A unified approach to invariants of plane elasticity tensors

Sandra Forte · Maurizio Vianello

Abstract The action of the orthogonal group $O(2)$ on the space of plane elasticity tensors has been the subject of some recent investigations. It is shown here that the approach based on the "harmonic decomposition" technique, which is also used in a three-dimensional setting, gives a unified perspective on this issue. We construct explicit relationships between invariants and quantities derived from such an approach and what was found earlier by Tsai and Pagano and, more recently, through the "polar method" and the use of complex variables.

Keywords Plane linear elasticity · Anisotropy · Invariants

1 Introduction

Recent years have seen some renewed interest about the material symmetries and orthogonal invariants of plane elasticity tensors, which linearly map the strain **E** into the stress **T** for infinitesimal deformations of elastic plane bodies. Such renewed interest is mostly concentrated on the class of *anisotropic* tensors, which are investigated in respect both to their subdivision into symmetry classes and to the set of functions which can be constructed from their components and shown to be invariants under the action of the plane orthogonal group.

As a significant and interesting example of this recent literature we mention some new articles by de Saxcé and Vallée [3], by Vannucci [8, 9] and by Vannucci and Verchery [11] (and quite a few other references mentioned therein).

This topic was also investigated in some detail in earlier work, as in the important contribution by Blinowski et al. [2] and the subsequent and strictly related article by Vianello [13], which are unfortunately missing among the references listed by de Saxcé and Vallée in [3].[1] In particular, as we shall see, a part of the treatment and some results found in [3] were anticipated in [2] and [13].

The goal of this research is to put some of this literature into a proper perspective and, above all, to show that most results scattered here and there can be shown to be nothing but special cases of what is obtained through the application of the powerful method of harmonic decomposition for elasticity tensors, which, in a three-dimensional context, has

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[1] Thus, we cannot fully agree with Ref. [3, Sect. 1] where it is stated that: "Curiously, not much attention has been paid in the literature to it, except Vannucci et al. [10], Vincenti et al. [14], Vannucci [8] and Vannucci et al. [11], following the polar method proposed by Verchery [12] using SU(2)."

been successfully exploited by Backus [1] and later on, with the additional help of Cartan decomposition, by some other Authors (see additional references cited in [4]).

In particular, we show that the harmonic decomposition of a plane elasticity tensor, as suggested and presented in [2, 13], gives directly most of the quantities which were found, in other disguise, in some recent and less recent articles by means of alternative and, we believe, less transparent approaches.

We first summarize the basic ideas behind the harmonic decomposition of a plane elasticity tensor $\mathbb{C}$, which makes possible to split it into an isotropic part, depending on two (well-known) scalar invariants, and two completely symmetric and traceless tensors $\mathbf{H}$ (second-order) and $\mathbb{K}$ (fourth-order), known as the harmonic components of $\mathbb{C}$, belonging to the two-dimensional spaces $\mathbf{H}$rm and $\mathbb{H}$rm. Next, we show that the action of the plane orthogonal group $O(2)$ on both $\mathbf{H}$ and $\mathbb{K}$ can be described through rotations and a reflection acting on such spaces. This part of our presentation is included here only for completeness, and is borrowed with some modification from earlier work [13]. Indeed, such results originated outside the field of continuum mechanics (see, e.g., [5]) and, basically, are just applications of group representation theory carried over from harmonic polynomials to harmonic tensors.

The central part of this research comes next, when, first, it is shown that a complete set of invariants for $\mathbb{C}$ can be deduced quite easily and naturally through an explicit geometric view of the action of $O(2)$ on $\mathbf{H}$rm and $\mathbb{H}$rm and, second, that from such a straightforward approach we can find, as already shown in [13], most, and perhaps all, of the relevant quantities deduced elsewhere, as in [3, 8, 9, 11].

Indeed, it is possible to establish a precise connection between quantities which appear in different contexts under different names. We shall go into some, but not too many, details and shall be content of comparing our deduction with what can be read in the classical treatise by Tsai and Hahn [7], in the articles about the "polar method" as introduced by Verchery [12], and in the recent and interesting contribution by de Saxcé and Vallée [3], which has a strict relation with what was found in [13] and elsewhere.

2 Plane elasticity tensors

We use quite standard notation: small $(\mathbf{a}, \mathbf{b}, \ldots)$ and capital $(\mathbf{A}, \mathbf{B}, \ldots)$ boldface latin letters denote vectors and tensors of a *two*-dimensional Euclidean space $\mathcal{V}$, while a blackboard bold font $(\mathbb{S}, \mathbb{C}, \mathbb{H}, \ldots)$ is used for fourth order tensors, elements of the space $\mathbb{L}$in. An orthonormal basis for $\mathcal{V}$ is $\mathcal{B} = \{\mathbf{e}_i\}$ $(i = 1, 2)$, while $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is a basis for Lin, the space of second-order tensors, where the symbol $\otimes$ denotes the tensor product. The set of orthogonal tensors $\mathbf{Q}$ is $O(2)$ and the subgroup of rotations (orthogonal tensors with determinant equal to one) is written as $SO(2)$. Sym is the set of symmetric second-order tensors (a three-dimensional subspace of Lin) while $\mathbb{S}$ym is the subspace of $\mathbb{L}$in of all tensors which are symmetric with respect to any permutation of the indexes of their cartesian components.

Cartesian components of a second- and fourth-order tensor $\mathbf{T}$ and $\mathbb{C}$ are given by

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j, \quad C_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbb{C}[\mathbf{e}_k \otimes \mathbf{e}_l],$$

where the same symbol "$\cdot$" is used for the inner product between vectors and between tensors.

When needed, we may look at a second- or fourth-order tensor as multilinear maps on the space of vectors, through

$$\mathbf{T}[\mathbf{a}, \mathbf{b}] = T_{ij} a_i b_j,$$
$$\mathbb{C}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}] = \mathbf{a} \otimes \mathbf{b} \cdot \mathbf{C}[\mathbf{c} \otimes \mathbf{d}] = C_{ijkl} a_i b_j c_k d_l.$$

By $\mathbf{I}$ we denote the identity, with components δ_{ij}, and by $\mathbf{Q}(\theta)$ the rotation such that

$$\mathbf{Q}\mathbf{e}_1 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \quad \mathbf{Q}\mathbf{e}_2 = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2. \tag{1}$$

Finally, $\tilde{\mathbf{Q}}$ is the reflection with respect to $\mathbf{e}_1$:

$$\tilde{\mathbf{Q}}\mathbf{e}_1 = \mathbf{e}_1, \quad \tilde{\mathbf{Q}}\mathbf{e}_2 = -\mathbf{e}_2.$$

The group $O(2)$ is generated by $SO(2)$ and $\tilde{\mathbf{Q}}$ and is the disjoint union of two sets. Using the matrix representation of its elements with respect to the fixed basis $\mathcal{B}$:

$$O(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$$

with $0 \leq \theta < 2\pi$.

