I. INTRODUCTION

Bayes’ THEOREM, or Bayes’ rule, is a well-known principle in probability that allows one to merge a subjective degree of belief with the available evidence. It has been often used, in several branches of physics and engineering, to refine the knowledge about a particular phenomenon on the basis of the available experimental evidence [1]. In measurement applications, Bayes’ theorem has been widely used to refine the a priori knowledge about the measurand, coming either from theoretical assumptions or previously obtained measurement results, on the basis of a new information provided by a measurement result and its associated uncertainty, so that a more accurate a posteriori estimate of the measurand is obtained.

Of course, Bayes’ theorem is defined within the probability theory and, in its continuous formulation, it requires that both the a priori knowledge and the new information are described in terms of probability density functions (pdfs). Starting from these pdfs, the a posteriori conditional pdf can be obtained. Therefore, Bayes’ theorem can be applied only when measurement results are expressed in terms of probability distributions, as suggested by the standard approach to uncertainty defined by the guide to the expression of uncertainty in measurement (GUM) [2].

Recently, a more general approach to the expression of measurement uncertainty, based on possibility distributions (PDs), has been proposed [3]–[7]. Very briefly, following this approach, a measurement result can be represented, together with the associated uncertainty, using a random-fuzzy variable (RFV) [6], [7]. As shown in Fig. 1, an RFV is composed by two different PDs: the PD $r^{\text{int}}$, called internal PD, which represents all nonrandom contributions to uncertainty, including the uncompensated systematic ones, and the PD $r^{\text{ran}}$, called external PD, which represents all contributions to uncertainty. This last PD is obtained by combining $r^{\text{int}}$ with the random PD $r^{\text{ran}}$, which represents only the random contributions to uncertainty. As shown in [8], $r^{\text{int}}$ and $r^{\text{ran}}$ can be built starting from the available metrological information.

An in-depth discussion about the RFV approach to uncertainty is out of the scope of this paper, since it has been widely discussed in the available literature to which the reader is addressed for further details. This paper is aimed at extending Bayes’ theorem to the possibility domain, extending the already available concepts of conditional possibilities [9]–[11] to the RFVs, so that, starting from an a priori knowledge and a new information described in terms of RFVs, the a posteriori conditional RFV can be obtained. This paper is organized as follows. After having briefly recalled the definition of conditional pdfs in Section II, Section III proposes a definition of internal conditional PDs and random conditional PDs. Starting from these PDs, the definition of conditional RFVs is discussed in Section IV. Finally, a simple experimental example is considered in Section V to validate the proposed definition of conditional RFVs.
II. Conditional Probability Distributions

The a posteriori conditional pdf \( p_{X|Y}(x|y_m) \) represents the knowledge about the possible values of the random variable \( X \), revised according to the knowledge of the actual value \( y_m \) assumed by the random variable \( Y \). In mathematical terms, this pdf can be obtained by applying the well-known Bayes' theorem for continuous random variables:

\[
p_X(x) \cdot p_{Y|X}(y_m|x) = p_{X,Y}(x, y_m) = p_Y(y_m) \cdot p_{X|Y}(x|y_m)
\]

where \( p_X \) and \( p_Y \) are the marginal pdfs, \( p_{Y|X} \) and \( p_{X|Y} \) are the conditional pdfs, and \( p_{X,Y} \) is the joint pdf. If (1) is written in a different way, the meaning of Bayes’ theorem can be more easily perceived:

\[
p_{X,Y}(x, y_m) = p_X(x) \cdot p_{Y|X}(y_m|x) \quad (2)
\]

\[
p_{X|Y}(x|y_m) = p_{X,Y}(x, y_m)/p_Y(y_m).
\]

Starting from the a priori knowledge about \( X \) (\( p_X \)) and the relationship between \( Y \) and \( X \) (\( p_{Y|X} \)), the joint pdf \( p_{X,Y} \) can be evaluated. The a posteriori knowledge about \( X \) (\( p_{X|Y} \)) can be obtained dividing the joint pdf by a normalization factor \( (p_Y) \). This normalization factor can be obtained from the joint pdf through marginalization:

\[
p_Y(y_m) = \int_{-\infty}^{+\infty} p_{X,Y}(x, y_m)dx.
\]

Therefore, the a posteriori conditional pdf of \( X \) can be obtained only if the joint pdf can be evaluated. The joint pdf, in turn, depends on the a priori knowledge about \( X \) (\( p_X \)) and the relationship between \( Y \) and \( X \) (\( p_{Y|X} \)).

