), and the definition of $D_\emptyset$, we get

$$\begin{aligned}(4.9)\quad \operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,\pi_\emptyset(g)]|^2|D_\emptyset|^{-s/2}) &= \operatorname{tr}\left(|D_\emptyset|^{-s/2}\begin{pmatrix} S_gS_g^* & 0 \\ 0 & S_{g^*}^*S_{g^*}\end{pmatrix}|D_\emptyset|^{-s/2}\right) \\ &= \operatorname{tr}((\partial_\alpha^*\partial_\alpha)^{-s/4}S_{g^*}^*S_{g^*}(\partial_\alpha^*\partial_\alpha)^{-s/4}) + \operatorname{tr}((\partial_\alpha\partial_\alpha^*)^{-s/4}S_gS_g^*(\partial_\alpha\partial_\alpha^*)^{-s/4}).\end{aligned}$$

As a consequence, Lemma 3.6 implies

$$(4.10)\qquad 2\zeta(\alpha s)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} \le \operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,\pi_\emptyset(g)]|^2|D_\emptyset|^{-s/2}) \le 4\zeta(\alpha s)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}.$$

(ii) Let $\alpha > \alpha_0$, $g \in H^{\alpha_0}(\ell_\emptyset)\setminus H^\alpha(\ell_\emptyset)$. By [41] Thm. 5.1, there exists a finite energy function f on K such that $f|_{\ell_\emptyset} = g$. By (i), $\operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,\pi_\emptyset(f)]|^2|D_\emptyset|^{-s/2})$ is infinite, and the thesis follows. □

Next Theorem shows that condition $\alpha \le \alpha_0$, together with $\alpha > \frac{\log 2}{\log 12/5}$, is also sufficient for the recovery of the energy via formula (4.7).

Theorem 4.12. *Let $\alpha \in (0,\alpha_0]$, $\delta_D = \max\{\alpha^{-1}, d_E\}$, with $d_E := \frac{\log 12/5}{\log 2} \approx 1.26$. Then:*

- *(i) For any f with finite energy, $s > \delta_D$, $|D|^{-s/2}|[D,f]|^2|D|^{-s/2}$ is a trace class operator.*
- *(ii) For $\alpha \in (d_E^{-1},\alpha_0]$, so that $\delta_D = d_E$, and f with finite energy, the functional*

$$Z_{D,f}(s) := \operatorname{tr}(|D|^{-s/2}|[D,\pi(f)]|^2|D|^{-s/2}),$$

defined for $\Re s > d_E$, has abscissa of convergence d_E, where it has a simple pole, and there exists a constant A such that

$$(4.11)\qquad \operatorname{Res}_{s=d_E} Z_{D,f}(s) = A\ \mathcal{E}[f].$$

Proof. (i) According to formulas (4.8), (4.9), (4.10),

$$(4.12)\qquad \operatorname{tr}(|D|^{-s/2}|[D,\pi(f)]|^2|D|^{-s/2}) \le 4\zeta(\alpha s)\sum_\sigma 2^{(2-s)|\sigma|}\|\partial_\alpha\pi_\emptyset(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}.$$

By [41] Thm. 4.1, the restriction map from $\mathcal{F}$ to $H^\alpha(\ell_\emptyset)$ is continuous (for $\alpha \le \alpha_0$), implying in particular that there exists a constant $K_1 = K_{1,\alpha}$ such that

$$(4.13)\qquad \|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} \le K_1\mathcal{E}[g],\quad \forall g \in \mathcal{F}.$$

Hence,

(4.14)

$$\sum_{|\sigma|=n}\|\partial_\alpha\pi_\emptyset(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset)} \le K_1\sum_{|\sigma|=n}\mathcal{E}[f\circ w_\sigma] = K_1\left(\frac{3}{5}\right)^n\sum_{|\sigma|=n}\mathcal{E}_{C_\sigma}[f] = K_1\left(\frac{3}{5}\right)^n\mathcal{E}[f].$$

As a consequence, if $s > \delta_D = \max\{\alpha^{-1}, \frac{\log 12/5}{\log 2}\}$,

$$\mathrm{tr}(|D|^{-s/2}|[D,\pi(f)]|^2|D|^{-s/2}) \leq 4K_1\zeta(\alpha s)\sum_n \left(\frac{3}{5}2^{2-s}\right)^n \mathcal{E}[f] = \frac{4K_1\zeta(\alpha s)}{1-\frac{3}{5}2^{2-s}}\mathcal{E}[f], \tag{4.15}$$

proving the first statement of the theorem.

(ii) (a) We first determine the constant A. Let us observe that, up to a multiplicative constant, the matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ determines the unique non-degenerate positive semidefinite quadratic form $Q[v] = \sum_{i,j}|v_i - v_j|^2$ on $\mathbb{C}^3$, invariant under permutations of the components and vanishing on constant vectors. We then consider the linear map which associates to any vector $\vec{v} = (v_0, v_1, v_2) \in \mathbb{C}^3$, the 0-harmonic function $g = \varphi(\vec{v})$ on the gasket taking values v_i, $i = 0,1,2$, on the extreme points of the lacuna $\ell_\emptyset$. Then, the form $\vec{v} \to \|\partial_\alpha \pi_\emptyset(\varphi(\vec{v}))\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}$ possesses al the properties of the quadratic form on $\mathbb{C}^3$ induced by the matrix above. Indeed, for $g = \varphi(\vec{v})$, $\|\partial_\alpha\pi_\emptyset(g)\|^2 = \sum_p |p|^{2\alpha}|(\pi_\emptyset(g), e_p)|^2$, which is positive semidefinite and vanishes on constants. More precisely, it vanishes only if g is constant on $\ell_\emptyset$, which implies g is constant, since g is zero-harmonic. Finally, such form is invariant under isometries of the circle, hence, in particular, it is invariant under the symmetry group of the triangle, that is, for 0-harmonic functions, under the permutation group of the vertices.

Hence $\|\partial_\alpha\pi_\emptyset(\varphi(\vec{v}))\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}$ is a multiple of $Q[v]$, which in turn is $\frac{1}{25}\mathcal{E}[g]$, since g is 0-harmonic, namely there exists a non-zero constant $K_2 = K_{2,\alpha}$ such that

$$\|\partial_\alpha\pi_\emptyset(g)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} = K_2\mathcal{E}[g], \quad \forall \text{ 0-harmonic } g. \tag{4.16}$$

By formula (3.11), $\|S^*_{\pi_\emptyset(g)}\partial_\alpha e_k\|^2 = \frac{1}{4}\sum_p(|k|^{2\alpha} + |p|^{2\alpha} - |p-k|^{2\alpha})^2|(\pi_\emptyset(g), e_p)|^2$. As above, for $k \neq 0$, $g \to \|S^*_{\pi_\emptyset(g)}\partial_\alpha e_k\|^2$ is a non-degenerate positive semidefinite quadratic form on 0-harmonic functions, invariant under symmetries of the triangle and vanishing on constant vectors. Therefore it is again a multiple of the energy of g, namely $\forall k \neq 0\, \exists C_k > 0$ such that

$$\|S^*_{\pi_\emptyset(g)}\partial_\alpha e_k\|^2 = C_k\mathcal{E}[g], \quad \forall \text{ 0-harmonic } g. \tag{4.17}$$

The constant A is then defined as

$$A = \frac{2K_2\zeta(\alpha d_E) + C(d_E)}{\log 2},$$

where we set $C(s) = \sum_k C_k|k|^{-(s+2)\alpha}$. We note that, by formula (3.11), $0 < C_k \leq |k|^{2\alpha}$ for $k \neq 0$ and, by relations (3.11), (4.16), (4.17),

$$\mathcal{E}[g]\sum_k C_k|k|^{-(s+2)\alpha} = \sum_k |k|^{-(s+2)\alpha}\|S^*_g\partial_\alpha e_k\|^2 \leq 2\zeta(s\alpha)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} = 2K_2\zeta(s\alpha)\mathcal{E}[g],$$

for any 0-harmonic g, namely $C(s) \leq 2K_2\zeta(\alpha s)$. In particular, C is analytic for $s > 1/\alpha$.