We denote by $\mathbb{E}$la the space of (plane) elasticity tensors, which we see as symmetric linear maps from Sym into itself. Thus, an elasticity tensor is a linear map $\mathbb{C}$ from the set of infinitesimal strains $\mathbf{E} \in$ Sym into the space of stress tensors $\mathbf{T} \in$ Sym:

$$T_{ij} = C_{ijkl}E_{kl}, \quad \mathbf{T} = \mathbb{C}[\mathbf{E}] \tag{2}$$

for which it is required that

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}.$$

Similar (left) actions of a subgroup G of $O(2)$ on Lin and $\mathbb{L}$in are easily constructed through:

$$\begin{aligned} (\mathbf{Q} * \mathbf{A})_{pq} &= Q_{pi}Q_{qj}A_{ij} = (\mathbf{QAQ}^T)_{pq}, \\ (\mathbf{Q} * \mathbb{C})_{pqrs} &= Q_{pi}Q_{qj}Q_{rk}Q_{sl}C_{ijkl}, \end{aligned} \tag{3}$$

for each $\mathbf{Q} \in G$, $\mathbf{A} \in$ Lin and $\mathbb{C} \in \mathbb{L}$in.

The symmetry group $g(\mathbb{C})$ of an elasticity tensor $\mathbb{C}$ is defined as

$$g(\mathbb{C}) = \{\mathbf{Q} \in O(2) : \mathbf{Q} * \mathbb{C} = \mathbb{C}\},$$

the collection of all plane orthogonal transformation which leave it fixed under the action given in (3) and two such tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ are said to belong to the same symmetry class if $g(\mathbb{C}_1)$ and $g(\mathbb{C}_2)$ are orthogonally conjugate (for a through discussion see [4]). The classification of elasticity tensors into symmetry classes is a classical problem of linear elasticity.

In plane elasticity, a convenient representation of an elasticity tensor $\mathbb{C}$ is given by a symmetric 3×3 matrix (the so-called Voigt–Kelvin representation)

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

where

$$\begin{aligned} &c_{11} = C_{1111}, \quad c_{12} = C_{1122}, \quad c_{13} = \sqrt{2}C_{1112}, \\ &c_{22} = C_{2222}, \quad c_{23} = \sqrt{2}C_{2212}, \quad c_{33} = 2C_{1212}. \end{aligned}$$

Representing the stress tensor by $t \in \mathcal{R}^3$, with $t_1 = T_{11}$, $t_2 = T_{22}$, $t_3 = \sqrt{2}T_{12}$ and the infinitesimal strain by $e \in \mathcal{R}^3$, with $e_1 = E_{11}$, $e_2 = E_{22}$, $e_3 = \sqrt{2}E_{12}$ the constitutive law (2) can be rewritten into the matrix relation

$$t = ce. \tag{4}$$

3 Harmonic decomposition and a geometric view of isotropic invariants

Tensors (of any order) which are completely symmetric and traceless are sometime called "harmonic", in view of an isomorphism with the set of harmonic polynomials. Indeed, to any second-order tensor $\mathbf{H}$ which is symmetric $(H_{ij} = H_{ji})$ and traceless $(H_{ii} = 0)$, we can associate, in a one-to-one correspondence, a second-degree homogeneous polynomial $\psi(x, y)$ in two variables

$$\psi(x, y) = \mathbf{H}[\mathbf{r}, \mathbf{r}] \quad (\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2),$$

which can be easily shown to be harmonic: $\Delta\psi = 0$.

Similarly, let $\mathbb{H}$rm $\subset \mathbb{S}$ym be the subspace of symmetric fourth-order tensors which are traceless:

$$\mathbb{H}\text{rm} = \{\mathbb{H} \in \mathbb{S}\text{ym} : H_{iijk} = 0\}.$$

To each $\mathbb{H} \in \mathbb{H}$rm we associate, in a one-to-one correspondence, an element $\psi(x, y)$ of the space of fourth-degree homogeneous polynomials in two variables, defined as

$$\psi(x, y) = \mathbb{H}[\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}] \quad (\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2),$$

which is also harmonic: $\Delta\psi = 0$. Thus, since both $\mathbf{H}$rm and $\mathbb{H}$rm are in this sense isomorphic to spaces of homogeneous harmonic polynomials, then, for such reason, their elements are frequently denoted as "harmonic tensors".

Harmonic tensors play a fundamental role in the decomposition of $\mathbb{E}$la into a direct sum of subspaces which are *irreducible* under the action of $O(2)$. In fact, there is an $O(2)$-invariant isomorphism which maps $\mathbb{C} \in \mathbb{E}$la into a quadruplet $(\lambda, \mu, \mathbf{H}, \mathbb{K})$, where λ and μ are scalars, while $\mathbf{H}$ and $\mathbb{K}$ belong to $\mathbf{H}$rm and $\mathbb{H}$rm, respectively. The construction of such an isomorphism, which is called *harmonic decomposition*, is outlined in great detail in [2, 13] and, for the reader's convenience, we only present here the final result.

For a plane elasticity tensor $\mathbb{C} \in \mathbb{E}$la let λ and μ be defined as

$$\lambda = \frac{3}{8}C_{ppqq} - \frac{1}{4}C_{pqpq}, \quad \mu = \frac{1}{4}C_{pqpq} - \frac{1}{8}C_{ppqq},$$

and let $\mathbf{H} \in \mathbf{H}$rm be given by

$$H_{ik} = \left[2C_{ipkp} - C_{pqpq}\,\delta_{ik}\right]/12. \tag{5}$$

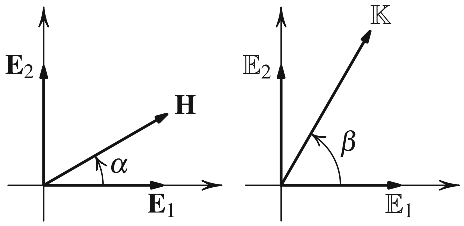

Fig. 1 Orthonormal bases for **H**rm and $\mathbb{H}$rm

Finally, for $\mathbb{K} \in \mathbb{H}$rm we take

$$K_{ijkl} = C_{ijkl} - [\delta_{ij}C_{kplp} + \delta_{kl}C_{ipjp} + \delta_{ik}C_{lpjp} + \delta_{lj}C_{ipkp} + \delta_{il}C_{jpkp} + \delta_{jk}C_{iplp}]/6 + C_{pqpq}(5\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})/12 - C_{ppqq}(3\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})/8. \tag{6}$$

Viceversa, the elasticity tensor $\mathbb{C}$ which corresponds to a given quadruplet $(\lambda, \mu, \mathbf{H}, \mathbb{K})$, is

$$C_{ijkl} = K_{ijkl} + [\delta_{ij}H_{kl} + H_{ij}\delta_{kl} + \delta_{ik}H_{lj} + H_{ik}\delta_{lj} + \delta_{il}H_{jk} + H_{il}\delta_{jk}]/6 + \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}).$$

The harmonic decomposition of an elasticity tensor $\mathbb{C}$ is described through the compact notation

$$\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K})$$

and it is said to be $O(2)$-invariant because, for each $\mathbb{C} \in \mathbb{E}$la and $\mathbf{Q} \in O(2)$,

$$\mathbf{Q} * \mathbb{C} = (\lambda, \mu, \mathbf{Q} * \mathbf{H}, \mathbf{Q} * \mathbb{K}), \tag{7}$$

while it is also called *irreducible* because **H**rm and $\mathbb{H}$rm do not have *proper* subspaces which are invariant under the action of $O(2)$.