Bayes’ theorem has been employed, in the measurement science, to define a probabilistic model of the measurement process [12]. In this model, the random variables \( Y \) and \( X \) represent the measured quantity and the measurand, respectively. Moreover, \( p_X \) represents the available a priori knowledge about the measurand and \( p_{Y|X} \) the distribution that describes the measurement process, that is, the process that generates a measured value \( (y_m) \) starting from the measurand value \( x \). Therefore, (2) yields the joint probability distribution and (3) the measurement result \( p_{X|Y} \), that is, the a posteriori knowledge about measurand \( X \) [12].

III. Conditional PDs

To extend the aforementioned concepts to the possibility domain, PDs have to be considered instead of pdfs. The a posteriori conditional PD \( r_{X|Y}(x|y_m) \) can be defined with a formally similar equation as (1)

\[
T[r_X(x), r_{Y|X}(y_m|x)] = r_{X,Y}(x, y_m) = T[r_Y(y_m), r_{X|Y}(x|y_m)]
\]

where \( r_X \) and \( r_Y \) are the marginal PDs, \( r_{Y|X} \) and \( r_{X|Y} \) are the conditional PDs, \( r_{X,Y} \) is the joint PD, and \( T \) is a class of algebraical operators called t-norms. t-norms are basically associative functions, satisfying specific mathematical constraints [13], [14]. A detailed survey of t-norms and t-norms parametrized families can be found in [13]–[15] to which the reader is addressed for further details. The main difference between (1) and (5) is related to this class \( T \). The algebraical operator product applied in (1) to combine the pdfs is replaced, in the possibility domain, by a general class of operators, which includes the product as a particular case, thus adding one degree of freedom to (5).

Having selected a particular t-norm \( T^* \) from class \( T \), and following the same approach as the one applied to pdfs, (5) can be written as

\[
r_{X,Y}(x, y_m) = T^*[r_X(x), r_{Y|X}(y_m|x)]
\]

\[
r_{X|Y}(x|y_m) = F^*[r_X(x, y_m), r_Y(y_m)]
\]

where \( F^* \) is the inverse algebraical operator of \( T^* \), provided that the employed t-norm \( T^* \) can be inverted [16]. Similar to the pdf case, starting from the a priori knowledge about \( X \) (\( r_X \)) and the relationship between \( Y \) and \( X \) (\( r_{Y|X} \)), the joint PD \( r_{X,Y} \) can be evaluated. The a posteriori knowledge about \( X \) (\( r_{X|Y} \)) can be obtained by applying the inverse operator \( F^* \) to the joint PD and the normalization factor \( r_Y \). The normalization factor \( r_Y \) can be obtained, also in this case, as the marginal distribution of the joint PD that, in the possibility domain, can be defined as

\[
r_Y(y_m) = \sup_{x \in [-\infty, +\infty]} r_{X,Y}(x, y_m).
\]

Therefore, the a posteriori conditional PD of \( X \) can be obtained only if the joint PD can be evaluated. The joint PD, in turn, depends on the a priori knowledge about \( X \) (\( r_X \)), the relationship between \( Y \) and \( X \) (\( r_{Y|X} \)), and the choice of a specific t-norm \( T^* \).

Let us consider a generic relationship between \( Y \) and \( X \), i.e., a generic measurement model:

\[
Y = f(X, U).
\]

This model is a functional relationship between the inputs of the measurement system \( X \) and \( U \), which represent the sets of possible measurand and possible uncertainty values, respectively, and the output of the measurement system \( Y \), which represents the set of the possible measured quantity values.

Also in the PD case, the measurement model leads to the construction of the conditional distribution \( r_{Y|X} \). \( r_{Y|X} \) is the possibility that \( Y \) assumes the fixed value \( y_m \), given that \( X \) assumes the fixed value \( x \). When fixed values are assigned to the measurand and the measured quantity, according to the measurement model (9), also uncertainty \( U \) assumes a fixed value \( u \). This value can be found as \( u = f^{-1}(x, y_m) \), where \( f^{-1} \) is the inverse of the measurement model (9). The fact that uncertainty assumes a known value implies the possibility that \( Y = y_m \), given that \( X = x \), is equal to the possibility that \( U = u \). Since this last possibility, namely, the uncertainty distribution \( r_U \), is considered to be known by assumption, \( r_{Y|X} \) is also known. In mathematical terms

\[
r_{Y|X}(y_m|x) = r_U(u).
\]

As far as the choice of the t-norm \( T^* \) is concerned, it depends on the nature of the uncertainty contributions represented by the PDs, as stated in [17] and [18]. Since, as briefly recalled in Section I, the internal and the random PDs represent different uncertainty contributions, different t-norms
shall be considered for these two PDs. The following two sections will address separately the conditioning of internal and random PDs. For the sake of simplicity, internal PDs are conditioned assuming that no random PDs are present, and vice versa. This assumption will be removed in Section IV.