(b) formula (4.11) for q-harmonic functions. According to formulas (4.8), (4.9), (3.12), and (3.13), we have

$$Z_{D,f}(s) = \sum_\sigma 2^{(2-s)|\sigma|}\left(2\zeta(\alpha s)\|\partial_\alpha\pi_\emptyset(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} + \sum_{k\neq 0}|k|^{-(s+2)\alpha}\|S^*_{\pi_\emptyset(f\circ w_\sigma)}\partial_\alpha e_k\|^2_{L^2(\ell_\emptyset)}\right). \tag{4.18}$$

Assume now f to be q-harmonic, and $s > d_E$. Then, when $|\sigma| \geq q$, $f \circ w_\sigma$ is 0-harmonic. Hence, making use of the equalites in (4.16), (4.17),

$$
\begin{aligned}
&\sum_{|\sigma|\geq q} \mathrm{tr}(|D_\sigma|^{-s/2}|[D_\sigma,\pi_\sigma(f)]|^2|D_\sigma|^{-s/2})\\
&= \sum_{|\sigma|\geq q} 2^{(2-s)|\sigma|}\left(2K_2\zeta(\alpha s)\mathcal{E}[f\circ w_\sigma] + \sum_{k\neq 0} C_k|k|^{-(s+2)\alpha}\mathcal{E}[f\circ w_\sigma]\right)\\
&= 2K_2\zeta(\alpha s)\sum_{n\geq q}2^{(2-s)n}\sum_{|\sigma|=n}\mathcal{E}[f\circ w_\sigma] + \sum_{n\geq q}2^{(2-s)n}\sum_{k\neq 0}C_k|k|^{-(s+2)\alpha}\sum_{|\sigma|=n}\mathcal{E}[f\circ w_\sigma]\\
&= \mathcal{E}[f]\,(2K_2\zeta(\alpha s)+C(s))\left(\frac{3}{5}2^{2-s}\right)^q\left(1-\frac{3}{5}2^{2-s}\right)^{-1}.
\end{aligned}
\tag{4.19}
$$

As a consequence,

$$
\begin{aligned}
\mathrm{Res}_{s=d_E} Z_{D,f}(s) &= \lim_{s\to d_E^+}(s-d_E)Z_{D,f}(s)\\
&= \lim_{s\to d_E^+}(s-d_E)\Big(\sum_{|\sigma|<q}\mathrm{tr}(|D_\sigma|^{-s/2}|[D_\sigma,\pi_\sigma(f)]|^2|D_\sigma|^{-s/2})\\
&\quad + \mathcal{E}[f]\,(2K_2\zeta(\alpha s)+C(s))\left((3/5)2^{2-s}\right)^q\left(1-(3/5)2^{2-s}\right)^{-1}\Big)\\
&= \frac{2K_2\zeta(\alpha\, d_E)+C(d_E)}{\log 2}\,\mathcal{E}[f] = A\,\mathcal{E}[f].
\end{aligned}
\tag{4.20}
$$

(c) formula (4.11) for general functions in $\mathcal{F}$. Let us observe that, for any $s > d_E$, the functional

$$
f \to N_s(f) := Z_{D,f}^{1/2}(s)
$$

is a seminorm on $\mathcal{F}$. We now choose $f\in\mathcal{F}$, and let g be a finitely-harmonic function. Then,

$$
\begin{aligned}
&|(s-d_E)^{1/2}N_s(f) - A^{1/2}\mathcal{E}^{1/2}[f]|\\
&\leq (s-d_E)^{1/2}|N_s(f)-N_s(g)| + |(s-d_E)^{1/2}N_s(g) - A^{1/2}\mathcal{E}^{1/2}[g]| + A^{1/2}|\mathcal{E}^{1/2}[g]-\mathcal{E}^{1/2}[f]|\\
&\leq (s-d_E)^{1/2}N_s(f-g) + |(s-d_E)^{1/2}N_s(g) - A^{1/2}\mathcal{E}^{1/2}[g]| + A^{1/2}\mathcal{E}^{1/2}[f-g]\\
&\leq \left((4K_1\zeta(\alpha s))^{1/2}\left(\frac{s-d_E}{1-\frac{3}{5}2^{2-s}}\right)^{1/2} + A^{1/2}\right)\mathcal{E}^{1/2}[f-g] + |(s-d_E)^{1/2}N_s(g) - A^{1/2}\mathcal{E}^{1/2}[g]|\\
&\longrightarrow \left(\left(\frac{4K_1\zeta(\alpha\, d_E)}{\log 2}\right)^{1/2} + A^{1/2}\right)\mathcal{E}^{1/2}[f-g],\quad s\to d_E^+
\end{aligned}
$$

where for the last inequality we used inequality (4.15). Since g varies among finitely-harmonic functions, the last term may be made arbitrarily small, namely

$$
\exists \lim_{s\to d_E^+}|(s-d_E)^{1/2}N_s(f) - A^{1/2}\mathcal{E}^{1/2}[f]| = 0,
$$

and the thesis is proved. □

4.6. **Standard Dirichlet form and Dixmier traces.** In this section we reconstruct the standard Dirichlet form on the Sierpinski gasket using the Dixmier trace. In particular, the self-similar energy of a function in a suitable form core coincides with the evaluation, by the Dixmier trace, of the square of the modulus of its commutator with the Dirac operator D

times a symmetrized weight proportional to a negative power of $|D|$. Throughout all this section we shall assume $\alpha \in (d_E^{-1}, \alpha_0]$, so that $\delta_D = d_E$.

Definition 4.13. For any $\varepsilon > 0$, we shall consider the set $\mathcal{B}_\varepsilon$ defined as follows:

$$\mathcal{B}_\varepsilon := \{f \in \mathcal{F} : \exists c_f > 0 \text{ such that } \mathcal{E}_{C_\sigma}[f] \leq c_f \, e^{-\varepsilon|\sigma|} \mathcal{E}[f], \sigma \in \Sigma\},$$

and set $\mathcal{B} := \cup_{\varepsilon>0} \mathcal{B}_\varepsilon$.

Lemma 4.14. *Let f be a k-harmonic function. Then $f \in \mathcal{B}_\varepsilon$ for $\varepsilon \leq \log(5/3)$. More precisely, for any $\sigma \in \Sigma$, $\mathcal{E}_{C_\sigma}[f] \leq (3/5)^{(|\sigma|-k)} \mathcal{E}[f]$.*

Proof. It is easy to check that if f is a harmonic function in the interior of a cell C and C_1 is one of its three sub-cells, then $\mathrm{Osc}(f)(C_1) \leq \frac{3}{5} \mathrm{Osc}(f)(C)$ (see for example [55] Chapter 1 Exercise 1.3.6). From this the thesis follows.

□

Proposition 4.15. *Each $\mathcal{B}_\varepsilon$, $\varepsilon > 0$, is a vector space, $\mathcal{B}$ is an algebra.*

Proof. We first prove additivity. The case $f_1 + f_2 = const$ being trivial, we assume $\mathcal{E}[f_1 + f_2] \neq 0$. Then

$$\begin{aligned} \mathcal{E}_{C_\sigma}[f_1 + f_2] &\leq 2\mathcal{E}_{C_\sigma}[f_1] + 2\mathcal{E}_{C_\sigma}[f_2] \\ &\leq 2(c_1 \mathcal{E}[f_1] + c_2 \mathcal{E}[f_2]) e^{-\varepsilon|\sigma|} = c \, e^{-\varepsilon|\sigma|} \mathcal{E}[f_1 + f_2], \end{aligned}$$

where $c = \mathcal{E}[f_1+f_2]^{-1} 2(c_1 \mathcal{E}[f_1] + c_2 \mathcal{E}[f_2])$. As for multiplicativity, assuming as before $\mathcal{E}[f_1 f_2] \neq 0$, we get

$$\begin{aligned} \mathcal{E}_{C_\sigma}[f_1 f_2] &\leq \|f_2|_{C_\sigma}\|_\infty \mathcal{E}_{C_\sigma}[f_1] + \|f_1|_{C_\sigma}\|_\infty \mathcal{E}_{C_\sigma}[f_2] \\ &\leq \|f_2\|_\infty c_1 \mathcal{E}[f_1] e^{-\varepsilon_1|\sigma|} + \|f_1\|_\infty c_2 \mathcal{E}[f_2] e^{-\varepsilon_2|\sigma|} = c \, e^{-\varepsilon|\sigma|} \mathcal{E}[f_1 f_2], \end{aligned}$$

where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $c = \mathcal{E}[f_1 f_2]^{-1} (\|f_2\|_\infty c_1 \mathcal{E}[f_1] + \|f_1\|_\infty c_2 \mathcal{E}[f_2])$. □

Remark 4.16. Let us note that the inequalities in the proof of the following Lemma are very close to those in (4.15), the Hilbert-Schmidt norm being replaced by the uniform norm. For functions in $\mathcal{B}$, this allows us to prove an estimate for values of s below d_E.

Lemma 4.17. *Assume $0 < \varepsilon < (d_E - \alpha^{-1}) \log 2$, $f \in \mathcal{B}_\varepsilon$. Then, for $s \geq d_E - \varepsilon/\log 2$, $[D, f]|D|^{-\frac{1}{2}s}$ is bounded.*

Proof. Making use of (4.10), (4.13), we get:

$$\begin{aligned}
\|[D,\pi(f)]\ |D|^{-\frac{1}{2}s}\|^2 &= \sup_\sigma \|[D_\sigma,\pi_\sigma(f)]|\,|D_\sigma|^{-\frac{1}{2}s}\|^2 \\
&= \sup_\sigma 2^{|\sigma|(2-s)}\|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma)]|\,|D_\emptyset|^{-\frac{1}{2}s}\|^2 \\
&\le \sup_\sigma 2^{|\sigma|(2-s)}\operatorname{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma)]|^2|D_\emptyset|^{-s/2}) \\
&\le 4\sup_\sigma 2^{|\sigma|(2-s)}\zeta(\alpha s)\|\partial_\alpha\pi_\emptyset(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} \\
&\le 4K_1\zeta(\alpha s)\sup_\sigma 2^{|\sigma|(2-s)}\mathcal{E}[f\circ w_\sigma] \\
&\le 4K_1\zeta(\alpha s)\sup_\sigma 2^{|\sigma|(2-s)}\left(\frac{3}{5}\right)^{|\sigma|}\mathcal{E}_{C_\sigma}[f] \\
&\le 4K_1\zeta(\alpha s)\sup_n\left(\frac{12}{5}2^{-s}\right)^n\max_{|\sigma|=n}\mathcal{E}_{C_\sigma}[f] \\
&\le 4c_fK_1\zeta(\alpha s)\sup_n\left(\frac{12}{5}2^{-s}e^{-\varepsilon}\right)^n\mathcal{E}[f].
\end{aligned}$$

We get a non trivial bound when $-s\log 2+\log(12/5)-\varepsilon\le 0$ and $\alpha s>1$, namely

$$s\ge\max\left\{\alpha^{-1},d_E-\frac{\varepsilon}{\log 2}\right\}.$$

Since $\varepsilon<(d_E-\alpha^{-1})\log 2$, this amounts to $s\ge d_E-\frac{\varepsilon}{\log 2}$. ☐

Remark 4.18. Let us notice that the actual bound on the norm of $[D,f]|D|^{-\frac{1}{2}s}$ does not play any role, see inequality (4.21).