In three dimensional elasticity harmonic (and Cartan) decomposition of elasticity tensors have been applied to reach different objectives in the study of material symmetry properties (see, e.g. [1, 4]). An application of harmonic decomposition to two dimensional elasticity is presented in [13], a publication which is strongly related to the present research.

Both spaces **H**rm and $\mathbb{H}$rm are two-dimensional. Indeed, an element of Sym has three independent components and the condition that it be traceless adds a linear restriction: thus, we are left with a space **H**rm of dimension $2 = 3 - 1$. Similarly, an element of $\mathbb{S}$ym has 5 independent components and the condition that puts its trace equal to zero gives 3 linear restrictions, leaving us with a space $\mathbb{H}$rm of dimension $2 = 5 - 3$.

It is important to define appropriate bases for both such spaces. For **H**rm we choose

$$\mathbf{E}_1 = \frac{\sqrt{2}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \quad \mathbf{E}_2 = \frac{\sqrt{2}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{8}$$

Similarly, a basis for $\mathbb{H}$rm can be constructed from $\mathbf{E}_1$ and $\mathbf{E}_2$ as

$$\mathbb{E}_1 = \frac{\sqrt{2}}{2}(\mathbf{E}_1 \otimes \mathbf{E}_1 - \mathbf{E}_2 \otimes \mathbf{E}_2), \quad \mathbb{E}_2 = \frac{\sqrt{2}}{2}(\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1). \tag{9}$$

More explicitly, by substitution of (8) into (9) we find that

$$\mathbb{E}_1 = \frac{\sqrt{2}}{4}(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1),$$

and

$$\mathbb{E}_2 = \frac{\sqrt{2}}{4}(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2).$$

(Fig. 1 gives a graphical representation of such bases for **H**rm and $\mathbb{H}$rm, and introduces angles α and β which are of importance for later developments).

The components of **H** and $\mathbb{K}$ in the two dimensional spaces **H**rm and $\mathbb{H}$rm, defined through

$$\mathbf{H} = H_1\mathbf{E}_1 + H_2\mathbf{E}_2, \quad \mathbb{K} = K_1\mathbb{E}_1 + K_2\mathbb{E}_2, \tag{10}$$

can be obtained as

$$H_1 = \mathbf{H} \cdot \mathbf{E}_1 = \frac{\sqrt{2}}{2}(c_{11} - c_{22}), \quad H_2 = \mathbf{H} \cdot \mathbf{E}_2 = c_{13} + c_{23}, \tag{11}$$

and

$$K_1 = \mathbb{K} \cdot \mathbb{E}_1 = \frac{\sqrt{2}}{4}[(c_{11} + c_{22}) - 2(c_{12} + c_{33})], \quad K_2 = \mathbb{K} \cdot \mathbb{E}_2 = c_{13} - c_{23}. \tag{12}$$

The angle α, $0 \leq \alpha < 2\pi$, between $\mathbf{H}$ and the horizontal axis directed as $\mathbf{E}_1$ in $\mathbf{H}$rm, and the angle β, $0 \leq \beta < 2\pi$, between $\mathbb{K}$ and the horizontal axis directed as $\mathbb{E}_1$ in $\mathbb{H}$rm, are characterized through the trigonometric functions

$$\cos\alpha = \frac{1}{|\mathbf{H}|}H_1 = \frac{\sqrt{2}}{2|\mathbf{H}|}(c_{11} - c_{22}),$$
$$\sin\alpha = \frac{1}{|\mathbf{H}|}H_2 = \frac{1}{|\mathbf{H}|}(c_{13} + c_{23}),$$
$$\cos\beta = \frac{1}{|\mathbb{K}|}K_1 = \frac{\sqrt{2}}{4|\mathbb{K}|}[(c_{11} + c_{22}) - 2(c_{12} + c_{33})],$$
$$\sin\beta = \frac{1}{|\mathbb{K}|}K_2 = \frac{1}{|\mathbb{K}|}(c_{13} - c_{23}).$$

(see Fig. 1).

We notice that $\mathbb{K}$ is a traceless, completely symmetric, fourth order tensor, and it can be seen as a map from $\mathbf{H}$rm into itself. Thus, since

$$\mathbb{K} = K_1\mathbb{E}_1 + K_2\mathbb{E}_2 = K_1\frac{\sqrt{2}}{2}(\mathbf{E}_1 \otimes \mathbf{E}_1 - \mathbf{E}_2 \otimes \mathbf{E}_2) + K_2\frac{\sqrt{2}}{2}(\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1),$$

the 2×2 matrix representation $\tilde{K}$ of such map $\mathbb{K}$ is given by

$$\tilde{K} = \begin{bmatrix} \frac{\sqrt{2}}{2}K_1 & \frac{\sqrt{2}}{2}K_2 \\ \frac{\sqrt{2}}{2}K_2 & -\frac{\sqrt{2}}{2}K_1 \end{bmatrix}$$

so that

$$\tilde{K}_{11} = \frac{1}{4}[(c_{11} + c_{22}) - 2(c_{12} + c_{33})],$$
$$\tilde{K}_{12} = \frac{\sqrt{2}}{2}(c_{13} - c_{23}),$$
$$\tilde{K}_{21} = \frac{\sqrt{2}}{2}(c_{13} - c_{23}),$$
$$\tilde{K}_{22} = -\frac{1}{4}[(c_{11} + c_{22}) - 2(c_{12} + c_{33})].$$

Later on, in Sect. 6, we shall find that the 2×2 matrix $\tilde{K}$ is coincident with a quantity derived by Saxcé and Vallée [3].

The important point is to understand how the action of $\mathbf{Q}(\theta)$ transforms both spaces of harmonic tensors $\mathbf{H}$rm and $\mathbb{H}$rm, so that we can have an immediate and geometrical straightforward interpretation of the orbits of $\mathbb{C}$.

The rotation $\mathbf{Q}(\theta)$ acting as in (1) transforms $\mathbf{E}_1$ into $\mathbf{E}_1'$ given by

$$\begin{aligned}\mathbf{E}_1' &= \mathbf{Q}(\theta) * \mathbf{E}_1 \\ &= \frac{\sqrt{2}}{2}(\mathbf{Q}(\theta)\mathbf{e}_1 \otimes \mathbf{Q}(\theta)\mathbf{e}_1 - \mathbf{Q}(\theta)\mathbf{e}_2 \otimes \mathbf{Q}(\theta)\mathbf{e}_2) \\ &= \frac{\sqrt{2}}{2}[(\cos^2\theta - \sin^2\theta)(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) \\ &\quad + 2\sin\theta\cos\theta(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)] \\ &= \cos(2\theta)\mathbf{E}_1 + \sin(2\theta)\mathbf{E}_2\end{aligned} \tag{13}$$

and, similarly, $\mathbf{E}_2$ into $\mathbf{E}_2'$, given by

$$\begin{aligned}\mathbf{E}_2' &= \mathbf{Q}(\theta) * \mathbf{E}_2 \\ &= \frac{\sqrt{2}}{2}(\mathbf{Q}(\theta)\mathbf{e}_1 \otimes \mathbf{Q}(\theta)\mathbf{e}_2 + \mathbf{Q}(\theta)\mathbf{e}_2 \otimes \mathbf{Q}(\theta)\mathbf{e}_1) \\ &= \frac{\sqrt{2}}{2}[-2\sin\theta\cos\theta(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) \\ &\quad + (\cos^2\theta - \sin^2\theta)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)] \\ &= -\sin(2\theta)\mathbf{E}_1 + \cos(2\theta)\mathbf{E}_2.\end{aligned} \tag{14}$$

