Mean square stability of a second-order parametric linear system excited by a colored Gaussian noise

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1. Introduction

The analysis of stability of linear second-order differential equations with random parametric excitation has attracted the attention of the scientists from many years [1–41]. Part of the studies have a pure mathematical character, while the others are more engineering oriented. In fact, a second-order linear differential equation describes various engineering systems, such as the motion of a beam, the oscillations of an electric circuit, the wave propagation in one-dimensional random medium and so on (for the former case see Appendix A). Thus, studying the stability of the response is important from both theoretical and applicative points of view. It is recalled that the loss of stability leads to system failure. As a consequence, the stability of the trivial zero solution is very important.

The excitation of a dynamic system is called parametric, or multiplicative, which causes time variation of the system parameters. The varying of the parameters drives the system, it may cause amplification of the system oscillation in particular at the resonant frequencies of the system, and in some cases it may lead to instability. When the parametric excitation is a stochastic process, the structural response is a stochastic process too so that the problem must be founded in

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the context of stochastic dynamics. If a second-order system is excited by parametric excitations, exact analytical solutions for the joint probability density function (PDF) $p_{X\dot{X}}$ of the displacement X and the velocity $\dot{X}$ exist only when strict relations among the system parameters and the spectral level of the excitation are satisfied. These relations are rarely met in practice. Thus, the analyst must resort to methods that contain something of approximation and depend on the type of excitation.

There are different definitions of stochastic stability, for which the reader is referred to [41] and to [42, Chapter 6]. Probably, the almost sure (sample) stochastic stability and the stability in the response statistical moments of order p are the object of the majority of the studies: the fundamental principles of both will be given in Section 2. Some studies regard the almost sure stochastic stability [4,7,9,11,12,14,22,25,27-29,32], while in [6,8,10,13,15-19,21,23,26,30,35,36,38] reference is made to the moment stability. Some other articles consider both type of stability [20,24,27,31,33,34,37]. The studies on the stochastic stability differ even for the nature of the stochastic excitation: all the above cited studies consider Gaussian multiplicative excitations that in many instances are assumed to be Gaussian white noise processes [2,3,6,14,27]. Excitations in the form of colored Gaussian processes are considered explicitly in [10,33-35,38,40]: this point will be addressed forward. For brevity's sake, the studies in which the excitations are not Gaussian agencies and those regarding nonlinear systems are omitted from the literature review.

Khas'minskii [7] considered dynamical system parametrically excited by Gaussian white noises, and gave necessary and sufficient conditions of almost sure stability. Mitchell and Kozin [14] moved in the stream of Khas'minskii. However, both these methods are difficult to apply. Because of the difficulty of the problem, some authors only gave sufficient conditions of stability [1,4,8,9,12,16,20,22,28,29,32]. Among the authors of the last group Caughey and Gray [4] looked for a sufficient condition of almost sure sample stability of Eq. (1) specializing the analysis to the second-order systems $\ddot{X}+2\zeta\dot{X}+[1+f(t)]X=0$. Other authors [9,11,12,22,28,29,32] tried to improve Caughey and Gray's stability bounds that were overly conservative, that is they underestimated the stability region by a large extent. Infante's result, which is also named Infante's theorem, for the previous cited system reads as $E[|f(t)|]\le 2\zeta\sqrt{1-\zeta^2}$ when $\zeta<\sqrt{2}/2$ [9]. This stability bound is very popular, but it too underestimates the stability domain. The bounds given in [11,12,22,28,29,32] are sharper, and probably those of [32] are the best. However, since all these stability bounds are only sufficient conditions, they are lower bounds of the true stability bounds.

In order to circumvent the difficulties of the problem, other studies [15,21,23,31,33,34,38] make the assumption of weak excitation. This assumption allows using the stochastic averaging method [51,52] as in [15,21,23], which simplifies the problem. However, in many instances the assumption of weakly excitation is not valid.

In this paper, the multiplicative excitation is given by a colored Gaussian process $F(t)$ that affects the stiffness of the oscillator, it is generated by a linear filter excited by a stationary Gaussian white noise, and it is not restricted to be weak. This choices are motivated by the following reasons: (1) many real agencies are not white processes such as ground motion, wind turbulence, and sea waves; (2) if the colored Gaussian process is obtained by filtering a white noise, the Markov methods of stochastic dynamics apply inasmuch as the augmented state vector $\{X\dot{X}F\}$ is globally a Markov vector; (3) a non-weak excitation is a more general case. It is recalled that an excitation $Q(t)$ with a non-flat power spectral density can be obtained in another way [34,38,40]: the colored process $Q(t)$ is given by $b\cos[\omega t+\theta+kL(t)]$, where $L(t)$ is a Gaussian process; clearly, $Q(t)$ is not Gaussian as it is obtained through a nonlinear transformation of a Gaussian process.

The asymptotic second-order moment stability is considered here. The first step consists in writing the differential equations that govern the time evolution of the response statistical moments. These equations are written by means of Itô's differential rule [43-45]. Because of parametric colored excitation, the moment equations form an infinite hierarchy: in other words, the moment equations that are written for computing the moments of an order s contain moments of order larger than s. Thus, the hierarchy must be closed. For doing it, various closure methods have been proposed: [43-52]. For the problem under examination Bolotin [10,55] uses the cumulant neglect closure method; Wedig [26] makes a stochastic state transformation, and truncates an infinite determinant. Dimentberg and Bucher [39] apply the stochastic averaging method. All these authors in some way operate in the contest of Markov methods of stochastic dynamics. On the contrary, Bobryk et al. [35,38] do not make use of the Markov methods in analyzing the stability in the second moments of a linear oscillator excited by a colored second-order Gaussian process. As a consequence, the derivation of the moment equations is cumbersome, and the closure order is very high, say the 40th or 50th order, so that a large number of equations must be taken into account (in [10] the closure is made at the third order, while in [26] it is at the fourth order).

Herein, as a mean square stability analysis is performed, the moment equations are written for the second moments. These equations contain moments of the third order that are called hierarchical. The cumulant neglect closure method [49,50] is adopted for closing the hierarchy. The third-order cumulants are assumed to be zero, and expressions for the third-order moments are obtained from the relationships linking moments and cumulants. In the numerical analyses a special attention is devoted to detect the phenomenon of the stochastic resonance [66] that lowers the stability bounds considerably.

The paper is organized in this way: in Section 2 some fundamental concepts on stochastic stability are recalled. Section 3 is devoted to the analytical formulation, while Sections 4 and 5 contain the numerical analyses and the conclusions, respectively. Two appendices complete the paper.

2. Fundamentals of stochastic stability

In this section some fundamental concepts on the stochastic stability will be recalled with reference to the almost sure (sample) stability, to the stability in the response statistical moments, and to the Lyapunov exponents for stochastic stability. Moreover, some stability criteria will be illustrated, which will be used in the applications for comparison's sake.