Theorem 4.19. *(i)* $\forall\varepsilon>0\,\exists M_\varepsilon\in\mathbb{R}:\forall f\in\mathcal{B}_\varepsilon,\operatorname{tr}_\omega\left(|D|^{-d_E/2}|[D,f]|^2|D|^{-d_E/2}\right)\le M_\varepsilon\mathcal{E}[f]$. *When* $0<\varepsilon<(d_E-\alpha^{-1})\log 2$ *we may choose* $M_\varepsilon=4eK_1\varepsilon^{-1}\zeta(\alpha d_E-\frac{\alpha\varepsilon}{\log 2})$.
(ii) In particular, $|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E}\in\mathcal{L}^{1,\infty}(\mathcal{H})$, $f\in\mathcal{B}$.

Proof. It is enough to give the proof for $0<\varepsilon<(d_E-\alpha^{-1})\log 2$. We shall use Lemma 4.5 in [27] with the contraction U given by the operator $[D,f]|D|^{-\frac{1}{2}s}$ suitably normalized, the positive operator T given by $|D|^{-(d_E-s)}$, and the convex function $f(x)=x^{1+t}$, with $t>0$. Then, we take $d_E-\frac{\varepsilon}{\log 2}\le s<d_E$, so that $[D,f]|D|^{-\frac{1}{2}s}$ is bounded and $|D|^{-\frac{1}{2}(d_E+t(d_E-s))}|[D,f]|^2|D|^{-\frac{1}{2}(d_E+t(d_E-s))}$ is trace class.

$$\begin{aligned}
&\operatorname{tr}\left((|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E})^{1+t}\right) \\
&\qquad= \operatorname{tr}\left((|D|^{-\frac{1}{2}(d_E-s)}|D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}|D|^{-\frac{1}{2}(d_E-s)})^{1+t}\right) \\
&\qquad= \||D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}\|^{1+t}\operatorname{tr}\left((T^{\frac{1}{2}}U^*UT^{\frac{1}{2}})^{1+t}\right) \\
&\qquad= \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\operatorname{tr}\left((UTU^*)^{1+t}\right) \\
&\qquad\le \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\operatorname{tr}(UT^{1+t}U^*) \\
&\qquad= \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\operatorname{tr}(T^{\frac{1}{2}(1+t)}U^*UT^{\frac{1}{2}(1+t)}) \\
&\qquad= \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t}\operatorname{tr}(|D|^{-\frac{1}{2}(d_E-s)(1+t)}|D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}|D|^{-\frac{1}{2}(d_E-s)(1+t)}) \\
&\qquad= \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t}\operatorname{tr}(|D|^{-\frac{1}{2}(d_E+t(d_E-s))}|[D,f]|^2|D|^{-\frac{1}{2}(d_E+t(d_E-s))}).
\end{aligned}$$

The previous inequality, together with equation (4.15), gives

$$\operatorname{tr}\left((|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E})^{1+t}\right) \leq \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t} 4K_1\zeta(\alpha s)\mathcal{E}[f]\left(1-\frac{3}{5}2^{2-d_E-t(d_E-s)}\right)^{-1}$$

hence, for $f \in \mathcal{B}_\varepsilon$, $\limsup_{t\to 0} t \operatorname{tr}\left((|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E})^{1+t}\right) \leq \frac{4K_1\zeta(\alpha s)}{(d_E-s)\log 2}\mathcal{E}[f]$. By Lemma 4.17, we may choose $s = d_E - \frac{\varepsilon}{\log 2}$, hence

$$\limsup_{t\to 0} t \operatorname{tr}\left((|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E})^{1+t}\right) \leq 4K_1\varepsilon^{-1}\zeta(\alpha d_E - \frac{\alpha\varepsilon}{\log 2})\mathcal{E}[f]. \tag{4.21}$$

By Theorem 4.5 (i) in [12], we get

$$\limsup_n \frac{1}{\log n}\sum_{k=1}^n \mu_k(|D|^{-\frac{1}{2}d_E}|[D,f]|^2|D|^{-\frac{1}{2}d_E}) \leq M_\varepsilon \mathcal{E}[f].$$

(i) follows by the definition of tr_ω, (ii) follows by the definition of $\mathcal{L}^{1,\infty}(\mathcal{H})$. □

Lemma 4.20. *Let $0 < \delta < d$, and B be a densely defined, positive (possibly unbounded) operator on $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})_+$ such that $T^s \in \mathcal{L}^1(\mathcal{H})$ for $s > d$, and $T^{s/2}BT^{s/2} \in \mathcal{L}^1(\mathcal{H})$ for $s > \delta$. Then*

$$\limsup_{s\to\delta^+}(s-\delta)\operatorname{tr}(T^{s/2}BT^{s/2}) \leq d\cdot \limsup_{r\to\infty}\frac{1}{r}\ \operatorname{tr}(T^{\delta/2}BT^{\delta/2})^{1+\frac{1}{r}}. \tag{4.22}$$

If $\lim_{s\to\delta^+}(s-\delta)\operatorname{tr}(T^{s/2}BT^{s/2})$ exists and $\limsup_{r\to\infty}\frac{1}{r}\ \operatorname{tr}(T^{\delta/2}B^2T^{\delta/2})^{1+\frac{1}{r}}$ is finite, then, for any dilation invariant state ω on ℓ^∞ vanishing on c_0,

$$\lim_{s\to\delta^+}(s-\delta)\operatorname{tr}(T^{s/2}BT^{s/2}) \leq d\cdot \operatorname{tr}_\omega(T^{\delta/2}BT^{\delta/2}). \tag{4.23}$$

Proof. For $r > 0$, Hölder inequality ([25],Thm 6) for the exponents $1+\frac{1}{r}$, $2(r+1)$, $2(r+1)$ gives

$$\operatorname{tr}(T^{s/2}BT^{s/2}) = \operatorname{tr}\left(T^{(s-\delta)/2}(T^{\delta/2}BT^{\delta/2})T^{(s-\delta)/2}\right) \leq \left(\operatorname{tr}(T^{\delta/2}BT^{\delta/2})^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\left(\operatorname{tr}(T^{(s-\delta)(r+1)})\right)^{\frac{1}{r+1}}.$$

Setting $r = \dfrac{d+\varepsilon+\delta-s}{s-\delta}$ for $\varepsilon > 0$, i.e. $(s-\delta)(r+1) = d+\varepsilon$, we get

$$\begin{aligned}\limsup_{s\to\delta^+}(s-\delta)\operatorname{tr}(T^{s/2}BT^{s/2}) &\leq \limsup_{r\to\infty}\frac{d+\varepsilon}{r+1}\left(\operatorname{tr}(T^{\delta/2}BT^{\delta/2})^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\left(\operatorname{tr}(T^{(d+\varepsilon)})\right)^{\frac{1}{r+1}}\\ &\leq (d+\varepsilon)\limsup_{r\to\infty}\frac{r}{r+1}\left(\frac{1}{r}\ \operatorname{tr}(T^{\delta/2}BT^{\delta/2})^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\\ &= (d+\varepsilon)\limsup_{r\to\infty}\frac{1}{r}\ \operatorname{tr}(T^{\delta/2}BT^{\delta/2})^{1+\frac{1}{r}}.\end{aligned}$$

Inequality (4.22) follows by the arbitrariness of ε. With the same notations as in [13], we may replace the $\limsup$ in the argument above with a $\tilde{\omega}-\lim$. Then (4.23) follows by [13], Thm 3.1. □

Proposition 4.21. *The quadratic form $f \to \operatorname{tr}_\omega(|D|^{-d_E/2}|[D,f]|^2|D|^{-d_E/2})$ defined on continuous functions with values in $[0,\infty]$ is self-similar with parameter $5/3$ and invariant under the symmetries of the triangle.*

Proof. Let us prove self-similarity.

$$\sum_{i=1,2,3} \operatorname{tr}_\omega(|D|^{-d_E/2}|[D,\pi(f\circ w_i)]|^2|D|^{-d_E/2}) = \sum_{i=1,2,3} \operatorname{tr}_\omega\left(\bigoplus_\sigma |D_\sigma|^{-d_E/2}|[D_\sigma,\pi_\sigma(f\circ w_i)]|^2|D_\sigma|^{-d_E/2}\right)$$

$$= \operatorname{tr}_\omega\left(\bigoplus_{i=1,2,3}\bigoplus_\sigma 2^{(2-d_E)|\sigma|}|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f\circ w_i\circ w_\sigma)]|^2|D_\emptyset|^{-d_E/2}\right)$$

$$= \operatorname{tr}_\omega\left(\bigoplus_{\tau\neq\emptyset} 2^{(2-d_E)(|\tau|-1)}|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f\circ w_\tau)]|^2|D_\emptyset|^{-d_E/2}\right)$$

$$= 2^{-(2-d_E)}\big(\operatorname{tr}_\omega(|D|^{-d_E/2}|[D,\pi(f)]|^2|D|^{-d_E/2}) - \operatorname{tr}_\omega(|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f)]|^2|D_\emptyset|^{-d_E/2})\big)$$

$$= \frac{3}{5}\operatorname{tr}_\omega(|D|^{-d_E/2}|[D,\pi(f)]|^2|D|^{-d_E/2}),$$

where $\operatorname{tr}_\omega(|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f)]|^2|D_\emptyset|^{-d_E/2})$ vanishes since $|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f)]|^2|D_\emptyset|^{-d_E/2}$ is trace class, as shown in Lemma 4.11.