Thus, $\mathbf{Q}(\theta)$ acts as a rotation of 2θ on $\mathbf{H}$rm:

$$\begin{cases} \mathbf{E}_1' = \cos(2\theta)\mathbf{E}_1 + \sin(2\theta)\mathbf{E}_2 \\ \mathbf{E}_2' = -\sin(2\theta)\mathbf{E}_1 + \cos(2\theta)\mathbf{E}_2 \end{cases} \tag{15}$$

In view of (9) and (15), and in full analogy with (13) and (14), we conclude that $\mathbf{Q}(\theta)$ acts on $\mathbb{H}$rm as a rotation of 4θ

$$\begin{cases} \mathbb{E}_1' = \mathbf{Q}(\theta) * \mathbb{E}_1 = \cos(4\theta)\mathbb{E}_1 + \sin(4\theta)\mathbb{E}_2 \\ \mathbb{E}_2' = \mathbf{Q}(\theta) * \mathbb{E}_2 = -\sin(4\theta)\mathbb{E}_1 + \cos(4\theta)\mathbb{E}_2 \end{cases}$$

Figure 2 shows such an action of $\mathbf{Q}(\theta)$ on both $\mathbf{H}$rm and $\mathbb{H}$rm). Notice that, as θ varies in $[0, 2\pi)$, $\mathbf{Q}(\theta) * \mathbb{C}$ describes the orbit of $\mathbb{C}$ twice, so, without loss of generality, we can limit the interval of variation of θ to $[0, \pi)$.

In order to complete the picture, we are left to consider of the action of $\tilde{\mathbf{Q}}$. Since $\tilde{\mathbf{Q}}$ changes the sign of $\mathbf{e}_2$, in view of (8) and (9)

$$\begin{cases} \tilde{\mathbf{Q}} * \mathbf{E}_1 = \mathbf{E}_1 \\ \tilde{\mathbf{Q}} * \mathbf{E}_2 = -\mathbf{E}_2 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\mathbf{Q}} * \mathbb{E}_1 = \mathbb{E}_1 \\ \tilde{\mathbf{Q}} * \mathbb{E}_2 = -\mathbb{E}_2 \end{cases}$$

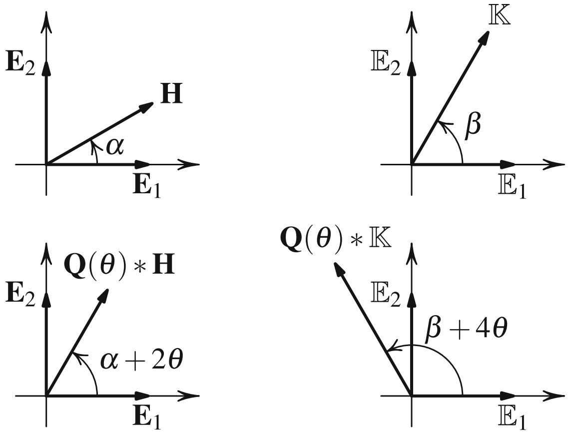

Fig. 2 The action of $\mathbf{Q}(\theta)$ on $\mathbf{H}$rm and $\mathbb{H}$rm

Hence $\tilde{\mathbf{Q}}$ acts in both planes $\mathbf{H}$rm and $\mathbb{H}$rm as a reflection with respect to the "horizontal" axes spanned by $\mathbf{E}_1$ and $\mathbb{E}_1$.

In a sense, we might say that Fig. 2 provides a graphical representation of all the basic ideas behind the approach developed here, and should make all further details in the discussion a straightforward consequence.

How can we construct a set of invariants which characterize the orbits of a given tensor $\mathbb{C}$? It was shown in [13] that through the geometric interpretation of the action of the orthogonal group $O(2)$ on $\mathbb{E}$la (briefly outlined here above) it is almost straightforward to obtain a polynomial integrity basis.

Two linear invariants are deduced as a straightforward consequence of (7):

$$I_1 = \lambda = \frac{1}{8}(c_{11} + c_{22} + 6c_{12} - 2c_{33}),$$
$$I_2 = \mu = \frac{1}{8}(c_{11} + c_{22} - 2c_{12} + 2c_{33}). \tag{16}$$

We also immediately notice that the magnitude of the "vectors" $\mathbf{H}$ and $\mathbb{K}$ respectively, does not change under the action of $O(2)$. Therefore, there are two additional invariants, which are quadratic polynomial functions of the components of $\mathbb{C}$

$$I_3 = |\mathbf{H}|^2 = H_1^2 + H_2^2 = \frac{1}{2}(c_{11} - c_{22})^2 + (c_{13} + c_{23})^2,$$
$$I_4 = |\mathbb{K}|^2 = K_1^2 + K_2^2 = \frac{1}{8}[(c_{11} + c_{22}) - 2(c_{12} + c_{33})]^2 + (c_{13} - c_{23})^2.$$

Since a rotation $\mathbf{Q}(\theta)$ acts as a rotation of 2θ on $\mathbf{H}$rm and as a rotation of 4θ on $\mathbb{H}$rm, the angle $\gamma = \beta - 2\alpha$ is not changed by the action of $SO(2)$. Indeed

$$\gamma = \beta - 2\alpha \mapsto \beta + 4\theta - 2(\alpha + 2\theta) = \beta - 2\alpha = \gamma$$

(see Fig. 2).

Since $\tilde{\mathbf{Q}}$ acts in both planes as a reflection with respect to the "horizontal" axes directed as $\mathbf{e}_1$ and $\mathbb{E}_1$ from Fig. 2 $\beta - 2\alpha$ becomes $-(\beta - 2\alpha)$. The cosine is an even function of its argument and gives a further invariant under the action of $O(2)$

$$\begin{aligned} I_5 &= |\mathbf{H}|^2 |\mathbb{K}| \cos(\beta - 2\alpha) \\ &= |\mathbf{H}|^2 |\mathbb{K}| [\cos\beta(\cos^2\alpha - \sin^2\alpha) + 2\sin\beta\cos\alpha\sin\alpha] \\ &= K_1(H_1^2 - H_2^2) + 2K_2H_1H_2 \\ &= \frac{\sqrt{2}}{8}\Big\{[(c_{11} + c_{22}) - 2(c_{12} + c_{33})] \\ &\quad \left[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2\right] \\ &\quad + 8(c_{13}^2 - c_{23}^2)(c_{11} - c_{22})\Big\}, \end{aligned}$$

which is a cubic polynomial function of the components of $\mathbb{C}$.