A. Internal PDs

The internal PDs represent the uncertainty contributions due to all kinds of nonrandom effects, including the uncompensated systematic ones. This has suggested [17], [19] that a minimum specificity principle (or maximum entropy principle) could be followed for the determination of the algebraical operator for their aggregation, i.e., the associated t-norm. The maximum entropy in their aggregation is obtained choosing, as \( T^* \), the min operator [17], [19], that is a particular t-norm (\( T_{\text{min}} \)) [13], [14]. Therefore, (6) becomes

\[
\hat{r}_{X|Y}(x, y_m) = T_{\text{min}}\left[r_{X}(x), r_{Y|x}(y_m|x)\right].
\]  

(11)

The conditional PD \( r_{X|Y}^{\text{int}}(x|y_m) \) can be obtained by applying, in (7), the inverse algebraical operator \( T_{\text{min}} \) [9] of \( T_{\text{min}} \):

\[
r_{X|Y}^{\text{int}}(x|y_m) = F_{\text{min}}\left[r_{X,Y}(x, y_m), r_{Y|x}^{\text{int}}(y_m)\right] = \begin{cases} \hat{r}_{X,Y}(x, y_m) & \text{if } r_{Y|x}^{\text{int}}(y_m) > r_{X,Y}^{\text{int}}(x, y_m) \\ \hat{r}_{X,Y}^{\text{int}}(x, y_m) & \text{if } r_{Y|x}^{\text{int}}(y_m) = r_{X,Y}^{\text{int}}(x, y_m). \end{cases}
\]

(12)

According to (12), \( r_{X|Y}^{\text{int}} \) is determined only if the joint PD is dominated by \( r_{Y|x}^{\text{int}} \) or it is equal to one. To avoid this ambiguity, the least specific solution of (12) has to be considered, as proposed in [10]

\[
r_{X|Y}^{\text{int}}(x|y_m) = \begin{cases} \hat{r}_{X,Y}(x, y_m) & \text{if } r_{Y|x}^{\text{int}}(y_m) > r_{X,Y}^{\text{int}}(x, y_m) \\ 1 & \text{if } r_{Y|x}^{\text{int}}(y_m) = r_{X,Y}^{\text{int}}(x, y_m). \end{cases}
\]

(13)

This solution has the advantage of providing a normalized conditional PD (\( \sup_x r_{X|Y}^{\text{int}}(x|y_m) = 1 \)) for every measured value \( y_m \). Interestingly, if (13) is applied with \( r_{Y|x}^{\text{int}}(y_m) = 0 \), i.e., when the measured value is considered impossible due to an incompatible \textit{a priori} knowledge, a totally uninformative \( r_{X|Y}^{\text{int}}(x|y_m) = 1 \) for every \( x \) value is obtained.\(^1\)

B. Random PDs

The random PDs represent the uncertainty contributions due to the sole random effects. This has suggested [18]–[20] that a maximum specificity principle (or minimum entropy principle) should be followed in the determination of the algebraical operator for their aggregation, i.e., the associated t-norm. The minimum entropy in their aggregation is obtained choosing, as \( T^* \) in (6), a t-norm selected among the Frank parametric family [13], [15], such that (6) becomes

\[
r_{X,Y}^{\text{ran}}(x, y_m) = T_F\left[r_{X}^{\text{ran}}(x), r_{Y|x}^{\text{ran}}(y_m|x)\right] = \log \left[1 + \frac{(\gamma r_{X}^{\text{ran}}(x) - 1) \cdot (\gamma r_{Y|x}^{\text{ran}}(y_m|x) - 1)}{\gamma - 1}\right]
\]

(14)

where \( \gamma \) is the parameter value (\( \gamma \notin [0, 1, +\infty] \)). The optimal \( \gamma = 0.05 \) value to correctly aggregate random contributions to uncertainty has been proposed in [18] and is here employed.

