LONGTIME BEHAVIOR OF NONLOCAL CAHN-HILLIARD EQUATIONS

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1. INTRODUCTION

The Cahn-Hilliard equation was proposed in [7] as a model for (isothermal) phase separation phenomena in binary alloys. Since then it was analyzed by many authors and used in several different contexts (see, e.g., [9, 40] and references therein). The basic form of such an equation is the following

$$(1.1) \qquad \partial_t \varphi = \nabla \cdot [\kappa(\varphi) \nabla \mu],$$

1

where φ is the relative difference of the two phases (or the concentration of one phase), and μ is the so-called chemical potential given by

$$(1.2) \qquad \mu = -\epsilon \Delta \varphi + \frac{1}{\epsilon} F'(\varphi).$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with *Lipschitz* boundary $\Gamma = \partial \Omega$. Here κ is the mobility coefficient, $\epsilon > 0$ is a given (small) parameter related to the thickness of the interface separating the two phases, and F is the (density) of potential energy. A physically relevant choice for F is the following

$$(1.3) \qquad F(r) = (1 + r) \log(1 + r) + (1 - r) \log(1 - r) - \lambda r^2, \quad \lambda \geq 1,$$

which is often approximated by a polynomial double well-potential, typically

$$(1.4) \qquad F(r) = (r^2 - 1)^2.$$

In the literature, it is common to distinguish between singular potentials, which are defined on finite intervals like (1.3), and regular ones as (1.4), defined on $\mathbb{R}$.

We recall that equations (1.1)-(1.2) have been deduced phenomenologically, i.e., as the (conserved) gradient flow associated with the Fréchet derivative of the free energy functional

$$\mathcal{L}(\varphi) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx.$$

In [29, 30], starting from a microscopic model, the authors rigorously derived a macroscopic equation for phase segregation phenomena. This is a nonlocal version of the Cahn-Hilliard equation, namely, the chemical potential is given by

$$(1.5) \qquad \mu = a_\epsilon(x)\varphi - \epsilon J_\epsilon * \varphi + \frac{1}{\epsilon} F'(\varphi),$$

where J is a (sufficiently smooth) interaction kernel such that $J(x) = J(-x)$ and

$$(J_\epsilon * \varphi)(x) := \int_\Omega J_\epsilon(x - y)\varphi(y)dy, \quad a_\epsilon(x) := \epsilon \int_\Omega J_\epsilon(x - y)dy,$$

where $J_\epsilon(x) = \epsilon^{-d} J(\epsilon^{-1}x)$. By using formal asymptotic analysis, the authors also showed that the interface evolution problems associated with such equation as ϵ goes to 0 are exactly the ones associated with the standard Cahn-Hilliard equation (i.e., Stefan-like and Mullins-Sekerka problems). In addition, also the nonlocal version can be viewed as the conserved gradient flow associated with the first variation of the free energy functional

$$\mathcal{N}(\varphi) = \int_{\Omega \times \Omega} \frac{\epsilon}{4} J_\epsilon(x - y)|\varphi(x) - \varphi(y)|^2 dxdy + \int_\Omega \frac{1}{\epsilon} F(\varphi)dx.$$

As a consequence, we can observe (formally) that the nonlocal interaction term can be locally approximated by the square gradient, provided that J is sufficiently concentrated around 0. That is, the functional $\mathcal{L}$ can be viewed as a local approximation of $\mathcal{N}$. This was already noted by Van der Waals (see [41]). Thus the nonlocal Cahn-Hilliard equation seems well justified and more general than the classical one, though the related literature is far less abundant. In particular, most of the theoretical results are devoted to well-posedness, but very few are concerned with the longtime behavior of solutions. The main reason is related to the eventual boundedness and regularization of the order parameter which are needed to prove the precompactness of trajectories in some convenient topology. Well-posedness and regularity issues were firstly analyzed in [30] on a three-dimensional torus with

degenerate mobility and logarithmic potential. A similar equation endowed with no flux boundary condition was studied in [23] (cf. also [11, 20, 21] and, for viscous versions, [33, 34]). For this case, the convergence to a single stationary state of a given trajectory was proven in [35] through a suitable Łojasiewicz-Simon inequality. This fact required to show preliminarily that a solution stays eventually strictly away from the pure phases: the so-called separation property.

For the constant mobility case and regular potentials, some existence, uniqueness and regularity results were obtained in [5] (see also [6, 32]). In that paper the existence of bounded absorbing sets was also established. Nevertheless, no results were known about the existence of more interesting invariant objects like, e.g., global attractors (cf. [38] and its references). Only recently, the existence of a (connected) global attractor has been proven in [18] for constant mobility and regular potentials (see [19] for singular ones). This has been done by exploiting the energy identity as a by-product of a result related to a more complicated model for phase separation in binary fluids. A natural question now arises: does the global attractor have *finite* (fractal) dimension? Here we give a positive answer and we actually prove more, namely, the existence of an exponential attractor (see again [38] for details). More precisely, taking for simplicity $\kappa = \epsilon = 1$, we consider the following nonlocal Cahn-Hilliard equation

(1.6) $$\partial_t \varphi = \Delta \mu, \quad \text{in } \Omega \times (0, \infty),$$

(1.7) $$\mu = a\varphi - J * \varphi + F'(\varphi) + \alpha \partial_t \varphi, \quad \text{in } \Omega \times (0, \infty),$$

subject to the no-flux boundary condition

(1.8) $$\partial_{\mathbf{n}} \mu = 0, \quad \text{on } \Gamma \times (0, \infty)$$

and to the initial condition

(1.9) $$\varphi(0) = \varphi_0, \quad \text{in } \Omega.$$

Here the coefficient $\alpha \geq 0$ characterizes the possible influences of internal microforces (see, e.g., [39]). The presence of this term is not necessary in the case of regular potentials, while it is crucial in the case of singular ones. In fact, in the former case, in order to prove our main result we need to first establish the eventual boundedness of φ. This boundedness is, say, built-in in the latter case, but we need to show that φ has the separation property uniformly with respect to the initial data. This feature is an open problem even for the classical *local* Cahn-Hilliard equation with constant mobility in dimension three (see [37]).

The paper is organized as follows. Section 2 is devoted to the nonviscous case with a regular potential, while Section 3 is concerned with the viscous equation with a singular potential. Provided suitable global bounds are obtained (this is the most technical part), the existence of an exponential attractor is proven through a short trajectory type technique devised in [15]. We also show that, in both cases, each solution converges to a single equilibrium by using a suitable version of the Łojasiewicz-Simon inequality, provided that F is real analytic. In the final Section 4, we consider the (nonviscous) equation with degenerate mobility and logarithmic potential. On account of the validity of the separation property, we can still prove the existence of an exponential attractor.

2. The nonviscous case with regular potential

2.1. **Some preliminary results.** We begin with some basic notation and assumptions. Let us first set $H := L^2(\Omega)$ and $V := H^1(\Omega)$. For every $\psi \in V'$, V' the dual space of V, we denote by $\langle \psi \rangle$ the average of ψ over Ω, that is,

$$\langle \psi \rangle = \frac{1}{|\Omega|} \langle \psi, 1 \rangle$$

where $|\Omega|$ stands for the Lebesgue measure of Ω and $\langle \cdot, \cdot \rangle$ is the duality product. Then we introduce the spaces $V_0 := \{\psi \in V : \langle \psi \rangle = 0\}$, $V_0' := \{\psi \in V' : \langle \psi, 1 \rangle = 0\}$, and the operator $A_N : V \to V'$, $A_N \in \mathcal{L}(V, V')$, defined by,

$$\langle A_N u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in V.$$

With these definitions, it is well known that $A_{N|V_0}$ maps V_0 into V_0' isomorphically, and that the inverse map $\mathcal{N} = A_N^{-1} : V_0' \to V_0$, is defined by

$$A_N \mathcal{N} \psi = \psi, \quad \forall \psi \in V_0', \qquad \mathcal{N} A_N f = f, \quad \forall f \in V_0.$$

These maps also satisfy the following well-known relations:

$$(2.1) \qquad \langle A_N u, \mathcal{N} v \rangle = \langle u, v \rangle, \quad \forall u \in V, v \in V_0',$$
$$\langle u, \mathcal{N} v \rangle = \langle v, \mathcal{N} u \rangle, \quad \forall u, v \in V_0'.$$

The assumptions listed below are the same as in [5] (see also [10]).

(H1) $J \in W^{1,1}(\mathbb{R}^d)$, $J(-x) = J(x)$, $a(x) := \int_\Omega J(x-y) \, dy \geq 0$ a.e. in Ω.

(H2) $F \in C_{\mathrm{loc}}^{2,1}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + \inf_{x \in \Omega} a(x) \geq c_0, \quad \forall s \in \mathbb{R}.$$

(H3) There exist $c_1 > \frac{1}{2} \|J\|_{L^1(\mathbb{R}^d)}$ and $c_2 \in \mathbb{R}$ such that

$$F(s) \geq c_1 s^2 - c_2, \quad \forall s \in \mathbb{R}.$$

(H4) There exist $c_3 > 0$, $c_4 \geq 0$ and $p \in (1,2]$ such that

$$|F'(s)|^p \leq c_3 |F(s)| + c_4, \quad \forall s \in \mathbb{R}.$$

(H5) $F \in C^2(\mathbb{R})$ and there exist $c_5, c_6 > 0$ and $q > 0$ such that

$$F''(s) + \inf_{x \in \Omega} a(x) \geq c_5 s^{2q} - c_6, \quad \forall s \in \mathbb{R}.$$

Remark 2.1. Note that the operator $\psi \mapsto J * \psi$ is self-adjoint and compact from H to itself, provided that (H1) is satisfied. Also, it is easy to realize that it is compact from $L^\infty(\Omega)$ to $C^0(\overline{\Omega})$ and that $a \in L^\infty(\Omega)$.

We report the following result (see [18, Corollary 1 and Proposition 5], cf. also [10]).

Theorem 2.2. *Let $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$, and assume (H1)-(H4), and (H5) with $q \geq \frac{1}{2}$ and $p \in \left(\frac{6}{5}, 2\right]$ when $d = 3$. Then, there exists a unique weak solution of (1.6)-(1.9) on $[0,T]$, for any $T > 0$, such that*

$$(2.2) \qquad \begin{aligned} &\varphi \in C([0,T]; H) \cap L^2([0,T]; V) \cap L^\infty([0,T]; L^{2+2q}(\Omega)), \\ &\partial_t \varphi \in L^2([0,T]; V'), \, \mu \in L^2([0,T]; V), \\ &F(\varphi) \in L^\infty([0,T]; L^1(\Omega)) \end{aligned}$$

and $\varphi(0) = \varphi_0$, $\langle \varphi(t) \rangle = \langle \varphi_0 \rangle$, for all $t \in [0, T]$. Furthermore, setting

$$(2.3) \quad \mathcal{E}(\varphi(t)) := \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\varphi(x, t) - \varphi(y, t))^2 \, dx dy + \int_\Omega F(\varphi(x, t)) \, dx,$$

the following equality holds for all $t \geq 0$,

$$(2.4) \qquad\qquad \mathcal{E}(\varphi(t)) + \int_0^t \|\nabla \mu(s)\|_H^2 \, ds = \mathcal{E}(\varphi_0).$$

Remark 2.3. We say that φ is a weak solution of (1.6)-(1.9) on $(0, T)$ if φ belongs to the class of functions (2.2), and for every $\psi \in V$,

$$(2.5) \qquad\qquad \langle \partial_t \varphi, \psi \rangle + (\nabla \mu, \nabla \psi) = 0, \quad \text{a.e. in } (0, T),$$

see [18, Definition 1]. Equivalently, we can write (2.5) in the following form:

$$(2.6) \qquad \langle \partial_t \varphi, \psi \rangle + (\nabla \Phi, \nabla \psi) = (\nabla J * \varphi, \nabla \psi), \quad \text{a.e. in } (0, T),$$

where $\Phi(x, \varphi) := a(x)\varphi + F'(\varphi)$ (cf. [5, Section 2]).

Remark 2.4. It is not difficult to check that a Gaussian $J(x) = c_J e^{-\xi |x|^2}$, $\xi > 0$, or a Newtonian interaction kernel $J(x) = c_J |x|^{-1}$, if $d = 3$, $J(x) = -c_J \log(|x|)$ if $d = 2$, fulfills (H1). Note that (H5) is slightly stronger than (H2) and is necessary to establish the energy identity (2.4). The further restriction on $p \in \left(\frac{6}{5}, 2\right]$ when $d = 3$ is also required for (2.4) to hold. Otherwise, we would only have an energy inequality. On the other hand, (H1)-(H4) are enough to establish the existence of at least one global weak solution (cf. [10]). Observe that assumption (H4) is fulfilled by a potential of arbitrary polynomial growth. Besides, (H2)-(H5) are certainly satisfied, for instance, by (1.4).

The next result can also be found in [18, Theorem 1].

Proposition 2.5. *Let $m \geq 0$ be given. Then every weak solution to (1.6)-(1.9) satisfies the dissipative estimate:*

$$(2.7) \qquad\qquad \mathcal{E}(\varphi(t)) \leq \mathcal{E}(\varphi_0) e^{-kt} + C(m), \quad \forall t \geq 0,$$

provided that $|\langle \varphi_0 \rangle| \leq m$, where k and C are positive constants independent of time and initial data, but which depend on the other structural parameters of the problem.

Remark 2.6. The proof of (2.7) does not require the validity of the energy identity (2.4), and so it holds also outside the range $p \in \left(\frac{6}{5}, 2\right]$ when $d = 3$, see [19].

Let us now set

$$\mathcal{Y}_m := \left\{ \psi \in H : F(\psi) \in L^1(\Omega), |\langle \varphi \rangle| \leq m \right\},$$

and endow $\mathcal{Y}_m$ with the following metric

$$(2.8) \qquad d(\psi_1, \psi_2) := \|\psi_1 - \psi_2\|_H + \left| \int_\Omega F(\psi_1) \, dx - \int_\Omega F(\psi_2) \, dx \right|^{1/2},$$

for any $\psi_1, \psi_2 \in \mathcal{Y}_m$. Thanks to (2.7) and Theorem 2.2, we can associate with problem (1.6)-(1.9) the solution semiflow

$$(2.9) \qquad\qquad S(t) : \mathcal{Y}_m \to \mathcal{Y}_m, \quad \varphi_0 \mapsto S(t)\varphi_0 = \varphi(t),$$

where $\varphi(t)$ is the unique weak solution of (1.6)-(1.9).

Concerning the long-term behavior, there holds (see [18, Theorem 4])

Theorem 2.7. *Let the assumptions of Theorem 2.2 be satisfied. The dynamical system $(S(t), \mathcal{Y}_m)$ possesses a connected global attractor $\mathcal{A}$.*

2.2. **Exponential attractors.** The main results of this section are contained in the following

Theorem 2.8. *For every fixed $m \geq 0$, there exists an exponential attractor $\mathcal{M} = \mathcal{M}(m)$ bounded in $V \cap C^\alpha(\overline{\Omega})$ for the dynamical system $(\mathcal{Y}_m, S(t))$ which satisfies the following properties:*
  (i) Semi-invariance: $S(t)\mathcal{M} \subset \mathcal{M}$, for every $t \geq 0$.
  (ii) Exponential attraction:

$$dist_{C^\alpha(\overline{\Omega}) \cap H^{1-\nu}(\Omega)}(S(t)\mathcal{Y}_m, \mathcal{M}) \leq Ce^{-\kappa t}, \quad \forall t \geq 0,$$

for some positive constants C_m and κ, for any $\nu \in (0,1)$ and some $\alpha \in (0,1)$.
  (iii) Finite dimensionality:

$$\dim_F \left(\mathcal{M}, C^\alpha(\overline{\Omega}) \right) \leq C_{m,\alpha} < \infty.$$

Thus we can immediately deduce the

Corollary 2.9. *The global attractor $\mathcal{A}$ is bounded in $V \cap C^\alpha(\overline{\Omega})$ and has finite fractal dimension:*

$$\dim_F \left(\mathcal{A}, C^\alpha(\overline{\Omega}) \right) < \infty.$$

To prove Theorem 2.8 we first need to derive a number of properties of the semigroup solution. The first result gives a dissipative estimate in the space $L^\infty(\Omega)$.