The problem of the stability of the linear differential equations with stochastic coefficients deserved the attention of the scientists since the late 1950s, but the different definitions of stochastic stability were proposed later. To explain the two definitions of stochastic stability considered here, reference is made to the stochastic differential equation in $\Re^n$

$$\dot{\boldsymbol{X}} = [\boldsymbol{A} + \boldsymbol{F}(t)]\boldsymbol{X}, \tag{1}$$

where $\boldsymbol{X}$ is a column vector $n \times 1$, $\boldsymbol{A}$ is an $n \times n$ constant matrix, and $\boldsymbol{F}(t)$ is an $n \times n$ matrix, whose elements $F_{ij}(t)$ are strictly stationary ergodic stochastic processes defined on a probability space $\{\Omega, F, P\}$. The solution $\boldsymbol{X}$ of Eq. (1) is said to be almost surely asymptotically sample stable (Lyapunov stability), if

$$\boldsymbol{P}\left[\lim_{t\to\infty} \|\boldsymbol{X}\| = 0 | \boldsymbol{X}(0) = \boldsymbol{x}_0\right] = 1. \tag{2}$$

The solution $\boldsymbol{X}$ of Eq. (1) is said to be asymptotically stable in the pth statistical moments, if

$$\lim_{t\to\infty} \boldsymbol{E}\left[\|\boldsymbol{X}(t)\|^p | \boldsymbol{X}(0) = \boldsymbol{x}_0\right] = 0, \tag{3}$$

where x_0 is the initial condition for $\boldsymbol{X}$. In Eqs. (2) and (3) $\|\bullet\|$ denotes a suitable norm. However, the two definitions of stochastic stability are linked as was highlighted by Kozin [5] and by Arnold [24].

The concept of the Lyapunov exponents was extended to the stochastic stability. The sample Lyapunov exponent is defined as

$$\lambda_{\boldsymbol{X}} = \lim_{t\to\infty} \frac{\log\|\boldsymbol{X}(t)\|}{t}. \tag{4}$$

The dynamic system is almost sure sample stable, if $\lambda_{\boldsymbol{X}}$ is negative. This means that the sample paths do not diverge. However, they may be so spread that the mean square values diverge.

This fact is better taken into account by the moment Lyapunov exponent that is defined as

$$\lambda_{\boldsymbol{X}^p} = \lim_{t\to\infty} \frac{\log(E[\|\boldsymbol{X}(t)\|^p])}{t}. \tag{5}$$

If $\lambda_{\boldsymbol{X}^p} < 0$, then $E[\|\boldsymbol{X}(t)\|^p] \to 0$ as $t \to \infty$. For the same $\boldsymbol{X}$, a mathematical relationship links the two exponents: $\lambda'_{\boldsymbol{X}^p}(0) = \lambda_{\boldsymbol{X}}$, which was established by Arnold [24], and where the apex denotes derivative with respect to p. An analytically exact and easy to apply method to compute the Lyapunov exponents does not exist. Probably, Khas'minskii [7] was the first who proposed a procedure to compute λ. Khas'minskii's considers the case in which the parametric excitations are Gaussian white noises: the formulation is exact, but approximations or numerical tools are necessary in performing the computations.

As regards the stability analysis in the response moments, simple and exact results are available when the excitation is a Gaussian white noise process. Caughey and Dienes [2] gave the stability bounds with respect to the moments of every order in the case of a scalar system; in the case of second-order systems they obtained the stability bounds with respect to the moments of the first and second order. On the contrary, as previously advanced, analyzing the stochastic stability under a colored Gaussian excitation is much more complicated as the moment equations form an infinite hierarchy. Now, we shall recall three stability criteria that will be used for comparison in the applications.

In order to circumvent the hierarchy, the stochastic averaging method is used in [15,21,23] (for this method see [42,53,54]). In [15] a general analysis is accomplished for the linear matrix system $\dot{\boldsymbol{X}} = [\boldsymbol{A}^0 + \varepsilon^2\boldsymbol{B}^0 + \varepsilon\boldsymbol{P}^0 f(t)]\boldsymbol{X}$, where $\boldsymbol{A}^0$, $\boldsymbol{B}^0$ and $\boldsymbol{P}^0$ are constant matrices, ε is a small parameter, and $f(t)$ is a wide-band stationary stochastic process. The analysis is specialized to the second-order oscillator

$$\ddot{X} + 2\varepsilon^2\rho_0\omega_0\dot{X} + \left[\omega_0^2 + \varepsilon f(t)\right]X = 0. \tag{6}$$

The following condition of mean square stability is obtained:

$$\rho_0 > \frac{\pi S_{ff}(2\omega_0)}{2\omega_0^3}, \tag{7}$$

where $S_{ff}(\bullet)$ is the power spectral density of $f(t)$.

In the 1970s the researchers at the Massachusetts Institute of Technology studied the stochastic stability of linear systems with Gaussian parametric excitations either white or colored [16-20]. Among them, Martin [16] found a sufficient mean square stability criterion for the second-order oscillator $\ddot{X}(t) + 2\zeta\Omega\dot{X}(t) + \Omega^2\left[1 + f(t)\right]X(t) = 0$, when the process $f(t)$ is an Orstein-Uhlenbeck one, that is its autocorrelation is $R_{ff}(\tau) = \sigma_f^2 \exp(-\alpha|\tau|)$. It reads as

$$\lim_{t\to+\infty} E[X^2(t)] = 0 \quad \text{if} \quad \frac{2\sigma_f^2}{\alpha} < \frac{\beta(\zeta)}{\gamma^2(\zeta)}. \tag{8}$$

As a first approximation, it is proposed $\beta(\zeta) = \zeta,\ \gamma^2(\zeta) = \left(1 - \zeta^2\right)^{-1}\ \zeta < 1$. In this way, the criterion is very similar to Infante's one [9]. A better approximation can be obtained by solving a constrained optimization problem for the ratio $\beta(\zeta)/\gamma^2(\zeta)$.

Xie studies the stochastic stability of the oscillator [33]

$$\frac{d^2q}{dt^2}+2\beta\frac{dq}{dt}+\left[\Omega^2-\varepsilon_0V(t)\right]q=0 \tag{9}$$

with the condition that $\varepsilon=\varepsilon_0/\sqrt{\Omega^2-\beta^2}$ is small, that is the assumption of weak excitation is made. $V(t)$ is an Ornstein–Uhlenbeck process as in the present paper. Using methods of functional analysis, the moment Lyapunov exponent of the previous system is given by

$$\lambda_{q^p}=-p\beta+\sqrt{\Omega^2-\beta^2}\Lambda_x(p), \tag{10}$$

where $\Lambda_x(p)=\Lambda_2\varepsilon^2+\Lambda_4\varepsilon^4+\Lambda_6\varepsilon^6+O(\varepsilon^8)$ (the reader is referred to [33] for the expressions of the Λ_i's). In order to derive an explicit criterion of stability from Eq. (10), this is equated to zero, and it is solved for the strength K of the white noise that generates the Ornstein–Uhlenbeck process. Keeping the second-order term only, one obtains

$$K_{cr}=\frac{2\zeta_0\Omega}{\pi\left(1-\zeta^2\right)}(\alpha^2+4), \tag{11}$$

where $\alpha=a/(\Omega\sqrt{1-\zeta_0^2})$ ($\alpha\cong a/\Omega$ for small damping). The expression of K_{cr} with the fourth-order term is not reported for the sake of brevity. However, it has been found that the inclusion of the fourth-order term has a very small effect on K_{cr}.