The conditional PD \( r_{X|Y}^{\text{ran}} \) can be obtained by applying, in (7), the inverse algebraical operator \( T_F \) of the Frank t-norm family:

\[
r_{X|Y}^{\text{ran}}(x|y_m) = F_F\left[r_{X,Y}^{\text{ran}}(x, y_m), r_{Y|x}^{\text{ran}}(y_m)\right] = \log \left(1 + \frac{(\gamma r_{X,Y}^{\text{ran}}(x, y_m) - 1) \cdot (\gamma - 1)}{\gamma r_{Y|x}^{\text{ran}}(y_m) - 1}\right).
\]

(15)

In this case, (15) cannot be applied with \( r_{Y|x}^{\text{ran}}(y_m) = 0 \), i.e., when the measured value is considered impossible according to the \textit{a priori} knowledge. Similar to the case of internal PDs, an uninformative \( r_{X|Y}^{\text{ran}}(x|y_m) = 1 \) for every \( x \) value shall be assumed when \( r_{Y|x}^{\text{ran}}(y_m) = 0 \).

IV. CONDITIONAL RFVs

When random and nonrandom effects are simultaneously present in the \textit{a priori} estimate and in the new measurement result, single PDs are no longer adequate to represent the \textit{a posteriori} estimate, and conditional RFVs shall be considered. Conditional RFVs are those distributions expressing the \textit{a posteriori} possibility that the measurand assumes a specific value due to nonrandom contributions only (conditional internal PD) and all contributions (conditional external PD). For the sake of clarity, the \textit{a posteriori} conditional PDs obtained in the presence of both random and nonrandom effects will be labeled as \( r_{X|Y} \).

The \textit{a posteriori} external PD can be found by combining the \textit{a posteriori} random and internal PDs. The combination can be more easily explained starting from a particular interpretation of the random and internal PDs: the distribution \( \hat{r}_{X|Y}^{\text{ran}}(x|y_m) \) represents the \textit{a posteriori} possibility that \( X = x \) due to the random contributions only and, assuming that it is unimodal,\(^2\) its peak value identifies the \textit{a posteriori} mode value

\[
x^* = \arg \max_x \hat{r}_{X|Y}^{\text{ran}}(x|y_m)
\]

(16)

Therefore, \( x^* \) is the \( x \) value for which \( \hat{r}_{X|Y}^{\text{ran}}(x|y_m) \) attains its maximum value. On the other hand, the distribution \( \hat{r}_{X|Y}^{\text{int}}(x'|y_m) \) represents the \textit{a posteriori} possibility that the mode value \( x^* \) of \( \hat{r}_{X|Y}^{\text{ran}} \) is shifted, due to the nonrandom contributions, such that the new mode value is \( x' \), i.e., that the final \textit{a posteriori} distribution is \( \hat{r}_{X|Y}^{\text{ext}}(X = x' + x^*|y_m) \). Starting from these considerations, the \textit{a posteriori} external PD can be found as

\[
\hat{r}_{X|Y}^{\text{ext}}(x|y_m) = \sup_{x'} T_{\text{min}}\left[\hat{r}_{X|Y}^{\text{ran}}(x - x' + x^*|y_m), \hat{r}_{X|Y}^{\text{int}}(x'|y_m)\right]
\]

(17)

i.e., for each \( x \) value, the \textit{a posteriori} external possibility is the sup, on all new mode values \( x' \), of \( \hat{r}_{X|Y}^{\text{ran}} \) centered on \( x' \), limited by the possibility that \( x' \) is the new mode value.

\(^1\)This means, in the possibility framework, that no \textit{a posteriori} knowledge about \( x \) is provided by the measured value \( y_m \), so that a PD representing total ignorance is consistently obtained [7].

\(^2\)In the following, only unimodal random distributions are considered, since they represent the most frequent cases in measurement. The extension of this method to multimodal distributions is still an open issue.
For the sake of simplicity, (17) is written with simplified notation as

$$r_{x|ym} = \sup_{x'} T_{min} \{ r_{x|ym|x'\prime} r_{x'\prime|ym} \}. \quad (18)$$

According to this formulation, the a posteriori external PD $r_{x|ym}$ is the marginal distribution over $x$ of the joint PD $r_{(x,y)|ym}$. The minimum specificity principle can be applied also in this case, so that the min $t$-norm has been selected to aggregate $r_{x|ym|x'\prime}$ and $r_{x'\prime|ym}$ [14]. In this case, this assumption is justified because the internal and random PDs are originated by independent uncertainty contributions and, therefore, they are independent.