Since $\sin(\beta - 2\alpha)$ changes its sign under the action of $\tilde{\mathbf{Q}}$, but it is left unchanged by $SO(2)$, we can introduce an $SO(2)$ invariant I_6

$$\begin{aligned} I_6 &= |\mathbf{H}|^2 |\mathbb{K}| \sin(\beta - 2\alpha) \\ &= |\mathbf{H}|^2 |\mathbb{K}| \big(\sin\beta(\cos^2\alpha - \sin^2\alpha) - 2\cos\beta\cos\alpha\sin\alpha\big) \\ &= K_2(H_1^2 - H_2^2) - 2K_1H_1H_2 \\ &= \frac{1}{2}\Big\{(c_{13} - c_{23})\left[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2\right] \\ &\quad - [(c_{11} + c_{22}) - 2(c_{12} + c_{33})](c_{11} - c_{22})(c_{13} + c_{23})\Big\}. \end{aligned}$$

We notice that I_6 is linked to I_5 by a syzygy, consequence of the well known trigonometric relation $\sin^2\theta + \cos^2\theta = 1$.

We could reach the same result, if we consider $\mathbb{K}$ as a linear map in Sym: $\mathbf{H} \mapsto \mathbb{K}[\mathbf{H}]$

$$\begin{aligned} \mathbb{K}[\mathbf{H}] &= \mathbb{K}[H_1\mathbf{E}_1 + H_2\mathbf{E}_2] \\ &= (\sqrt{2}/2)[(K_1H_1 + K_2H_2)\mathbf{E}_1 + (K_2H_1 - K_1H_2)\mathbf{E}_2] \\ &= (\sqrt{2}/2)|\mathbf{H}|\,|\mathbb{K}|[(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\mathbf{E}_1 \\ &\quad + (\sin\beta\cos\alpha - \cos\beta\sin\alpha)\mathbf{E}_2] \\ &= (\sqrt{2}/2)|\mathbf{H}|\,|\mathbb{K}|[\cos(\beta - \alpha)\mathbf{E}_1 + \sin(\beta - \alpha)\mathbf{E}_2]. \end{aligned}$$

Now, $\mathbb{K}[\mathbf{H}]$ and $\mathbf{H}$ belong to the same (two dimensional) space $\mathbf{H}$rm and the action of $SO(2)$ on $\mathbf{H}$rm does not vary the angle γ between the two "vectors", where $\gamma = \beta - 2\alpha$. Thus we realize that $\cos(\beta - 2\alpha)$ does not vary under the action of $O(2)$. This suggests an additional invariant $\hat{I}_5$ for the action of $O(2)$:

$$\begin{aligned}\hat{I}_5 &= \mathbb{K}[\mathbf{H}] \cdot \mathbf{H} \\ &= \frac{\sqrt{2}}{2} |\mathbf{H}|^2 \, |\mathbb{K}| (\cos(\beta - \alpha)\cos\alpha + \sin(\beta - \alpha)\sin\alpha) \\ &= \frac{\sqrt{2}}{2} |\mathbf{H}|^2 \, |\mathbb{K}| \cos(\beta - 2\alpha) \\ &= \frac{\sqrt{2}}{2} I_5\end{aligned}$$

which, as noted, turns out to be just a multiple of I_5.

The geometric interpretation of the collection of invariants $(I_1, \ldots, I_5)$ makes possible to deduce that for plane linearly elastic materials there are only symmetry classes corresponding to groups Z_2, D_2, D_4 and $O(2)$. We recall this result (derived differently in [2]), as stated and proved in [13].

Proposition 1 For $G \subset O(2)$ let $\mathbb{E}\mathrm{la}(G)$ be the set of plane elasticity tensors $\mathbb{C}$ such that $g(\mathbb{C})$ is conjugate with G. Then, the only non empty such sets are characterized as

$$\begin{aligned}&\mathbb{C} \in \mathbb{E}\mathrm{la}(O(2)) \Leftrightarrow I_3 = I_4 = 0, \\ &\mathbb{C} \in \mathbb{E}\mathrm{la}(D_4) \Leftrightarrow I_3 = 0, \quad I_4 \neq 0, \\ &\mathbb{C} \in \mathbb{E}\mathrm{la}(D_2) \Leftrightarrow \begin{cases} I_3 \neq 0, & I_4 = 0, \\ I_3 \neq 0, & I_4 \neq 0, \quad I_5^2 - I_3^2 I_4 = 0, \end{cases} \\ &\mathbb{C} \in \mathbb{E}\mathrm{la}(Z_2) \Leftrightarrow I_3 \neq 0, \quad I_4 \neq 0, \quad I_5^2 - I_3^2 I_4 \neq 0.\end{aligned}$$

The goal of the next sections, where in a sense the core content of this research lies, is to show explicitly which kind of relationships exist between some sets of quantities introduced in significant articles and books about linear plane elasticity and what was presented and commented upon here above. Our main point is that, in one way or another, very direct connections exist between the invariants I_j, the horizontal and vertical components of $\mathbf{H}$ and $\mathbb{K}$ (as defined in (10) in view of Fig. 1) and the quantities introduced by Tsai and Pagano (see, e.g., [7]), by Verchery (see, e.g. [12]) and, more recently, by de Saxcé and Vallée (see [3]).

As we shall quickly see, all such quantities have a direct and straightforward interpretation in terms of the "harmonic decomposition" of a given plane elasticity tensor $\mathbb{C}$. Indeed, it is the sum of such observations which led us to think that it might be appropriate to think of such an approach as "unified" or "unifying", since it seems to give a coherent and geometrically easily interpretable point of view on many problems related with the algebraic and group-theoretical aspects of the space of plane elasticity tensors and the action of $O(2)$ on it.

4 The parameters of Tsai and Pagano

A classical representation of plane anisotropy, widely used in the field of design of composite laminates, is due to Tsai and Pagano [7].

They investigated the variation of the components of $\mathbb{C}$, as the plane is rotated through an angle θ. The functions $C'_{ijkl} = [\mathbf{Q}(\theta) * \mathbb{C}]_{ijkl}$ reveal an interesting dependence on the angle of rotation. It was noticed that the components C'_{ijkl} may be expressed, by use of trigonometric identities, as linear combinations of sine and cosine of even multiples of the angle of rotation θ, thus providing an invariant form of C'_{ijkl} in terms of seven parameters U_i (known as the "parameters of Tsai and Pagano"),

$$\begin{aligned}C'_{1111} &= U_1 + U_2\cos 2\theta + 2U_6\sin 2\theta + U_3\cos 4\theta \\ &\quad + U_7\sin 4\theta, \\ C'_{1122} &= U_4 - U_3\cos 4\theta - U_7\sin 4\theta, \\ C'_{1112} &= 2U_6\cos 2\theta - U_2\sin 2\theta + 2U_7\cos 4\theta \\ &\quad - 2U_3\sin 4\theta, \\ C'_{2222} &= U_1 - U_2\cos 2\theta - 2U_6\sin 2\theta + U_3\cos 4\theta \\ &\quad + U_7\sin 4\theta, \\ C'_{1222} &= 2U_6\cos 2\theta - U_2\sin 2\theta - 2U_7\cos 4\theta \\ &\quad + 2U_3\sin 4\theta, \\ C'_{1111} &= U_5 - U_3\cos 4\theta - U_7\sin 4\theta,\end{aligned} \tag{17}$$

where the U_i's are linear functions of the cartesian components of the elasticity tensor $\mathbb{C}$, given by

$$U_1 = \frac{1}{8}(3C_{1111} + 2C_{1122} + 3C_{2222} + 4C_{1212}),$$
$$U_2 = \frac{1}{2}(C_{1111} - C_{2222}),$$
$$U_3 = \frac{1}{8}(C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212}),$$
$$U_4 = \frac{1}{8}(C_{1111} + 6C_{1122} + C_{2222} - 4C_{1212}),$$
$$U_5 = \frac{1}{8}(C_{1111} - 2C_{1122} + C_{2222} + 4C_{1212}),$$
$$U_6 = \frac{1}{2}(C_{1112} + C_{1222}),$$
$$U_7 = \frac{1}{2}(C_{1112} - C_{1222}).$$