Lemma 2.10. *Let the assumptions of Theorem 2.2 be satisfied. For every $\tau > 0$, there exists a constant $C_{m,\tau} > 0$ such that*

$$(2.10) \qquad \sup_{t \geq 2\tau} \|\varphi(t)\|_{L^\infty(\Omega)} \leq C_{m,\tau}.$$

Moreover, there exists $R_0 > 0$ (independent of time, τ and initial data) such that $S(t)$ possesses an absorbing ball $\mathcal{B}_{L^\infty(\Omega)}(R_0)$, bounded in $L^\infty(\Omega)$.

Proof. Our proof of (2.10) relies on an iterative argument as in [24]. The estimates will be derived assuming sufficiently smooth solutions to (1.6)-(1.9) so that the function $|\varphi|^{p-1}\varphi$ is also L^2-summable for each $p > 1$. The scheme we employ is as follows: let $\varphi_{0\varepsilon} \in L^\infty(\Omega)$ such that $\varphi_{0\varepsilon} \to \varphi_0$ in H, and such that $F(\varphi_{0\varepsilon}) \to F(\varphi_0)$ in $L^1(\Omega)$ as $\varepsilon \to 0$. In this case, we can exploit the existence proof of Theorem 2.2 (see [10]) one more time and an a priori L^∞-estimate from [5, Theorem 2.1] to deduce the existence of a weak solution φ_ε satisfying (2.2) with the additional *essential* property

$$(2.11) \qquad \varphi_\varepsilon \in L^\infty(\mathbb{R}_+ \times \Omega), \quad \forall \varepsilon > 0.$$

Also for practical purposes, C denotes from now on a positive constant that is independent of t, ε, φ and initial data, but which only depends on the other structural parameters. Such a constant may vary even from line to line. Further dependencies of this constant on other parameters will be pointed out as needed.

For $p > 1$, omitting the subscript ε, we multiply equation (1.6) by $|\varphi|^{p-1}\varphi$ and integrate over Ω, to obtain

$$(2.12) \qquad \frac{d}{dt}\int_\Omega |\varphi|^{p+1}\, dx + \frac{2pC}{p+1}\int_\Omega \left| \nabla |\varphi|^{\frac{p+1}{2}} \right|^2 dx \leq C(p+1)^2 \int_\Omega |\varphi|^{p+1}\, dx,$$

where $C > 0$ is independent of p and $\varepsilon > 0$, owing to the assumptions (H1)-(H2) (cf. [5, Theorem 2.1, (2.8)-(2.16)]). Note that the regularity (2.11) is key in proving this estimate. Setting now $p = 2^k - 1$, $k \geq 0$, then

$$x_k(t) := \int_\Omega |\varphi(t)|^{2^k} \, dx, \quad k \geq 0,$$

and having established (2.12), we can now exploit the scheme in [24, Theorem 3.2, (3.8)-(3.10)] (see also [25, Theorem 2.3]) to derive the following inequality:

$$(2.13) \qquad x_k(t) \leq C_\xi \left(2^k\right)^\sigma \left(\sup_{s \geq t - \xi/2^k} x_{k-1}(s)\right)^2, \quad \forall k \geq 1,$$

where t, ξ are two positive constants such that $t - \xi/2^k > 0$, and C_ξ, σ are positive constants independent of k; the constant C_ξ is bounded if ξ is bounded away from zero. We can iterate in (2.13) reasoning exactly as in, e.g., [24, Theorem 3.2] (cf., also, [25, Theorem 2.3]). For the sake of completeness, we report a sketch of the argument. Choose any numbers $\tau' > \tau > 0$ such that $\xi = (\tau' - \tau)$, $t_0 = \tau'$ and $t_k = t_{k-1} - \xi/2^k$, $k \geq 1$. Thus, in view of (2.13) we have

$$(2.14) \qquad \sup_{t \geq t_{k-1}} x_k(t) \leq C_\xi \left(2^k\right)^\sigma \left(\sup_{s \geq t_k} x_{k-1}(s)\right)^2, \quad k \geq 1.$$

Next, define

$$(2.15) \qquad C_H := \sup_{s \geq t_1} x_0(s) = \sup_{s \geq t_1} \|\varphi(s)\|_H^2.$$

Thus, we can iterate in (2.14) with respect to $k \geq 1$ and obtain that

$$(2.16) \qquad \sup_{t \geq t_0} x_k(t) \leq \sup_{t \geq t_{k-1}} x_k(t) \leq C_\xi^{A_k} 2^{\sigma B_k} \left(C_H\right)^{2^k},$$

where

$$(2.17) \qquad A_k := 1 + 2 + 2^2 + \ldots + 2^k \leq 2^k \sum_{i \geq 1} \frac{1}{2^i}$$

$$(2.18) \qquad B_k := k + 2(k-1) + 2^2(k-2) + \ldots + 2^k \leq 2^k \sum_{i \geq 1} \frac{i}{2^i}.$$

Therefore, taking the 2^k-root on both sides of (2.16) and then letting $k \to +\infty$ (note that the series in (2.17)-(2.18) are convergent), we deduce

$$(2.19) \qquad \sup_{t \geq t_0 = \tau'} \|\varphi(t)\|_{L^\infty(\Omega)} \leq \limsup_{k \to +\infty} \sup_{t \geq t_0} \left(x_k(t)\right)^{1/2^k} \leq C_\xi \left(C_H\right),$$

for some positive constant C_0 independent of t, k, φ, ε, ξ and initial data.

In order to prove the first assertion of lemma, we observe that a simple argument [18, Proposition 4, (3.21)-(3.22)] yields, on account of (2.7), that

$$(2.20) \qquad C_H = \sup_{t \geq \frac{3\tau}{2}} \|\varphi(t)\|_H^2 \leq C_m \left(1 + \mathcal{E}(\varphi_0)\right).$$

Thus, setting $\tau' = 2\tau$ so that $\xi = \tau$, we readily obtain the first claim (2.10) of lemma from (2.20). On the other hand, the same argument as in [18, Proposition 4, (3.21)-(3.22)] yields a bounded absorbing ball in H. Indeed, in light of (2.7), it is not difficult to see that, for any bounded set $\mathcal{B} \subset \mathcal{Y}_m$, there exists a time $t_* = t_*(\mathcal{B}) > 0$ such that $S(t)\mathcal{B} \subset H$, for all $t \geq t_*$. Next, we can choose

$\tau' = \tau + 2\xi$ with $\tau = t_*$ and $\xi = 1$, so that C_H and C_ξ are bounded uniformly with respect to initial data as $t \geq t_*$. Hence, the L^2-L^∞ smoothing property (2.19) immediately entails the second assertion of lemma. $\qquad\square$

We also have

Lemma 2.11. *Let the assumptions of Theorem 2.2 be satisfied. Then, for every $\tau > 0$, there exists a constant $C_{m,\tau,\alpha} > 0$ such that*

$$(2.21) \qquad \sup_{t \geq 2\tau} \|\varphi\|_{C^{\alpha/2,\alpha}([t,t+1]\times\overline{\Omega})} \leq C_{m,\tau,\alpha},$$

for some $\alpha \in (0,1)$. Thus, there exists $R_1 > 0$ (independent of time, τ and initial data) such that $S(t)$ possesses an absorbing ball $\mathcal{B}_{C^\alpha(\overline{\Omega})}(R_1)$, bounded in $C^\alpha(\overline{\Omega})$.

Proof. We can rewrite the system (1.6)-(1.8) in the following form

$$(2.22) \qquad \partial_t \varphi = \mathrm{div}\,(a\,(x,\varphi,\nabla\varphi)), \quad (a\,(x,\varphi,\nabla\varphi)\cdot\mathbf{n})_{|\Gamma} = 0,$$

where

$$(2.23) \qquad a\,(x,\varphi,\nabla\varphi) := (a\,(x) + F''\,(\varphi))\,\nabla\varphi + (\nabla a)\,\varphi - \nabla J * \varphi.$$

Since $J \in W^{1,1}(\mathbb{R}^d)$ and φ is bounded by Lemma 2.10, using the fact that $a\,(x) + F''\,(\varphi) \geq c_0$, by (H2), it is easy to check that

$$a\,(x,\varphi,\nabla\varphi)\,\nabla\varphi \geq \frac{c_0}{2}\,|\nabla\varphi|^2 - C_1, \quad |a\,(x,\varphi,\nabla\varphi)| \leq C_2\,|\nabla\varphi| + C_3,$$

for some positive constants C_i which depend only on $\|J\|_{W^{1,1}}$ and (2.10). Thus, the desired estimate in (2.21) follows from the application of [13, Corollary 4.2]. The proof is finished. $\qquad\blacksquare$

In view of Lemma 2.10 and the proof of [5, Theorem 4.3, (4.19)-(4.39)], the following result is now straightforward.

Lemma 2.12. *Let the assumptions of Theorem 2.2 be satisfied. Then, for every $\tau > 0$, there exists a constant $C_{m,\tau} > 0$ such that*

$$(2.24) \qquad \sup_{t \geq 3\tau}\left[\|\varphi\,(t)\|_V + \|\partial_t\varphi\|_{L^2([t,t+1]\times\Omega)}\right] \leq C_{m,\tau}.$$

Moreover, for any bounded set $\mathcal{B} \subset \mathcal{Y}_m$, there exists a time $t_\# = t_\#\,(\mathcal{B}) > 0$ such that $S\,(t)\,\mathcal{B} \subset V$, for all $t \geq t_\#$.

The following result shows that the semigroup is strongly continuous with respect to the V'-metric.

Proposition 2.13. *Let φ_i, $i = 1,2$, be a pair of weak solutions according to the assumptions of Theorem 2.2. Then the following estimate holds:*

$$(2.25) \qquad \|\varphi_1\,(t) - \varphi_2\,(t)\|_{V'}^2 + c_0\int_0^t \|\varphi_1\,(s) - \varphi_2\,(s)\|_H^2\,ds$$

$$\leq \|\varphi_1\,(0) - \varphi_2\,(0)\|_{V'}^2\,e^{\kappa t} + Ce^{\kappa t}\,|M_1 - M_2|,$$

for all $t \geq 0$, where $M_i := \langle\varphi_i\,(0)\rangle$, for some positive constants κ, C which depend on c_0 and J but are independent of $\varphi_i\,(0)$.

Proof. We have that $\varphi := \varphi_1 - \varphi_2$ (formally) satisfies the problem:

$$(2.26) \qquad \partial_t \varphi = \Delta \overline{\mu}, \ \overline{\mu} = a(x) \varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2),$$

subject to the boundary and initial conditions

$$(2.27) \qquad \partial_{\mathbf{n}} \overline{\mu}_{|\Gamma} = 0, \ \varphi_{|t=0} = \varphi_1(0) - \varphi_2(0) \ \text{in} \ \Omega.$$

Also, observe that (2.26)-(2.27) yields $\langle \varphi(t) \rangle = M_1 - M_2$, for all $t \geq 0$. Consider now the operator $A_N = -\Delta_N$, with domain $D(A_N) = \{ \varphi \in H^2(\Omega) : \partial_{\mathbf{n}} \varphi_{|\Gamma} = 0 \}$. Test the first equation of (2.26) with $A_N^{-1}(\varphi(t) - \langle \varphi(t) \rangle)$, then integrate by parts exploiting the relations (2.1). We obtain, thanks to the assumptions (H1)-(H2) and arguing as in [18, Proposition 5, (4.2)-(4.3)], the following estimate:

$$(2.28) \qquad \frac{d}{dt} \| \varphi(t) \|_{V'}^2 + 2c_0 \| \varphi(t) \|_H^2$$
$$\leq 2 \left| (J * \varphi(t), \varphi(t)) \right| + 2 \left| \langle \overline{\mu}(t) \rangle \right| |\Omega| \left| \langle \varphi(t) \rangle \right|$$
$$\leq 2 \left\| A_N^{1/2}(J * \varphi(t)) \right\|_H \left\| A_N^{-1/2} \varphi(t) \right\|_H$$
$$+ 2 \left| \langle \overline{\mu}(t) \rangle \right| |\Omega| |M_1 - M_2|$$
$$\leq c_0 \| \varphi(t) \|_H^2 + C \| \varphi(t) \|_{V'}^2 + C \left| \langle \overline{\mu}(t) \rangle \right| |M_1 - M_2|,$$

for all $t \geq 0$. Applying Gronwall's inequality to (2.28) and using the estimate

$$(2.29) \qquad \int_0^t \left| \langle \overline{\mu}(s) \rangle \right| ds \leq C_m(1+t), \ \forall t \geq 0$$

(this follows easily due to estimate (2.7) and assumptions (H3)-(H5)), we obtain estimate (2.25). $\qquad \square$

The crucial step in order to establish the existence of an exponential attractor is the validity of so-called smoothing property for the difference of two solutions (see [37]). In the present case, such a property is a consequence of the following two lemmas. The first result establishes that the semigroup $S(t)$ is some kind of contraction map, up to the term $\| \varphi_1 - \varphi_2 \|_{L^2([3\tau, t]; V')}$.

Lemma 2.14. *Let the assumptions of Proposition 2.13 hold. Then, for every $\tau > 0$, we have:*

$$(2.30) \qquad \| \varphi_1(t) - \varphi_2(t) \|_{V'}^2 + C |M_1 - M_2|$$
$$\leq e^{-\kappa t} \left(\| \varphi_1(0) - \varphi_2(0) \|_{V'}^2 + C |M_1 - M_2| \right)$$
$$+ C_{m,\tau} \int_0^t \left(\| \varphi_1(s) - \varphi_2(s) \|_{V'}^2 + |M_1 - M_2| \right) ds,$$

for all $t \geq 3\tau$, for some positive constants $C, C_{m,\tau}, \kappa$ which depend on c_0, Ω and J.

Proof. First, we observe that, due to the estimates (2.10) and (2.24), there holds:

$$(2.31) \qquad \sup_{t \geq 3\tau} \left| \langle \overline{\mu}(t) \rangle \right| \leq C_{m,\tau},$$

for every (weak) solutions φ_1, φ_2. Thus, combining (2.28) together with Poincaré's inequality

$$\left\| A_N^{-1/2}(\varphi - \langle \varphi \rangle) \right\|_H^2 + \langle \varphi \rangle^2 \leq C_\Omega \| \varphi \|_H^2,$$

we deduce from (2.28) and (2.31) the following inequality:

$$(2.32) \qquad \frac{d}{dt} \left(\|\varphi(t)\|_{V'}^2 + C \, |M_1 - M_2| \right) + c_0 \left(\|\varphi(t)\|_{V'}^2 + C \, |M_1 - M_2| \right)$$
$$\leq C \, \|\varphi(t)\|_{V'}^2 + C_{m,\tau} \, |M_1 - M_2|,$$

for all $t \geq 3\tau$. Thus, Gronwall's inequality entails the desired estimate (2.30).

We now need some compactness for the term $\|\varphi_1 - \varphi_2\|_{L^2([3\tau,t];V')}$ on the right-hand side of (2.30). This is given by

Lemma 2.15. *Let the assumptions of Proposition 2.13 hold. Then, for every $\tau > 0$, the following estimate holds:*

$$(2.33) \qquad \|\partial_t \varphi_1 - \partial_t \varphi_2\|_{L^2([3\tau,t];D(A_N)')}^2 + c_0 \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_H^2 \, ds$$
$$\leq C_{m,\tau} e^{\kappa t} \|\varphi_1(0) - \varphi_2(0)\|_{V'}^2 + C e^{\kappa t} |M_1 - M_2|,$$

for all $t \geq 3\tau$, where $C_{m,\tau}, C$ and $\kappa > 0$ also depend on c_0, Ω and J.

