A Weak-L^p Prodi–Serrin Type Regularity Criterion for the Navier–Stokes Equations

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Communicated by R. Shvydkoy

Abstract. We give simple proofs that a weak solution u of the Navier–Stokes equations with $\mathbf{H}^1$ initial data remains strong on the time interval $[0, T]$ if it satisfies the Prodi–Serrin type condition $u \in L^s(0, T; \mathbf{L}^{r,\infty}(\Omega))$ or if its $L^{s,\infty}(0, T; \mathbf{L}^{r,\infty}(\Omega))$ norm is sufficiently small, where $3 < r \leq \infty$ and $(3/r) + (2/s) = 1$.

Mathematics Subject Classification. 35Q30, 76D03, 76D05.

Keywords. Navier–Stokes equations, weak solutions, strong solutions, blow-up, regularity criteria, weak Lebesgue spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be either a bounded domain with smooth boundary $\partial\Omega$ or the whole space $\mathbb{R}^3$, and let $T > 0$ be fixed but arbitrary. In $\Omega \times (0, T)$, we consider the dimensionless form of the Navier–Stokes equations describing the flow of a homogeneous incompressible fluid

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p \\ \nabla \cdot u = 0, \end{cases}$$

where u represents the velocity field, $\nu > 0$ the kinematic viscosity, and p the pressure. The system is supplemented with the no-slip boundary condition

$$u(x, t)_{|x \in \partial\Omega} = 0.$$

Notation Let $\mathcal{V}$ be the space of divergence-free vector fields belonging to $\mathbf{C}_0^\infty(\Omega)$. We denote by $\mathbf{L}^2_{\text{div}}(\Omega)$ and $\mathbf{H}^1_{0,\text{div}}(\Omega)$ the closures of $\mathcal{V}$ in the norms of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1_0(\Omega)$, respectively. For $p \in [1, \infty]$, let $\| \cdot \|_p$ be the standard norm in $L^p(\Omega)$. In addition, given $p \in [1, \infty)$ and a measurable set $M \subset \mathbb{R}^n$, we denote by $L^{p,\infty}(M)$ the space of weak-L^p functions on M, and we set

$$\|f\|_{p,\infty} = \sup_{t>0} t \big[\mu \{ \tau \in [0, T] : |f(\tau)| > t \} \big]^{\frac{1}{p}},$$

where μ is the Lebesgue measure on M.

Introducing the Stokes operator A on $\mathbf{L}^2_{\text{div}}(\Omega)$

$$Au = -P\Delta u \qquad \text{with domain} \qquad \mathbf{H}^1_{0,\text{div}}(\Omega) \cap \mathbf{H}^2(\Omega),$$

where P is the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_{\text{div}}(\Omega)$, the Navier–Stokes system takes the form

$$\partial_t u + \nu A u + P(u \cdot \nabla) u = 0. \tag{NS}$$

After the works of Leray [5] and Hopf [4], it is well known that for any initial condition $u(0) \in \mathbf{L}^2_{\text{div}}(\Omega)$ the Eq. (NS) has at least one weak solution $u \in L^\infty(0, T; \mathbf{L}^2_{\text{div}}(\Omega)) \cap L^2(0, T; \mathbf{H}^1_{0\ ,\text{div}}(\Omega))$. At the same time, whenever $u(0) \in \mathbf{H}^1_{0,\text{div}}(\Omega)$ there exists $T_\star \in (0, \infty]$ such that (NS) admits a unique strong solution

JCR is supported by an EPSRC Leadership Fellowship EP/G007470/1.

$u \in L^\infty(0,T;\mathbf{H}^1_{0,\mathrm{div}}(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ provided that $T < T_\star$. If $T_\star < \infty$ then the solution must blow up in $\mathbf{H}^1$, i.e.

$$\lim_{t\to T_\star} \|\nabla u(t)\|_2 = \infty.$$

According to a result of Prodi [7] (see also Serrin [8]), the existence of a strong solution to (NS) on the whole interval $[0,T]$ is guaranteed if

$$u \in L^s(0,T;\mathbf{L}^r(\Omega)),$$

where (r,s) is a Prodi–Serrin pair, that is, $r \in (3,\infty]$ and $s \in [2,\infty)$ satisfy

$$\frac{3}{r} + \frac{2}{s} = 1.$$

The proof in [2] of a similar result when $u \in L^\infty(0,T;\mathbf{L}^3(\mathbb{R}^3))$ is significantly more involved.

The aim of this note is to provide a short proof of the following two generalisations of the result for $r \in (3,\infty]$ using weak Lebesgue spaces.