According to (17), the a posteriori external PD, and consequently the conditional RFV, can be obtained starting from the a posteriori random and internal PDs. As proposed in [21], the simplest, though not always strict way to obtain the a posteriori random and internal PDs, in the presence of both random and nonrandom effects, is to assume that random and internal PDs can be conditioned separately, following the equations discussed in the previous section. Therefore

$$r_{X|Y}(x|ym) = r_{X|Y}(x|ym) \quad (x|ym)$$

$$r_{X|Y}(x|ym) = r_{X|Y}(x|ym). \quad (19)$$

Of course, this assumption is valid until the conditioning of the internal PDs is not influenced by the presence of the random PDs, and vice versa, i.e., until there is a predominant uncertainty source between random and nonrandom effects.

When random and nonrandom effects lead to comparable uncertainty contributions, random and internal PDs cannot be conditioned separately and a different method shall be devised for the determination of the a posteriori random and internal PDs. A more general method can be obtained by adapting and situations, as done in (17). Due to $r_{X|Y}$, the mode of $r_{X|Y}$ can be shifted in the new mode values $x'$. For a given $x'$ value, the distribution $r_{x|x'}$ can be defined as

$$r_{x|x'} = r_{X}(x - x' + x'). \quad (20)$$

Moreover, due to $r_{Y|X}$, the mode of $r_{Y|X}$ can be shifted in the new mode values $x'$, $y'$. For given $x'$, $y'$ values, a distribution $r_{Y(x'|x')}$ can be defined as

$$r_{Y(x'|x')}(x' - y' + y'|x - x' + x'). \quad (21)$$

Therefore, for fixed new mode values $x'$ and $y'$, $r_{x|x'}$ and $r_{Y(x'|x')}(x',y')$ still represent random PDs and, therefore, they can be conditioned independently from the internal PDs. The a posteriori random PD, given $x'$ and $y'$, can be obtained by applying the conditioning method for the random part discussed in Section III-B. Applying the conditioning rules, the a posteriori random PD, given $x'$ and $y'$, becomes

$$r_{(x|ym)|(x',y')} = r_{X|Y}(x' - y' + y'|ym - y' + y'). \quad (22)$$

i.e., the new mode values $x'$ and $y'$ produce a simple translation of the a posteriori random PD. In this respect, the shape of the a posteriori random PD is not influenced by the presence of the internal PD, but only the position of its mode value is affected. The a posteriori position of the mode, given $x'$ and $y'$, can be defined, similar to (16), as

$$x_{pos} = \arg \max_{x} r_{x|ym}(x'|y'). \quad (23)$$

Since the shape of $r_{(x|ym)|(x',y')}$ is not influenced by $x'$, $y'$, the final a posteriori random PD $r_{y|ym|x'}$ can be defined as

$$r_{y|ym}(x'|y') \quad (x'|y')$$

as proved by (17), a different position of $r_{y|ym}(x'|y')$ does not affect the a posteriori external PD, and the final a posteriori RFV. In other words, the a posteriori mode values of $r_{y|ym}$ are imposed by the distribution $r_{y|ym}$. Therefore, it can be assumed as

$$r_{y|ym}(x|ym) \triangleq r_{y|ym}(x'=x',y'=y') = r_{y|ym}(x|ym). \quad (24)$$

Equation (24) states that the conditioning of the random PDs, originated by random contributions only, is never influenced by the presence of the internal PDs, i.e., by the presence of nonrandom contributions to uncertainty.

On the other hand, the a posteriori internal PD $r_{y|ym}$ represents the a posteriori distribution of the mode. Therefore, for given $x'$, $y'$ values, this PD is influenced both by the a posteriori position of the mode ($y_{pos}$) and the possibility that $y_{pos}$ is the actual a posteriori mode value. Since $y_{pos}$ depends on $x'$, $y'$, this possibility corresponds to the a posteriori possibility of $x'$, $y'$ ($r_{y|x'}|ym|ym)$. Starting from these considerations, the a posteriori internal PD can be found as

$$r_{y|x'}|ym(x|ym) = \sup_{(x',y')} r_{y|x'}|ym(x',y'|ym). \quad (25)$$

i.e., the a posteriori internal PD is the sup of the a posteriori possibility of $y_{pos}$, on all $x'$, $y'$ values which lead to $y_{pos}$. Equation (25) states that, in general, the conditioning of the internal PDs is always influenced by the presence of the random PDs that affect the a posteriori position of the mode and the presence of internal PDs that affect the a posteriori possibility of the mode.