We now prove symmetry invariance. We observe that the symmetry group of the triangle is generated by the reflections along the axes. Hence, it is enough to show that, for a reflection T along an axis of the gasket, and setting $f^T(x) = f(Tx)$, the operator $|D|^{-d_E/2}|[D,\pi(f^T)]|^2|D|^{-d_E/2}$ and the operator $|D|^{-d_E/2}|[D,\pi(f)]|^2|D|^{-d_E/2}$ are unitary equivalent. Since

$$|D|^{-d_E/2}|[D,\pi(f)]|^2|D|^{-d_E/2} = \bigoplus_\sigma 2^{(2-d_E)|\sigma|}|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma)]|^2|D_\emptyset|^{-d_E/2},$$

$|\sigma^T| = |\sigma|$, and $f^T\circ w_\sigma = f\circ w_{\sigma^T}\circ T$, where $\sigma\to\sigma^T$ denotes the natural action of T on the word σ,

$$|D|^{-d_E/2}|[D,\pi(f^T)]|^2|D|^{-d_E/2} = \bigoplus_\sigma 2^{(2-d_E)|\sigma|}|D_\emptyset|^{-d_E/2}|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma\circ T)]|^2|D_\emptyset|^{-d_E/2}.$$

Moreover, as in (4.9),

$$|D_\emptyset|^{-d_E/2}|[D_\emptyset,g]|^2|D_\emptyset|^{-d_E/2} = \begin{pmatrix} (\partial_\alpha^*\partial_\alpha)^{-d_E/4}S_{g^*}^*S_{g^*}(\partial_\alpha^*\partial_\alpha)^{-d_E/4} & 0 \\ 0 & (\partial_\alpha\partial_\alpha^*)^{-d_E/4}S_gS_g^*(\partial_\alpha\partial_\alpha^*)^{-d_E/4}\end{pmatrix}.$$

Therefore, it is enough to show that, for any $g\in H^\alpha(\ell_\emptyset)$, $(\partial_\alpha^*\partial_\alpha)^{-d_E/4}\ S_{g^*\circ T}^*S_{g^*\circ T}\ (\partial_\alpha^*\partial_\alpha)^{-d_E/4}$ and $(\partial_\alpha^*\partial_\alpha)^{-d_E/4}\ S_{g^*}^*S_{g^*}\ (\partial_\alpha^*\partial_\alpha)^{-d_E/4}$ are unitary equivalent, and $(\partial_\alpha\partial_\alpha^*)^{-d_E/4}\ S_{g\circ T}S_{g\circ T}^*\ (\partial_\alpha\partial_\alpha^*)^{-d_E/4}$ and $(\partial_\alpha\partial_\alpha^*)^{-d_E/4}\ S_gS_g^*\ (\partial_\alpha\partial_\alpha^*)^{-d_E/4}$ are unitary equivalent.

Let us consider the (self-adjoint) unitary operator $U_{T,0}$ on $L^2(\ell_\emptyset)$ given by $(U_{T,0}\xi)(x) = \xi(Tx)$ and the (self-adjoint) unitary operator $U_{T,1}$ on $L^2(\ell_\emptyset\times\ell_\emptyset)$ given by $(U_{T,1}\eta)(x,y) = \eta(Tx,Ty)$. A direct computation shows that $U_{T,1}\partial_\alpha = \partial_\alpha U_{T,0}$ and $M_{g^T} = U_{T,0}M_gU_{T,0}$, hence $S_{g^T} = U_{T,1}S_gU_{T,0}$. As a consequence, for $g\in H^\alpha(\ell_\emptyset)$, we get

$$(\partial_\alpha^*\partial_\alpha)^{-d_E/4}S_{g^*\circ T}^*S_{g^*\circ T}(\partial_\alpha^*\partial_\alpha)^{-d_E/4} = U_{T,0}(\partial_\alpha^*\partial_\alpha)^{-d_E/4}S_{g^*}^*S_{g^*}(\partial_\alpha^*\partial_\alpha)^{-d_E/4}U_{T,0}\,,$$

$$(\partial_\alpha\partial_\alpha^*)^{-d_E/4}S_{g\circ T}S_{g\circ T}^*(\partial_\alpha\partial_\alpha^*)^{-d_E/4} = U_{T,1}(\partial_\alpha\partial_\alpha^*)^{-d_E/4}S_gS_g^*(\partial_\alpha\partial_\alpha^*)^{-d_E/4}U_{T,1}\,,$$

and the thesis follows. □

Remark 4.22. With the same argument as in the Proposition above, we may show that the quadratic form $f \to \mathrm{tr}_\omega(|D|^{-\delta/2}|[D,f]|^2|D|^{-\delta/2})$ is self-similar with parameter $2^{2-\delta}$. As a consequence the value d_E is uniquely determined by requiring self-similarity with scaling parameter $5/3$. Hence the energy dimension d_E is not just an abscissa of convergence, but is completely determined by the structure of the Dirac operator and the scaling of the energy under self-similarity.

We state here a simple variant of a well known uniqueness result, cf. e.g. [53, 24].

Lemma 4.23. *A quadratic form $\mathcal{G}$ which is finite on finitely harmonic functions on the gasket, vanishes only on constants, is self-similar with parameter $5/3$, and invariant under the symmetries of the triangle coincides with a multiple of the standard Dirichlet form on finitely harmonic functions.*

Proof. Let us consider the linear map which associates to any vector $\vec{v} = (v_0, v_1, v_2) \in \mathbb{C}^3$, the 0-harmonic function $g = \varphi(\vec{v})$ on the gasket taking values v_i, $i = 0, 1, 2$, on the extreme points of the lacuna $\ell_\emptyset$. Then, the quadratic form $\vec{v} \to \mathcal{G}[g]$ is positive semidefinite on $\mathbb{C}^3$, vanishes only on constant vectors, since $g = \varphi(\vec{v})$ is constant iff $\vec{v}$ is constant, and is invariant under permutations of the components, because of the symmetry invariance in the assumptions. As in the proof of Theorem 4.12 $(ii)(a)$, there exists a constant k such that $\mathcal{G}[g] = k\mathcal{E}[g]$ for any 0-harmonic function g. Let now h be n-harmonic, so that $h \circ w_\sigma$ is 0-harmonic for $|\sigma| = n$. Then, by self-similarity,

$$\mathcal{G}[h] = \left(\frac{5}{3}\right)^n \sum_{|\sigma|=n} \mathcal{G}[h \circ w_\sigma] = k\left(\frac{5}{3}\right)^n \sum_{|\sigma|=n} \mathcal{E}[h \circ w_\sigma] = k\mathcal{E}[h].$$

The thesis follows. □

Corollary 4.24. *On the algebra $\mathcal{B}$, the standard Dirichlet form $\mathcal{E}$ and the quadratic form $f \to \mathcal{E}_D[f] := \mathrm{tr}_\omega(|D|^{-d_E/2}|[D,f]|^2|D|^{-d_E/2})$ coincide up to a multiplicative constant, namely there exists a constant B_ω, which may depend on the generalized limit ω, such that,*

$$\mathcal{E}_D[f] = B_\omega \mathcal{E}[g], \quad \forall g \in \mathcal{B}. \tag{4.24}$$

Proof. We first prove that, for $f \in \mathcal{B}$, the quadratic form in the statement is bounded from above and from below by multiples of the standard Dirichlet form on the gasket. In the inequality (4.25) below the upper bound seems to depend on ε, but, as soon as the statement of the theorem is proved, the smallest M_ε will work on the whole $\mathcal{B}$.

Let $f \in \mathcal{B}_\varepsilon$. We may then apply inequality (4.23) with $T = |D|^{-1}$, $B = |[D,f]|^2$, $\delta = d_E$. In fact, condition $\exists \lim_{s\to\delta^+}(s-\delta)\,\mathrm{tr}(T^{s/2}BT^{s/2})$ is satisfied by Theorem 4.12, and condition $\limsup_{r\to\infty} \frac{1}{r}\,\mathrm{tr}(T^{\delta/2}B^2T^{\delta/2})^{1+\frac{1}{r}} < +\infty$ follows by (4.21). Then, Theorems 4.12, 4.19 and inequality (4.23) give, for sufficiently small ε,

$$\frac{A}{d}\mathcal{E}[f] \le \mathcal{E}_D[f] \le M_\varepsilon \mathcal{E}[f]. \tag{4.25}$$

In particular, by Lemma 4.14, previous inequalities hold for finitely harmonic functions.

Then, we observe that (4.24) holds for finitely harmonic functions. Indeed, by inequality (4.25) and Proposition 4.21, the assumptions of Lemma 4.23 are satisfied.