Theorem 1.1. *Assume that $u(0) \in \mathbf{H}^1_{0,\mathrm{div}}(\Omega)$ and that u is a weak solution to (NS) with this initial condition that satisfies $u \in L^s(0,T;\mathbf{L}^{r,\infty}(\Omega))$ for some Prodi–Serrin pair (r,s). Then u remains strong on $[0,T]$ and is therefore unique.*

Theorem 1.2. *For every Prodi–Serrin pair (r,s), there is a constant $c > 0$ depending only on r and Ω such that if $u(0) \in \mathbf{H}^1_{0,\mathrm{div}}(\Omega)$ and if u is a weak solution to (NS) with this initial condition that satisfies the estimate*

$$\|u\|_{L^{s,\infty}(0,T;\mathbf{L}^{r,\infty}(\Omega))} \leqslant c\nu^{1-\frac{1}{s}},$$

then u remains strong on $[0,T]$ and is therefore unique.

Theorem 1.1 in the whole space $\Omega = \mathbb{R}^3$ can be found in [1], whereas Theorem 1.2 is proved in [9] for a small constant $c > 0$, although the value of this constant is not explicit. On the contrary, in our proof the value of c can be in principle explicitly calculated.

2. Proof of Theorem 1.1

First, we establish a suitable estimate for the nonlinear term appearing in the equation.

Lemma 2.1. *Let $u \in \mathbf{H}^1_{0,\mathrm{div}}(\Omega) \cap \mathbf{H}^2(\Omega)$ be given, and let (r,s) be a Prodi–Serrin pair. Then*

$$\|(u\cdot\nabla)u\|_2 \leqslant C_r \|u\|_{r,\infty} \|\nabla u\|_2^{\frac{2}{s}} \|Au\|_2^{1-\frac{2}{s}}.$$

Proof. Take $\epsilon \in (0,1)$; its value will be chosen later. Applying the Hölder inequality, we easily deduce

$$\|(u\cdot\nabla)u\|_2 \leqslant \|u\|_{r+\epsilon} \|\nabla u\|_{\frac{2(r+\epsilon)}{r+\epsilon-2}}.$$

We need to estimate the two terms appearing on the right hand side. To this end, we make use of the interpolation inequality holding in weak-L^p spaces (see [3])

$$\|u\|_{r+\epsilon} \leqslant \Big(\frac{r+\epsilon}{\epsilon}\Big)^{\frac{1}{r+\epsilon}} \|u\|_{r,\infty}^{\frac{r}{r+\epsilon}} \|u\|_\infty^{\frac{\epsilon}{r+\epsilon}}.$$

In particular, the constant is easily seen to be uniformly bounded for $r > 3$, for any fixed ϵ. We also recall the following Gagliardo–Nirenberg type inequality, valid both on the whole space and on bounded domains where a Poincaré type inequality is true:

$$\|\nabla u\|_p \leqslant C\|u\|_q^{1-\alpha} \|A^\sigma u\|_2^\alpha.$$

Here, the exponents satisfy the relations

$$\frac{1}{p} = \frac{1}{3} + \Big(\frac{1}{2} - \frac{2\sigma}{3}\Big)\alpha + \frac{1-\alpha}{q} \qquad \text{and} \qquad \frac{1}{2\sigma} \leqslant \alpha \leqslant 1.$$

Since $\epsilon < 1$, the Gagliardo–Nirenberg inequality above reduces to

$$\|\nabla u\|_{\frac{2(r+\epsilon)}{r+\epsilon-2}} \leqslant C_{r,\epsilon}\|u\|_{r+\epsilon}^{\frac{\epsilon}{r}}\|A^\sigma u\|_2^{\frac{r-\epsilon}{r}},$$

where σ is given by

$$\sigma = \frac{2r-3\epsilon+6}{4(r-\epsilon)}.$$

We observe that $\sigma > \frac{1}{2}$. We now fix

$$\epsilon < 2r-6,$$

so that $\sigma < 1$. Finally, the L^2-norm of fractional powers of the Stokes operator satisfies the interpolation inequality

$$\|A^\sigma u\|_2 \leqslant C\|\nabla u\|_2^{2(1-\sigma)}\|Au\|_2^{2\sigma-1}, \quad \forall\sigma \in \left[\tfrac{1}{2}, 1\right].$$