3. Formulation

Consider the second-order oscillator with stochastic perturbation in the stiffness:

$$\ddot{X}(t)+2D\dot{X}(t)+\Omega^2[1+V(t)]X(t)=0, \tag{12}$$

where the dots mean derivative with respect to the time. The stochastic process $V(t)$ is chosen to be a colored Gaussian process defined on a probability space $\{\Omega, F, P\}$ and obtained by passing a Gaussian white noise $W(t)$ through a linear filter, namely

$$\frac{d^nV}{dt^n}+a_{n-1}\frac{d^{n-1}V}{dt^{n-1}}+\cdots+a_1\frac{dV}{dt}+a_0V=W(t). \tag{13}$$

In this way, the response of Eq. (12) is a diffusive Markov vector $\{X\,\dot{X}\}^t$, the apex denoting transpose, and the stochastic differential calculus becomes applicable [43-45].

In the field of stochastic dynamics the model of Gaussian white noise excitation is very popular as in this case analytical solutions are available. However, the white noise is a mathematical idealization. In fact, in the time domain it is completely uncorrelated, that is the values of the process in two instants of time t_1 and t_2 are independent as far as the difference t_2-t_1 is small. In frequency domain, the power spectral density $S_{WW}(\omega)$ is constant on the whole real axis. Clearly, these features are physically unrealizable. On the other hand, in many cases the white noise idealization is worthwhile, especially when the excitation is broad-banded.

However, many excitations of engineering structures, such as wind turbulence, sea waves, and ground motion, have power spectral densities markedly non constant. As the Markov methods of stochastic dynamics are applicable only when the primary excitation is a Gaussian white noise, the colored excitation is approximated by means of the output of a linear filter excited by a white noise such as Eq. (13) or equivalently by a cascade of linear filters [60-62]: the higher the order of the filter, the better the approximation of original spectrum. For simplicity's sake, herein attention will be restricted to first and second-order filters, that is $n=1$ or $n=2$ in Eq. (13), respectively. In the field of stochastic stability filtered Gaussian excitations are considered in [10,16,18,26,33-35,39,55].

The stochastic stability of Eq. (12) in the response second statistical moments is concerned [Eq. (3) with $p=2$]: the equations ruling the time evolution of these quantities are easily written by applying Itô's stochastic differential calculus. Given a non-homogeneous initial condition $\{X_0\,\dot{X}_0\}^t$, if the oscillator is stable, the response moments tend to zero, otherwise they grow without limits, and the oscillator is unstable. The equations for the moments constitute a set of first-order differential equations. In this way, the analysis of stability becomes the deterministic problem of studying the stability of a set of first-order differential equations.

3.1. First-order filter

Let the filter that generates the process $V(t)$ in Eq. (12) be a first-order filter (Langevin equation), that is $V(t)$ is an Ornstein–Uhlenbeck process:

$$\dot{V}(t)=-aV(t)+\sqrt{2\pi K}W(t), \tag{14}$$

where a and K are real positive constants, while $W(t)$ is a normalized stationary Gaussian white noise. It is recalled that the power spectral density of $V(t)$ has the largest value for zero frequency; then, it decays monotonically. This power spectral density was proposed for the turbulence [63].

Introducing the state variables $z_1 = X,\ \ z_2 = \dot{X},\ z_3 = V$, the augmented system of Eqs. (12) and (14) is recast in Itô's form as

$$\begin{aligned} dz_1 &= z_2 dt \\ dz_2 &= -\left(2Dz_2 + \Omega^2 z_1 + \Omega^2 z_1 z_3\right)dt, \\ dz_3 &= -az_3\ dt + \sqrt{2\pi K}\ dB \end{aligned} \tag{15}$$

where dB is the increment of a unit Wiener process $B(t)$ on $\mathfrak{R}^+ = [0,\ \ +\infty)$ with autocorrelation function $R_{BB}(t,t') = \min(t,\ t')$. The derivative of $B(t)$ in the sense of the mathematical distributions is the white noise $W(t)$. It is noted that the Itô's and Stratonovich interpretations of Eq. (15) coincide as the white noise acts in the filter only. Thus, the Wong–Zakai–Stratonovich corrective terms [57,58] are zero in this case.

Clearly, zero solution $z_1 = z_2 = z_3 = 0\ \forall t$ satisfies Eq. (15). Zero statistical moments $E\left[z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\right](\alpha_1 + \alpha_2 + \alpha_3 = r)$ correspond to it. The stability of zero solution is analyzed in the moments of second order (mean square stability). To write the differential equations for them, we introduce the new variables $y_1 = X^2,\ \ y_2 = X\dot{X},\ \ y_3 = \dot{X}^2$, obtaining:

$$\begin{aligned} dy_1 &= 2y_2\ dt \\ dy_2 &= \left[-\Omega^2 y_1 - 2Dy_2 + y_3 - \Omega^2 V(t) y_1\right]dt, \\ dy_3 &= \left(-2\Omega^2 y_2 - 4Dy_3 - \Omega^2 V(t) y_2\right)dt \end{aligned} \tag{16}$$

where again the Wong–Zakai–Stratonovich corrective terms are zero.

For the differential equations ruling the time evolution of the moments $\mu_i = E\left[y_i\right]$ to be written, Itô's differential rule is used [43-45], which reads as

$$d\psi = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial z_i} dz_i + \frac{1}{2}\frac{\partial^2\psi}{\partial z_i \partial z_j} dz_i\ dz_j, \tag{17}$$

where ψ is a non-anticipating function of the state variables z_i, the summation rule is adopted, and only the terms of order dt are retained (it is recalled that $dB \propto \sqrt{dt}$). It is chosen $\psi = y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ that is inserted in Eq. (17). Retaining the terms of order dt only, using the expressions of the dy_i's from Eq. (16), dividing by dt, applying the averaging operator $E[\bullet]$, and keeping into account that because of the non-anticipating property of Itô's calculus $E[y_1^i y_2^j y_3^k\ dB] = E[y_1^i y_2^j y_3^k]E[dB] = 0$, the differential equations ruling the time evolution of the moments μ_i are obtained as