Finally, we may proceed as in the proof of Theorem 4.12 $(ii)(c)$. Choose $f \in \mathcal{B}_\varepsilon$ and let g be a finitely harmonic function. Then,

$$\begin{aligned} &|\mathcal{E}_D^{1/2}[f] - B_\omega^{1/2}\mathcal{E}^{1/2}[f]| \\ &\qquad \leq |\mathcal{E}_D^{1/2}[f] - \mathcal{E}_D^{1/2}[g]| + |\mathcal{E}_D^{1/2}[g] - B_\omega^{1/2}\mathcal{E}^{1/2}[g]| + B_\omega^{1/2}|\mathcal{E}^{1/2}[g] - \mathcal{E}^{1/2}[f]| \\ &\qquad \leq \mathcal{E}_D^{1/2}[f-g] + B_\omega^{1/2}\mathcal{E}^{1/2}[f-g] \leq (M_\varepsilon^{1/2} + B_\omega^{1/2})\mathcal{E}^{1/2}[f-g]. \end{aligned}$$

Since g varies among finitely-harmonic functions, the last term may be made arbitrarily small, and the Theorem is proved. □

5. A two parameter deformation of spectral triple: the (β, α) plane

In this subsection we consider a deformation of our construction. Namely, after having chosen $\alpha \in (0,1]$, we introduce a further parameter $\beta > 0$, and, for any $\sigma \in \cup_n \{0,1,2\}^n$, consider the triple $(\mathcal{C}(K), \mathcal{H}_\sigma, D_\sigma)$, where $\mathcal{H}_\sigma = \mathcal{H}_\emptyset$, $D_\sigma = 2^{\beta|\sigma|}D_\emptyset$, and the algebra $\mathcal{C}(K)$ acts via the representation π_σ, with $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$. Here, we only give the statements concerning this two-parameter family of spectral triples. On the one hand, the proofs are more or less direct extensions of the proofs for the case $\beta = 1$. On the other hand, detailed proofs are contained in a previous version of this paper, available on the arXiv as http://arxiv.org/abs/1112.6401v2.

Definition 5.1. Let us consider for values of the parameters $\alpha \in (0,1]$ and $\beta > 0$, the triple $(\mathcal{A}, \mathcal{H}, D)$, where the Hilbert space and the self-adjoint operator are given by $\mathcal{H} = \oplus_{\sigma\in\Sigma}\mathcal{H}_\sigma$, $D = \oplus_{\sigma\in\Sigma}D_\sigma$ and $\mathcal{A}$ is the subalgebra of $\mathcal{C}(K)$ consisting of functions with bounded commutator with D, acting on $\mathcal{H}$ via the representation $\pi = \oplus_{\sigma\in\Sigma}\pi_\sigma$.

As customary in Noncommutative Geometry, we denote by $\oint f := tr_\omega(f|D|^{-d})$ the functional on the algebra $\mathcal{C}(K)$ obtained through the Dixmier logarithmic trace tr_ω, where d is the abscissa of convergence of the function $s \to \mathrm{tr}(|D|^{-s})$.

As first result in this section, we describe how the dimensional spectrum of the triple depends on $\beta > 0$ and that, independently on $0 < \beta < \alpha\frac{\log 3}{\log 2}$, the measure induced on K by the functional $f \mapsto \oint f$ is a suitable multiple of its Hausdorff measure.

Theorem 5.2. *The volume zeta function $\mathcal{Z}_D$ of the triple $(\mathcal{A}, \mathcal{H}, D)$, i.e. the meromorphic extension of the function $\mathbb{C} \ni s \mapsto \mathrm{tr}(|D|^{-s})$ initially defined for $s \in \mathbb{C}$ having large real part, is given by*

$$\mathcal{Z}_D(s) = \frac{4\zeta(\alpha s)}{1 - 3\cdot 2^{-\beta s}},$$

where ζ denotes the Riemann zeta function. Therefore, the dimensional spectrum of the triple is

$$\mathcal{S}_{dim} = \{\alpha^{-1}\} \cup \left\{\frac{\log 3}{\beta \log 2}\left(1 + \frac{2\pi i}{\log 3}k\right) : k \in \mathbb{Z}\right\} \subset \mathbb{C}.$$

As a consequence, the metric dimension d_D of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, namely the abscissa of convergence of its volume zeta function, is $\max\{\alpha^{-1}, \beta^{-1}d_H\}$.

When $0 < \beta < \alpha\, d_H$, i.e. $d_D = \beta^{-1}d_H$, $\mathcal{Z}_D$ has a simple pole in d_D, and the measure associated via Riesz theorem with the functional $\mathcal{C}(K) \ni f \to \oint f$ coincides with a multiple of the Hausdorff measure H_{d_H}:

$$\mathrm{vol}(f) \equiv \int_K f\, d\,\mathrm{vol} := tr_\omega(f|D|^{-d_D}) = \frac{4d_D}{\log 3}\frac{\zeta(d_D)}{(2\pi)^{d_D}}\int_K f\, dH_{d_H} \qquad f \in C(K).$$

The next result says that for $\alpha, \beta \in (0,1]$ the above triple is a spectral triple according to Connes [22], the associated seminorm is a Lip-morm on $\mathcal{C}(K)$ in the sense of Rieffel [52] and the induced topology on K coincides with the original one.
If moreover $\alpha < \beta < 1$, we obtain that the associated Connes metric is bi-Lipschitz w.r.t. the root $(\rho_{geo})^\beta$ of the geodesic metric on K.

Corollary 5.3. *For any $\alpha, \beta \in (0,1]$ the triple $(\mathcal{A}, \mathcal{H}, D)$ is a finitely summable spectral triple, and the seminorm $f \to \|[D,f]\|$ is a Lip-norm according to Rieffel* [52]. *Therefore, the Connes metric*

$$\rho_D(x,y) = \sup\{|f(x) - f(y)| : f \in \mathcal{A}\,, \|[D,f]\| \le 1\}$$

induces the original topology on K. Let ρ_{geo} denote the Euclidean geodesic metric on K. Then, if $\beta > \alpha$, the seminorm $\|[D,f]\|$ and the Hölder seminorm $\|f\|_{C^{0,\beta}(K)}$ are equivalent, and the metric ρ_D is bi-Lipschitz w.r.t. the metric $(\rho_{geo})^\beta$ on K.

Let $\alpha, \beta \in (0,1]$, and denote by $(\mathcal{A}, \mathcal{H}, D)$ the spectral triple for the gasket considered above. Let F be the phase of D, and, for any $\sigma \in \Sigma$, denote by $\gamma_\sigma, \varepsilon_\sigma, S_\sigma$, a copy of the operators γ, ε, S associated, by Proposition 3.9, to the lacuna ℓ_σ, identified with $\mathbb{T}$. Finally, let $\gamma = \oplus_{\sigma\in\Sigma}\gamma_\sigma$, $\varepsilon = \oplus_{\sigma\in\Sigma}\varepsilon_\sigma$, $S = \oplus_{\sigma\in\Sigma}S_\sigma$.

Theorem 5.4. *The quintuple $\mathcal{F} = (\mathcal{H}, \pi, F, \gamma, \varepsilon)$ is a tamely degenerate 1-graded Fredholm module on $\mathcal{A}$. The ungraded Fredholm module $\mathcal{F}^+ = (\mathcal{H}^+, \pi^+, F^+)$ associated to it by Proposition 2.12 is a tamely degenerate module on $\mathcal{A}$, and $F^+ = S$. The module $\mathcal{F}^+$ is non trivial in K-homology. In particular, it pairs non trivially with the generators of the (odd) K-theory of the gasket associated with the lacunas.*

As a last result in this section, we compute how the energy dimension of the triple depends on $\beta > 0$ and that, independently of the values of α and β in suitable ranges, the induced quadratic form on K, defined as the residue of a suitable energy functional, is a multiple of its standard Dirichlet form.

Theorem 5.5. *Assume as above that $\beta > 0$, $\frac{1}{2} < \alpha \le \alpha_0$, with $\alpha_0 = \frac{\log(10/3)}{\log 4} \approx 0.87$, and assume f has finite energy* [1]. *Then the abscissa of convergence of the energy functional*

$$\mathrm{tr}(|D|^{-s/2}|[D,f]|^2|D|^{-s/2})$$

is given by $\delta_D = \max\{\alpha^{-1}, 2 - \frac{\log 5/3}{\beta \log 2}\}$. Whenever $\beta(2-\alpha^{-1}) > \frac{\log(5/3)}{\log 2}$, the energy functional has a simple pole at δ_D and its residue coincides with the value of the standard Dirichlet form on f

$$\mathrm{Res}_{s=\delta_D}\, \mathrm{tr}(|D|^{-s/2}|[D,f]|^2|D|^{-s/2}) = const \cdot \mathcal{E}[f]. \tag{5.1}$$

Finally, notice that, up to a multiplicative constant, the residue of the energy functional coincides with the standard Dirichlet form for any β in a suitable range, so that Corollary 4.24 remains true as it is.

Remark 5.6. We note here that β rescales the Euclidean geodesic metric ρ_{geo} of K essentially to ρ_{geo}^β, therefore it has to be expected that the associated metric dimension scales from d_H to $\beta^{-1}d_H$, while the corresponding volume measure remains the same up to a multiplicative constant. A similar effect occurs for the energy: the energy dimension passes from $2 - \frac{\log 5/3}{\log 2}$ to $2 - \beta^{-1}\frac{\log 5/3}{\log 2}$, the energy form remaining the same up to a multiplicative constant.

[1] The conditions $\beta > 0$ and $2 - \frac{\log 5/3}{\beta \log 2} > \alpha^{-1}$ indeed imply $\alpha > \frac{1}{2}$.