Using all the results recalled above, we can easily prove the statement of the Lemma by arguing as follows:

$$\begin{aligned}
\|(u\cdot\nabla)u\|_2 \leqslant & \|u\|_{r+\epsilon}\|\nabla u\|_{\frac{2(r+\epsilon)}{r+\epsilon-2}} \\
\leqslant & C_r\|u\|_{r+\epsilon}^{\frac{r+\epsilon}{r}}\|A^{\frac{2r-3\epsilon+6}{4(r-\epsilon)}}u\|_2^{\frac{r-\epsilon}{r}} \\
\leqslant & C_r\|u\|_{r,\infty}\|u\|_\infty^{\frac{\epsilon}{r}}\|\nabla u\|_2^{\frac{(2r-\epsilon-6)}{2r}}\|Au\|_2^{\frac{6-\epsilon}{2r}} \leqslant C_r\|u\|_{r,\infty}\|\nabla u\|_2^{\frac{r-3}{r}}\|Au\|_2^{\frac{3}{r}},
\end{aligned}$$

where in the last line we exploited the Agmon-type inequality

$$\|u\|_\infty \leqslant C\|\nabla u\|_2^{\frac{1}{2}}\|Au\|_2^{\frac{1}{2}}.$$

Recalling the definition of (r,s), we are done.

This estimate on the nonlinear term leads to an energy estimate for (NS).

Lemma 2.2. *For every Prodi–Serrin pair (r,s), the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 \leqslant C_r\nu^{1-s}\|u\|_{r,\infty}^s\|\nabla u\|_2^2$$

holds while the solution remains strong.

Proof. While the solution remains strong it has sufficient regularity that we can multiply (NS) by Au. Then on account of Lemma 2.1 we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + \nu\|Au\|_2^2 &= -\langle(u\cdot\nabla)u, Au\rangle \\
&\leqslant \|(u\cdot\nabla)u\|_2\|Au\|_2 \leqslant C_r\|u\|_{r,\infty}\|\nabla u\|_2^{\frac{2}{s}}\|Au\|_2^{2-\frac{2}{s}}.
\end{aligned}$$

By means of Young's inequality we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + \nu\|Au\|_2^2 \leqslant C_r\nu^{1-s}\|u\|_{r,\infty}^s\|\nabla u\|_2^2,$$

which yields the desired estimate.

To complete the proof of Theorem 1.1 we argue by contradiction. Suppose that the solution remains strong only on the interval $[0,T')$ with $T' < T$. By virtue of Lemma 2.2, it follows from the classical Gronwall lemma that $\|\nabla u(t)\|_2$ remains bounded on $[0,T]$. But if $[0,T')$ is the maximal interval of existence for a strong solution then $\|\nabla u(t)\|_2 \to \infty$ as $t \to T'$. It follows that the solution remains strong on $[0,T)$, and a further application of Lemma 2.2 and the Gronwall lemma guarantee that the solution is strong on $[0,T]$.

3. Proof of Theorem 1.2

To prove Theorem 1.2 we will use a generalised Gronwall inequality of the following form.

Lemma 3.1. *Let φ be a measurable positive function defined on the interval $[0,T]$. Suppose that there exists an $\epsilon_0 > 0$ and a constant $\kappa > 0$ such that for all $0 < \epsilon < \epsilon_0$ and a.e. $t \in [0,T]$, φ satisfies the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi \leqslant \kappa \lambda^{1-\epsilon}\varphi^{1+2\epsilon}, \tag{3.1}$$

where $\lambda \in L^{1,\infty}(0,T)$ with

$$\kappa \|\lambda\|_{1,\infty} < \frac{1}{2}.$$

Then φ is bounded on $[0,T]$.

Proof. First we note that if $\kappa\|\lambda\|_{1,\infty} < \frac{1}{2}$ then

$$\kappa \limsup_{\epsilon \to 0} \epsilon \int_0^T \lambda^{1-\epsilon}(s)\,\mathrm{d}s < \frac{1}{2}. \tag{3.2}$$

Indeed, a straightforward computation yields

$$\epsilon \int_0^T \lambda^{1-\epsilon}(t)\,\mathrm{d}t = \epsilon(1-\epsilon)\int_0^\infty \frac{1}{t^\epsilon}\,\mu\left\{\tau \in [0,T] : \lambda(\tau) > t\right\}\,\mathrm{d}t$$

$$\leqslant \epsilon T + \epsilon(1-\epsilon)\|\lambda\|_{1,\infty}\int_1^\infty \frac{1}{t^{1+\epsilon}}\,\mathrm{d}t = \epsilon T + (1-\epsilon)\|\lambda\|_{1,\infty},$$

from which (3.2) follows. Now choose $\delta > 0$ such that

$$\limsup_{\epsilon \to 0} 2\kappa\epsilon \int_0^T \lambda^{1-\epsilon}(s)\,\mathrm{d}s < 1 - 3\delta.$$

If we integrate (3.1) from 0 to $t < T$ then we obtain

$$-\varphi^{-2\epsilon}(t) + \varphi^{-2\epsilon}(0) \leq 2\kappa\epsilon \int_0^t \lambda^{1-\epsilon}(s)\,\mathrm{d}s \leq 2\kappa\epsilon \int_0^T \lambda^{1-\epsilon}(s)\,\mathrm{d}s.$$

Now we choose ϵ sufficiently small that

$$2\kappa\epsilon \int_0^T \lambda^{1-\epsilon}(s)\,\mathrm{d}s < 1 - 2\delta \quad \text{and} \quad \varphi^{-2\epsilon}(0) > 1 - \delta$$

from which it follows that $-\varphi^{-2\epsilon}(t) < -\delta$, i.e. $\varphi(t) \leq \delta^{-1/2\epsilon}$ for all $t < T$.