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= -2D\mu_2 - \Omega^2 m_1 + \mu_3 - \Omega^2 E\left[y_1 V\right], \\ \dot{\mu}_3 &= -2\Omega^2 \mu_2 - 4D\mu_3 - \Omega^2 E\left[y_2 V\right] \end{aligned} \tag{18}$$

where $\mu_i = E[y_i]\ \ (i = 1, 2, 3)$. System (18) is not close since it contains the cross-moments $E\left[y_1 V\right] = \mu_{14}$ and $E\left[y_2 V\right] = \mu_{24}$. Hence, specific equations are required for them, which are written by applying Itô's rule again with $dV = dy_4 = -aV\ dt + \sqrt{2\pi K}\ dB$. They are:

$$\begin{aligned} \dot{\mu}_{14} &= 2\mu_{24} - a\ \mu_{14} \\ \dot{\mu}_{24} &= -2D\ \mu_{24} - \Omega^2 \mu_{14} + \mu_{34} - \Omega^2 \mu_{144} - a\ \mu_{24}. \\ \dot{\mu}_{34} &= -4D\mu_{34} - 2\Omega^2 \mu_{24} - 2\Omega^2 \mu_{244} - a\ \mu_{34} \end{aligned} \tag{19}$$

By inspecting Eq. (19), it is noted that the set is not closed as there are the moments $\mu_{144} = E\lfloor y_1 V^2\rfloor = E\lfloor X^2 V^2\rfloor$ and $\mu_{244} = E\lfloor y_2 V^2\rfloor = E\lfloor X\dot{X}V^2\rfloor$, which are moments of fourth order in the original variables. Since the equations that are written for the moments of second order contain moments of higher order, the moment equations for system (15) constitute an infinite hierarchy. Thus, it is necessary to close it. Several methods have been proposed to perform the closure: see [26,39,46-52]. The cumulant neglect closure method is chosen here. The use of this method in the present problem was proposed in [10], where some applications were given. However, the rationale of the cumulant neglect closure method was given some years later in [49,50]. This choice is based on the relative simplicity of applying it in the present problem, while the closure methods proposed in [26,35] are cumbersome particularly from a computational point of view.

The moments μ_{144} and μ_{244} that are in Eq. (19) are expressed by putting the corresponding third-order cumulants to zero. The relationship between a third cumulant and the corresponding third-order moment is

$$\kappa_3(X_i X_j X_k) = E[X_i X_j X_k] - \sum_{1\ldots 3} E[X_i]E[X_j X_k] + E[X_i]E[X_j]E[X_k]. \tag{20}$$

By equating the left-hand-side of Eq. (20) to zero, and specializing the right-hand-side, it is obtained for the moments μ_{144} and μ_{244}:

$$\begin{aligned} \mu_{144} &= \mu_1 \mu_{44} + 2\mu_4 \mu_{14} - \mu_1 \mu_4^2 \\ \mu_{244} &= \mu_2 \mu_{44} + 2\mu_4 \mu_{24} - \mu_2 \mu_4^2, \end{aligned} \tag{21}$$

where $\mu_4 = E[V]$, $\mu_{44} = E[V^2]$. The right-hand-sides of Eq. (21) are no longer linear. Thus, they are linearized by giving the steady-state values to the moments of $V(t)$, that is $\mu_4 = E[V] = 0$, and $\mu_{44} = E[V^2] = \pi K/a$. In this way, Eq. (21) change into

$$\begin{aligned} \mu_{144} &\cong \mu_1 \mu_{44} = \frac{\pi K}{a}\mu_1 \\ \mu_{244} &\cong \mu_2 \mu_{44} = \frac{\pi K}{a}\mu_2. \end{aligned} \tag{22}$$

Inserting Eq. (22) into Eq. (19), and putting Eqs. (18) and (19) together, a linear system of first-order differential equation is obtained, which in compact matrix form reads as

$$\dot{\boldsymbol{m}}(t) = \boldsymbol{A}\boldsymbol{m}(t), \tag{23}$$

where $\boldsymbol{m}(t)$ is 6×1 vector collecting all the unknown second moments of the system states, and the cross-moments among the system states and the filter output. The solution to Eq. (23) is

$$\boldsymbol{m}(t) = \boldsymbol{m}_0 \exp(\boldsymbol{A}t), \tag{24}$$

where $\boldsymbol{m}_0$ is the vector whose entries are the initial conditions for the moments. These constitute the perturbation to zero solution for system (12) with filter (14).

The 6×6 matrix $\boldsymbol{A}$ of the coefficients has the form

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -\Omega^2 & -2D & 1 & -\Omega^2 & 0 & 0 \\ 0 & -2\Omega^2 & -4D & 0 & -2\Omega^2 & 0 \\ 0 & 0 & 0 & -a & 2 & 0 \\ -\pi K\Omega^2/a & 0 & 0 & -\Omega^2 & -(2D+a) & 1 \\ 1 & -\pi K\Omega^2/a & 0 & 0 & -2\Omega^2 & -(4D+a) \end{bmatrix}. \tag{25}$$

It is noted that the filter parameter a and the white noise strength K are present in the elements a_{44}, a_{51}, a_{55}, a_{62}, a_{66} only.

Whenever the matrix $\boldsymbol{A}$ has negative real eigenvalues and complex eigenvalues with negative real part, the statistical moments tend to zero as t grows. Since the matrix $\boldsymbol{A}$ depends on the system and filter parameters as well as on the strength of the white noise, there exist critical values of these quantities for which the real part of almost an eigenvalue becomes zero. Increasing the parameters, a real part becomes positive, and the moments grow to infinity. Thus, the problem of the stochastic stability is led to an ordinary problem of the stability of a first-order differential system. To achieve this aim, the characteristic equation is formed:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0, \tag{26}$$

where $\boldsymbol{I}$ is the unit matrix of sixth order. Eq. (26) is a sixth-order algebraic equation: for brevity's sake it is not reported, but it is quite cumbersome. Algebraic equations of order larger than the fourth have no analytical solutions so that they must be solved numerically. From a theoretical point of view, one might resort to the Routh–Hurwitz criterion of stability [59]: using this criterion, starting from the characteristic equation, as many matrices as the order of $\boldsymbol{A}$ are constructed, and the signs of their determinants must be studied so that again equations of order larger than the fourth are to be analyzed. Thus, the use of numerical tools is compulsory. It has been preferred to study the roots of Eq. (26) directly: in this way only one equation is analyzed. The study was performed by writing suitable programs in MAPLE language: a parameter is varied till an eigenvalue becomes nearly zero, while the other parameters are kept constant.

3.2. Second-order filter

The second-order filtered excitation $V(t)$ is now generated by the filter

$$\ddot{V} + 2\zeta_f \omega_f \dot{V} + \omega_f^2 V(t) = \sqrt{2\pi K}W(t). \tag{27}$$

Again, because of the linearity, $V(t)$ is a Gaussian process, whose power spectral density is variable along the ω axis [see Eq. (32) and Figs. 3, 4]. The parameters ω_f, ζ_f allow adapting the shape of its spectrum: a wider discussion is in Section 4.2. Filters of form (27) have been used for reproducing the spectrum of the ground motion [64] and that of wind turbulence [65].