Let us summarize the properties of the family of triples introduced above, for the different values of the parameters α and β.

- The whole construction makes sense only if $0 < \alpha \le 1$, $\beta \in \mathbb{R}$.
- If $\beta > 0$, the inverse of D on the orthogonal complement of the kernel is compact.
- if $\beta > 0$ and $\alpha > \beta/d_H$, the noncommutative volume measure coincides (up to a constant factor) with the Hausdorff measure H_{d_H}, $d_H = \frac{\log 3}{\log 2}$ being the Hausdorff (or similarity) dimension. The metric dimension is $d_D = \frac{d_H}{\beta}$.
- If $0 < \beta \le 1$, $\|[D,f]\|$ is a densely defined Lip-norm, $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ is a 1-graded Fredholm module. The latter has non-trivial pairing with the topological K-theory group $K^1(K)$ of the gasket.
- If $d_{E,\beta}^{-1} < \alpha \le \alpha_0 < \beta \le 1$, where $d_{E,\beta} = 2 - \frac{\log(5/3)}{\log 2}\beta^{-1}$, then the residue at $s = d_{E,\beta}$ of the energy functional $\mathrm{tr}(|D|^{-s/2}[D,f]^2|D|^{-s/2})$ coincides, up to a multiplicative factor, with the standard Dirichlet form $\mathcal{E}[f]$, for all finite energy functions.

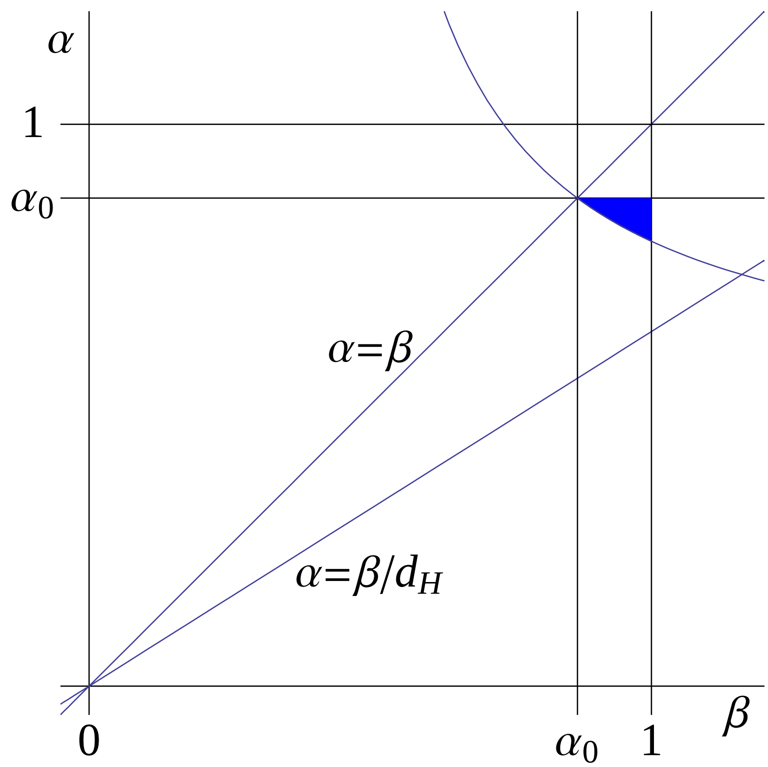


FIGURE 4. The (β, α) plane

Appendix A. Estimates on the Clausen function

According to ([49], p. 236, [26] section 1.11) the analytic extension of the polylogarithm function of order s on the whole complex plane with the line $[1, +\infty)$ removed is given by

$$\mathrm{Li}_s(z) = -\frac{z\Gamma(1-s)}{2\pi i} \int_\gamma \frac{(-t)^{s-1}}{e^t - z}\, dt,$$

where γ is a path as in figure 5.

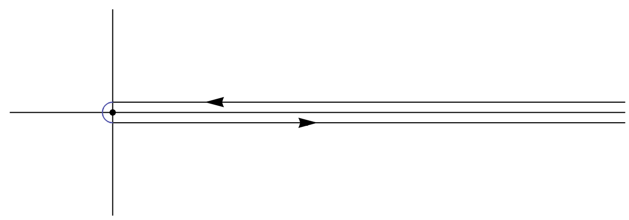


Figure 5. Path used for the analytic extension of polylogarithm.

Therefore the Clausen cosine function $\mathrm{Ci}_s(t)$ can be defined as

$$\mathrm{Ci}_s(\vartheta) = -\operatorname{Re} \frac{\Gamma(1-s)}{2\pi i} \int_\gamma \frac{(-t)^{s-1}}{e^{t-i\vartheta} - 1}\, dt,$$

Lemma A.1. *When* $\operatorname{Re} s < 1$, $\lim_{t\to 0^\pm} |t|^{1-s} \mathrm{Li}_s(e^{it}) = \Gamma(1-s) e^{\pm i\pi(1-s)/2}$, *as a consequence, for* $\alpha \in (0,1]$,

$$\lim_{t\to 0} |t|^{1+2\alpha} \mathrm{Ci}_{-2\alpha}(t) = -\Gamma(1+2\alpha)\sin\pi\alpha.$$

Moreover, when $\alpha \in [\frac{1}{2}, 1)$ *and* $|t| \le \frac{\pi}{4}$, $\mathrm{Ci}_{-2\alpha}$ *is strictly negative, and*

$$|\mathrm{Ci}_{-2\alpha}(t) + \Gamma(1+2\alpha)\sin\pi\alpha\, |t|^{-(2\alpha+1)}| \le \frac{31}{2\pi^2}\Gamma(1+2\alpha)\sin\pi\alpha, \tag{A.1}$$

$$\frac{1}{32}\sin(\pi\alpha)\Gamma(1+2\alpha) \le -\mathrm{Ci}_{-2\alpha}(t)|t|^{2\alpha+1} \le \frac{63}{32}\sin(\pi\alpha)\Gamma(1+2\alpha). \tag{A.2}$$

Finally, when $|t| \ge \pi/4$,

$$|\mathrm{Ci}_{-2\alpha}(t)|\, |t|^{2\alpha+1} \le 23.$$

Proof. Let $0 < |\vartheta| \le \pi/4$. Then we may choose γ in figure 5 as $\gamma_0 - \sigma$ where γ_0 is made of the half lines $\sqrt{\pi^2 - \varepsilon^2} + t \pm i\varepsilon$, $t > 0$, and (most of) the circle of radius π centered at the origin, and σ is a suitably small positively oriented cycle surrounding the point $i\vartheta$. Then

$$\begin{aligned}
\mathrm{Li}_s(e^{i\vartheta}) &= -\frac{e^{i\vartheta}\Gamma(1-s)}{2\pi i} \int_{\gamma_0} \frac{(-t)^{s-1}}{e^t - e^{i\vartheta}}\, dt + \frac{e^{i\vartheta}\Gamma(1-s)}{2\pi i} \int_\sigma \frac{(-t)^{s-1}}{e^t - e^{i\vartheta}}\, dt \\
&= -\frac{\Gamma(1-s)}{2\pi i} \int_{\gamma_0} \frac{(-t)^{s-1}}{e^{(t-i\vartheta)} - 1}\, dt + \Gamma(1-s)\operatorname{Res}_{t=i\vartheta} \frac{(-t)^{s-1}}{e^{(t-i\vartheta)} - 1} \\
&= -\frac{\Gamma(1-s)}{2\pi i} \int_{\gamma_0} \frac{(-t)^{s-1}}{e^{(t-i\vartheta)} - 1}\, dt + \Gamma(1-s) e^{i\,\mathrm{sgn}(\vartheta)\pi(1-s)/2} |\vartheta|^{s-1}.
\end{aligned}$$

In particular,

$$\mathrm{Ci}_{-2\alpha}(\vartheta) = \operatorname{Re}\mathrm{Li}_{-2\alpha}(e^{i\vartheta}) = -\frac{\Gamma(1+2\alpha)}{2\pi} \operatorname{Im}\left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)} - 1}\, dt\right) - \Gamma(1+2\alpha)\sin\pi\alpha\ |\vartheta|^{-(2\alpha+1)},$$

from which the first relations hold. Moreover,

$$-\operatorname{Ci}_{-2\alpha}(\vartheta)=\Gamma(1+2\alpha)\sin\alpha\pi\;|\vartheta|^{-2\alpha-1}+\frac{\Gamma(1+2\alpha)}{2\pi}\;\operatorname{Im}\left(\int_{\gamma_0}\frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right),$$

hence

$$|\operatorname{Ci}_{-2\alpha}(\vartheta)+\Gamma(1+2\alpha)\sin\alpha\pi\;|\vartheta|^{-2\alpha-1}|=\frac{\Gamma(1+2\alpha)}{2\pi}\;\left|\operatorname{Im}\left(\int_{\gamma_0}\frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right)\right|.$$

We now assume $\alpha\geq\frac{1}{2}$, and observe that the part of the path constituted by the half lines $\sqrt{\pi^2-\varepsilon^2}+t\pm i\varepsilon$, $t>0$ is invariant under reflection w.r.t. to the real axis, which sends the variable of integration to its conjugate. Therefore,

$$\int_{\gamma_0}\frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt=\int_{|z|=\pi}\frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz+2i\sin(2\pi\alpha)\int_{\pi}^{\infty}\frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\;.$$