Proof. The second term on the left-hand side of (2.33) can be easily controlled by (2.25). Thus we only need to estimate the time derivative. Recall that φ satisfies (2.26). Furthermore, in light of Lemmas 2.10 and 2.12, recall that we have

$$(2.34) \qquad \sup_{t \geq 3\tau} \|\varphi_i(t)\|_{V \cap L^\infty(\Omega)} \leq C_{m,\tau}, \; i = 1, 2.$$

Thus, for any test function $\psi \in D(A_N)$, using the weak formulation (2.5), there holds

$$(2.35) \qquad \langle \partial_t \varphi(t), \psi \rangle = (\nabla \overline{\mu}(t), \nabla \psi) = \langle \overline{\mu}(t), \Delta_N \psi \rangle$$
$$\leq \|\overline{\mu}(t)\|_H \|\psi\|_{D(A_N)} \leq C_{m,\tau} \|\varphi\|_H \|\psi\|_{D(A_N)}.$$

This estimate together with (2.25) gives the desired estimate on the time derivative in (2.33). $\qquad \square$

We now show that the semigroup $S(t)$ is actually uniformly Hölder continuous in the C^α-norm with respect to the initial data.

Lemma 2.16. *Let $\varphi_i(t) = S(t) \varphi_i(0)$, with $\varphi_i(0) \in \mathcal{Y}_m$ such that $M_i = \langle \varphi_i(0) \rangle$, $i = 1, 2$. Then, for every $\tau > 0$, the following estimate is valid:*

$$(2.36) \quad \|\varphi_1(t) - \varphi_2(t)\|_{C^{\alpha/2}(\overline{\Omega})} \leq C_{m,\tau} e^{\kappa t} \left(\|\varphi_1(0) - \varphi_2(0)\|_{V'}^\beta + |M_1 - M_2|^{\frac{\beta}{2}} \right),$$

for all $t \geq 3\tau$, where the constants $C_{m,\tau}$, κ and $\beta < 1$ are independent of the initial data.

Proof. Using the interpolation $[V, V']_{1/2,2} = H$, we deduce from estimates (2.25) and (2.34) that

$$(2.37) \qquad \|\varphi_1(t) - \varphi_2(t)\|_H \leq C_{m,\tau} e^{\kappa t} \left(\|\varphi_1(0) - \varphi_2(0)\|_{V'}^{1/2} + |M_1 - M_2|^{1/4} \right),$$

for all $t \geq 3\tau$. On the other hand, due to the boundedness of $\varphi_i(t) \in L^\infty(\Omega) \cap V$ for $t \geq 3\tau$, $i = 1, 2$ (cf. 2.34), the nonlinearity $f = F'$ is controlled by the linear part of the equation (2.26) (no matter how fast it grows) and obtaining the L^2-L^∞ smoothing property for our dynamical system is actually reduced to the same standard procedure we used in the proof of Lemma 2.10. Indeed, we already have an estimate of the L^∞-norm of the solution $\varphi(t)$ (due to (2.10)) and, consequently, we

do not need to worry about the growth of $f = F'$. In particular, estimate (2.12) also holds for the difference of solutions $\varphi = \varphi_1 - \varphi_2$. This observation combined with (2.37) and a proper interpolation inequality between $C^\alpha\left(\overline{\Omega}\right) \subset C^{\alpha/2}\left(\overline{\Omega}\right) \subset L^\infty\left(\Omega\right)$ implies the desired inequality (2.36).

The last ingredient we need is the uniform Hölder continuity of $t \mapsto S(t)\varphi_0$ in the C^α-norm, namely,

Lemma 2.17. *Let the assumptions of Theorem 2.2 be satisfied. Consider $\varphi\left(t\right) = S\left(t\right)\varphi_0$ with $\varphi_0 \in \mathcal{Y}_m$. Then, for every $\tau > 0$, there holds*

$$(2.38) \qquad \|\varphi\left(t\right) - \varphi\left(s\right)\|_{C^{\alpha/2}\left(\overline{\Omega}\right)} \leq C_{m,\tau}\left|t - s\right|^\beta, \quad \forall t, s \geq 3\tau,$$

where $\beta < 1$ and the positive constant $C_{m,\tau}$ is independent of initial data, φ and t, s.

Proof. According to (2.34), the following bound holds for μ:

$$\sup_{t \geq 3\tau} \|\mu\left(t\right)\|_V \leq C_{m,\tau}.$$

Consequently, by comparison in (1.6), we have that

$$\sup_{t \geq 3\tau} \|\partial_t \varphi\left(t\right)\|_{V'} \leq C_{m,\tau},$$

which entails

$$(2.39) \qquad \|\varphi\left(t\right) - \varphi\left(s\right)\|_{V'} \leq C_{m,\tau}\left|t - s\right|, \quad \forall t, s \geq 3\tau.$$

Estimate (2.38) now follows from (2.39), $[V, V']_{1/2,2} = H$, the L^2-L^∞ smoothing property of the solutions in $[3\tau, \infty)$ and the interpolation inequality

$$(2.40) \qquad \|\varphi\|_{C^{\alpha/2}\left(\overline{\Omega}\right)} \leq C\|\varphi\|_{C^\alpha\left(\overline{\Omega}\right)}^{1-\zeta}\|\varphi\|_{L^\infty\left(\Omega\right)}^\zeta,$$

for some $\zeta = \zeta\left(\alpha\right) > 0$. $\blacksquare$

We report for the reader's convenience the following abstract result on the existence of exponential attractors [15, Proposition 4.1] which will be used in the following proof and in the other sections as well.

Proposition 2.18. *Let $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$ be Banach spaces such that the embedding $\mathcal{V}_1 \subset \mathcal{V}$ is compact. Let B be a closed bounded subset of $\mathcal{H}$ and let $\mathbb{S} : B \to B$ be a map. Assume also that there exists a uniformly Lipschitz continuous map $\mathbb{T} : B \to \mathcal{V}_1$, i.e.,*

$$(2.41) \qquad \|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}_1} \leq L\|b_1 - b_2\|_{\mathcal{H}}, \quad \forall b_1, b_2 \in B,$$

for some $L \geq 0$, such that

$$(2.42) \qquad \|\mathbb{S}b_1 - \mathbb{S}b_2\|_{\mathcal{H}} \leq \gamma\|b_1 - b_2\|_{\mathcal{H}} + K\|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}}, \quad \forall b_1, b_2 \in B,$$

for some $\gamma < \frac{1}{2}$ and $K \geq 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_d \subset B$ of the semigroup $\{\mathbb{S}(n) := \mathbb{S}^n, n \in Z+\}$ with discrete time in the phase space $\mathcal{H}$.

Proof of Theorem 2.8. In order to apply Proposition 2.18, it is sufficient to verify the existence of an exponential attractor for the restriction of $S(t)$ on some properly chosen semi-invariant absorbing set in $\mathcal{Y}_m$. Recall that, by Lemmas 2.10 and 2.12, the ball $B_0 := B_{C^a\left(\overline{\Omega}\right) \cap V}\left(R_0\right)$ will be absorbing for $S\left(t\right)$, provided that $R_0 > 0$

is sufficiently large. Since we want this ball to be semi-invariant with respect to the semigroup, we push it forward by the semigroup, by defining first the set $B_1 = [\cup_{t \geq 0} S(t) B_0]_{V'_m}$, where $[\cdot]_{V'_m}$ denotes closure in the space

$$V'_m := \{\psi \in V' : |\langle \psi \rangle| \leq m\},$$

and then the set $\mathbb{B} = S(1) B_1$. Thus, $\mathbb{B}$ is a semi-invariant compact (for the metric $d_{V'_m}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_{V'} + |\langle \varphi_1 - \varphi_2 \rangle|^{1/2}$) subset of the phase space $\mathcal{Y}_m$. On the other hand, due to the results proven in this section, we have

$$\sup_{t \geq 0} \left(\|\varphi(t)\|_{C^a(\overline{\Omega}) \cap V} + \|\mu(t)\|_V + \|\partial_t \varphi(t)\|_{V'} \right) \leq C_m,$$

for every trajectory φ originating from $\varphi_0 \in \mathbb{B}$, for some positive constant C_m which is independent of the choice of $\varphi_0 \in \mathbb{B}$. We can now apply the abstract result above to the map $\mathbb{S} = S(T)$ and $\mathcal{H} = V'_m$, for a fixed $T > 0$ such that $e^{-\kappa T} < \frac{1}{2}$, where $\kappa > 0$ is the same as in Lemma 2.14. To this end, we introduce the functional spaces

$$(2.43) \quad \mathcal{V}_1 := L^2\left([0,T]; L^2(\Omega)\right) \cap H^1\left([0,T]; D(A_N)'\right), \quad \mathcal{V} := L^2\left([0,T]; V'_m\right),$$

and note that $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$. Finally, we introduce the operator $\mathbb{T} : \mathbb{B} \to \mathcal{V}_1$, by $\mathbb{T}\varphi_0 := \varphi \in \mathcal{V}_1$, where φ solves (1.6)-(1.9) with $\varphi(0) = \varphi_0 \in \mathbb{B}$. We claim that the maps $\mathbb{S}$, $\mathbb{T}$, the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$ thus defined satisfy all the assumptions of Proposition 2.18. Indeed, the global Lipschitz continuity (2.41) of $\mathbb{T}$ is an immediate corollary of Lemma 2.15, and estimate (2.42) follows from estimate (2.30). Therefore, due to Proposition 2.18, the semigroup $\mathbb{S}(n) = S(nT)$ generated by the iterations of the operator $\mathbb{S} : \mathbb{B} \to \mathbb{B}$ possesses a (discrete) exponential attractor $\mathcal{M}_d$ in $\mathbb{B}$ endowed by the topology of V'_m. In order to construct the exponential attractor $\mathcal{M}$ for the semigroup $S(t)$ with continuous time, we note that, due to Lemma 2.13, this semigroup is Lipschitz continuous with respect to the initial data in the topology of V'_m. Moreover, by (2.36) and (2.38) the map $(t, \varphi_0) \mapsto S(t) \varphi_0$ is also uniformly Hölder continuous on $[0,T] \times \mathbb{B}$, where $\mathbb{B}$ is endowed with the metric topology of V'_m. Hence, the desired exponential attractor $\mathcal{M}$ for the continuous semigroup $S(t)$ can be obtained by the standard formula

$$(2.44) \qquad\qquad \mathcal{M} = \bigcup_{t \in [0,T]} S(t) \mathcal{M}_d.$$

In order to finish the proof of the theorem, we only need to verify that $\mathcal{M}$ defined as above will be the exponential attractor for $S(t)$ restricted to $\mathbb{B}$ not only with respect to the V'_m-metric, but also in with respect to a stronger metric. This is an immediate corollary of the following facts: $\mathbb{B}$ is bounded in $V \cap C^\alpha(\overline{\Omega})$, the L^2-$C^\alpha(\overline{\Omega})$ smoothing property of the map $\varphi_0 \mapsto S(t) \varphi_0$, and the interpolation inequalities given by (2.40) and

$$\|u\|_{H^{1-\nu}(\Omega)} \leq C_s \|u\|_H^s \|u\|_V^{1-s}, \quad \nu \in (0,1),$$

for some $s = s(\nu) \in (0,1)$. Theorem 2.8 is now proved.

Remark 2.19. The methods used in this section can also be applied to other nonlocal problems which have a variational structure similar to the nonlocal Cahn-Hilliard

equation. An interesting case (see [5, Sec. 5] and its references) is a model related to interacting particle systems with Kawasaki dynamics, namely,

$$(2.45) \qquad \begin{cases} \partial_t \varphi = \Delta \left(\varphi - \tanh(\beta J(x) * \varphi) \right) & \text{in } \Omega \times (0, \infty), \\ \partial_{\mathbf{n}} \left(\varphi - \tanh(\beta J(x) * \varphi) \right) = 0 & \text{on } \Gamma \times (0, \infty), \end{cases}$$

for some constant β. In fact, in this case the L^2-L^∞ smoothing property proven in Lemma 2.10 holds again regardless of the value of β. The existence of an absorbing set in $V \cap L^\infty(\Omega)$ for the solution map $\varphi_0 \in H \mapsto \varphi(t) \in H$ of (2.45) can be also established as in [5, Section 5]. Hence, the existence of an exponential attractor for the dynamical system associated with (2.45) can be proven arguing as in Theorem 2.8.

2.3. **Convergence to a single equilibrium.** Let φ be a weak solution to (1.6)-(1.9) according to Theorem 2.2. In this section we aim to show that that the ω-limit set,

$$\omega[\varphi] = \{\varphi_* : \exists t_n \to \infty \text{ such that } \varphi(t_n) \to \varphi_* \text{ in } H\}$$

is a singleton, where φ_* is a solution of the stationary problem:

$$(2.46) \qquad \begin{cases} a(x)\varphi_* - J * \varphi_* + F'(\varphi_*) = \mu_*, & \text{a.e. in } \Omega, \\ \mu_* = \text{constant}, \quad \langle \varphi_* \rangle = \langle \varphi_0 \rangle \end{cases}$$

(see, e.g., [5, Theorem 4.5]). We employ a generalized version of the Łojasiewicz-Simon theorem proved in [22, Theorem 6] (cf. also [35, Section 4]). The version that applies to our case is formulated in the following.

Lemma 2.20. *Let J satisfy (H1) and $F \in C^2(\mathbb{R})$ be a real analytic function satisfying (H2). Then, there exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\varepsilon > 0$ such that the following inequality holds:*

$$(2.47) \qquad |\mathcal{E}(\varphi) - \mathcal{E}(\varphi_*)|^{1-\theta} \leq C \|\mu - \langle \mu \rangle\|_H,$$

for all $\varphi \in L^\infty(\Omega) \cap \mathcal{Y}_m$ provided that $\|\varphi - \varphi_*\|_H \leq \varepsilon$.

Proof. We will now apply the abstract result [22, Theorem 6] to the energy functional $\mathcal{E}(\varphi)$, which according to (2.3) is the sum of a double-well potential and an interface energy term. In contrast to this feature, we shall split $\mathcal{E}(\varphi)$ into the sum of a convex (entropy) functional $\Phi : H = L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$, with a suitable effective domain, and a non-local interaction functional $\Psi : H \to \mathbb{R}$. To this end, we define the lower-semicontinuous and strongly convex functional

$$\Phi(\varphi) := \begin{cases} \int_\Omega \left(F(\varphi) + \frac{a(x)}{2}\varphi^2 \right) dx, & \text{if } \varphi \in L^\infty(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

with closed effective domain $\text{dom}(\Phi) = L^\infty(\Omega) \cap \mathcal{Y}_m$, and the quadratic functional $\Psi : H \to \mathbb{R}$, given by

$$\Psi(\varphi) := \frac{1}{4} \int_\Omega \int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2 \, dx \, dy + \int_\Omega \frac{a(x)}{2}(2\varphi - \varphi^2) \, dx$$

$$= \int_\Omega \left(-\frac{1}{2}\varphi(J * \varphi) + a(x)\varphi \right) dx$$

(the last equality is a direct computation). We have that Φ is Fréchet differentiable on any open subset $\overline{U}$ of $U_m := \{\psi \in L^\infty(\Omega) : |\langle \psi \rangle| \leq m\}$, with Fréchet derivative

$D\Phi : \overline{U} \to L^{\infty}(\Omega)$ having the form

$$\langle D\Phi(\varphi), \xi \rangle = \int_{\Omega} (F'(\varphi) + a(x)\varphi) \cdot \xi dx,$$

for all $\varphi \in \overline{U}$ and $\xi \in L^{\infty}(\Omega)$. The analyticity of $D\Phi$ as a mapping on $L^{\infty}(\Omega)$ is standard and can be proved exactly as in, e.g., [16, Theorem 5.1]. Moreover, due to assumption (H2), there holds

$$\langle D\Phi(\varphi_1) - D\Phi(\varphi_2), \varphi_1 - \varphi_2 \rangle \geq c_0 \|\varphi_1 - \varphi_2\|_H^2,$$

for all $\varphi_1, \varphi_2 \in \overline{U}$, and

$$\|D\Phi(\varphi_1) - D\Phi(\varphi_2)\|_{H^*} \leq \gamma \|\varphi_1 - \varphi_2\|_H,$$

for some positive constant γ. Moreover, computing the second Fréchet derivative $D^2\Phi$ of Φ,

$$\langle D^2\Phi(\varphi)\xi_1, \xi_2 \rangle = \int_{\Omega} (F''(\varphi) + a(x))\xi_1 \cdot \xi_2 dx$$

yields that $D^2\Phi \in \mathcal{L}(L^{\infty}(\Omega), L^{\infty}(\Omega))$ is an isomorphism for every $\varphi \in \overline{U}$. Concerning the (quadratic) function Ψ, we see that

$$\Psi(\varphi) = \frac{1}{2}\langle -J * \varphi, \varphi \rangle + \langle a(x), \varphi \rangle, \quad a \in L^{\infty}(\Omega), \forall \varphi \in H.$$

We recall that the linear operator $\psi \mapsto J * \psi$ is self-adjoint and compact from H to itself and is also compact from $L^{\infty}(\Omega)$ to $C^0(\bar{\Omega})$ (cf. Remark 2.1). On the other hand, we also have the following (orthogonal) sum decomposition of $H = H_0 \oplus H_1$, where

$$H_0 := \{\varphi \in H : \langle \varphi \rangle = 0\}, \ H_1 := \{\varphi \in H : \varphi = \text{const}\}$$

Thus the annihilator of H_0 is the one-dimensional subspace of constant functions $H_0^0 := \{ch \in H^* : c \in \mathbb{R}\}$, where $h \in H^* \simeq H$ is given by $\langle h, u \rangle = \frac{1}{|\Omega|}\int_{\Omega} udx$, $u \in H$. Hence, the hypotheses of [22, Theorem 6] are satisfied and the sum

$$\mathcal{E} = \Phi + \Psi : H \to \mathbb{R} \cup \{\infty\}$$

is a well defined, bounded from below functional with nonempty, closed, and convex effective domain $\text{dom}(F) = \text{dom}(\Phi)$. Unravelling notation in [22, Theorem 6], and observing that the Fréchet derivative

$$D\mathcal{E}(\varphi) = a(x)\varphi - J * \varphi + F'(\varphi) = \mu,$$

we have

$$|\mathcal{E}(\varphi) - \mathcal{E}(\varphi_*)|^{1-\theta} \leq C \inf_{\varphi \in H} \left\{ \|D\mathcal{E}(\varphi) - \mu_*\|_{L^2(\Omega)} : \mu_* \in H_0^0 \right\}$$
$$= C \|\mu - \langle \mu \rangle\|_{L^2(\Omega)},$$

from which (2.47) follows.