The a posteriori possibility of $x'$, $y'$ can be defined by applying once again the conditioning process described in Section III-A. The joint PD among $ym$, $x'$, and $y'$ can be found, according to (11), as

$$r_{ym,x',y'} = T_{min} \{ r_{ym}(x',y'), r_{x,x'}(y') \}. \quad (26)$$

Since, for a given $y'$, the possibility of the measured value $ym$ is not affected by the position of the mode value along $x$, it is possible to write

$$r_{ym|y'} = r_{ym}(y'). \quad (27)$$

Similar to (18), once the mode $y'$ of $r_{ym|y'}$ is assumed, only random contributions affect the possible values taken by $ym$ about $y$. Hence

$$r_{ym}(y') = r_{y|ym}(ym - y' + y'). \quad (28)$$

On the other hand, similar to (18), the a priori possibility of $x'$, $y'$ can be simply obtained as

$$r_{x',y'} = r_{x|y}(x',y'). \quad (29)$$
Therefore, the joint PD \( r_{ym,x',y'} \) in (26) becomes, due to (28) and (29)
\[
r_{ym,x',y'} = T_{min}[r^{am}_{ym}(y_m - y^n + y'), r^{int}_{x',y'}(x', y')]. \tag{30}
\]
Finally, starting from the marginal distribution of the joint PD \( r_{ym,x',y'} \) that is given by \( r_{ym} = sup_{x',y'} r_{ym,x',y'} \), the \textit{a posteriori} possibility of \( x', y' \) can be obtained, according to (13), as
\[
r_{(x',y')|ym} = F_{min}[r_{ym,x',y'}, r_{ym}]. \tag{31}
\]

In conclusion, it is worth noting that, despite the complexity of the whole mathematical derivation, the practical implementation of (17), (24), and (25) is relatively simple.

V. MEASUREMENT EXAMPLE

In this section, the proposed method for the definition of the \textit{a posteriori} conditional RFVs is applied to a practical measurement example: a temperature measurement based on a common platinum resistance temperature detector (RTD) (Pt-100). The measured temperature is the temperature of the water contained in a bath tank, which is maintained stable during the measurement process using a Julabo 12-ED heating circulator. A picture of the experimental setup is shown in Fig. 2. The temperature set point of the heating circulator, together with all available information about the accuracy and stability of the temperature control of the circulator, represents the \textit{a priori} knowledge about water temperature. In this respect, the accuracy range of the temperature control is estimated as \( \Delta \theta = \pm 0.01 \cdot \theta_{\text{set}} \) °C, where \( \theta_{\text{set}} \) is the temperature set point, and the standard deviation quantifying the stability is \( \sigma_\theta = 0.03 \) °C, according to the experimental variability of the temperature measured by the temperature control. Considering \( \Delta \theta \) as the quantification of nonrandom contributions to uncertainty and \( \sigma_\theta \) as the quantification of random contributions, the \textit{a priori} internal and random PDs associated to temperature \( \theta \) can be built. Fig. 3 shows the \textit{a priori} internal and external PDs \( r_0 \) for the set point \( \theta_{\text{set}} = 55 \) °C.

Water temperature is measured also using the RTD with known manufacturer specifications. For this sensor, the measurement model specifies the functional relationship between temperature \( \theta \) (the measurand) and resistance \( R \) (the measured quantity):
\[
R = R_0 \cdot [1 + a \theta + b \theta^2] \tag{32}
\]
where typical values for \( R_0, a, \) and \( b \) in the temperature range \( 0 \) °C \( \leq \theta < 850 \) °C are \( R_0 = 100 \) Ω, \( a = 3.9083 \cdot 10^{-3} \) °C\(^{-1} \), and \( b = -5.775 \cdot 10^{-7} \) °C\(^{-2} \), respectively. For the case \( \theta_{\text{set}} = 55 \) °C, a value \( R_m = 121.16 \) Ω has been measured, which corresponds to a temperature value \( \theta_m = 54.56 \) °C.

Of course, uncertainty \( U \) is associated to the resistance measured value, and this should be made explicit in the measurement model (32) to evaluate the conditional PDs correctly. Therefore, the measurement model shall be written in terms of RFVs as
\[
\tilde{R} = R_0 \cdot [1 + a \tilde{\theta} + b \tilde{\theta}^2] + \tilde{U} \tag{33}
\]
where \( \tilde{R} \) is the RFV representing the possible measured value of the RTD resistance, \( \tilde{\theta} \) is the RFV representing the possible values of the measurand, and \( \tilde{U} \) is the RFV representing the possible values of uncertainty contributions. \( \tilde{U} \) can be obtained starting from the specifications of the RTD and of the multimeter used for the resistance measurement.