As for the second integral, we have

$$\begin{aligned}
&\left|2\sin(2\pi\alpha)\;\operatorname{Im}\left(i\int_{\pi}^{\infty}\frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right)\right|\\
&\qquad=2|\sin(2\pi\alpha)|\;\left|\int_{\pi}^{\infty}\frac{\left(e^t\cos\vartheta-1\right)t^{-(1+2\alpha)}}{|e^{(t-i\vartheta)}-1|^2}\,dt\right|\\
&\qquad\leq 2|\sin(2\pi\alpha)|\,\pi^{-(1+2\alpha)}\int_{\pi}^{\infty}\frac{e^t}{e^{2t}+1-2e^t\cos\vartheta}\,dt\\
&\qquad\leq 2|\sin(2\pi\alpha)|\,\pi^{-(1+2\alpha)}\int_{\pi}^{\infty}\frac{e^t}{(e^t-1)^2}\,dt\\
&\qquad\leq 4|\sin(\pi\alpha)|\,\pi^{-(1+2\alpha)}(e^{\pi}-1)^{-1},
\end{aligned}\tag{A.3}$$

where in the first inequality we used $|\vartheta|\leq\frac{\pi}{4}$, which implies $|e^t\cos\vartheta-1|\leq e^t$ for $t\geq\pi$. We now come to the first integral. Let us observe that, when $\alpha\in\mathbb{Z}$, it is a contour integral of a meromorphic function, therefore it may be computed via residues. In particular, when $\alpha\neq 0$, the only residue comes from $z=i\vartheta$, whose real part vanishes, as shown above. To get an estimate which is small for α close to 1, we set

$$\psi(\alpha,\vartheta)=\operatorname{Im}\left(\int_{|z|=\pi}\frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz\right),$$

so that we have

$$|\psi(\alpha,\vartheta)|=\left|\int_{\alpha}^{1}\frac{\partial\psi}{\partial\alpha}(s,\vartheta)ds\right|\leq\int_{\alpha}^{1}\left|\frac{\partial\psi}{\partial\alpha}(s,\vartheta)\right|ds\leq(1-\alpha)\sup_{\alpha\leq s\leq 1}\left|\frac{\partial\psi}{\partial\alpha}(s,\vartheta)\right|.\tag{A.4}$$

Moreover,

$$\begin{aligned}
|\partial_\alpha\psi(s,\vartheta)|&=\left|\operatorname{Im}\left(\int_{|z|=\pi}-2\log(-z)\frac{(-z)^{-(1+2s)}}{e^{(z-i\vartheta)}-1}\,dz\right)\right|\\
&=\left|\operatorname{Im}\left(\int_{0}^{2\pi}-2(\log\pi+i(t-\pi))\pi^{-2s}\frac{e^{-i(t-\pi)(1+2s)}}{e^{(\pi e^{it}-i\vartheta)}-1}i\,e^{it}\,dt\right)\right|\\
&\leq 4\pi^{1-2\alpha}(\log\pi+\pi)\left(\min_{t\in[0,2\pi]}|e^{(\pi e^{it}-i\vartheta)}-1|^2\right)^{-1/2},\quad \alpha\leq s\leq 1.
\end{aligned}$$

We now consider the two cases $\pi|\sin t| \le |\vartheta| + \pi/2$, and $\pi|\sin t| \ge |\vartheta| + \pi/2$.
If $\pi|\sin t| \le |\vartheta| + \pi/2$, $|\cos t| \ge (1 - (|\vartheta|/\pi + 1/2)^2)^{1/2}$, and

$$\begin{aligned} |e^{(\pi e^{it} - i\vartheta)} - 1|^2 &= e^{2\pi \cos t} + 1 - 2e^{\pi \cos t} \cos(\pi \sin t - \vartheta) \\ &\ge (e^{\pi \cos t} - 1)^2 \ge (1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}})^2. \end{aligned}$$

If $\pi|\sin t| \ge |\vartheta| + \pi/2$, $\frac{3}{2}\pi \ge |\pi \sin t - \vartheta| \ge |\pi \sin t| - |\vartheta| \ge \pi/2$, therefore $\cos(\pi \sin t - \vartheta) \le 0$, and

$$|e^{(e^{it} - i\vartheta)} - 1|^2 = e^{2\pi \cos t} + 1 - 2e^{\pi \cos t} \cos(\pi \sin t - \vartheta) \ge 1.$$

We have proved that

$$|\psi(\alpha, \vartheta)| \le 4(1 - \alpha)\pi^{1-2\alpha}(\log \pi + \pi)(1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}})^{-1}, \tag{A.5}$$

hence, by inequalities (A.3), (A.4), (A.5), and since $\alpha \ge 1/2$ implies $2(1 - \alpha) \le \sin \pi\alpha$,

$$\left| \mathrm{Im} \left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t - i\vartheta)} - 1} \, dt \right) \right| \le \frac{2\sin(\pi\alpha)}{\pi^{(1+2\alpha)}} \Big(\frac{2}{e^{\pi} - 1} + \frac{\pi^2(\log \pi + \pi)}{1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}}} \Big). \tag{A.6}$$

Then,

$$\left| \frac{\mathrm{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1}}{\sin(\pi\alpha)\Gamma(1 + 2\alpha)} + 1 \right| \le \frac{|\vartheta|^{2\alpha+1}}{2\pi \sin(\pi\alpha)} \left| \mathrm{Im} \left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t - i\vartheta)} - 1} \, dt \right) \right| \le \left(\frac{|\vartheta|}{\pi} \right)^{2\alpha+1} h\left(\frac{|\vartheta|}{\pi} \right),$$

where the function $h(r)$, $r \in (0, 1/4]$ is given by

$$h(r) = \frac{\pi(\log \pi + \pi)}{1 - \exp\left(-\pi \sqrt{1 - (r + 1/2)^2} \right)} + \frac{2}{\pi(e^{\pi} - 1)}.$$

Since h is increasing, it attains its maximum for $r = \frac{1}{4}$, where $h(\frac{1}{4}) < \frac{31}{2}$. Hence,

$$\left(1 - \frac{31}{2} \left(\frac{|\vartheta|}{\pi} \right)^{2\alpha+1} \right) \le \frac{-\mathrm{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1}}{\sin(\pi\alpha)\Gamma(1 + 2\alpha)} \le \left(1 + \frac{31}{2} \left(\frac{|\vartheta|}{\pi} \right)^{2\alpha+1} \right), \tag{A.7}$$

which implies (A.1). Now, since $|\vartheta| \le \pi/4$ and $\alpha \ge 1/2$, we get $\frac{31}{2}(|\vartheta|/\pi)^{2\alpha+1} \le \frac{31}{32}$, hence

$$\frac{1}{32} \sin(\pi\alpha)\Gamma(1 + 2\alpha) \le -\mathrm{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1} \le \frac{63}{32} \sin(\pi\alpha)\Gamma(1 + 2\alpha),$$

showing in particular that $-\mathrm{Ci}_{-2\alpha}(\vartheta)$ is strictly positive for $|\vartheta| \le \pi/4$.

We finally estimate $|\mathrm{Ci}_{-2\alpha}(\vartheta)|$ for $|\vartheta| \ge \frac{\pi}{4}$. We simply choose the contour γ as the circle of radius $\lambda|\vartheta|$ around the origin and the half lines $\sqrt{\lambda^2\vartheta^2 - \varepsilon^2} + t \pm i\varepsilon$, $t > 0$, for $\frac{1}{2} < \lambda < 1$. As for the first integral, we get

$$\left| \int_{|z| = \lambda|\vartheta|} \frac{(-z)^{-(1+2\alpha)}}{e^{(z - i\vartheta)} - 1} \, dz \right| \le 2\pi\lambda|\vartheta|(\lambda|\vartheta|)^{-(2\alpha+1)} \left(\min_{t \in [0, 2\pi]} |e^{\lambda|\vartheta| e^{it} - i\vartheta} - 1|^2 \right)^{-1/2}.$$

Since $|\lambda \sin t - 1| \, |\vartheta| \ge (1 - \lambda) \, |\vartheta|$, we get $\cos((\lambda \sin t - 1)\vartheta) \le \cos((1 - \lambda) \, \vartheta)$, therefore

$$\begin{aligned} |e^{\lambda|\vartheta| e^{it} - i\vartheta} - 1|^2 &= e^{2\lambda|\vartheta| \cos t} + 1 - 2e^{\lambda|\vartheta| \cos t} \cos((\lambda \sin t - 1)\vartheta) \\ &\ge e^{2\lambda|\vartheta| \cos t} + 1 - 2e^{\lambda|\vartheta| \cos t} \cos((1 - \lambda)\vartheta) \\ &\ge \sin^2[(1 - \lambda) \, |\vartheta|]. \end{aligned}$$

As a consequence,

$$\left|\int_{|z|=\lambda|\vartheta|} \frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz\right| \le (\lambda|\vartheta|)^{-(2\alpha+1)}\frac{2\pi\lambda|\vartheta|}{\sin[(1-\lambda)\,|\vartheta|]} \le (\lambda|\vartheta|)^{-(2\alpha+1)}\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}$$