The augmented system of Eqs. (12) and (27) has the four state variables: $z_1 = X$, $z_2 = \dot{X}$, $z_3 = V$, $z_4 = \dot{V}$. The Itô's equations are

$$\begin{aligned} dz_1 &= z_2\, dt \\ dz_2 &= -\left(2Dz_2 + \Omega^2 z_1 + \Omega^2 z_1 z_3\right) dt \\ dz_3 &= z_4\, dt \\ dz_4 &= -\left(\beta_f z_4 + \omega_f^2 z_3\right) dt + \sqrt{2\pi K}\, dB \end{aligned}, \tag{28}$$

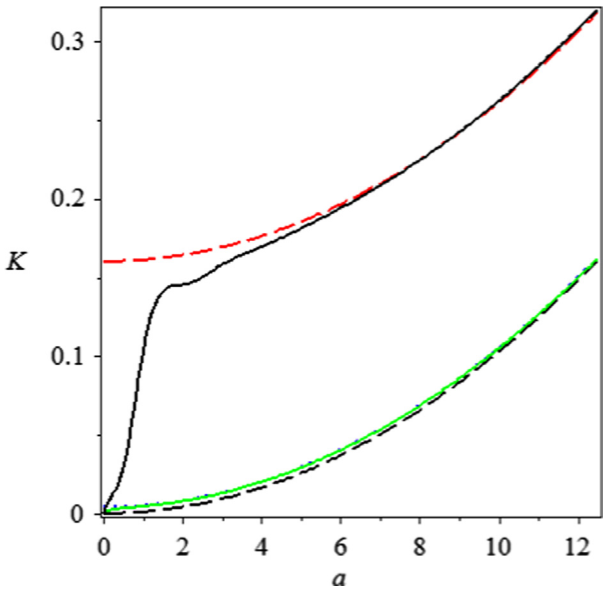

Fig. 1. System with first-order filter: plot of the critical value K of the white noise strength against the filter parameter a for $D=0.02\pi$: black line present approach, red line Ariaratnam and Tam criterion [Eq. (31)], green line Eq. (11) and black dotted line Xie's criterion with the fourth-order term, dashed black line Martin's criterion [Eq. (8)].

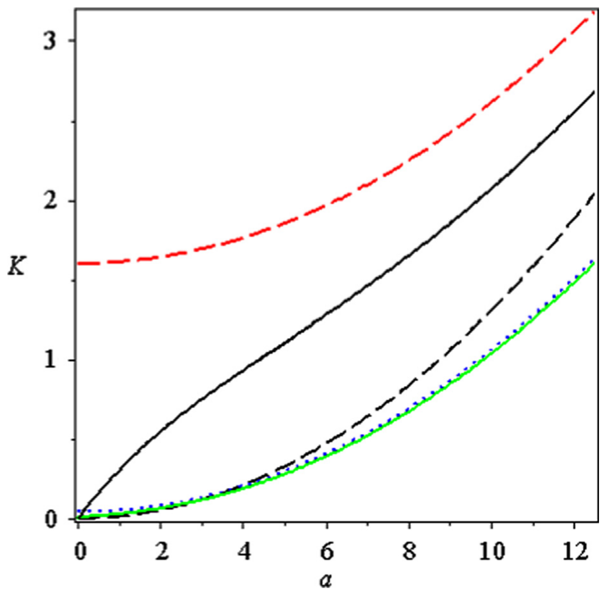

Fig. 2. System with first-order filter: plot of the critical value K of the white noise strength against the filter parameter a for $D=0.20\pi$: keys as in Fig. 1.

where $\beta_f = 2\zeta_f\omega_f$. Again, we make use of the variables $y_1 = X^2$, $y_2 = X\dot{X}$, $y_3 = \dot{X}^2$, so that Eqs. (16) and (18) are still valid. However, the equations for the cross moments among system and filter variables must be added:

$$\begin{aligned}
&\dot{m}_{14} = m_{15} + m_{24}\\
&\dot{m}_{15} = -\omega_f^2 m_{14} - \beta_f m_{15} + 2m_{25}\\
&\dot{m}_{24} = -\Omega^2 m_{14} - 2Dm_{24} + m_{34} + m_{25} - \Omega^2 m_{144}\\
&\dot{m}_{25} = -\Omega^2 m_{15} - 2Dm_{25} - \beta_f m_{25} + m_{35} - \Omega^2 m_{145}\\
&\dot{m}_{34} = -2\Omega^2 m_{24} - 4Dm_{34} + m_{35} - \Omega^2 m_{244}\\
&\dot{m}_{35} = -2\Omega^2 m_{25} - \omega_f^2 m_{34} - 4Dm_{35} - \beta_f m_{35} - \Omega^2 m_{245}
\end{aligned}, \tag{29}$$

where $\beta_f = \zeta_f\omega_f$, $m_{144} = E[y_1 V^2]$, $m_{145} = E[y_1 V\dot{V}]$, etc.

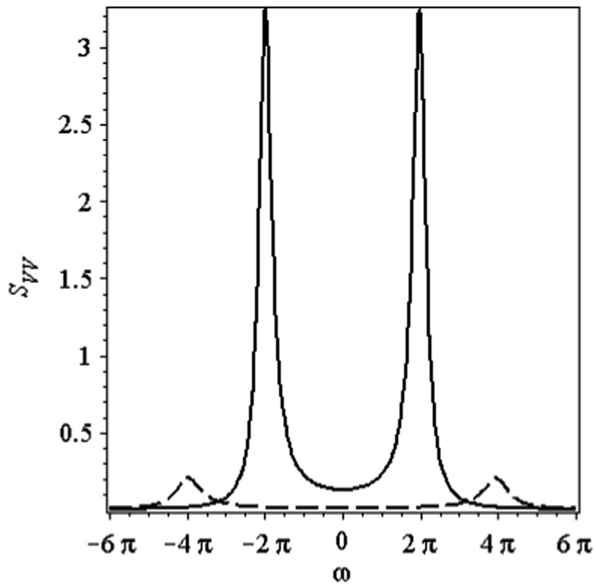

Fig. 3. Plot of the PSD of $V(t)$ output of filter (27): solid line $\omega_f = 2\pi$, dashed line $\omega_f = 4\pi$ ($\zeta_f = 0.10$ in both cases).