We can now prove the following convergence result.

Theorem 2.21. *Let the assumptions of Theorem 2.2 hold and, in addition, assume that F is real analytic. Then, any weak solution φ to problem (1.6)-(1.9) belonging to the class (2.2) satisfies*

(2.48)
$$\lim_{t \to \infty} \|\varphi(t) - \varphi_*\|_{L^{\infty}(\Omega)} = 0,$$

where φ_* is solution to (2.46).

Proof. Before we begin the proof, we note that by virtue of Lemma 2.10 and Lemma 2.12, all $\varphi_* \in \omega[\varphi]$ are bounded in $L^\infty(\Omega) \cap V$. Besides, recalling also the energy identity (2.3), we have

$$\mathcal{E}(\varphi(t)) \to \mathcal{E}_\infty, \text{ as } t \to \infty,$$

and the limit energy $\mathcal{E}_\infty$ is the same for every steady-state solution $\varphi_* \in \omega[\varphi]$. Moreover, we can integrate the energy equality (2.3),

$$\frac{d}{dt}\mathcal{E}(\varphi(t)) = -\|\nabla\mu(t)\|_H^2$$

over (t, ∞) to get

(2.49) $$\int_t^\infty \|\nabla\mu(s)\|_H^2 \, ds = \mathcal{E}(\varphi(t)) - \mathcal{E}_\infty = \mathcal{E}(\varphi(t)) - \mathcal{E}(\varphi_*).$$

By virtue of Lemma 2.20 (cf. also Remark 2.1), we have

$$|\mathcal{E}(\varphi(t)) - \mathcal{E}(\varphi_*)|^{1-\theta} \le C\|\mu(t) - \langle\mu(t)\rangle\|_{L^2(\Omega)} \le C\|\nabla\mu(t)\|_H^2$$

provided that

(2.50) $$\|\varphi - \varphi_*\|_H \le \varepsilon.$$

This, combined with the previous identity, yields

(2.51) $$\int_t^\infty \|\nabla\mu(s)\|_H^2 \, ds \le C\|\nabla\mu(t)\|_H^{\frac{1}{(1-\theta)}},$$

for all $t > 0$, for as long as (2.50) holds. Note that, in general, the quantities θ, C and ε above may depend on φ_*. Let us set

$$M = \cup\{\mathcal{I} : \mathcal{I} \text{ is an open interval on which } (2.50) \text{ holds}\}.$$

Clearly, M is nonempty since $\varphi_* \in \omega[\varphi]$. We can now use (2.51), the fact that $\|\nabla\mu(t)\|_H \in L^2(0, \infty)$ (cf. (2.3)), and exploit [17, Lemma 7.1] with $\alpha = 2(1-\theta)$ to deduce that $\|\nabla\mu(\cdot)\|_H \in L^1(M)$ and

(2.52) $$\int_M \|\nabla\mu(s)\|_H \, ds \le C(\varphi_*) < \infty.$$

Consequently, using the bound (2.52) and the main equation (1.6), we also obtain

(2.53) $$\int_M \|\partial_t\varphi(s)\|_{V'} \, ds < \infty.$$

In order to finish the proof of the convergence result in (2.48) it suffices to show that it holds in H-norm. Indeed, in this case (2.48) will become an immediate consequence of the L^2-$(L^\infty \cap V)$ smoothing property of the weak solutions and all $\varphi_* \in \omega[\varphi]$ (see Lemmas 2.10 and 2.12). We claim that we can find a sufficiently large time $\tau > 0$ such that $(\tau, \infty) \subset M$. To this end, recalling (2.49) and the above bounds, we also have that $\partial_t\varphi \in L^2(0, \infty; V')$, $\nabla\mu \in L^2(0, \infty; H^d)$ and, furthermore, for any $\delta > 0$ there exists a time $t_* = t_*(\delta) > 0$ such that

(2.54) $\|\partial_t\varphi\|_{L^1(M\cap(t_*,\infty);V')} \le \delta$, $\|\partial_t\varphi\|_{L^2((t_*,\infty);V')} \le \delta$, $\|\nabla\mu\|_{L^2((t_*,\infty);H^d)} \le \delta$.

Next, observe that by Lemma 2.12 and Lemma 2.10, there is a time $t_\# > 0$ such that

(2.55) $$\sup_{t \ge t_\#} \|\varphi(t)\|_{V \cap L^\infty(\Omega)} \le C.$$

Now, let $(t_0, t_2) \subset M$, for some $t_2 > t_0 \geq t_* (\delta)$, $|t_0 - t_2| \geq 1$ such that (2.55) holds (w.l.o.g., we shall assume that $t_* \geq t_\#$). This claim is an immediate consequence of the aforementioned L^2-$(L^\infty \cap V)$ smoothing property and bounds (2.54). Using (2.54) and (2.55), we obtain

$$(2.56) \qquad \|\varphi(t_0) - \varphi(t_2)\|_H^2 = 2 \int_{t_0}^{t_2} \langle \partial_t \varphi(s), \varphi(s) - \varphi(t_0) \rangle \, ds$$

$$\leq 2 \int_{t_0}^{t_2} \|\partial_t \varphi(s)\|_{V'} (\|\varphi(s)\|_V + \|\varphi(t_0)\|_V) \, ds$$

$$\leq C \|\partial_t \varphi\|_{L^1(t_0, t_2; V')} \left(\|\varphi\|_{L^\infty(t_*, \infty; V)} + 1 \right) \leq C\delta.$$

Therefore we can choose a time $t_* (\delta) = \tau < t_0 < t_2$, such that

$$(2.57) \qquad \qquad \qquad \|\varphi(t_0) - \varphi(t_2)\|_H < \frac{\varepsilon}{3}$$

provided that (2.50) holds for all $t \in (t_0, t_2)$. Since $\varphi_* \in \omega[\varphi]$, a large (redefined) τ can be chosen such that

$$(2.58) \qquad \qquad \qquad \|\varphi(\tau) - \varphi_*\|_H < \varepsilon/3;$$

hence, (2.57) yields $(\tau, \infty) \subset M$. Indeed, taking

$$\overline{t} = \inf \{ t > \tau : \|\varphi(t) - \varphi_*\|_H \geq \varepsilon \},$$

we have $\overline{t} > \tau$ and $\|\varphi(\overline{t}) - \varphi_*\|_H \geq \varepsilon$ if $\overline{t}$ is finite. On the other hand, in view of (2.57) and (2.58), we have

$$\|\varphi(t) - \varphi_*\|_H \leq \|\varphi(t) - \varphi(\tau)\|_H + \|\varphi(\tau) - \varphi_*\|_H < \frac{2}{3}\varepsilon,$$

for all $\overline{t} > t \geq \tau$, and this leads to a contradiction. Therefore, $\overline{t} = \infty$ and by (2.54) the integrability of $\partial_t \varphi$ in $L^1(\tau, \infty; V')$ follows. Hence, $\omega[\varphi] = \{\varphi_*\}$ and (2.48) holds on account of the L^2-$(L^\infty \cap V)$ smoothing property. The proof is finished. □

Remark 2.22. Exploiting the L^2-$(C^\alpha \cap V)$ smoothing property of the weak and stationary solutions again, and the inequality (2.47) it is be possible the show the convergence rate:

$$\|\varphi(t) - \varphi_*\|_{C^{a/2}(\overline{\Omega})} \sim (1 + t)^{-\frac{1}{\rho}}, \quad \text{as } t \to \infty,$$

for some positive constant $\rho = \rho(\alpha, \theta, \varphi_*) \in (0, 1)$.

3. The viscous case with singular potential

3.1. **Well-posedness result.** Here we consider the viscous case, but we suppose that the potential is singular. More precisely, following [19], we assume that F can be written as $F = F_1 + F_2$, where $F_1 \in C^{(2+2q)}(-1, 1)$, with q a fixed positive integer and $F_2 \in C^2[-1, 1]$. Such functions are subject to the following hypotheses:

(H7) There exist $c_1 > 0$ and $\epsilon_0 > 0$ such that

$$F_1^{(2+2q)}(s) \geq c_1, \qquad \forall s \in (-1, -1 + \epsilon_0] \cup [1 - \epsilon_0, 1).$$

(H8) There exists $\epsilon_0 > 0$ such that, for each $k = 0, 1, \cdots, 2 + 2q$ and each $j = 0, 1, \cdots, q$,

$$F_1^{(k)}(s) \geq 0, \qquad \forall s \in [1 - \epsilon_0, 1),$$
$$F_1^{(2j+2)}(s) \geq 0, \qquad F_1^{(2j+1)}(s) \leq 0, \qquad \forall s \in (-1, -1 + \epsilon_0].$$

(H9) There exists $\epsilon_0 > 0$ such that $F_1^{(2+2q)}$ is non-decreasing in $[1 - \epsilon_0, 1)$ and non-increasing in $(-1, -1 + \epsilon_0]$.

(H10) There exist $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > - \min\limits_{[-1,1]} F_2''$ such that

$$F_1''(s) \geq \alpha, \qquad \forall s \in (-1, 1), \quad a(x) \geq \beta, \text{ a.e. } x \in \Omega.$$

(H11) $\lim_{s \to \pm 1} F'(s) = \pm\infty$.

Assumptions (H7)-(H11) are satisfied, e.g., when F is of the form (1.3), for any fixed positive integer q. In particular, setting

$$F_1(s) = (1 + s) \log(1 + s) + (1 - s) \log(1 - s), \qquad F_2(s) = -\lambda s^2,$$

assumption (H10) is satisfied if and only if $\beta > \gamma - 1$.

The notion of weak solution to problem (1.6)-(1.9) is given by

Definition 3.1. *Let $0 < T < +\infty$ be given. Suppose $\varphi_0 \in L^\infty(\Omega)$, $|\langle\varphi_0\rangle| < 1$ with $F(\varphi_0) \in L^1(\Omega)$. A function φ is a weak solution of (1.6)-(1.9) on $[0, T]$, if*

(3.1) $\varphi \in L^\infty(0, T; H) \cap L^\infty(0, T; L^{2+2q}(\Omega))$;

(3.2) $\partial_t \varphi \in L^2(0, T; V'), \quad \sqrt{\alpha}\partial_t \varphi \in L^2(0, T; H)$;

(3.3) $\mu = a\varphi - J * \varphi + F'(\varphi) + \alpha\partial_t\varphi \in L^2(0, T; V)$;

(3.4) $\varphi \in L^\infty(Q), \quad |\varphi(x, t)| < 1 \quad a.e. \ (x, t) \in Q := \Omega \times (0, T)$;

and φ satisfies the weak formulation (2.5).

The following existence result holds.

Theorem 3.2. *Suppose that J obeys (H1) and assume that (H7)-(H11) are satisfied for some fixed positive integer q. Let $\varphi_0 \in L^\infty(\Omega)$ be such that $|\langle\varphi_0\rangle| < 1$ and $F(\varphi_0) \in L^1(\Omega)$. Then, for every $T > 0$, there exists a weak solution φ of (1.6)-(1.9) in the sense of Definition 3.1. In addition, for all $t \geq 0$, we have $(\varphi(t), 1) = (\varphi_0, 1)$ and the energy identity holds:*

(3.5) $$\mathcal{E}(\varphi(t)) + \int_0^t \left(\|\nabla\mu(s)\|_H^2 + \alpha \|\partial_t\varphi(s)\|_H^2 \right) ds = \mathcal{E}(\varphi_0), \ \forall t \geq 0,$$

where $\mathcal{E}(\varphi)$ is given by (2.3).

Proof. The argument for $\alpha = 0$ was given in [19, Theorem 1 and Corollary 1]. In the case $\alpha > 0$ the proof goes essentially as in [19] with some minor modifications (see *Step 1* below). Indeed, the whole idea is to approximate (1.6)-(1.9) by a problem P_ϵ which is obtained from (1.6)-(1.9) by replacing the singular potential F with a smooth (regular) potential of polynomial growth $F_\epsilon = F_{1\epsilon} + \overline{F}_2$, where $F_{1\epsilon}$ is defined in such a way (cf. [19, Lemma 1 and Lemma 2]) that $F_\epsilon' \to F'$ uniformly on every compact interval included in $(-1, 1)$, and such that the following properties hold:

(i) There exist $c_q, d_q > 0$, which depend on q but are independent of ϵ, and $\epsilon_0 > 0$ such that

(3.6) $$F_\epsilon(s) \geq c_q |s|^{2+2q} - d_q, \qquad \forall s \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_0].$$

(ii) Setting $c_0 := \alpha + \beta + \min_{[-1,1]} F_2'' > 0$, there exists $\epsilon_1 > 0$ such that

(3.7) $$F_\epsilon''(s) + a(x) \geq c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \quad \forall \epsilon \in (0, \epsilon_1].$$

The approximating problem P_ϵ, $\epsilon > 0$, then takes the following form: find a weak solution φ_ϵ to

(3.8) $$\begin{cases} \partial_t \varphi_\epsilon = \Delta \widetilde{\mu}_\epsilon, \quad \widetilde{\mu}_\epsilon = \mu_\epsilon + \alpha \partial_t \varphi_\epsilon, & \text{in } \Omega \times (0, T), \\ \mu_\epsilon = a \varphi_\epsilon - J * \varphi_\epsilon + F_\epsilon'(\varphi_\epsilon), & \text{in } \Omega \times (0, T) \\ \partial_{\mathbf{n}} \widetilde{\mu}_\epsilon = 0, & \text{on } \Gamma \times (0, T), \\ \varphi_\epsilon(0) = \varphi_0, & \text{in } \Omega. \end{cases}$$

Step 1. We will now briefly explain how to deduce the existence of at least one weak solution to problem P_ϵ for a given $\epsilon > 0$. We first take an initial datum φ_0 such that $|\langle \varphi_0 \rangle| < 1$, $F_\epsilon(\varphi_0) \in L^1(\Omega)$ and