The parameters U_i are also known as "material invariants", even if such name is quite inappropriate. Indeed, only U_1, U_4, U_5 are really invariant: U_4, U_5 coincide with $I_1 = \lambda$ and $I_2 = \mu$, appearing in the harmonic decomposition of $\mathbb{C}$ (16) and U_1 is nothing but

$$U_1 = U_4 + 2U_5 = \lambda + 2\mu = I_1 + 2I_2.$$

The remaining parameters are *not* invariant, but are related to the components of $\mathbf{H}$, and $\mathbb{K}$. More precisely, U_2 and U_6 are proportional to H_1 and H_2 respectively (11)

$$U_2 = \frac{1}{2}(c_{11} - c_{22}) = \frac{\sqrt{2}}{2} H_1,$$
$$U_6 = \frac{\sqrt{2}}{4}(c_{13} + c_{23}) = \frac{\sqrt{2}}{4} H_2,$$

while U_3 and U_7 are proportional, respectively, to K_1 and K_2 (12)

$$U_3 = \frac{1}{8}(c_{11} - 2c_{12} + c_{22} - 2c_{33}) = \frac{\sqrt{2}}{4} K_1,$$
$$U_7 = \frac{\sqrt{2}}{4}(c_{13} - c_{23}) = \frac{\sqrt{2}}{4} K_2.$$

Interestingly, the geometric perspective provided through Fig. 2 makes obvious the reason why such quantities U_2, U_6, U_3, U_7 are *not* invariant.

The parameters of Tsai and Pagano U_i are then shown to be strictly related to the harmonic components of $\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K})$. In fact U_4 and U_5 are nothing but the linear invariants λ and μ, $U_1 = \lambda + 2\mu$, U_2 and U_6 are proportional to the components of $\mathbf{H}$ and, finally, U_3 and U_7 are proportional to the components of $\mathbb{K}$.

We thus conclude that (17) is just a manifestation of the harmonic decomposition of $\mathbb{E}$la.

5 The polar method by Verchery

In order to describe, through invariant quantities, the behaviour of a plane anisotropic elastic material, Verchery introduced in 1978 the so called "Polar Method" [12], which makes use of complex variables. Here, we summarize this method, with an approach which follows a more recent critical review by Vannucci [8]. The technique is based upon a clever complex variable change, which was used before by Green and Zerna [6]. In the complex plane let $z = x + iy$ and

$$X^1 = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} z = \frac{x + y - i(x - y)}{2},$$
$$X^2 = \bar{X}^1 = \frac{x + y + i(x - y)}{2},$$

(this transformation can be interpreted as a change of frame).

When applied to a tensor $\mathbb{C} \in \mathbb{E}$la, the transformation of Verchery leads to

$$\begin{bmatrix} \tilde{C}^{1111} \\ \tilde{C}^{1112} \\ \tilde{C}^{1122} \\ \tilde{C}^{1212} \\ \tilde{C}^{1222} \\ \tilde{C}^{2222} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & -4i & 2 & 4 & 4i & -1 \\ -i & 2 & 0 & 0 & 2 & i \\ 1 & 0 & -2 & 4 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ i & 2 & 0 & 0 & 2 & -i \\ -1 & 4i & 2 & 4 & -4i & -1 \end{bmatrix} \begin{bmatrix} C_{1111} \\ C_{1112} \\ C_{1122} \\ C_{1212} \\ C_{1222} \\ C_{2222} \end{bmatrix}.$$

Notice that the components $\tilde{C}^{1111}$ and $\tilde{C}^{1112}$ are complex, while $\tilde{C}^{1122}$ and $\tilde{C}^{1212}$ are real. Moreover, $\tilde{C}^{1222}$ is the complex conjugate of $\tilde{C}^{1112}$ and $\tilde{C}^{2222}$ is the complex conjugate of $\tilde{C}^{1111}$.

An important feature of the complex variable change of Verchery lies in the transformation rule of the components $\tilde{C}^{ijkl}$ under a rotation. For $r = e^{-i\theta}$, the equation $z' = rz$ describes a rotation in the complex plane through an angle θ. It can be proved that the rotation matrix is diagonal, namely

$$\begin{bmatrix} \tilde{C}'^{1111} \\ \tilde{C}'^{1112} \\ \tilde{C}'^{1122} \\ \tilde{C}'^{1212} \\ \tilde{C}'^{1222} \\ \tilde{C}'^{2222} \end{bmatrix} = \begin{bmatrix} r^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{r}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{r}^4 \end{bmatrix} \begin{bmatrix} \tilde{C}^{1111} \\ \tilde{C}^{1112} \\ \tilde{C}^{1122} \\ \tilde{C}^{1212} \\ \tilde{C}^{1222} \\ \tilde{C}^{2222} \end{bmatrix}. \tag{18}$$

From Eq. (18) it is quite easy to recognize six polynomial invariants, which are denoted by capital letters L, Q

and C, where L stands for linear, Q for quadratic and C for cubic (as functions of the components of $\mathbb{C}$):

$$\begin{aligned} &L_1 = \tilde{C}^{1122}, \quad L_2 = \tilde{C}^{1212},\\ &Q_1 = \tilde{C}^{1111}\tilde{C}^{2222}, \quad Q_2 = \tilde{C}^{1222}\tilde{C}^{1112},\\ &C_1 + iC_2 = \tilde{C}^{1111}\left(\tilde{C}^{1222}\right)^2. \end{aligned} \tag{19}$$

The six invariants are linked by the syzygy

$$C_1^2 + C_2^2 = Q_1 Q_2^2.$$

The invariants (19) can be rewritten, with respect to the cartesian components of $\mathbb{C}$, as:

$$\begin{aligned} L_1 =& \frac{1}{4}(C_{1111} - 2C_{1122} + 4C_{1212} + C_{2222})\\ =& \frac{1}{4}(c_{11} - 2c_{12} + 2c_{33} + c_{22}),\\ L_2 =& \frac{1}{4}(C_{1111} + 2C_{1122} + C_{2222}) = \frac{1}{4}(c_{11} + 2c_{12} + c_{22}),\\ Q_1 =& \frac{1}{16}(C_{1111} + C_{2222} - 2C_{1122} - 4C_{1212})^2\\ &+ (C_{1112} - C_{1222})^2\\ =& \frac{1}{16}(c_{11} + c_{22} - 2c_{12} - 2c_{33})^2 + \frac{1}{2}(c_{13} - c_{23})^2,\\ Q_2 =& \frac{1}{16}(C_{1111} - C_{2222})^2 + \frac{1}{4}(C_{1112} + C_{1222})^2\\ =& \frac{1}{16}(c_{11} - c_{22})^2 + \frac{1}{8}(c_{13} + c_{23})^2,\\ C_1 =& \frac{1}{64}(C_{1111} + C_{2222} - 2C_{1122} - 4C_{1212})\\ &[(C_{1111} - C_{2222})^2 - 4(C_{1112} + C_{1222})^2]\\ &+ \frac{1}{4}(C_{1112}^2 - C_{1222}^2)(C_{1111} - C_{2222})\\ =& \frac{1}{64}(c_{11} + c_{22} - 2c_{12} - 2c_{33})\\ &[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2]\\ &+ \frac{1}{8}(c_{13}^2 - c_{23}^2)(c_{11} - c_{22}),\\ C_2 =& \frac{1}{16}\Big\{(C_{1112} - C_{1222})[(C_{1111} - C_{2222})^2\\ &-4(C_{1112} + C_{1222})^2] - (C_{1112} + C_{1222})\\ &(C_{1111} - C_{2222})(C_{1111} + C_{2222} - 2C_{1122} - 4C_{1212})\Big\}\\ =& \frac{1}{16\sqrt{2}}\Big\{(c_{13} - c_{23})\left[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2\right]\\ &-(c_{13} + c_{23})(c_{11} - c_{22})(c_{11} + c_{22} - 2c_{12} - 2c_{33})\Big\}. \end{aligned}$$

A direct comparison with the invariants obtained from the harmonic decomposition of $\mathbb{E}$la shows that L_1 and L_2 are linear combination of the invariants λ and μ, the quadratic invariants Q_1 and Q_2 are multiples of $|\mathbf{H}|^2$, and $|\mathbb{K}|^2$ respectively and finally the cubic invariants C_1 and C_2 are multiples of I_5 and I_6 respectively. More precisely

$$\begin{aligned} &L_1 = 2\mu, \quad L_2 = \lambda + \mu,\\ &Q_1 = \frac{1}{2}|\mathbb{K}|^2 = \frac{1}{2}I_4, \quad Q_2 = \frac{1}{8}|\mathbf{H}|^2 = \frac{1}{8}I_3,\\ &C_1 = \frac{1}{8\sqrt{2}}I_5, \quad C_2 = \frac{1}{8\sqrt{2}}I_6. \end{aligned} \tag{20}$$

Again, we see that most quantities introduced in [8, 12] are without any doubts explicitly and directly related with what was derived through the method of harmonic decomposition. We wish to emphasize, however, how the approach presented in Sect. 3 gives a more complete and easily interpretable picture.

An interesting link between the polar method of Verchery and the approach through harmonic decomposition lies in the representation of elasticity tensors through vectors in the complex plane. Verchery introduced what he called the *polar components* T_0, T_1, R_0, R_1, Φ_0, Φ_1 of an elasticity tensor $\mathbb{C}$, where T_0 and T_1 are defined by

$$T_0 = L_1/2, \quad T_1 = L_2/2,$$

and, in view of (20), are obvious combinations of the basic invariants: $T_0 = \mu$ and $T_1 = (\lambda + \mu)/2$.

The polar components R_0, Φ_0, R_1, and Φ_1 are defined by Verchery through two complex numbers

$$v_0 = R_0 e^{4i\Phi_0} = -\frac{1}{2}\tilde{C}^{1111}, \quad v_1 = R_1 e^{2i\Phi_1} = \frac{1}{2}i\tilde{C}^{1112}.$$

We notice, here, that Verchery's complex numbers v_0 and v_1 are just multiples of the "vectors" $\mathbb{K}$ and $\mathbf{H}$, defined in (5) and (6) from the harmonic decomposition of $\mathbb{C}$. Indeed, once we identify both with a number in the complex plane, by direct comparison with (11) and (12) we have

$$\begin{aligned}v_0 &= \frac{1}{8}[(C_{1111} - 2C_{1122} - 4C_{1212} + C_{2222}) \\ &\quad + 4i(C_{1112} - C_{1222})] \\ &= \frac{\sqrt{2}}{4}\left[\frac{\sqrt{2}}{4}(c_{11} - 2c_{12} - 2c_{33} + c_{22}) + i(c_{13} - c_{23})\right] \\ &= \frac{\sqrt{2}}{4}(K_1 + iK_2), \\ v_1 &= \frac{1}{8}[(C_{1111} - C_{2222}) + 2i(C_{1112} + C_{1222})] \\ &= \frac{\sqrt{2}}{8}\left[\frac{\sqrt{2}}{2}(c_{11} - c_{22}) + i(c_{13} + c_{23})\right] \\ &= \frac{\sqrt{2}}{8}(H_1 + iH_2).\end{aligned}$$

Thus, the modulus of Verchery's numbers are proportional to the magnitude of vectors $\mathbb{K}$ and $\mathbf{H}$

$$R_0^2 = \frac{1}{8}|\mathbb{K}|^2, \quad R_1^2 = \frac{1}{8}|\mathbf{H}|^2,$$

and, moreover, recalling that, as shown in Fig. 2, β is the angle between $\mathbb{K}$ and the "horizontal" axis in $\mathbb{H}$rm and α is the angle between $\mathbf{H}$ and the "horizontal" axis in $\mathbf{H}$rm, it follows that

$$4\Phi_0 = \beta, \quad 2\Phi_1 = \alpha.$$

From (18), it can be shown that a rotation of amplitude θ acts on v_0 as a rotation of 4θ and on v_1 as a rotation of 2θ, making the identification of Verchery's numbers with $\mathbf{H}$ and $\mathbb{K}$ an $O(2)$-invariant isomorphism.

We finally recall an interesting result which can be found in the literature (for references see [8]). The three complex numbers $\mu + v_0$, $3\mu + \lambda + 2v_1$, and $3\mu + \lambda + 4v_1$, describe, under a rotation, three circles, called "generalized Mohr's circles". Just like Mohr's circle for the Cauchy stress tensor, the generalized Mohr's circles give a graphical representation of the transformation law for the cartesian components of $\mathbb{C}$ under a rotation of frame.

6 A comparison with recent results by De Saxcé and Vallée

In a recent paper [3] de Saxcé and Vallée obtain an irreducible invariant decomposition of $\mathbb{E}$la. In this section we show the equivalence between the harmonic decomposition of $\mathbb{E}$la (as summarized here in Sect. 3) and the irreducibles subspaces used in [3]. In order to avoid any source of confusion, we use a superscript SV for the set of invariants I_i as found in [3], to make a distinction with the invariants introduced here. As we shall see, there is a close relationship between the sets $\{I_i\}$ and $\{I_i^{SV}\}$.