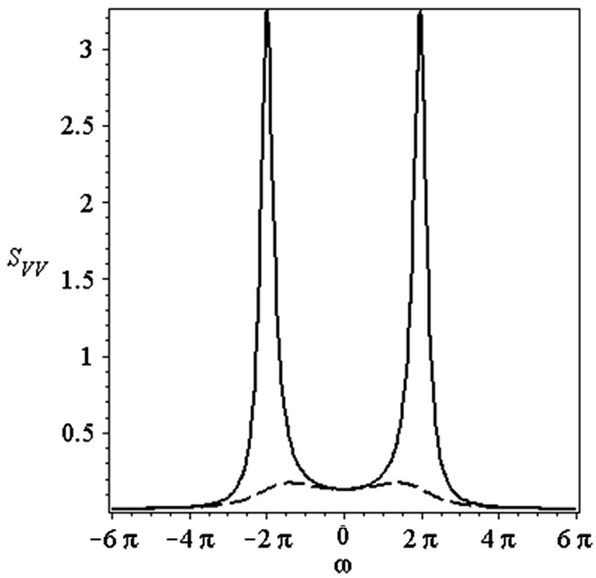

Fig. 4. Plot of the PSD of $V(t)$ output of filter (27): solid line $\zeta_f = 0.10$, dashed line $\zeta_f = 0.50$ ($\omega_f = 2\pi$ in both cases).

In four out of six of Eq. (29) there are hierarchical terms, that is third-order moments m_{ijk} that are of fourth order in the original variables. Again, they are expressed by equating the corresponding cumulants to zero, and giving the steady-state values to moments of the filter variables. It is obtained

$$m_{144} \cong \sigma_V^2 m_1, \quad m_{145} \cong 0, \quad m_{2144} \cong \sigma_V^2 m_2, \quad m_{245} \cong 0, \tag{30}$$

as $E[V] = E[\dot{V}] = 0$ and $E[V^2] = \sigma_V^2 = \pi K/(2\zeta_f \omega_f^3)$.

Proceeding in this way, the moment equations form a linear system of nine first-order ordinary differential equations as Eq. (23). Thus, the matrix $\boldsymbol{A}$ of the coefficients is a 9×9 one, and it is not reported for brevity's sake; however, the study is more complex. It runs through the same steps as in previous case: the characteristic equation $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$ is formed, and its eigenvalues are studied.

4. Numerical analyses

4.1. System with first-order filter

The second-order oscillator of Eq. (12) have a stochastic perturbation in the stiffness given by the output $V(t)$ of the filter (14). The numerical analyses are aimed at finding the critical value of K, the strength of the white noise; it is recalled that the variance σ_V^2 of $V(t)$ is related to K as $\sigma_V^2 = \pi K / a$. The other parameters are kept constant, and they take the values: $\Omega = 2\pi$, $D = \zeta_0 \Omega = 0.02\pi$ or 0.2π, being ζ_0 the ratio of critical damping. The parameter a is varied from 0.1 to 10: for each value of a K is increased till the real part of an eigenvalue of the matrix $\boldsymbol{A}$ in Eq. (25) becomes zero. Each analysis requires few seconds of computation on a work station with 2.5 MHz of clock speed.

The stability curves are plotted in Fig. 1 for $D = 0.02\pi$ $(\zeta_0 = 0.01)$, and in Fig. 2 for $D = 0.2\pi$ $(\zeta_0 = 0.10)$: for each curve the stable region is below it. The results of the present approach are denoted by a black continuous line, and are compared with those deriving from the stability criteria by Ariaratnam and Tam [15], by Martin [16], and by Xie [33]. As regards the first, the oscillator for which it is derived coincides with Eq. (12) if $\varepsilon = \Omega^2$ and $\rho_0 = \zeta_0 / \Omega^4$. Taking into account that the power spectral density of $V(t)$ is $K/(a^2 + \omega^2)$, and solving Eq. (7) for the critical value of K, it is obtained

$$K_{cr} = \frac{2\zeta_0}{\pi\Omega}\left(a^2 + 4\Omega^2\right). \tag{31}$$

In Figs. 1 and 2 Eq. (31) is the red dashed line. In applying Martin's criterion, the black dashed lines in the figures, the right-hand-side of Eq. (8) assumes the values that are obtained by an optimization procedure.

For both values of the damping K_{cr} increases with a. At a parity of a a larger damping causes a larger critical value of K_{cr}. For small damping (Fig. 1) the present approach agrees well with Eq. (31) except for values of a lesser than 1 when Eq. (31) surely overestimates K_{cr}. The other criteria predict definitely smaller values of K_{cr}, and their curves superimpose. It is outlined that Martin's criterion is a sufficient condition only so that a lower bound of the stability limit is obtained. Xie's criterion relies on the hypothesis of weak excitation that is not made here: thus, the comparison is not completely significant. The inclusion of higher order terms in the expression of the moment Lyapunov exponent has a negligible effect: the curve that takes the fourth-order term into account is nearly coincident with that deriving from the second-order term only.

In the case of a higher value of damping (Fig. 2) the curves of the different criteria are more spread. However, a ratio of critical damping of 0.10 undermines the hypothesis of weak excitation more. In other words, probably the criterion of Ariaratnam and Tam is not applicable for this value of damping, and it tends to overestimate K_{cr}.

4.2. System with second-order filter

$V(t)$ in Eq. (12) is now the output of filter (27). The power spectral density and the variance of $V(t)$ are, respectively

$$S_{VV}(\omega) = \frac{2K}{(\omega_f^2 - \omega^2)^2 + 4\zeta_f^2 \omega_f^2 \omega^2}, \tag{32}$$

$$E[V^2] = \sigma_V^2 = \frac{\pi K}{2\zeta_f \omega_f^3}, \tag{33}$$

where ω denotes the angular frequency. As the filter has two parameters ω_f and ζ_f, there is sufficient flexibility to accommodate spectra of different shapes. Two spectra are compared in Fig. 3, the one with $\omega_f = 2\pi$, the other with $\omega_f = 4\pi$, being K the same: as the peak frequency ω_f increases, the area below the curve is smaller, that is the process $V(t)$ has less power. Moreover, the process $V(t)$ tends to be narrow-banded, while a white noise has infinite band. In Fig. 4 the PSD's that are compared have the same peak frequency $\omega_f = 2\pi$ and two different damping ratios, $\zeta_f = 0.10$ and $\zeta_f = 0.50$. The power again decreases with the damping ratio, and σ_V^2 decreases accordingly.

The first group of numerical investigations is aimed at finding the critical value of excitation, the critical value of the mean square σ_V^2 of $V(t)$ or the critical value of the white noise strength K_{cr}. In both cases it is studied the dependence on filter frequency ω_f that varies in the interval [0.5, 18], while the other parameters are fixed: $\Omega = 2\pi$, $\zeta_0 = 0.01$ $(D = \zeta_0 \Omega)$, $\zeta_f = 0.10$. In Fig. 5 the critical value of σ_V^2 is plotted against ω_f. There is a qualitative agreement with Bolotin's plots [10,55]: in fact there are two minima and a maximum. However, the curves by Bolotin have cusps in correspondence of the extrema, while the present results show a continuous variation which is more justified theoretically. The curve has a pronounced minimum when the filter frequency ω_f is tuned to 2Ω, that is when $\omega_f = 4\pi$. This phenomenon is termed stochastic resonance (e.g. see [66]). On the whole, the curves presented in [35,38] look different, but these curves too have the minimum for 2Ω.