(3.9) $$\varphi_0 \in L^\infty(\Omega) \cap V.$$

We also approximate the interaction kernel J with, say, $J \in W^{1,\infty}(\mathbb{R}^d)$. To prove the existence of a solution $\varphi_\epsilon(t)$ to problem (3.8) (cf. Remark 2.3) corresponding to the initial datum φ_0 we can perform the same estimates as in [10, proof of Theorem 2.2] (cf. also [19, Section 3]) using a Galerkin-type argument. Note that, this argument is actually independent of whether $\alpha > 0$ or $\alpha = 0$, since $\langle \partial_t \varphi(t) \rangle = 0$, for all $t \geq 0$, so that $\langle \widetilde{\mu}_\epsilon(t) \rangle = \langle \mu_\epsilon(t) \rangle$, for all $t \geq 0$. Thus, using the assumptions (H8)-(H9) on F and exploiting (i)-(ii) above, one can argue exactly word by word as in [19, (3.20)-(3.38)] to deduce from (3.5), for every $T > 0$, the following estimates on the Galerkin approximating solutions (indices are omitted)

(3.10) $$\|\varphi_\epsilon\|_{L^\infty(0,T;L^{2+2q}(\Omega))} \leq C,$$
$$\|\nabla \widetilde{\mu}_\epsilon\|_{L^2(0,T;H)} \leq C,$$
$$\sqrt{\alpha} \|\partial_t \varphi_\epsilon\|_{L^2(0,T;H)} \leq C,$$
$$\|F_\epsilon'(\varphi_\epsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$

for some positive constant C which depends on the initial data $\varphi_0 \in L^\infty(\Omega)$, but is independent of t, T, ϵ, and $\alpha \geq 0$. These estimates and the Poincaré-Wirtinger inequality entail:

(3.11) $$\|\widetilde{\mu}_\epsilon\|_{L^2(0,T;V)} \leq C.$$

Hence, in view of (3.11), by comparison in (3.8) we also have the bound:

(3.12) $$\|\partial_t \varphi_\epsilon\|_{L^2(0,T;V')} \leq C.$$

Next, we need another estimate for $\varphi_\epsilon(t)$ in the V-norm. For this we can choose $\psi = \varphi_\epsilon(t)$ as a test function in (2.5). We obtain

$$\frac{d}{dt} \left(\|\varphi_\epsilon(t)\|_H^2 + \alpha \|\nabla \varphi_\epsilon(t)\|_H^2 \right)$$
$$+ 2 \left((a(x) + F_\epsilon''(\varphi_\epsilon)) \nabla \varphi_\epsilon(t), \nabla \varphi_\epsilon(t) \right)$$
$$= 2 \left(\nabla J * \varphi_\epsilon(t) - \varphi_\epsilon(t) \nabla a, \nabla \varphi_\epsilon(t) \right).$$

Recalling (ii) and estimate (3.10), we get

$$(3.13) \qquad \frac{d}{dt}\left(\|\varphi_\epsilon(t)\|_H^2 + \alpha\|\nabla\varphi_\epsilon(t)\|_H^2\right) + c_0\|\nabla\varphi_\epsilon(t)\|_H^2 \le C_J,$$

for some appropriate constant $C_J > 0$ which is independent of ϵ but depends on $\|\nabla J\|_{L^\infty(\mathbb{R}^d)}$. Estimate (3.13) yields on account of a suitable Gronwall's inequality,

$$(3.14) \qquad \|\varphi_\epsilon(t)\|_V^2 \le \|\varphi_0\|_V^2 \, e^{-\gamma t} + C_{\alpha,J}, \quad \forall t \ge 0,$$

for some positive constant γ independent of ϵ. This further estimate ensures that we have strong convergence in $L^2(0,T;H)$ for some subsequence so that we can identify the nonlinear term in the continuous limit. The existence of a solution to P_ϵ is proven in the case of a smooth initial datum.

We can now establish the existence of a solution with an initial datum $\varphi_0 \in H$ such that $|\langle\varphi_0\rangle| < 1$ and $F_\epsilon(\varphi_0) \in L^1(\Omega)$ with a sequence $\varphi_{0j} \subset V$ with the same properties and such that

$$(3.15) \qquad \varphi_{0j} \to \varphi_0 \text{ in } H\text{-norm},$$

as $j \to \infty$. Also we take a sequence $\{J_j\} \in W^{1,\infty}(\mathbb{R}^d)$ which satisfies (H1) and strongly converges to J in $W^{1,1}(\mathbb{R}^d)$ as $j \to \infty$.

Let $\{\varphi_{j,\epsilon}\}$ be a sequence of solutions associated with $\{\varphi_{j,0}\}$. Arguing as in the proof of Lemma 3.3 below, we can get the estimate

$$(3.16) \qquad \begin{aligned} &\|\varphi_{j,\epsilon}(t) - \varphi_{i,\epsilon}(t)\|_{V'}^2 + \alpha\|\varphi_{j,\epsilon}(t) - \varphi_{i,\epsilon}(t)\|_H^2 \\ &\le \left(\|\varphi_{j,\epsilon}(0) - \varphi_{i,\epsilon}(0)\|_{V'}^2 + \alpha\|\varphi_{j,\epsilon}(0) - \varphi_{i,\epsilon}(0)\|_H^2\right) e^{\kappa t} \\ &\quad + C\left|\langle\varphi_{j,\epsilon}(0)\rangle - \langle\varphi_{i,\epsilon}(0)\rangle\right| e^{\kappa t}, \end{aligned}$$

for some positive constants C, κ which depend on $\|J_j\|_{W^{1,1}(\mathbb{R}^d)}$ and Ω, but are independent of ϵ and i,j. This yields, on account of (3.15), that as $j \to \infty$, we have the strong convergence of the sequence of solutions $\varphi_{j,\epsilon}$ for $\alpha > 0$, to some function φ_ϵ, i.e.,

$$(3.17) \qquad \varphi_{j,\epsilon} \to \varphi_\epsilon \text{ strongly in } C\left([0,T];H\right),$$

for every $\epsilon > 0$. Finally, from the preceding estimates (3.10)-(3.12), we can infer that (up to subsequences)

$$(3.18) \qquad \begin{cases} \partial_t\varphi_{j,\epsilon} & \to \partial_t\varphi_\epsilon \text{ weakly in } L^2(0,T;H), \\ \varphi_{j,\epsilon} & \to \varphi_\epsilon \text{ weakly-star in } L^\infty(0,T;L^{2+2q}(\Omega)), \\ \widetilde{\mu}_{j,\epsilon} & \to \widetilde{\mu}_\epsilon \text{ weakly in } L^2(0,T;V), \end{cases}$$

as $j \to \infty$. Therefore, arguing as in [10], the convergence properties (3.17)-(3.18) allow us to show that φ_ϵ is a weak solution to problem P_ϵ, with φ_0 satisfying the assumptions of Theorem 3.2 and J fulfilling (H1).

Step 2. The passage to limit as $\epsilon \to 0$ is actually easier. Indeed, we have already observed that estimates (3.10)-(3.12) and (3.16) hold with constants independent of $\epsilon > 0$. The strong convergence can still be deduced by using the continuous dependence estimate proven here below (see (3.19)). Thus a similar argument works when ϵ goes to zero. We will only mention that in order to pass to the limit in the variational formulation for problem P_ϵ, we need to show that $|\varphi| < 1$ almost everywhere in $Q = \Omega \times (0,T)$. This can be done by adapting an argument from [14, Section 4]. We refer the reader to [19, Section 3] for further details. $\qquad\square$

Uniqueness is an immediate consequence of the following result whose proof goes essentially as in Lemma 2.14.

Lemma 3.3. *Let φ_i, $i = 1, 2$, be a pair of weak solutions according to the assumptions of Theorem 3.2. Then the following estimate holds:*

$$(3.19) \quad \|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 + \alpha \|\varphi_1(t) - \varphi_2(t)\|_H^2$$

$$\leq \left(\|\varphi_1(0) - \varphi_2(0)\|_{V'}^2 + \alpha \|\varphi_1(0) - \varphi_2(0)\|_H^2 \right) e^{\kappa t} + C e^{\kappa t} |M_1 - M_2|,$$

for all $t \geq 0$, where $M_i := \langle \varphi_i(0) \rangle$, for some positive constants C, κ which depend on c_0 and J, but are independent of $\alpha \geq 0$.

Proof. We see that φ (formally) satisfies the problem:

$$(3.20) \qquad \partial_t \varphi = \Delta \overline{\mu}, \quad \overline{\mu} = a(x) \varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2) + \alpha \partial_t \varphi,$$

subject to the boundary and initial conditions

$$(3.21) \qquad\qquad \partial_{\mathbf{n}} \overline{\mu}_{|\Gamma} = 0, \; \varphi_{|t=0} = \varphi_1(0) - \varphi_2(0) \text{ in } \Omega.$$

Arguing as in Lemma 2.14, we obtain, on account of (H10), the following estimate:

$$(3.22) \qquad \frac{d}{dt} \left(\|\varphi(t)\|_{V'}^2 + \alpha \|\varphi(t)\|_H^2 \right) + 2c_0 \|\varphi(t)\|_H^2$$

$$\leq c_0 \|\varphi(t)\|_H^2 + \kappa \|\varphi(t)\|_{V'}^2 + C |\langle \overline{\mu}(t) \rangle| |M_1 - M_2|,$$

where κ, C depend on c_0, Ω and J, but are independent of $\alpha \geq 0$. Observe now that we also have $F'(\varphi) \in L^\infty(0, T; L^1(\Omega))$. Therefore we can still deduce (2.29) and the application of Gronwall's inequality to (3.22) entails the desired estimate (3.19) exactly as in Lemma 2.14. $\qquad\square$

On account of the previous results, we can define a dynamical system on the metric space

$$\mathcal{Y}_m := \{ \psi \in L^\infty(\Omega) : |\psi| < 1, \text{ a.e. in } \Omega, \; F(\psi) \in L^1(\Omega), \; |\langle \psi \rangle| \leq m \}$$

where $m \in [0, 1)$ is fixed and the metric is given by (2.8). Then, for each $\alpha \geq 0$ we can also define a semigroup

$$(3.23) \qquad\qquad S(t) : \mathcal{Y}_m \to \mathcal{Y}_m, \quad S(t) \varphi_0 = \varphi(t),$$

where $\varphi(t)$ is the unique weak solution of (1.6)-(1.9). In fact, arguing as in [19, Section 4, Theorem 2], we deduce the following

Theorem 3.4. *Let the assumptions of Theorem 3.2 hold and assume that F is bounded in $(-1, 1)$. Then the dynamical system $(\mathcal{Y}_m, S(t))$ possesses a connected global attractor $\mathcal{A}$.*

3.2. **Exponential attractors.** Note that, according to Theorem 3.4, a global attractor $\mathcal{A}$ exists for any $\alpha \geq 0$. However, we are able to show its finite dimensionality only in the case $\alpha > 0$. This assumption is intimately connected with the aforementioned separation property which will allow to handle F' on a closed interval of the form $[-1 + \delta, 1 - \delta]$. We have the following.

Theorem 3.5. *Let the assumptions of Theorem 3.2 be satisfied. If $\alpha > 0$ then, for every fixed $m \geq 0$, there exists an exponential attractor $\mathcal{M} = \mathcal{M}(m)$, bounded in $L^\infty(\Omega)$ and compact in H, for the dynamical system $(\mathcal{Y}_m, S(t))$ which satisfies the following properties:*

(i) Semi-invariance: $S(t)\mathcal{M} \subset \mathcal{M}$, *for every* $t \geq 0$.

(ii) Separation property: there exists $\delta_0 = \delta_0(m, \alpha) \in (0, 1)$ *such that*

$$\|\mathcal{M}\|_{L^\infty(\Omega)} \leq 1 - \delta_0.$$

(iii) Exponential attraction:

$$dist_{L^s(\Omega)}(S(t)\mathcal{Y}_m, \mathcal{M}) \leq Ce^{-\kappa t},$$

for some positive constants C_m *and* κ, *for any* $s \in [2, \infty)$.

(iv) Finite dimensionality:

$$\dim_F(\mathcal{M}, H) \leq C_m < \infty.$$

Remark 3.6. Note that, thanks to the separation property, the assumption that F is bounded on $(-1, 1)$ (see Theorem 3.4) is not needed.

Corollary 3.7. *Let the assumptions of Theorem 3.5 be satisfied. The global attractor* $\mathcal{A}$ *has finite fractal dimension*

$$\dim_F(\mathcal{A}, H) < \infty$$

and satisfies $\|\mathcal{A}\|_{L^\infty(\Omega)} \leq 1 - \delta_0$, *for some* $\delta_0 = \delta_0(m, \alpha) \in (0, 1)$.

First, we derive some (uniform in time) a priori estimates for the weak solutions. For the next result, we also assume that the boundary Γ is of class $\mathcal{C}^2$ (we note that Lemma 3.8 is the only place where this assumption is used; however, see Remark 3.9).

Lemma 3.8. *Let the assumptions of Theorem 3.5 be satisfied. For every* $\tau > 0$, *there exists a positive constant* $C_{m,\tau} \sim 1 + \tau^{-n_0}$ *(for some* $n_0 > 0$*) such that the following estimate holds:*

$$(3.24) \qquad \sup_{t \geq \tau} \left(\|\mu(t)\|_{H^2(\Omega)} + \alpha \|\partial_t \varphi(t)\|_H \right) \leq C_{m,\tau}.$$

Proof. To rigorously prove (3.24), one has to employ the regularization procedure introduced in Theorem 3.2 and to exploit the fact that all the estimates below hold *uniformly* in $\epsilon > 0$ (we can also employ a Faedo-Galerkin scheme for (3.8) to ensure that the approximate solutions φ_ϵ are smooth enough).

To this end, set $\zeta := \partial_t \varphi$ and note that $\langle \zeta(t) \rangle = 0$, for all $t \geq 0$. According to (2.5), the function ζ satisfies the following weak formulation:

$$(3.25) \qquad \langle \partial_t \zeta, \psi \rangle + (\nabla \eta, \nabla \psi) = 0, \text{ a.e. in } (0, T),$$

for every $\psi \in V$, where

$$(3.26) \qquad \eta = (a(x) + F''(\varphi))\zeta - J * \zeta + \alpha \partial_t \zeta, \text{ a.e. in } \Omega \times (0, T).$$

As we mentioned above, we note that (3.25) is actually intended to be satisfied by a standard Galerkin approximation of φ_ε, in which we should have at least $\partial_t \zeta \in L^2(0, T; L^2(\Omega))$. The required regularity in (3.24) will be then obtained by passing to the limit in the subsequent estimates. Thus, in what follows we shall proceed formally. Testing (3.25) with $\psi = \mathcal{N}\zeta(= A_N^{-1}\zeta)$, then integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|_{V'}^2 = -(\eta - \langle \eta \rangle, \zeta)_H$$
$$= -\left(a(x) + F''(\varphi), \zeta^2\right)_H + (J * \zeta, \zeta)_H - \alpha(\partial_t \zeta, \zeta)_H,$$

which yields, thanks to assumptions (H1) and (H10),

$$(3.27) \qquad \frac{d}{dt}\left(\|\zeta\|_{V'}^2 + \alpha \|\zeta\|_H^2 \right) \leq C \|\zeta\|_H^2 ,$$

for some positive constant C which depends only on c_0 and J. Thus, using this inequality and exploiting the basic energy identity (3.5), we have

$$(3.28) \qquad \sup_{t \geq 0} \int_t^{t+1} \left(\|\zeta(s)\|_H^2 + \|\mu(s)\|_V^2 \right) ds \leq C_\alpha \sim \alpha^{-1}.$$

Thus, in view of the uniform Gronwall's lemma, we infer

$$(3.29) \qquad \sup_{t \geq \tau} \left(\|\zeta\|_{V'}^2 + \alpha \|\zeta\|_H^2 \right) \leq \mathcal{C} = \mathcal{C}_{\alpha,\tau}.$$

From this point on, the constant $\mathcal{C}$ will always denote a computable quantity whose expression is allowed to vary on occurrence, depending on the initial data, on $\alpha^{-1} > 0$, and on the other fixed parameters of the system. We shall again point out its dependence on various parameters whenever necessary. Therefore, by comparison in (1.6) and on account of (3.29), we deduce that

$$(3.30) \qquad \sup_{t \geq \tau} \|\Delta \mu(t)\|_H^2 \leq \mathcal{C}_{\alpha,\tau}.$$