The second integral is estimated, as above, by

$$\begin{aligned}\left|2i\sin(2\pi\alpha)\ \int_{\lambda|\vartheta|}^{\infty}\frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right| &\le 4\sin(\pi\alpha)(\lambda|\vartheta|)^{-(1+2\alpha)}\int_{\lambda|\vartheta|}^{\infty}\frac{e^t+1}{(e^t-1)^2}\,dt\\ &\le 8\sin(\pi\alpha)(\lambda|\vartheta|)^{-(1+2\alpha)}(e^{\lambda|\vartheta|}-1)^{-1} \le 8(\lambda|\vartheta|)^{-(1+2\alpha)}(e^{\frac{\lambda\pi}{4}}-1)^{-1}\end{aligned}$$

whence

$$|\operatorname{Ci}_{-2\alpha}(\vartheta)| \le \frac{\Gamma(1+2\alpha)}{2\pi}(\lambda|\vartheta|)^{-(2\alpha+1)}\left(\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}+\frac{8}{e^{\frac{\lambda\pi}{4}}-1}\right).$$

Finally, for any $\lambda\in(0,1)$,

$$\begin{aligned}|\operatorname{Ci}_{-2\alpha}(\vartheta)|\,|\vartheta|^{2\alpha+1} &\le \frac{\Gamma(1+2\alpha)}{2\pi}\lambda^{-(2\alpha+1)}\left(\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}+\frac{8}{e^{\frac{\lambda\pi}{4}}-1}\right)\\ &\le \lambda^{-3}\left(\frac{2\lambda\pi}{\sin[(1-\lambda)\pi]}+\frac{8}{\pi(e^{\frac{\lambda\pi}{4}}-1)}\right).\end{aligned}$$

Suitably choosing λ, one gets $|\operatorname{Ci}_{-2\alpha}(\vartheta)|\,|\vartheta|^{2\alpha+1}<23$. □

Proposition A.2. *Let f be an even $\mathcal{C}^\infty$ function on $\mathbb{T}$ vanishing in 0. Then, for $\alpha\in(0,1)$,*

$$\int_{-\pi}^{\pi}\operatorname{Ci}_{-2\alpha}(\vartheta)f(\vartheta)\,d\vartheta=\pi(\{k^{2\alpha}\},\{f_k\})_{\ell^2(\mathbb{N})},$$

where $f_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(k\vartheta)f(\vartheta)\,d\vartheta$.

Proof. Let us set $\operatorname{Ci}(s,\rho,\vartheta):=\operatorname{Re}(\operatorname{Li}_s(\rho e^{i\vartheta}))$. Then, reasoning as in the proof of Lemma A.1, it is not difficult to show that

$$\sup_{\rho\in[0,1],|\vartheta|\le\pi}|\operatorname{Ci}(-2\alpha,\rho,\vartheta)|\vartheta|^{2\alpha+1}|<\infty. \tag{A.8}$$

We may assume that f is real valued, namely $f(\vartheta)=\sum_{k\ge0}f_k\cos(k\vartheta)$, with $f_k\in\mathbb{R}$. The other properties of f amount to f_k rapidly decreasing and $\sum_k f_k=0$. Since f is even, it has indeed a zero of order 2 in $\vartheta=0$, hence, by (A.8), $\operatorname{Ci}(-2\alpha,\rho,\vartheta)f(\vartheta)$ is uniformly $L^1(\vartheta)$, for $\rho\in[0,1]$. By dominated convergence,

$$\begin{aligned}\int_{-\pi}^{\pi}\operatorname{Ci}_{-2\alpha}(\vartheta)f(\vartheta)\,d\vartheta &= \lim_{\rho\to1}\int_{-\pi}^{\pi}\operatorname{Ci}(-2\alpha,\rho,\vartheta)f(\vartheta)\,d\vartheta\\ &= \lim_{\rho\to1}\operatorname{Re}\left(-i\int_{|z|=\rho}\operatorname{Li}_{-2\alpha}(z)\left(\frac12\sum_{k=0}^{\infty}f_k(\rho^{-k}z^k+\rho^kz^{-k})\right)\frac{dz}{z}\right)\\ &= \lim_{\rho\to1}\operatorname{Re}\left(\frac{-i}{2}\sum_{k=0}^{\infty}f_k\rho^k\sum_{n=1}^{\infty}n^{2\alpha}\int_{|z|=\rho}z^{n-k}\frac{dz}{z}\right)\\ &= \pi\lim_{\rho\to1}\sum_{k=0}^{\infty}\rho^kf_k\ k^{2\alpha}=\pi\sum_{k=0}^{\infty}f_k\ k^{2\alpha}.\end{aligned}$$

□

Proposition A.3. *Let* $\alpha \in (0,1)$*, and consider the seminorm* $p_\alpha(f)$*,* $f \in C(\mathbb{T})$*, given by*

$$p_\alpha(f)^2 = \frac{1}{2\pi} \sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy < +\infty,$$

where $\varphi_\alpha(t) = -2\pi \operatorname{Ci}_{-2\alpha}(t)$*, and denote by* $\|f\|_{0,\alpha} = \sup_{x,y} \dfrac{|f(x)-f(y)|}{d(x,y)^\alpha}$ *the Hölder seminorm. Then,*

(i) $\forall \varepsilon > 0$*,* $p_\alpha(f) \le c_\varepsilon \|f\|_{0,\alpha+\varepsilon}$*, where* $c_\varepsilon = \frac{1}{\sqrt{\varepsilon}} \left(\frac{\pi}{4}\right)^\varepsilon \left(4 + 23(4^{2\varepsilon}-1)\right)^{1/2}$*,*

(ii) for $\alpha \ge \frac{1}{2}$*,* $\tilde{c}_\alpha \|f\|_{0,\alpha} \le p_\alpha(f)$*, where* $\tilde{c}_\alpha = \frac{\sqrt{3\sin(\pi\alpha)}}{16\sqrt{2}}$*.*

Proof.

(i) If f is $(\alpha+\varepsilon)$-Hölder then

$$\begin{aligned}\sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy &\le \|f\|^2_{0,\alpha+\varepsilon} \sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y) d(x,y)^{2(\alpha+\varepsilon)}\, dy \\ &= 2\|f\|^2_{0,\alpha+\varepsilon} \int_0^\pi \varphi_\alpha(t) t^{2(\alpha+\varepsilon)}\, dt\,.\end{aligned}$$

Making use of the estimates in Lemma A.1, one gets

$$\begin{aligned}\int_0^\pi \varphi_\alpha(t) t^{2(\alpha+\varepsilon)}\, dt &= \int_0^{\pi/4} \varphi_\alpha(t) t^{2(\alpha+\varepsilon)}\, dt + \int_{\pi/4}^\pi \varphi_\alpha(t) t^{2(\alpha+\varepsilon)}\, dt \\ &\le 2\pi \frac{63}{32} \sin(\pi\alpha)\Gamma(1+2\alpha) \int_0^{\pi/4} t^{2\varepsilon-1}\, dt + 2\pi \cdot 23 \int_{\pi/4}^\pi t^{2\varepsilon-1}\, dt \\ &\le 4\pi \sin(\pi\alpha)\Gamma(1+2\alpha) \frac{(\pi/4)^{2\varepsilon}}{2\varepsilon} + 46\pi \frac{\pi^{2\varepsilon} - (\pi/4)^{2\varepsilon}}{2\varepsilon} \\ &\le \frac{\pi}{\varepsilon} \left(\frac{\pi}{4}\right)^{2\varepsilon} \left(4 + 23(4^{2\varepsilon}-1)\right).\end{aligned}$$

(ii) Assume $p_\alpha(f) < \infty$, let $x, y \in \mathbb{T}$, and denote by σ the distance between x and y, and by I_σ the arc of length σ with end-points x and y. By Lemma A.1, $\varphi_\alpha(t) > 0$ for $|t| \le \frac{\pi}{4}$.

Then, for $\sigma \le \frac{\pi}{4}$,

$$\begin{aligned}\left| f(x) - \frac{1}{\sigma} \int_{I_\sigma} f(z)\, dz \right| &\le \frac{1}{\sigma} \int_{I_\sigma} |f(x) - f(z)|\, dz \\ &= \sigma^{-1} \int_{I_\sigma} |f(x)-f(z)| \varphi_\alpha(x-z)^{1/2} \varphi_\alpha(x-z)^{-1/2}\, dz \\ &\le \sigma^{-1} p_\alpha(f)\sqrt{2\pi} \cdot \left(\int_0^\sigma \frac{1}{\varphi_\alpha(t) t^{1+2\alpha}} t^{1+2\alpha}\, dt \right)^{\frac{1}{2}} \\ &\le (2+2\alpha)^{-1/2} \left(\sup_{0<t\le\frac{\pi}{4}} \frac{2\pi}{\varphi_\alpha(t) t^{1+2\alpha}} \right)^{1/2} \sigma^\alpha\, p_\alpha(f) \\ &\le (2+2\alpha)^{-1/2} \left(\frac{32}{\sin(\pi\alpha)\Gamma(1+2\alpha)} \right)^{1/2} \sigma^\alpha\, p_\alpha(f) \\ &\le 4\big((1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)\big)^{-1/2} \sigma^\alpha p_\alpha(f).\end{aligned}$$

Therefore, using the triangle inequality we obtain, for all x, y, such that $d(x,y) \leq \frac{\pi}{4}$,

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq \frac{8}{\sqrt{(1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)}}\, p_\alpha(f).$$

A direct computation then shows

$$\|f\|_{0,\alpha} \leq \frac{32 \cdot 4^{-\alpha}}{\sqrt{(1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)}}\, p_\alpha(f) \leq \frac{16\sqrt{2}}{\sqrt{3\sin(\pi\alpha)}}\, p_\alpha(f).$$

□

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