In Fig. 6 the critical value of the white noise strength K_{cr} is plotted against ω_f. From Eq. (33) one obtains $K_{cr} = 2\zeta_f \omega_f^3 (\sigma_V^2)_{cr} / \pi$. The curve of the present approach is compared with Ariaratnam and Tam's criterion, which because of the power spectral density (32) results in

$$\sigma_{V_{cr}}^2 = \frac{2\zeta_0\left[(\omega_f^2 - 4\Omega^2)^2 + 16\zeta_f^2 \omega_f^2 \Omega^2\right]}{\pi\Omega}. \tag{34}$$

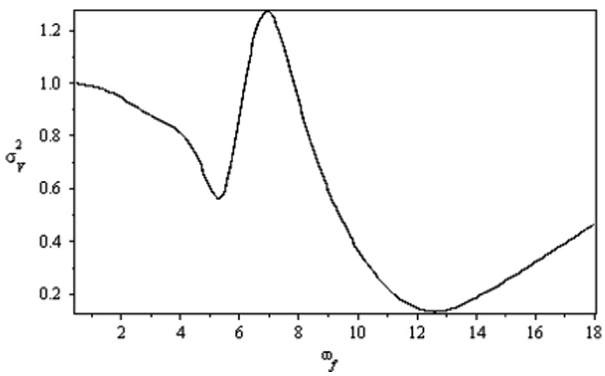

Fig. 5. Second-order parametric excitation $V(t)$ of Eq. (27): plot of the critical value of the mean square σ_V^2 against the filter frequency ω_f; the other parameters are $\Omega = 2\pi$, $\zeta_0 = 0.01$, $\zeta_f = 0.10$.

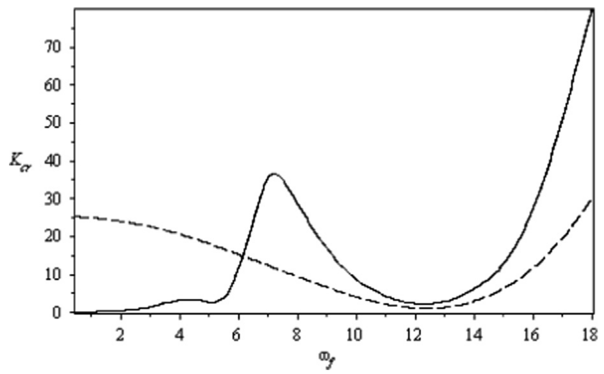

Fig. 6. Second-order parametric excitation $V(t)$ of Eq. (27): plot of the critical value of the white noise strength K_{cr} against the filter frequency ω_f: continuous line present approach, dashed line Eq. (34); the other parameters are as in Fig. 5.

As expected, the critical curve is similar to that of Fig. 5 with the severe minimum for $\omega_f = 4\pi$, but the first minimum is faded. Eq. (34) agrees with the present approach only in the neighbors of $\omega_f = 4\pi$, while it underestimates K_{cr} for $\omega_f < 4\pi$, and it overestimates K_{cr} for $\omega_f > 4\pi$. Again, it is recalled that Ariaratnam and Tam's criterion relies on the hypothesis of weak excitation, which is not formulated here.

Figs. 7 and 8 plot the critical values of the white noise strength K_{cr} against the nominal pulsation Ω of the oscillator, while the other parameters are fixed: $\zeta_0 = 0.01$, $\zeta_f = 0.10$, $\omega_f = 2\pi$ in Fig. 7, and $\omega_f = 4\pi$ in Fig. 8. The dashed lines are the critical curves deriving from Ariaratnam and Tam's criterion. According to this K_{cr} is given by the right-hand-side of Eq. (34) multiplied by $(2\zeta_f\omega_f^3)/\pi$. Both curves obtained by the proposed approach have a deep minimum: in Fig. 7 the minimum is at $\Omega = \pi$, while in Fig. 8 it is at $\Omega = 2\pi$. In this way, the location of the minimum is at $\Omega = \omega_f/2$. Hence, the minimum depends on the filter frequency, what was not clearly highlighted in Refs. [10, 35, 55]. When $\omega_f = 4\pi$ the parametric excitation $V(t)$ has few power (see Fig. 3), but the minimum is equally small: K_{cr} is worth 2.20676 only. When $\omega_f = 2\pi$ Ariaratnam and Tam's criterion agrees with the present approach in the neighborhood of the minimum only. On the other hand, when $\omega_f = 4\pi$ the present approach and Ariaratnam and Tam's criterion agrees except for $\Omega < 4$ rad/s.

In the last group of analyses the effect of the critical damping ratio ζ_0 is investigated. It is varied between 0.005 and 0.75, while the other parameters always hold the values: $\Omega = 2\pi$, $\omega_f = 2\pi$, $\zeta_f = 0.10$. The results are in Fig. 9, where K_{cr} is plotted against ζ_0. When ζ_0 is in the range (0.005, 0.25), roughly K_{cr} increases linearly. Then, the slope diminishes, and the increase is less than linear. However, in practical cases damping ratios larger than 0.20–0.30 are hardly conceivable. In this plot the stability limit proposed by Ariaratnam and Tam is not reported as it gives acceptable estimates of K_{cr} only when ζ_0 does not exceed 0.05: larger values of ζ_0 violate the principles of the stochastic averaging procedure so that K_{cr} is heavily overestimated.

5. Conclusions

In the present paper, the stochastic stability of a second-order linear oscillator with a stochastic perturbation in the stiffness is investigated with reference to the mean square stability. The parametric excitation is a zero mean stationary,

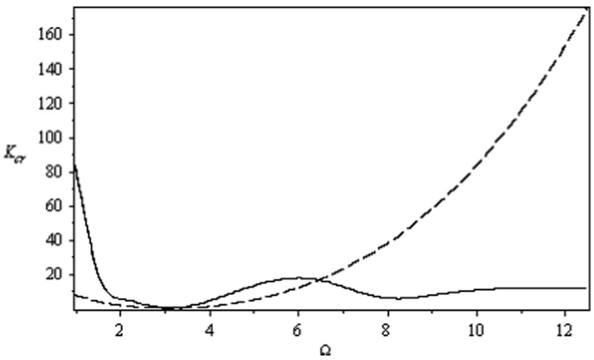

Fig. 7. Second-order parametric excitation $V(t)$ of Eq. (27): plot of the critical value of the white noise strength K_{cr} against the oscillator pulsation Ω: continuous line present approach, dashed line Eq. (34); ($\zeta_0=0.01,\ \ \omega_f=2\pi,\ \ \zeta_f=0.10$).