Next, let us test (2.5) by $\psi = \mathcal{N}\left(F'(\varphi) - \langle F'(\varphi) \rangle \right)$ to obtain

$$\langle F'(\varphi) - \langle F'(\varphi) \rangle, \mathcal{N}\partial_t \varphi \rangle = - \langle \mu, F'(\varphi) - \langle F'(\varphi) \rangle \rangle.$$

Then note that

$$\langle \mu, F'(\varphi) - \langle F'(\varphi) \rangle \rangle = \langle a\varphi - J * \varphi + F'(\varphi) - \langle F'(\varphi) \rangle, F'(\varphi) - \langle F'(\varphi) \rangle \rangle$$
$$\geq \frac{1}{2} \|F'(\varphi) - \langle F'(\varphi) \rangle\|_H^2 - \frac{1}{2}\|a\varphi - J * \varphi\|_H^2$$
$$\geq \frac{1}{2} \|F'(\varphi) - \langle F'(\varphi) \rangle\|_H^2 - C_J\|\varphi\|_H^2.$$

Therefore, on account of (3.5) and (3.29), the above estimate allows us to infer

$$\|F'(\varphi) - \langle F'(\varphi) \rangle\|_H \leq C(\|\mathcal{N}\partial_t \varphi\|_H + 1) \leq \mathcal{C}_{\alpha,\tau}.$$

We can now easily argue as in the proof of Theorem 3.2, see (3.10) (cf. [19, Section 3, Theorem 1] and [37, Proposition A.2]) to deduce

$$\sup_{t \geq \tau} |\langle \mu(t) \rangle|^2 \leq \mathcal{C}_{\alpha,\tau}.$$

This estimate together with (3.29) and (3.30) yield the desired inequality (3.24). $\qquad \square$

Remark 3.9. The assumption on $\Gamma \in \mathcal{C}^2$ can be dispensed with so that the result below in Lemma 3.10 also holds for bounded domains with *Lipschitz* boundary Γ. Indeed, on account of known elliptic regularity theory (cf., e.g., [12]) for problem (1.6), (1.8), we can deduce that $\mu(t) \in L^\infty\left([\tau, \infty); H^{1+\gamma}(\Omega)\right)$, for any $\gamma \in \left(\frac{1}{2}, 1\right)$. Note that we cannot take $\gamma = 1$ without further assumptions on Γ (see [12]). Since $\Omega \subset \mathbb{R}^d$, $d \leq 3$, we have $H^{1+\gamma}(\Omega) \subset L^\infty(\Omega)$ in the range provided for γ and the argument below in (3.36) still applies. Thus, we can conclude the validity of Lemma 3.10 in the case of a bounded domains with Lipschitz boundary as well.

We now show the separation property. The restriction $\alpha > 0$ allows us to apply a comparison argument. Unfortunately, these bounds are not uniform as $\alpha \to 0^+$.

Lemma 3.10. *Let the assumptions of Theorem 3.5 be satisfied and let $\alpha > 0$. Let $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1 - \overline{\delta}$, for some $\overline{\delta} > 0$. Then, the solution $\varphi(t) = S(t)\varphi_0$ is instantaneously bounded, i.e., for every $\tau > 0$, we have*

$$(3.31) \qquad \sup_{t \geq \tau} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_{\alpha,\tau,\overline{\delta}} \left(\|\varphi_0\|_{\mathcal{Y}_m}\right),$$

where the constant $\delta_{\alpha,\tau,\overline{\delta}} > 0$ depends on α^{-1}, τ, $\overline{\delta}$ and the initial data φ_0 in $\mathcal{Y}_m$. Moreover, there exists a time $t_0 = t_0 \left(\|\varphi_0\|_{\mathcal{Y}_m}\right) > 0$, depending on the initial data, and there are constants $C'_\alpha, \delta_\alpha > 0$, independent of the initial data, such that

$$(3.32) \qquad \sup_{t \geq t_0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_\alpha.$$

In particular, the separation property $\|F'(\varphi(t))\|_{L^\infty(\Omega)} \leq C'_\alpha$ holds for all $t \geq t_0$.

Proof. Step 1. To prove the instantaneous boundedness (3.31), we rewrite equation (1.7) as a first-order ordinary differential equation:

$$(3.33) \qquad \alpha \partial_t \varphi + F'(\varphi) + a(x)\varphi = \mu + J * \varphi =: h_{\mu,\varphi}.$$

Recall that (3.33) is also subject to the initial condition

$$(3.34) \qquad \varphi(0) = \varphi_0, \quad \text{with } |\varphi_0| < 1, \quad \text{a.e. in } \Omega,$$

and that we have (cf. Theorem 3.2)

$$(3.35) \qquad |\varphi(t)| < 1, \quad \text{a.e. in } Q_+ = \mathbb{R}_+ \times \Omega.$$

Next, according to estimate (3.24) and using the embedding $H^2(\Omega) \subset L^\infty(\Omega)$, we have

$$(3.36) \qquad \sup_{t \geq \tau} \|\mu(t)\|_{L^\infty(\Omega)} \leq \mathcal{C} = \mathcal{C}_{\tau,\alpha},$$

with an appropriate positive constant $\mathcal{C}_{\tau,\alpha}$. Moreover using (3.35) we readily obtain

$$(3.37) \qquad \sup_{t \geq \tau} \|(J * \varphi)(t)\|_{L^\infty(\Omega)} \leq C_{\alpha,\tau,J},$$

for every $\tau > 0$, which in light of (3.36) and (3.37), yields

$$\sup_{t \geq \tau} \|h_{\mu,\varphi}(t)\|_{L^\infty(\Omega)} \leq C_{\alpha,\tau}.$$

Therefore, on account of assumptions (H10)-(H11), bound (3.31) follows from the application of the comparison principle (see, e.g, [37, Proposition A.3]) to (3.33)-(3.34) on $[\tau, \infty)$.

Step 2. In order to deduce the uniform estimate (3.32) we shall first derive the following dissipative estimate:

$$(3.38) \qquad \mathcal{E}(\varphi(t)) + \int_t^{t+1} \left(\|\nabla\mu(s)\|_H^2 + \alpha\|\partial_t\varphi(s)\|_H^2\right) ds \leq \mathcal{E}(\varphi_0) e^{-\kappa t} + C_m,$$

for all $t \geq 0$, for some positive constant C_m independent of the initial data, time and $\alpha \geq 0$, but which depends on $m \in (0,1)$ such that $|\langle\varphi_0\rangle| \leq m$. The proof of (3.38) follows the same lines of [19, Proposition 2] and [10, Corollary 2]. We briefly mention some details. Let us thus multiply equation $\mu = a\varphi - J * \varphi + F'(\varphi) + \alpha\partial_t\varphi$ by φ in $L^2(\Omega)$ and integrate over Ω. We obtain

$$(3.39) \quad \langle\mu,\varphi\rangle = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \langle F'(\varphi),\varphi\rangle + \frac{\alpha}{2}\frac{d}{dt}\|\varphi\|_H^2.$$

Observe now that, due to the singular character of F', we can find $C_F > 0$ such that

$$(3.40) \qquad F'(s)s \geq F(s) - C_F, \quad \forall s \in (-1,1),$$

Then, using (3.40), we obtain

$$(3.41) \qquad \langle \mu, \varphi \rangle \geq \frac{1}{4} \int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \int_\Omega F(\varphi(t)) dx$$

$$- C_F |\Omega| + \frac{\alpha}{2} \frac{d}{dt} \|\varphi\|_H^2 .$$

We also have (note that $\langle \partial_t \varphi \rangle = 0$)

$$\langle \mu, \varphi \rangle = \langle \mu - \langle \mu \rangle, \varphi \rangle + \langle \mu \rangle |\Omega| \langle \varphi \rangle \leq c_\Omega \|\nabla \mu\|_H \|\varphi\|_H + c_m,$$

and then, by means of (3.40), from (3.41) we have

$$\frac{1}{8} \int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \frac{1}{2} \int_\Omega F(\varphi) dx$$

$$+ \frac{c}{2} \int_\Omega |\varphi|^{2+2q} dx - c + \frac{\alpha}{4} \frac{d}{dt} \|\varphi\|_H^2$$

$$\leq \|\nabla \mu\|_H^2 + \frac{c_\Omega^2}{2} \|\varphi\|_H^2 + c_m,$$

for appropriate constants $c_m, c > 0$, independent of the initial data, time and α. Therefore, we deduce

$$(3.42) \qquad \frac{1}{8} \int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \frac{1}{2} \int_\Omega F(\varphi) dx + \frac{\alpha}{4} \frac{d}{dt} \|\varphi\|_H^2$$

$$\leq \|\nabla \mu\|_H^2 + c_m$$

and, hence, by virtue of (3.5) and (3.42), we get

$$(3.43) \qquad \frac{d}{dt} \left(\mathcal{E}(\varphi) + \alpha \|\varphi\|_H^2 \right) + c\mathcal{E}(\varphi) + \|\nabla \mu\|_H^2 + \alpha \|\partial_t \varphi\|_H^2 \leq c_m,$$

for all $t \geq 0$. By means of Gronwall's lemma we thus easily infer (3.38) from (3.43). From (3.38), we can now find a time $t_\# = t_\# \left(\mathcal{E}(\varphi_0) \right) > 0$ such that

$$(3.44) \qquad \sup_{t \geq t_\#} \left[\mathcal{E}(\varphi(t)) + \int_t^{t+1} \|\nabla \mu(s)\|_H^2 + \alpha \|\partial_t \varphi(s)\|_H^2 \, ds \right] \leq R_m^\#,$$

for some $R_m^\# > 0$, independent of t, α and the initial data. With estimate (3.44) at hand, we can now argue as in the proof of Lemma 3.8 to get the bound:

$$(3.45) \qquad \sup_{t \geq t_\# + 1} \left(\|\mu(t)\|_{H^2(\Omega)} + \alpha \|\partial_t \varphi(t)\|_H \right) \leq R_{\alpha,m},$$

for some positive constant $R_{\alpha,m}$ which depends on $R_m^\#$ and $\alpha^{-1} > 0$ only. Finally using (3.45) and then arguing as in *Step 1* above, we can easily arrive at the following inequality:

$$(3.46) \qquad \sup_{t \geq t_0} \|h_{\mu,\varphi}(t)\|_{L^\infty(\Omega)} \leq \mathcal{R}_{\alpha,m},$$

for some positive constant $\mathcal{R}_{\alpha,m}$ which only depends on $R_{\alpha,m}$, q, and the other fixed parameters of the problem. Here t_0 depends on $t_\#$. Thus, on account of (3.46), we can once again apply the comparison principle (see, e.g., [37, Corollary A.1]) to (3.33)-(3.34), to deduce the existence of a positive constant $\delta_1 = \delta_1 (\mathcal{R}_{\alpha,m})$

which is independent of φ_0 and time, such that $|\varphi(t)| \leq 1 - \delta_1$, a.e. in Ω, for all $t \geq t_0$. Inequality (3.32) is now proven. $\qquad\square$

In what follows, we derive as in Section 2.2 some basic properties of $S(t)$ which will be useful to establish the existence of an exponential attractor. The following proposition, whose proof goes as in Lemma 2.14, is immediate (see Lemma 3.3).

Proposition 3.11. *Let the assumptions of Lemma 3.3 hold. Then we have*

$$(3.47) \qquad \left(\|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 + \alpha \|\varphi_1(t) - \varphi_2(t)\|_H^2 + C|M_1 - M_2| \right)$$
$$\leq \left(\|\varphi_1(0) - \varphi_2(0)\|_{V'}^2 + \alpha \|\varphi_1(0) - \varphi_2(0)\|_H^2 + |M_1 - M_2| \right) e^{-\beta t}$$
$$+ C \int_0^t \left(\|\varphi_1(s) - \varphi_2(s)\|_{V'}^2 + |M_1 - M_2| \right) ds,$$

for all $t \geq 0$, where $M_i := \langle \varphi_i(0) \rangle$, for some positive constants β, C which depend on c_0 and J, but are independent of α.

The following one is also straightforward.

Proposition 3.12. *Let the assumptions of Lemma 3.3 be satisfied. Then, for every $\tau > 0$, the following estimate holds:*

$$(3.48) \qquad \|\partial_t \varphi_1 - \partial_t \varphi_2\|_{L^2([\tau,t];D(A_N)')}^2 + c_0 \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_H^2 ds$$
$$\leq C_{m,\tau} e^{\kappa t} \left(\|\varphi_1(0) - \varphi_2(0)\|_{V'}^2 + \alpha \|\varphi_1(0) - \varphi_2(0)\|_H^2 + |M_1 - M_2| \right),$$

for all $t \geq \tau$, where $C_{m,\tau}$ and $\kappa > 0$ also depend on c_0, Ω, $\alpha > 0$ and J.

Proof. In light of the separation property, the proof goes essentially as the one of Lemma 2.15. $\qquad\square$

The next lemma gives the uniform Hölder continuity of $t \mapsto S(t)\varphi_0$ with respect to the H-norm.

Lemma 3.13. *Let the assumptions of Theorem 3.2 be satisfied. Consider $\varphi(t) = S(t)\varphi_0$ with $\varphi_0 \in \mathcal{Y}_m$. Then, for every $\tau > 0$, there holds:*

$$(3.49) \qquad \|\varphi(t) - \varphi(s)\|_H \leq C_{m,\alpha,\tau} |t - s|, \ \forall t, s \geq \tau,$$

where the positive constant $C_{m,\alpha,\tau}$ is independent of initial data, φ and t, s.

Proof. According to (3.24), we have the bound

$$(3.50) \qquad \sup_{t \geq \tau} \|\partial_t \varphi(t)\|_H \leq C_{m,\alpha,\tau}.$$

Observing that

$$\varphi(t) - \varphi(s) = \int_s^t \partial_t \varphi(z) \, dz,$$

we readily deduce (3.49), thanks to (3.50).

Proof of Theorem 3.5. As in Section 2.2, we apply the abstract result of Proposition 2.18. In light of the separation property in Lemma 3.10, it is not difficult to realize that there exists an absorbing set of the following form

$$\mathbb{B}(\delta_\alpha, m) := \{\varphi \in \mathcal{Y}_m \ : \ -1 + \delta_\alpha \leq \varphi \leq 1 - \delta_\alpha, \ \text{a.e. in } \Omega\},$$

for a suitable constant δ_α. We endow $\mathbb{B}(\delta_\alpha, m)$ with the metric of $\mathcal{H} = H$, and reasoning as above (see Section 2.2), we can suppose that $\mathbb{B}(\delta_\alpha, m)$ is semi-invariant for $S(t)$ for $t \geq 0$. On the other hand, due to the results proven in this section, we have

$$\sup_{t \geq 0} \left(\|F'(\varphi(t))\|_{L^\infty(\Omega)} + \|\mu(t)\|_{H^2(\Omega)} + \|\partial_t \varphi(t)\|_H \right) \leq C_{m,\alpha},$$

for every trajectory $\varphi(t)$ originating from $\varphi_0 \in \mathbb{B}(\delta_\alpha, m)$, for some positive constant $C_{m,\alpha}$ which is independent of $\varphi_0 \in \mathbb{B}(\delta_\alpha, m)$. We can now apply Proposition 2.18 to the map $\mathbb{S} = S(T)$ and $\mathcal{H} = H$, with the same choice of the functional spaces $\mathcal{V}_1$, $\mathcal{V}$ as in (2.43), owing to Propositions 3.11, 3.12 and Lemma 3.13. Consequently, we obtain the (finite-dimensional) exponential attractor $\mathcal{M}$ for $S(t)$ restricted to $\mathbb{B}(\delta_\alpha, m)$ in the H-metric. The attraction property (iii) of Theorem 3.5 is again a consequence of the separation property and the basic interpolation inequality

$$\|u\|_{L^s(\Omega)} \leq C_s \|u\|_H^{\nu_s} \|u\|_{L^\infty(\Omega)}^{1-\nu_s}, \quad \nu_s \in (0, 1).$$

Theorem 3.5 is thus proved.