In particular, it is known that the RTD is in class A (\( \Delta \theta = \pm (0.15 + 0.002 \cdot \theta) \) °C) up to 70 °C and in class B (\( \Delta \theta = \pm (0.3 + 0.005 \cdot \theta) \) °C) for higher temperatures. Moreover, the resistance value of the RTD is measured using a Fluke 8845A digital multimeter in a four-wire configuration. The accuracy range of the multimeter is \( \Delta R = \pm (0.01 + 0.0001 \cdot R_m) \) Ω. Starting from the accuracy range \( \Delta \theta \) of the RTD and \( \Delta R \) of the multimeter, the internal PD associated with uncertainty \( U \) can be found. To do so, the accuracy range \( \Delta \theta \) has to be expressed as an interval in the resistance scale exploiting (32), and it has to be added to \( \Delta R \) [8].

Moreover, during the test, the experimental variability of the resistance measured values has been evaluated and found to be \( \sigma_R = 7 \cdot 10^{-3} \) Ω. Starting from the experimental variability \( \sigma_R \), the random PD associated with uncertainty \( U \) can be found [20].
As discussed in Section III, the internal and random PDs $r_{R|\theta}$ can be obtained starting from the internal and random PDs associated with uncertainty, considering the measurement model. The PDs $r_{R|\theta}$ are shown in the upper plot of Fig. 4 for a single value $\theta = \theta_m$. If these PDs are plotted on the $\theta$ axis, exploiting again (32), the PDs $r_{\theta|\theta_m}$ shown in the lower plot of Fig. 4 are obtained. These last PDs, which express the estimate of $\theta$ only due to the new measurement result, are not necessary for the conditioning process, but are anyway shown here for a better clarification of this process.

Starting from the conditional PD $r_{\text{int}}^{\text{int}}_{R|\theta}$ and the a priori PD $r_{\theta}^{\text{int}}$, according to the procedures defined in Section III-A, it is possible to obtain the joint PD $r_{\text{int}}^{\text{int}}_{\theta,R}$ through (11) and the a posteriori conditional PD $r_{\text{int}}^{\text{int}}_{\theta|R}$ through (13). These distributions are shown in Fig. 5. On the other hand, starting from the conditional PD $r_{\text{ran}}^{\text{ran}}_{R|\theta}$ and the a priori PD $r_{\theta}^{\text{ran}}$, according to the procedures defined in Section III-B, it is possible to obtain the joint PD $r_{\text{ran}}^{\text{ran}}_{\theta,R}$ through (14) and the a posteriori conditional PD $r_{\text{ran}}^{\text{ran}}_{\theta|R}$ through (15). These distributions are shown in Fig. 6.

In this way, the nonrandom and random contributions are initially conditioned separately. However, as discussed in Section IV, to find the final a posteriori PDs $\tilde{r}_{\theta|R_m}$ composing the a posteriori RFV, the influence of the random PDs on the conditioning of the internal PDs has to be considered. In this respect, the a posteriori PD $\tilde{r}_{\text{int}}^{\text{int}}_{\theta|R_m}$ can be found by applying (25), and the a posteriori PD $\tilde{r}_{\text{ran}}^{\text{ran}}_{\theta|R_m}$ can be found by applying (24).

The resulting a posteriori PDs, composing the final a posteriori RFV, are shown in the lower plot of Fig. 7 (green lines). The a posteriori RFV is compared with the a priori RFV (blue lines) and with the RFV given by the measurement result (magenta lines) in the upper plot. It can be seen that the measured value refines the a priori knowledge about temperature reducing both the nonrandom and random contributions in the a posteriori estimate of $\theta$, i.e., excluding those temperature values that a posteriori are no longer possible.

The measurement result is therefore represented by the whole green RFV in the lower plot of Fig. 7, which provides all possible intervals, at all possible confidence levels $(1 - \alpha)$, where the temperature value is supposed to lie. If a single interval is desired, as in an usual measurement report, a single $\alpha$-cut of this RFV can be provided instead.
of the whole variable. For instance, 95% confidence interval is the α-cut for α = 0.05 and, in this case, would provide \( \theta = (54.7 \pm 0.29) ^\circ C \).

The a posteriori RFV is also compared with an RFV (black lines) provided by a Dostmann P 655-LOG high precision temperature probe, featuring an accuracy range \( \Delta \theta_{\text{ref}} = \pm 0.03 ^\circ C \) and showing an experimental variability with \( \sigma_{\theta_{\text{ref}}} = 0.01 ^\circ C \). According to the comparison, as expected, the a posteriori RFV is a better estimate of water temperature than the a priori RFV and the RFV representing the measurement result.