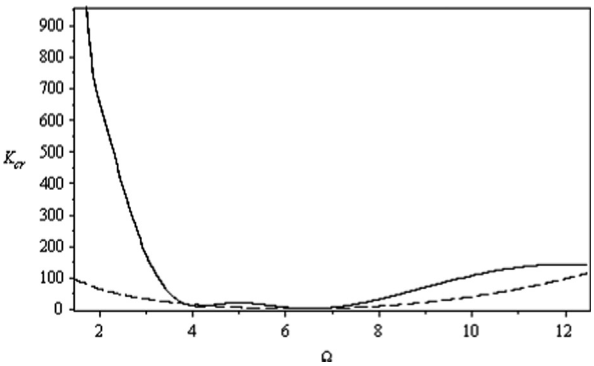

Fig. 8. Second-order parametric excitation $V(t)$ of Eq. (27): plot of the critical value of the white noise strength K_{cr} against the oscillator pulsation Ω: continuous line present approach, dashed line Eq. (34); ($\zeta_0=0.01,\ \ \omega_f=4\pi,\ \ \zeta_f=0.10$).

ergodic and colored stochastic Gaussian process that is obtained by passing a white noise through a first-order or a second-order linear filter. The stability of the second-order differential equations was widely studied in the past, but in the majority of the articles the parametric excitation is restricted to be a Gaussian white noise or it is amenable to a white noise [2,6,21,23], while there are few papers in which the parametric excitation is a colored Gaussian process [10,27,33,35,38,39]. In fact, the determination of the stability bound is relatively simple when the excitation is a white noise and it is much more complicated if the excitation is colored.

Constructing the excitation as is explained above, the Markov methods of the stochastic dynamics are applicable. In this way, the differential equations ruling the second moments of the response are formed by means of Itô's differential rule. The difficulty intrinsic to the case of colored Gaussian excitation is that the moment equations form an infinite hierarchy, that herein is closed by applying the cumulant neglect closure method. Then, the moment equations are linearized. There is stability in the second moments, if all the eigenvalues of the matrix of the coefficients of the moment equations have negative real parts. The limit of stability is reached when the real part of an eigenvalue becomes null. In this way, the stochastic stability study is reduced to the deterministic study of the eigenvalues of a matrix.

Extended numerical investigations have been performed to find the stability boundaries: searches were made for the critical values of the strength K of the white noise, which generates the parametric excitation $V(t)$ through the filter. Particular attention is devoted to detect the phenomenon of the stochastic resonance [66], namely whether there is a remarkable decrease of the critical value K_{cr} for some combination of the parameters.

Analyzing the results, it is found that: (1) this way of defining the excitation does not require the assumption that it is weak. (2) As is shown in Figs. 1 and 2, the stochastic resonance does not appear when the excitation is the output of a first-order filter (Langevin equation), which is caused by the monotonic decrease of the power spectral density of the excitation $S_{VV}(\omega)\left(=aK/\left(a^2+\omega^2\right)\right)$ that has a small value in correspondence of the nominal pulsation of the oscillator $\Omega=2\pi$ rad/s. (3) When the excitation is the output of a second-order filter, the power spectral density $S_{VV}(\omega)$ has two symmetric peaks in $\pm\omega_f$(see Figs. 3 and 4). If ω_f is tuned to 2Ω, there is the phenomenon of stochastic resonance as is shown in Figs. 5–8, that is K_{cr} has a marked minimum. (4) Increasing the oscillator damping enlarges the stability region so that the stochastic resonance can be considerably attenuated.

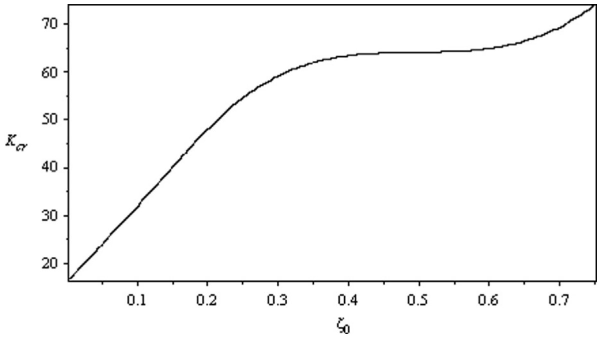

Fig. 9. Second-order parametric excitation $V(t)$ of Eq. (27): plot of the critical value of the white noise strength K_{cr} against the ratio of critical damping of the oscillator ζ_0 ($\Omega=2\pi,\ \ \omega_f=2\pi,\ \ \zeta_f=0.10$).

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Appendix A

The lateral vibration of a compressed Bernoulli–Navier beam is governed by the following equation [67]:

$$EI\frac{\partial^4 w}{\partial z^4}+bI\frac{\partial^5 w}{\partial z^4 \partial t}+[P_0+P(t)]\frac{\partial^2 w}{\partial z^2}+\rho\frac{\partial^2 w}{\partial t^2}=0, \tag{A.1}$$

where $w(t)$ is the lateral displacement, I is the second moment of the cross-section area, ρ is the mass for unit length of the column, P_0 is the static axial load, $P(t)$ is the time-varying stochastic load, E is Young's modulus and b is the damping coefficient.

Introducing the normal modes, the lateral deflection is expressed as

$$w(z,t)=\sum_0^\infty {}_k X_k(t)\ \phi_k(z). \tag{A.2}$$

In the case of a hinged–hinged beam $\phi_k(z)=\ \sin(k\pi/L)z$, where L is beam length (however the final equation is the same even for other end conditions). Inserting Eq. (A.2) into (A.1) it is obtained

$$\frac{\mathrm{d}^2 X_k}{\mathrm{d}t^2}+2\zeta_k\omega_k\frac{\mathrm{d}X_k}{\mathrm{d}t}+\omega_k^2\left[1-\frac{P_0+P(t)}{N_{cr_k}}\right]X_k=0, \tag{A.3}$$

where $N_{cr_k}=EI\lambda_k^2/L^2$ is the kth Euler's buckling load, $\omega_k=(\lambda_k^2/L^2)\sqrt{EI/\rho}$ is the kth natural frequency of the beam, and $\zeta_k=(\lambda_k^2/2L^2)\sqrt{b^2I/E\rho}$ is the ratio of critical damping of the kth mode. The eigenvalues λ_k depend on the end restraints, and they are equal to $k\pi$ for a hinged–hinged beam.

By making the positions $F_k(t)=P(t)/(N_{cr_k}-P_0)$, $\Omega_k=\omega_k\sqrt{1-(P_0/N_{cr_k})}$, $\overline{\zeta}_k=\zeta_k\sqrt{1-(P_0/N_{cr_k})}$, Eq. (A.3) is rewritten as

$$\frac{\mathrm{d}^2 X_k}{\mathrm{d}t^2}+2\overline{\zeta}_k\Omega_k\frac{\mathrm{d}X_k}{\mathrm{d}t}+\Omega_k^2[1-F_k(t)]X_k=0. \tag{A.4}$$

Clearly, $F_k(t)$ is a zero-mean stochastic Gaussian process: since the processes $F_k(t)$ and $-F_k(t)$ have the same statistical characteristics, Eq. (A.4) is equivalent to Eq. (12) that is studied in the present paper.

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