Remark 3.14. In contrast to the results proved in the case of regular potentials, we cannot show that $\varphi(t)$ is ultimately bounded in V-norm like in the nonviscous case $\alpha = 0$ (cf. Lemma 2.12 and (3.13)). This can also be understood by formally rewriting the original equation in the following form

$$\varphi_t = (I - \alpha \Delta)^{-1} \Delta (a\varphi - J * \varphi + F'(\varphi))$$

which shows that this equation is much closer to the nonlocal Allen-Cahn equation (see, for instance, [1, 2, 4, 8, 28] and references therein). Moreover, there is a close connection between the viscous nonlocal Cahn-Hilliard equation and the phase-field system investigated in [31], namely,

(3.51) $$\partial_t(\varsigma \vartheta + \varphi) = \Delta \vartheta,$$

(3.52) $$\alpha \partial_t \varphi - J * \varphi + a\varphi + F'(\varphi) = \vartheta,$$

in $\Omega \times (0, \infty)$, where ϑ denotes a rescaled relative temperature and $\varsigma > 0$. Indeed, if we let ς go to 0 formally, then we get

$$\partial_t \varphi = \Delta \vartheta.$$

Thus we obtain

(3.53) $$\partial_t \varphi = \Delta (\alpha \partial_t \varphi - J * \varphi + a\varphi + F'(\varphi)),$$

that is, the viscous nonlocal Cahn-Hilliard equation. It would be interesting to investigate the connections between the phase-field system (3.51)-(3.52) and equation (3.53) along the lines of what was done for the local equations (see, e.g., [27] and its references).

3.3. **Convergence to a single equilibrium.** Also in this case we have all the ingredients to show that each trajectory does converge to a single equilibrium. We can state the following version of the Łojasiewicz-Simon theorem whose proof goes exactly, with some minor modifications, as in Lemma 2.20 (cf. [22] also).

Lemma 3.15. *Let J satisfy (H1) and let F satisfy (H7)-(H11) and be real analytic on $[-1 + \delta, 1 - \delta]$. Then, there exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\varepsilon > 0$ such that the following inequality holds:*

(3.54) $$|\mathcal{E}(\varphi) - \mathcal{E}(\varphi_*)|^{1-\theta} \leq C \|\mu - \langle \mu \rangle\|_H,$$

for all

$$\varphi \in \{\psi \in L^\infty(\Omega) \cap \mathcal{Y}_m : -1 + \delta \leq \psi \leq 1 - \delta, \ a.e. \ in \ \Omega\},$$

provided that $\|\varphi - \varphi_*\|_H \leq \varepsilon.$

The analog of Theorem 2.21 in the case $\alpha > 0$ and singular potentials f is

Theorem 3.16. *Let the assumptions of Theorem 3.5 hold. Suppose in addition that F is real analytic on $[-1 + \delta, 1 - \delta]$. Then, any weak solution φ to (1.6)-(1.9) belonging to the class (3.1)-(3.4) satisfies*

$$\lim_{t \to \infty} \|\varphi(t) - \varphi_*\|_H = 0,$$

where φ_* is solution to (2.46).

Proof. The proof goes essentially along the lines of Theorem 3.16. Indeed, it is easier since by virtue of (3.54) and the energy identity (3.5), one can establish instead of (2.53) the bound:

$$\int_M \|\partial_t \varphi(s)\|_H \, ds \leq C_\alpha,$$

which entails the integrability of $\partial_t \varphi$ in $L^1(\tau, \infty; H)$. We leave the details to the interested reader (see, also, [16, Section 6]). $\qquad\square$

4. Degenerate mobility and logarithmic potential

In this section we consider the model proposed in [23] (see also [29, 30]). Thanks to the particular form of the mobility coefficient, the separation property holds even in absence of viscosity (see [35, 36]). As a consequence, we can prove the existence of an exponential attractor in this case as well.

Referring to [23] for details, we consider the following boundary value problem

(4.1)
$$\begin{cases} \mu = F'(\varphi) + w, & \text{in } Q, \\ w(x, t) = \int_\Omega J(x - y)(1 - 2\varphi(y, t)) \, dy, & (x, t) \in Q, \\ \partial_t \varphi = \nabla \cdot (\kappa(\cdot, \varphi) \nabla \mu), & \text{in } Q, \\ \kappa(\varphi) \partial_{\mathbf{n}} \mu = 0, & \text{on } \Gamma \times (0, T), \end{cases}$$

subject to the initial condition

(4.2)
$$\varphi_{|t=0} = \varphi_0, \quad \text{in } \Omega.$$

On account of [23, Section 2] we assume the following hypotheses:

(H12) $F(\varphi) = \varphi \log \varphi + (1 - \varphi) \log(1 - \varphi).$
(H13) The mobility κ has the form

(4.3)
$$\kappa(x, \varphi) = \frac{b(x, |\nabla \varphi|)}{F''(\varphi)},$$

where the Carathéodory function $b(x, |s|) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:

(4.4)
$$(b(x, |s_1|) s_1 - b(x, |s_2|) s_2)(s_1 - s_2) \geq \alpha_0 |s_1 - s_2|^2,$$

for all $s_1, s_2 \in \mathbb{R}^d$, a.e. in Ω, for some $\alpha_0 > 0$,

(4.5)
$$|b(x, |s_1|) s_1 - b(x, |s_2|) s_2| \leq \alpha_1 |s_1 - s_2|,$$

for all $s_1, s_2 \in \mathbb{R}^d$, a.e. in Ω, for some $\alpha_1 > 0$.

The notion of weak solution to problem (4.1)-(4.2) is given by

Definition 4.1. *Let $Q := \Omega \times (0, T)$. A function φ is called a solution of (4.1)-(4.2) if*

$$(4.6) \qquad\qquad \varphi \in L^\infty(Q) \cap L^2([\delta, T]; V)$$

with

$$(4.7) \qquad \partial_t \varphi \in L^2([\delta, T]; V'), \quad w \in L^\infty([\delta, T]; W^{1,\infty}(\Omega))$$

satisfy (4.1) for every $\delta > 0$, the chemical potential obeys

$$\int_0^T \int_\Omega \kappa(x, \varphi) |\nabla \mu|^2 \, dx ds < \infty,$$

and the following identity holds

$$(4.8) \qquad \langle \partial_t \varphi, \psi \rangle + (\kappa(\cdot, \varphi) \nabla \mu, \nabla \psi) = 0, \quad \forall \psi \in V, \quad a.e. \ on \ (0, T).$$

The following well-posedness result was proven in [23, Theorem 3.5 and Corollary 3.7] (see also [35, 36]).

Theorem 4.2. *Let assumptions (H1) and (H13)-(H14) be satisfied. Consider*

$$\mathcal{Y}_{m_1, m_2} := \{ \psi \in L^\infty(\Omega) : 0 \le \psi \le 1, \ a.e. \ in \ \Omega, \ F(\psi) \in L^1(\Omega),$$
$$0 < m_1 \le \langle \psi \rangle \le m_2 < 1 \}$$

where m_1, m_2 are fixed and the metric is given by (2.8). If $\varphi_0 \in \mathcal{Y}_{m_1, m_2}$, then there exists a unique solution to problem (4.1)-(4.2) in the sense of Definition 4.1 such that $\langle \varphi(t) \rangle = \langle \varphi_0 \rangle$, for all $t \ge 0$. Moreover, we have

$$(4.9) \quad F'(\varphi) \in L^\infty([\delta, T]; L^\infty(\Omega)), \quad \mu \in L^\infty([\delta, T]; L^\infty(\Omega)) \cap L^2([\delta, T]; V),$$

for all $\delta > 0$.

Remark 4.3. We can take $\delta = 0$ in (4.6), (4.7) and (4.9) provided that $\varphi_0 \in \mathcal{Y}_{m_1, m_2}$ and, *more importantly*, $F'(\varphi_0) \in L^\infty(\Omega)$ (see [23, Corollary 3.7]). Furthermore, by [35, Lemma 2.1] the solution $\varphi(t)$ also belongs to $C([0, T]; H)$ (see Proposition 4.10 below). Thus, in view of (4.9), $\varphi \in C([0, T]; L^p(\Omega))$ for every $p \in [2, \infty)$, using interpolation and the L^∞-bound of φ.

The following separation property for the (weak) solutions given by Theorem 4.2 was proven in [35, Theorem 2.1] (see also [36]).

Theorem 4.4. *Let assumptions of Theorem 4.2 and let φ be a weak solution. Then there exist positive constants T_0, C, δ (with δ, C indepedent of the initial data) such that*

$$(4.10) \qquad\qquad \delta \le \varphi(x, t) \le 1 - \delta, \quad a.e. \ on \ \Omega \times (T_0, \infty)$$

and

$$(4.11) \qquad\qquad \sup_{t \ge T_0} \|\mu(t)\|_{L^\infty(\Omega)} \le C.$$

Remark 4.5. The proof of Theorem 4.4 is given in [35, Lemma 3.1-Lemma 3.3] (see also [36], for a simplifying argument), assuming that $b(x, |s|) \equiv const$, by exploiting an Alikakos-Moser iteration scheme for suitable powers of the functions $\log(1 - \varphi)$ and $\log(\varphi)$. The special form of κ (see (4.3)) plays an *essential* role in the calculations. The proofs in [35, 36] can be easily adapted without too much difficulty to the case of nonconstant functions $b(x, |s|)$. Indeed, from assumptions

(4.4)-(4.5) there holds $0 < \alpha_0 \leq b\left(x, |s|\right) \leq \alpha_1$, for almost any $(x, s) \in \Omega \times \mathbb{R}^d$. The same bound (4.10) can be deduced for more general functions, that is,

$$f \in C^2\left(0, 1\right) : f \text{ strictly convex, Image}(f^{'})^{-1} = [0, 1]\,, \ \frac{1}{f^{''}} \text{ strictly concave,}$$

see [36].

Remark 4.6. Note that the separation property (4.10) implies that the problem (4.1)-(4.2) is non-degenerate for all $t \geq T_0$. Indeed, we have

(4.12) $\qquad \alpha_0 \delta\left(1 - \delta\right) \leq \kappa\left(\cdot, \varphi\right) \leq \alpha_1, \quad$ a.e. in $\Omega \times \left(T_0, \infty\right).$

It is also worth mentioning that (4.10) and (4.12) hold almost everywhere in $\Omega \times \left(\tau, \infty\right),$ uniformly with respect to bounded sets of initial data in $\mathcal{Y}_{m_1, m_2}$. More precisely, for every ball of radius R, there exists $T_0 = T_0(R) > 0$ such that (4.10) and (4.11) hold (see [35], [36]).

Theorem 4.7. *Let the assumptions of Theorem 4.2 be satisfied. Then there holds*

$$\sup_{t \geq T_0} \|\varphi\|_{C^{\alpha/2, \alpha}\left([t, t+1] \times \overline{\Omega}\right)} \leq C,$$

for some $\alpha \in (0, 1)\,.$

Proof. As in the proof of Lemma 2.11, we can rewrite (4.1) as (2.22) on $t \in \left(T_0, \infty\right),$ for the function

$$a\left(x, \varphi, \nabla \varphi\right) := b\left(x, |\nabla \varphi|\right) \nabla \varphi + \kappa\left(x, \varphi\right) \nabla w.$$

Notice that, from Remarks 4.5 and 4.6, we have $a\left(x, \varphi, \nabla \varphi\right) \nabla \varphi \geq \frac{\alpha_0}{2} |\nabla \varphi|^2 - C_\delta$ and $|a\left(x, \varphi, \nabla \varphi\right)| \leq \alpha_1 |\nabla \varphi| + C_\delta$; hence [13, Corollary 4.2] still applies and this entails the desired estimate. $\qquad \square$

The main result of this section is contained in the following

Theorem 4.8. *Let the assumptions of Theorem 4.2 be satisfied. There exists an exponential attractor* $\mathcal{M}$ *bounded in* $C^\alpha\left(\overline{\Omega}\right), \alpha \in (0, 1)$ *and compact in* H, *for the dynamical system* $\left(\mathcal{Y}_{m_1, m_2}, S\left(t\right)\right)$ *associated with (4.1)-(4.2), satisfying the following properties:*

(i) Semi-invariance: $S\left(t\right) \mathcal{M} \subset \mathcal{M}$, *for every* $t \geq 0$.

(ii) The separation property (4.10) holds for every $\varphi \in \mathcal{M}$.

(iii) Exponential attraction:

$$dist_{L^s(\Omega)}\left(S\left(t\right) \mathcal{Y}_{m_1, m_2}, \mathcal{M}\right) \leq C e^{-\lambda t},$$

for some positive constants C, λ *and any* $s \in [2, \infty)\,.$

(iv) Finite dimensionality:

$$\dim_F(\mathcal{M}, V') \leq C < \infty.$$

Consequently, we also have the following

Corollary 4.9. *Let the assumptions of Theorem 4.8 be satisfied. The problem (4.1)-(4.2) possesses a finite dimensional global attractor* $\mathcal{A}$, $\dim_F(\mathcal{A}, V') < \infty$.

In what follows, we derive as in Sections 2 and 3 some basic properties of $S\left(t\right)$ which will be useful in order to establish the existence of an exponential attractor. The following proposition shows that the semigroup $S\left(t\right)$ is Lipschitz continuous in the H-norm with respect to the initial data.

Proposition 4.10. *Let φ_i, $i = 1, 2$, be a pair of weak solutions corresponding to a pair of initial data $\varphi_i(0)$, satisfying the assumptions of Theorem 4.2. Then there holds*

$$(4.13) \quad \|\varphi_1(t) - \varphi_2(t)\|_H^2 + \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_V^2\, ds \le \lambda e^{\lambda t} \|\varphi_1(0) - \varphi_2(0)\|_H^2,$$

for all $t \ge 0$, for some positive constant λ independent of t.

Proof. According to Definition 4.1, we have that $\varphi := \varphi_1 - \varphi_2$ satisfies the following weak formulation

$$(4.14) \quad \langle \partial_t \varphi, \psi \rangle + (\kappa(\cdot, \varphi_1) \nabla \mu_1 - \kappa(\cdot, \varphi_2) \nabla \mu_2, \nabla \psi) = 0, \ \forall \psi \in V, \text{ a.e. on } (0, T),$$

where $\mu_i = F'(\varphi_i) + w_i$. Testing equation (4.14) with $\psi = \varphi_1 - \varphi_2$, we deduce

$$(4.15)$$
$$\frac{1}{2} \frac{d}{dt} \|\varphi_1 - \varphi_2\|_H^2 + (b(\cdot, |\nabla \varphi_1|) \nabla \varphi_1 - b(\cdot, |\nabla \varphi_2|) \nabla \varphi_2, \nabla \varphi_1 - \nabla \varphi_2)$$
$$= -(b(\cdot, |\nabla \varphi_1|) \varphi_1 (1 - \varphi_1) \nabla w_1 - b(\cdot, |\nabla \varphi_2|) \varphi_2 (1 - \varphi_2) \nabla w_2, \nabla \varphi_1 - \nabla \varphi_2),$$

for all $t \ge 0$. Recall that, on account of Theorem 4.2, φ_i are bounded. Therefore, thanks to (H1), we have the following a priori bounds:

$$(4.16) \qquad\qquad \sup_{t \ge 0} \|\nabla w_i(t)\|_{L^\infty(\Omega)} \le C, \ i = 1, 2,$$

for a suitable positive constant C. Thus, in view of assumptions (4.4) and (4.5) (see, also, Remark 4.5), and exploiting (4.16), we obtain

$$(4.17) \qquad \frac{1}{2} \frac{d}{dt} \|\varphi_1(t) - \varphi_2(t)\|_H^2 + \alpha_0 \|\nabla(\varphi_1(t) - \varphi_2(t))\|_H^2$$
$$\le C_{\alpha_0} \left(\|\nabla w_1\|_{L^\infty(\Omega)}^2 + \|\nabla w_2\|_{L^\infty(\Omega)}^2 \right) \|\varphi_1(t) - \varphi_2(t)\|_H^2$$
$$+ \frac{\alpha_0}{2} \|\nabla(\varphi_1(t) - \varphi_2(t))\|_H^2,$$

which easily yields (4.13) on account of Gronwall's inequality.