Fig. 7 shows also that, due to the presence of the random PDs, the resulting a posteriori internal PD has not rectangular shape. Of course, since in this case, the random PDs are much narrower than the internal PDs, their effect of modifying the shape of the a posteriori internal PD is almost negligible. Therefore, in this case, the a posteriori internal and random PDs can be also determined using the simplified approach (19), whose assumption is that random and internal PDs can be conditioned separately. The obtained a posteriori PDs \( r_\theta | R_m \) are shown in the lower plot of Fig. 8, where they are also compared with the PDs \( \bar{r}_\theta | R_m \). It can be observed that \( r_\theta | R_m \) and \( \bar{r}_\theta | R_m \) are almost overlapping. However, if the same comparison is performed starting, for instance, from experimental \( \sigma \) values multiplied by 10 with respect to the original \( \sigma \) values, as shown in Fig. 9, the difference between the simplified and the general approaches for the determination of the a posteriori conditional RFV becomes evident: when the internal and random PDs show similar magnitudes, the simplified approach, already proposed in [21], may lead to an underestimate of uncertainty in the a posteriori conditional RFV.

It is worth noting that, despite the shape of the internal PD of the conditional RFV can be affected by the original random PDs, it still represents the effects of the nonrandom contributions only. As graphically shown by the green and red-dashed internal PDs in the lower plot of Fig. 9, the proposed mathematical method may provide an internal PD (green line) larger than the narrower one (red-dashed line) obtained when random and nonrandom PDs are processed independently. However, the conditional internal PD is always limited [by definition (25)] by the max between the internal a priori (blue line in Fig. 9) and measured (magenta line in Fig. 9) PDs. This confirms that the resulting conditional internal PD preserves its nature of representing nonrandom effects only. Even if it is not strictly necessary to represent the a posteriori measurement result, which is globally represented by the external PD, it is nevertheless necessary if this result has to be employed in a subsequent conditioning process.

The a posteriori conditional RFV has been determined for different temperature set points of the heating circulator. As an example, Fig. 10 shows the resulting RFVs for \( \theta_{\text{set}} = 45 ^\circ C \). This figure simply shows that, when the conditioning process is applied starting from a noninformative a priori RFV, the a posteriori RFV does not provide any additional information with respect to the RFV given by the measurement result.

Figs. 11 and 12 show the resulting RFVs for \( \theta_{\text{set}} = 75 ^\circ C \). In Fig. 11, it has been erroneously assumed that for \( \theta_{\text{set}} = 75 ^\circ C \), the employed RTD is in class A. The figure shows that, starting from erroneous information about temperature, the conditioning process may lead again to an underestimation of uncertainty in the a posteriori RFV. The nonrandom contributions are excessively compensated in the a posteriori RFV which turns out to be incompatible with the reference...
temperature PD $r_{\theta \text{est}}$. On the other hand, assuming correctly that for $\theta_{\text{est}} = 75 \degree C$, the employed RTD is in class B as in Fig. 12, the conditioning process leads to an effective \textit{a posteriori} temperature estimate.

VI. CONCLUSION

This paper has presented an extension to the possibility domain of the well-known Bayes’ theorem, so that, starting from the RFV expressing the \textit{a priori} knowledge and the RFV associated with a new measurement result, the \textit{a posteriori} conditional RFV can be obtained. In particular, different conditional PDs have been obtained for the internal and random PDs composing an RFV, under the assumption that only nonrandom and random effects, respectively, are present. After removing this assumption, it has been shown that, in general, internal and random PDs cannot be conditioned separately, since the \textit{a posteriori} internal PD is influenced also by the presence of random effects.

The proposed method for the determination of the \textit{a posteriori} conditional RFV has been also applied to a simple experimental example to prove its usefulness and effectiveness. The results have shown that, starting from reliable \textit{a priori} knowledge about the measurand, it is possible to obtain a more accurate \textit{a posteriori} estimate of the measurand. They have also shown the effect of the random PDs on the shape of the \textit{a posteriori} internal PD: the more comparable the random PDs with respect to the internal PDs, the more severe the effect. Moreover, the results have confirmed that the simplified approach for the determination of the \textit{a posteriori} conditional RFV is a valid approach when nonrandom contributions to uncertainty are predominant with respect to random contributions, and vice versa.

REFERENCES