First, we repeat the main steps of the procedure found in [3]. Let

$$P^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the constitutive law (4), by a change of variables $\tilde{t} = P^{-1}t$, $\tilde{e} = P^{-1}e$, becomes

$$\tilde{t} = \tilde{c}\tilde{e},$$

where $\tilde{c} = P^{-1}cP$, with

$$\begin{aligned}\tilde{c}_{11} &= \frac{1}{2}(c_{11} + c_{22} + 2c_{12}), \\ \tilde{c}_{12} &= \frac{1}{2}(c_{11} - c_{22}), \\ \tilde{c}_{13} &= \frac{\sqrt{2}}{2}(c_{23} + c_{13}), \\ \tilde{c}_{22} &= \frac{1}{2}(c_{11} + c_{22} - 2c_{12}), \\ \tilde{c}_{23} &= \frac{\sqrt{2}}{2}(c_{13} - c_{23}), \\ \tilde{c}_{33} &= c_{33}.\end{aligned}$$

Then, the matrix $\tilde{c}$ is decomposed into blocks

$$\tilde{c} = \begin{bmatrix} \tilde{c}_{11} & \tilde{v}^T \\ \tilde{v} & \tilde{A} \end{bmatrix},$$

where $\tilde{v} \in \mathbb{R}^2$ and $\tilde{A}$ is a symmetric 2×2 real matrix:

$$\tilde{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \tilde{c}_{12} \\ \tilde{c}_{13} \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{c}_{22} & \tilde{c}_{23} \\ \tilde{c}_{23} & \tilde{c}_{33} \end{bmatrix}.$$

For $r_{2\theta}$ the matrix

$$\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

de Saxcé and Vallée show that the action of $\mathbf{Q}(\theta)$ on $\mathbb{C}$ translates into

$$\tilde{c}_{11} \mapsto \tilde{c}_{11}, \quad \tilde{v} \mapsto r_{2\theta}\,\tilde{v}, \quad \tilde{A} \mapsto r_{2\theta}\tilde{A}\,(r_{2\theta})^T. \tag{21}$$

In view of (21) they easily recognize that $\tilde{c}_{11}$ and $\operatorname{tr}\tilde{A}$ are linear invariants related to λ and μ, defined in (16), by

$$\tilde{c}_{11} = 2(\lambda+\mu), \quad \operatorname{tr}\tilde{A} = \tilde{c}_{22} + \tilde{c_{33}} = 4\lambda.$$

Next, the symmetric matrix $\tilde{c}$ is additively decomposed into its isotropic and deviatoric parts

$$\tilde{c}_0 = \begin{bmatrix} 2(\lambda+\mu) & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix}, \quad \tilde{c}_d = \begin{bmatrix} 0 & a & b \\ a & -2f & -2g \\ b & -2g & 2f \end{bmatrix},$$

where (after some computations)

$$f = \mu - \frac{1}{2}\tilde{c}_{22} = \frac{1}{8}(2(c_{12}+c_{33}) - (c_{11}+c_{22})),$$
$$g = -\frac{1}{2}\tilde{c}_{23} = -\frac{\sqrt{2}}{4}(c_{13}-c_{23}).$$

The action (21) shows that $\tilde{v}$ belongs to a 2-dimensional space E_1 on which a rotation $\mathbf{Q}(\theta)$ acts as a rotation of 2θ. We notice that the components of $\tilde{v} = (a, b)$ are related to the components (11) of $\mathbf{H}$ in $\mathbf{H}$rm, namely

$$a = \frac{\sqrt{2}}{2}H_1, \quad b = \frac{\sqrt{2}}{2}H_2.$$

The norm of $\tilde{v}$ gives the third invariant $I_3^{(\mathrm{SV})}$ of de Saxcé and Vallée $I_3^{(\mathrm{SV})}$, which is related to the third invariant I_3 obtained through the harmonic decomposition method,

$$I_3^{(\mathrm{SV})} = a^2 + b^2 = \frac{1}{2}(H_1^2 + H_2^2) = \frac{1}{2}I_3.$$

De Saxcé and Vallée decompose the deviatoric part of $\tilde{c}$ into blocks

$$\tilde{c}_d = \begin{bmatrix} 0 & \tilde{v}^T \\ \tilde{v} & \tilde{A}_0 \end{bmatrix},$$

where $\tilde{A}_0$ is a symmetric and traceless 2×2 real matrix

$$\tilde{A}_0 = \begin{bmatrix} -2f & -2g \\ -2g & 2f \end{bmatrix}.$$

We expect the existence of a relation between $\tilde{A}_0$ and $\mathbb{K}$ in $\mathbb{H}$rm. In fact, by a direct comparison, $\tilde{A}_0 = \tilde{K}$ and thus

$$f = \frac{\sqrt{2}}{4}K_1, \quad g = \frac{\sqrt{2}}{4}K_2.$$

De Saxcé and Vallée remark in [3] that (f, g) may be considered as the components of a vector $\tilde{k}$ in a two dimensional space E_2 on which $SO(2)$ acts as a rotation of $4\,\theta$, hence

$$I_4^{(\mathrm{SV})} = f^2 + g^2$$

is invariant and related to I_4 by

$$I_4^{(\mathrm{SV})} = \frac{1}{8}I_4.$$

Nevertheless, in order to characterize the planar elasticity tensors a fifth invariant is needed. De Saxcé and Vallée remark that the orbits of $\tilde{v}$ and $\tilde{k}$ are coupled, in the sense that a rotation of ϕ of (a, b) corresponds to a rotation of 2ϕ of (f, g). Using complex numbers, let $z_2 = a + ib$ and $z_4 = f + ig$. The action is given by

$$z_2' = e^{-2i\theta}z_2, \quad z_4' = e^{-4i\theta}z_4.$$

Eliminating the parameter θ they obtain the relation

$$(z_2')^2\,\bar{z}_4' = z_2^2\,\bar{z}_4,$$

that leads to the complex invariant

$$\xi = z_2^2\,\bar{z}_4 = (a+ib)^2(f-ig),$$

which is linked to the quadratic invariants by the syzygy

$$|\xi|^2 = I_4^{(\mathrm{SV})}\left(I_3^{(\mathrm{SV})}\right)^2.$$

(The arguments developed here, following [3], should be compared with the approach of Ref. [13, §4, pp. 204–205])

The complex invariant ξ is equivalent to two real cubic invariants

$$\begin{aligned} I_5^{(\mathrm{SV})} &= \xi_r = 2bga + f(a^2 - b^2) \\ &= \frac{1}{32}\Big[8\Big((c_{23})^2 - (c_{13})^2\Big)(c_{11}-c_{22}) \\ &\quad + (2c_{33} + 2c_{12} - c_{11} - c_{22})((c_{11}-c_{22})^2 \\ &\quad -2(c_{23}+c_{13})^2)\Big], \end{aligned}$$

$$\begin{aligned} I_6^{(\mathrm{SV})} &= \xi_i = 2bfa - g(a^2 - b^2) \\ &= \frac{1}{8\sqrt{2}}\Big[(c_{23}+c_{13})(2c_{33}+2c_{12}-c_{11}-c_{22}). \\ &\quad (c_{11}-c_{22}) - (c_{23}-c_{13})((c_{11}-c_{22})^2 \\ &\quad -2(c_{23}+c_{13})^2)\Big], \end{aligned}$$

It is easy to check by direct comparison that the cubic invariants of de Saxcé and Vallée are related to the cubic invariants obtained through the method of harmonic decomposition

$$I_5^{(SV)} = -\frac{1}{4\sqrt{2}} I_5, \qquad I_6^{(SV)} = \frac{1}{4\sqrt{2}} I_6.$$

In conclusion, de Saxcé and Vallée obtain an irreducible decomposition of the representation of $SO(2)$ into subspaces of $\mathbb{E}$la, which is equivalent to the harmonic decomposition, and derive the same set of invariants $(I_1, \ldots, I_6)$. In particular, their approach leads to results which are significantly equivalent to what was found in [2] and [13].

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