As already mentioned, the crucial step in order to establish the existence of an exponential attractor is the validity of so-called smoothing property for the difference of two solutions. This is given by

Lemma 4.11. *Let the assumptions of Proposition 4.10 hold. Then, for every $t \ge T_0$ the following estimates hold:*

$$(4.18) \qquad \|\varphi_1(t) - \varphi_2(t)\|_H^2 \le e^{-\lambda_0 t} \|\varphi_1(0) - \varphi_2(0)\|_H^2$$
$$+ C_\delta \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_H^2\, ds,$$

$$(4.19) \qquad \|\partial_t \varphi_1 - \partial_t \varphi_2\|_{L^2([T_0, t]; V')}^2 + \int_0^t \|\varphi_1(s) - \varphi_2(s)\|_V^2\, ds$$
$$\le C_\delta e^{\lambda t} \|\varphi_1(0) - \varphi_2(0)\|_H^2,$$

for some positive constants $C_\delta, \lambda, \lambda_0$ which depend only on α_0, α_1, δ, Ω and J.

Proof. First, we observe that, due to the inequality (4.17), the separation property (4.10) and the fact that $w_i \in L^\infty \left(T_0, \infty; W^{1,\infty}(\Omega) \right)$ uniformly with respect to time, it holds

$$(4.20) \qquad \frac{d}{dt} \left\| \varphi_1(t) - \varphi_2(t) \right\|_H^2 + \alpha_0 \left\| \nabla \left(\varphi_1(t) - \varphi_2(t) \right) \right\|_H^2 \le C_\delta \left\| \varphi_1(t) - \varphi_2(t) \right\|_H^2,$$

for every $t \ge T_0$. Thus, combining (4.20) together with Poincaré's inequality

$$\left\| \varphi \right\|_H^2 \le C_\Omega \left(\left\| \nabla \varphi \right\|_H^2 + \langle \varphi \rangle^2 \right),$$

and recalling the fact that $\langle \partial_t (\varphi_1 - \varphi_2) \rangle = 0$, we deduce from (4.20), the following inequality:

$$\frac{d}{dt} \left\| \varphi_1(t) - \varphi_2(t) \right\|_H^2 + C_{\Omega, \alpha_0} \left\| \varphi_1(t) - \varphi_2(t) \right\|_H^2 \le C_\delta \left\| \varphi_1(t) - \varphi_2(t) \right\|_H^2.$$

Thus, Gronwall's inequality entails the desired estimate (4.18). The second term on the left-hand side of (4.19) is estimated in (4.13). To estimate the time derivative in (4.19), recall that φ satisfies (4.14). Thus, arguing as in the proof of Proposition (4.10) (note that the mobility $\kappa(\varphi_i)$ also satisfies (4.12)) there holds

$$(4.21) \qquad \langle \partial_t (\varphi_1(t) - \varphi_2(t)), \psi \rangle = (\kappa(\cdot, \varphi_1) \nabla \mu_1 - \kappa(\cdot, \varphi_2) \nabla \mu_2, \nabla \psi)$$
$$\le C_\delta \left\| \nabla \psi \right\|_H \left\| \nabla \varphi_1(t) - \nabla \varphi_2(t) \right\|_H,$$

for any test function $\psi \in V$, for all $t \ge T_0$. This estimate together with (4.13) gives the desired estimate on the time derivative in (4.19). $\qquad \square$

The last ingredient we need is the Hölder continuity of $t \mapsto S(t)\varphi_0$ in the V'-norm, namely,

Lemma 4.12. *Let the assumptions of Proposition 4.10 be satisfied. Consider $\varphi(t) = S(t) \varphi_0$ with $\varphi_0 \in \mathcal{Y}_{m_1, m_2}$. Then, there holds*

$$(4.22) \qquad \left\| \varphi(t) - \varphi(s) \right\|_{V'} \le C \left| t - s \right|^{1/2}, \quad \forall t, s \in [T_0, T],$$

where the constant $C = C_{\delta, T, T_0} > 0$ is independent of initial data, φ and t, s.

Proof. Testing equation (4.8) with μ, then taking the inner product in H of $\mu = F'(\varphi) + w$ with $\partial_t \varphi$, and adding the resulting relations we obtain

$$\frac{1}{2} \frac{d}{dt} (F(\varphi(t)), 1) + \int_\Omega \kappa(x, \varphi) \left| \nabla \mu(t) \right|^2 dx = - (w, \partial_t \varphi), \quad \forall t \ge T_0.$$

By virtue of (4.10) and (4.12), we can integrate this relation over (t, T), exploit the basic interpolation inequality $[V, V']_{2, 1/2} = H$, and deduce the following inequality:

$$(4.23) \qquad C_\delta \int_t^T \left\| \nabla \mu(s) \right\|_H^2 ds \le C \int_t^T \left\| w(s) \right\|_V \left\| \partial_t \varphi(s) \right\|_{V'} ds + C_\delta$$
$$\le C \int_t^T \left\| w(s) \right\|_V \left\| \nabla \mu(s) \right\|_H ds + C_\delta,$$

for all $t \ge T_0$ (here, we have used (4.8) again to estimate the time derivative). Note that (H1) and (4.10) also give the estimate

$$\sup_{t \ge T_0} \left\| w(t) \right\|_{W^{1, \infty}(\Omega)} \le C_\delta.$$

Consequently, from (4.23) we deduce that

$$\sup_{t \geq T_0} \int_t^T \|\nabla \mu(s)\|_H^2 \, ds \leq C_\delta \left(1 + (T - T_0)\right),$$

which entails

(4.24)
$$\sup_{t \geq T_0} \int_t^T \|\partial_t \varphi(s)\|_{V'}^2 \, ds \leq C_\delta \left(1 + (T - T_0)\right).$$

Estimate (4.22) now follows from (4.24).

Proof of Theorem 4.8. We shall essentially argue as in Section 2.2 by applying Proposition 2.18. We briefly mention the details. In light of the separation property in Theorem 4.4, it is not difficult to realize that there exists a (semi-invariant) absorbing set of the following form

$$\mathbb{B}_\delta := \left\{ \varphi \in \mathcal{Y}_0 \cap C^\alpha\left(\overline{\Omega}\right) : \delta \leq \varphi \leq 1 - \delta, \ \text{a.e. in } \Omega \right\}.$$

Therefore, it is sufficient to verify the existence of an exponential attractor for $S(t)_{|\mathbb{B}_\delta}$. Note that due to the above results, we also have

$$\sup_{t \geq 0} \left(\|\varphi(t)\|_{C^\alpha(\overline{\Omega})} + \|\mu(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \right) \leq C_\delta,$$

for every trajectory φ originating from $\varphi_0 \in \mathbb{B}_\delta$, for some positive constant C_δ which is independent of the choice of $\varphi_0 \in \mathbb{B}_\delta$. We can now apply the abstract result above to the map $\mathbb{S} = S(T)$ and $\mathcal{H} = H$, for a fixed $T \geq T_0$ such that $e^{-\lambda_0 T} < \frac{1}{2}$, where $\lambda_0 > 0$ is the same as in Lemma 4.11. To this end, we introduce the functional spaces

$$\mathcal{V}_1 := L^2\left([0, T]; V\right) \cap H^1\left([0, T]; V'\right), \quad \mathcal{V} := L^2\left([0, T]; H\right),$$

and note that $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$. Finally, we introduce the operator $\mathbb{T} : \mathbb{B}_\delta \to \mathcal{V}_1$, by $\mathbb{T}\varphi_0 := \varphi \in \mathcal{V}_1$, where φ solves (4.1)-(4.2) with $\varphi(0) = \varphi_0 \in \mathbb{B}_\delta$. The maps $\mathbb{S}$, $\mathbb{T}$, the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$ thus defined satisfy all the assumptions of Proposition 2.18 on account of Lemma 4.11 (see (4.18)-(4.19)). Therefore, the semigroup $\mathbb{S}(n) = S(nT)$ generated by the iterations of the operator $\mathbb{S} : \mathbb{B}_\delta \to \mathbb{B}_\delta$ possesses a (discrete) exponential attractor $\mathcal{M}_d$ in $\mathbb{B}_\delta$ endowed by the topology of H. In order to construct the exponential attractor $\mathcal{M}$ for the semigroup $S(t)$ with continuous time, we note that, due to Lemma 4.12 and Proposition 4.10, this semigroup is Lipschitz continuous with respect to the initial data in the topology of H. Besides, the map $(t, \varphi_0) \mapsto S(t)\varphi_0$ is Hölder continuous on $[0, T] \times \mathbb{B}_\delta$, where $\mathbb{B}_\delta$ is endowed with the metric topology of V'. Hence, the desired exponential attractor $\mathcal{M}$ for the continuous semigroup $S(t)$ can be obtained by the same standard formula in (2.44). Theorem 4.8 is now proved.

Unfortunately, it does not seem possible to establish the finite dimensionality of $\mathcal{M}$ in Theorem 4.8 with respect to the stronger H-metric. This issue is ultimately connected to deriving the same uniform boundedness of $\varphi(t)$ in V-norm (cf. also Remark 3.14). However, in a special case at least, the following regularizing property holds

(4.25)
$$\sup_{t \geq T_0 + 1} \left(\|\varphi(t)\|_V + \|\partial_t \varphi\|_{L^2([t, t+1]; H)} \right) \leq C_\delta,$$

provided we assume in addition that

$$(4.26) \qquad b\left(\cdot,|s|\right) \equiv b_0\left(\cdot\right) \in L^\infty\left(\Omega\right), \quad J \in W^{2,1}(\mathbb{R}^d).$$

In this case, the exponential attraction (iii) and finite dimensionality (iv) of $\mathcal{M}$ also holds with respect to the $L^s \cap H^{1-\nu}$-metric for any $s \geq 2$ and $\nu \in (0,1)$, on account of (4.10) and (4.25).

Let us briefly explain how to get (4.25).

Proposition 4.13. *Let the assumptions of Theorem 4.2 hold. In addition, suppose (4.26). Then every weak solution φ of problem (4.1)-(4.2) also satisfies estimate (4.25).*

Proof. Let $h\left(\varphi\right) := F'\left(\varphi\right) + a$, where $a(x) = (1 * J)(x)$. According to (4.8), every weak solution φ satisfies

$$(4.27) \qquad \langle \partial_t \varphi, \psi \rangle + \left(\kappa\left(\cdot,\varphi\right)\nabla h, \nabla \psi\right) = 2\left(\kappa\left(\cdot,\varphi\right)\nabla J * \varphi, \nabla \psi\right),$$

for all $\psi \in V$ and almost everywhere in (T_0, ∞).

Testing this identity with $h\left(\varphi\right)$ yields

$$(4.28) \qquad \frac{d}{dt}\left[\left(F\left(\varphi\left(t\right)\right),1\right) + \left(a,\varphi\left(t\right)\right)\right] + \int_\Omega \kappa\left(x,\varphi\left(t\right)\right)\left|\nabla h\left(\varphi\left(t\right)\right)\right|^2 dx$$
$$= 2\left(\kappa\left(\cdot,\varphi\right)\nabla J * \varphi\left(t\right), \nabla h\left(t\right)\right),$$

for all $t \geq T_0$. Recalling once again that (4.10) and (4.11) hold uniformly in (T_0, ∞), we can integrate (4.28) over $(t, t+1)$ to deduce the following bound:

$$(4.29) \qquad \int_t^{t+1} \left\|\nabla h\left(\varphi\left(s\right)\right)\right\|_H^2 ds \leq C_\delta, \; \forall t \geq T_0.$$

This gives, on account of the separation property (4.10) and (H1), that

$$(4.30) \qquad \sup_{t \geq T_0} \int_t^{t+1} \left\|\nabla \varphi\left(s\right)\right\|_H^2 ds \leq C_\delta.$$

Finally, testing equation (4.8) with $\partial_t \varphi$ (this can be easily justified within an appropriate Galerkin scheme), we find

$$(4.31) \qquad \left\|\partial_t \varphi\right\|_H^2 + \int_\Omega \kappa\left(x,\varphi\right)F''\left(\varphi\right)\nabla \varphi \cdot \nabla \partial_t \varphi dx = \int_\Omega \kappa\left(x,\varphi\right)\nabla w \cdot \nabla \partial_t \varphi dx.$$

By observing that $\kappa\left(\cdot,\varphi\right)F''\left(\varphi\right) = b_0(\cdot) > 0$, by virtue of (4.10) and (4.11) we can further estimate

$$(4.32) \qquad \left\|\partial_t \varphi\left(t\right)\right\|_H^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega b_0\left(x\right)\left|\nabla \varphi\left(t\right)\right|^2 dx$$
$$\leq C_\delta \left\|\nabla w\left(t\right)\right\|_V \left\|\nabla \partial_t \varphi\left(t\right)\right\|_{V'}$$
$$\leq \frac{1}{2}\left\|\partial_t \varphi\left(t\right)\right\|_H^2 + C_\delta'\left\|\overrightarrow{\mathcal{V}}\right\|_V^2,$$

for all $t \geq T_0$, where

$$\overrightarrow{\mathcal{V}}\left(\cdot\right) := \nabla a\left(\cdot\right) - 2\nabla J * \varphi.$$

Recalling that $J \in W^{2,1}(\mathbb{R}^d)$ the second term on the right-hand side of (4.32) is also uniformly (in time) bounded by some positive constant $C_{\delta,J}$ (i.e., $\nabla \overrightarrow{\mathcal{V}} \in L^2\left(\mathbb{R}^d \times \mathbb{R}^d\right)$). Therefore, we may integrate (4.32) over $(t, t+1)$ and exploit (4.30) to deduce (4.25) from an application of the uniform Gronwall inequality.

Remark 4.14. Exploiting a suitable version of the Łojasiewicz-Simon inequality (see, e.g., Lemma 3.15), it was proven in [35, Theorem 2.2] that every weak solution φ of (4.1)-(4.2) converges in the H-metric as time goes to infinity to a single equilibrium

$$\varphi_* = 1/\left(e^{w_* - \mu_*} + 1\right), \qquad \langle \varphi_* \rangle = \langle \varphi_0 \rangle,$$
$$w_*(x) = a(x) - 2J * \varphi_*, \quad \mu_* = \text{constant}.$$

In view of (4.25), this can be improved to a convergence rate in the $L^s \cap H^{1-\nu}$-metric for any $s \geq 2$ and $\nu \in (0, 1)$, i.e.,

$$\|\varphi(t) - \varphi_*\|_{L^p \cap H^{1-\nu}} \sim (1+t)^{-\frac{1}{\rho}}, \text{ as } t \to \infty,$$

thanks to (4.10), and the additional assumption (4.26) (cf. also Remark 2.22).

Remark 4.15. The assumption on J in (4.26) can be actually relaxed to also include Newtonian and Bessel-like potentials. In general, the second derivatives of such potentials are not locally integrable, but one may still properly define $\nabla^2 J * \varphi$ as a bounded distribution on $L^p(\Omega)$, $1 < p < \infty$, using the Calderón–Zygmund theory. In particular, for such distributions there holds $\nabla \overrightarrow{\mathcal{V}} \in L^p$ for every $1 < p < \infty$ (see, e.g., [3, Lemma 2], [26, Lemma 2.1]), and thus we can also conclude (4.25) for such potentials.

Remark 4.16. It was proved in [36] that solutions of (4.1)-(4.2) also satisfy

$$\varphi(t) \in L^\infty\left([T_0, \infty); W^{2,2}(\Omega)\right) \cap L^\infty\left([T_0, \infty); C^\beta\left(\overline{\Omega}\right)\right), \ \beta < \frac{1}{2},$$

after the separation time $T_0 > 0$ provided that $\|\partial_t \varphi(T_0)\|_{(H^1)^*}$ is finite and, in addition,

$$\|J * u\|_{W^{2,2}(\Omega)} \leq C_J \|u\|_{W^{1,2}(\Omega)}.$$

This is a *conditional* result which requires *stronger* assumptions on the kernel J and on Ω. For instance, to prove the condition $\|\partial_t \varphi(T_0)\|_{(H^1)^*} < \infty$ one needs to show that $\|\varphi(T_0)\|_V < \infty$.

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