Observe that the constant $\frac{1}{2}$ appearing above is optimal. Indeed, setting $T = \kappa = 1$ for simplicity, consider for $t \in [0,1]$ the family of inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi \leqslant \lambda^{1-\epsilon}\varphi^{1+2\epsilon},$$

where $\lambda(t) = \alpha(1-t)^{-1}$ and $\varphi(0) = 1$. A straightforward computation gives $\|\lambda\|_{1,\infty} = \alpha$. An integration on $(0,t)$ of the above differential inequality at fixed ϵ gives

$$\varphi^{2\epsilon}(t) \geqslant \frac{1}{1 + 2\alpha^{1-\epsilon}(1-t)^\epsilon - 2\alpha^{1-\epsilon}}.$$

If $\alpha \geqslant \frac{1}{2}$ then no matter how small we take ϵ there is always a value of $t_\epsilon < 1$ for which the denominator in the right-hand side vanishes. Accordingly, $\varphi(t)$ blows up before $t = 1$.

In order to apply Lemma 3.1 to the present setting, we adapt slightly the result of Lemma 2.2, as follows (cf. [6, Lemma 9.3]).

Lemma 3.2. *For every Prodi–Serrin pair (r, s) and for any ϵ sufficiently small, while the solution u remains strong it satisfies the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 \leqslant C_r \nu^{1-s} \|u\|_{r,\infty}^{s(1-\epsilon)} \|\nabla u\|_2^{2(1+2\epsilon)}.$$

Proof. This is an immediate consequence of Lemma 2.2 if we choose

$$r_\epsilon = \frac{3s + 3\epsilon(4-s)}{s - 2 + \epsilon(4-s)} \qquad \text{and} \qquad s_\epsilon = s + \epsilon(4-s).$$

In particular, a standard interpolation gives

$$\|u\|_{r_\epsilon,\infty}^{s_\epsilon} \leqslant \|u\|_{r,\infty}^{s(1-\epsilon)} \|u\|_{6,\infty}^{4\epsilon} \leqslant C^\epsilon \|u\|_{r,\infty}^{s(1-\epsilon)} \|\nabla u\|_2^{4\epsilon},$$

from which we immediately deduce the claimed result. □

At this point, combining Lemmas 3.1 and 3.2, we readily obtain that the solution remains bounded in $\mathbf{H}^1_{0,\mathrm{div}}(\Omega)$ on $[0, T]$ provided that

$$C_r \nu^{1-s} \|u\|^s_{L^{s,\infty}(0,T;\mathbf{L}^{r,\infty}(\Omega))} < \frac{1}{2}.$$

Acknowledgments. We would like to thank Roman Shvydkoy for a close reading of the original version of this paper, which led in particular to a significant improvement to both the statement and proof of Lemma 3.1. The first author would also like to thank James Robinson for the warm invitation to the University of Warwick where this work originated.

References

[1] Bjorland, C., Vasseur, A.: Weak in space, log in me improvement of the Ladyženskaja–Prodi–Serrin criteria. J. Math. Fluid Mech. **13**, 259–269 (2011)
[2] Escriaza, L., Seregin, G., Šverák, V.: $L_{3,\infty}$-soluons of Navier–Stokes equaons and backwards uniqueness. Russ. Math. Surv. **58**, 211–250 (2003)
[3] Grafakos, L.: Classical Fourier Analysis. Springer, New York (2008)
[4] Hopf, E.: Uber die Anfangswertfgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. **4**, 213–231 (1950)
[5] Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
[6] Pata, V.: On the regularity of solutions to the Navier–Stokes equations. Commun. Pure Appl. Anal. **11**, 747–761 (2012)
[7] Prodi, G.: Un teorema di unicità per le equazioni di Navier–Stokes. Ann. Mat. Pura Appl. **48**, 173–182 (1959)
[8] Serrin, J.: On the interior regularity of weak soluons of the Navier–Stokes equations. Arch. Ration. Mech. Anal. **9**, 187–195 (1962)
[9] Sohr, H.: A regularity class for the Navier–Stokes equaons in Lorentz spaces. J. Evol. Equ. **1**, 441–467 (2001)

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(accepted: April 23, 2014; published online: July 29, 2014)