Krein–Langer Factorization and Related Topics in the Slice Hyperholomorphic Setting

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1 Introduction

In operator theory and in the theory of linear systems, the study of functions analytic and contractive in the open unit disk (Schur functions) is called Schur analysis. It includes, in particular, interpolation problems, operator models, and has been extended to various settings. See, for instance, [2, 29, 31, 35] for some related books. In [12, 13] we began a study of Schur analysis in the setting of slice hyperholomorphic functions. Following [12, 13], let us recall that a generalized Schur function is an $\mathbb{H}^{N\times M}$-valued function S slice hyperholomorphic in a neighborhood V of the origin and for which the kernel

$$K_S(p,q) = \sum_{n=0}^{\infty} p^n \big(I_N - S(p)S(q)^*\big)\overline{q}^n \tag{1.1}$$

has a finite number of negative squares in V, or more generally such that the kernel

$$\sum_{n=0}^{\infty} p^n \big(\sigma_2 - S(p)\sigma_1 S(q)^*\big)\overline{q}^n \tag{1.2}$$

has a finite number of negative squares in V, where $\sigma_1 \in \mathbb{H}^{M\times M}$ and $\sigma_2 \in \mathbb{H}^{N\times N}$ are signature matrices (i.e., both self-adjoint and invertible). Since this work is aimed at different audiences, it is worth mentioning that the classical counterparts of the kernels (1.2) originate with the theory of characteristic operator functions. In the indefinite case, such kernels have been studied by Krein and Langer; see, for instance, [39–42]. When $\sigma_1 = \sigma_2$ and when the kernel is positive definite, Potapov gave in the fundamental paper [46] the multiplicative structure of the corresponding functions S.

In [12] we studied the realization of such S in the case of positive definite kernels. In [13] we studied an interpolation problem, and began the study of the indefinite metric case, where the underlying spaces are Pontryagin spaces rather than Hilbert spaces. In this work we prove a Beurling–Lax type theorem in this setting and study the Krein–Langer factorization for slice hyperholomorphic generalized Schur functions. Slice hyperholomorphic functions turned out to be a natural tool to generalize Schur analysis to the quaternionic setting. Some references for this theory of functions, with no claim of completeness, are [16, 23, 33], the book [28], and the forthcoming [34].

The analogue of the resolvent operator in classical analysis is now the S-resolvent operator, and according to this resolvent, the spectrum has to be replaced by the S-spectrum. The relation between the S-spectrum and the right spectrum of a right linear quaternionic operator is important for the present paper. Indeed, in the literature there are several results on the right spectrum which is widely studied, especially for its application in mathematical physics; see, e.g., [1]. However, it is well known that the right spectrum is not associated with a right linear quaternionic operator; the eigenvectors associated with a given eigenvalue do not even form a linear space. The S-spectrum arises in a completely different setting; it is associated with a right linear operator and, quite surprisingly, the point S-spectrum coincides with the right spectrum. This fact and the fact that any right eigenvector is also an S-eigenvector, see Proposition 4.7, allow us to use for the S-spectrum various results which hold for the right spectrum; see Sects. 6 to 9.

The S-resolvent operator allows the definition of the quaternionic analogue of the operator $(I - zA)^{-1}$ that appears in the realization function $s(z) = D + zC(I - zA)^{-1}B$. It turns out that when A is a quaternionic matrix and p is a quaternion then $(I - pA)^{-1}$ has to be replaced by $(I - \bar{p}A)(|p|^2 A^2 - 2\,\mathrm{Re}(p)A + I)^{-1}$, which is equal to $p^{-1}S_R^{-1}(p^{-1}, A)$, where $S_R^{-1}(p^{-1}, A)$ is the right S-resolvent operator associated with the quaternionic matrix A. Moreover, the S-resolvent operator allows us to introduce and study the Riesz projectors and the invariant subspaces under a quaternionic operator.

The S-resolvent operator is also a fundamental tool to define the quaternionic functional calculus, and we refer the reader to [15, 17, 19, 25] for further discussions. Schur multipliers in the quaternionic setting have been studied also in [8–10], in a different setting, using the Cauchy–Kovalesvkaya product and series of Fueter polynomials. Since Schur analysis plays an important role in linear systems, we mention that papers [37, 44, 45] treat various aspects of a theory of linear systems in the quaternionic setting. We finally remark that it is possible to define slice hyperholomorphic functions with values in a Clifford algebra, [24, 26, 27], which admit a functional calculus for n-tuples of operators; see [18, 20, 22, 28].

The paper consists of eight sections besides the introduction, and its outline is as follows. Sections 2–4 are related to results on slice hyperholomorphic functions and the corresponding functional calculus. Sections 5–9 are related to Schur analysis. More precisely: Sects. 2 and 3 are of a survey nature on slice hyperholomorphic functions and the quaternionic functional calculus, respectively. Section 4 contains new results on the analogue of the Riesz projector for the quaternionic functional calculus. Moreover, it contains a discussion on the right spectrum, which has been widely studied in the literature both in linear algebra [49] and in mathematical physics [1], and which, as we have already pointed out, coincides with the point S-spectrum. These results will be used in the second part of the paper. A characterization of the number of negative squares of a slice hyperholomorphic kernel in terms of its power series coefficients is given in Sect. 5. In Sect. 6 we present some results on linear operators in quaternionic Pontryagin spaces. We show in particular that a contraction with no S-spectrum on the unit sphere has a maximal negative invariant subspace. In Sect. 7 we prove a version of the Beurling–Lax theorem, the so-called structure theorem, in the present setting. In Sect. 8 we discuss the counterparts of matrix-valued unitary rational functions. The last section considers a far-reaching result in the quaternionic framework, namely, the Krein–Langer factorization theorem for generalized Schur functions, in the finite dimensional case. It is interesting to note that the result is based on Blaschke products whose zeros and poles have a peculiar behavior when taking the slice hyperholomorphic reciprocal.

2 Slice Hyperholomorphic Functions

We begin this section by recalling the notation and some basic facts on the theory of slice hyperholomorphic functions that we will use in the sequel. We refer the reader to the papers [16, 23, 33] and the book [28] for more details. Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ whose elements satisfy the relations

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

We will denote a quaternion p as $p = x_0 + ix_1 + jx_2 + kx_3$, $x_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$, its conjugate as $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, its norm $|p|^2 = p\overline{p}$. The real part of a quaternion will be denoted with the symbols $\mathrm{Re}(p)$ or x_0, while $\mathrm{Im}(p)$ denotes the imaginary part of p. Let $\mathbb{S}$ be the 2-sphere of purely imaginary unit quaternions, i.e.,

$$\mathbb{S} = \left\{p = ix_1 + jx_2 + kx_3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1\right\}. \tag{2.1}$$

With each nonreal quaternion p it is possible to uniquely associate the element $I_p \in \mathbb{S}$ defined by

$$I_p = \frac{\mathrm{Im}(p)}{|\mathrm{Im}(p)|}.$$

The complex plane $\mathbb{C}_{I_p} = \mathbb{R} + I_p\mathbb{R} = \{x + I_q y | x, y \in \mathbb{R}\}$ is determined by the imaginary unit I_p, and $\mathbb{C}_{I_p}$ obviously contains p.

Definition 2.1 Given $p \in \mathbb{H}$, $p = p_0 + I_p p_1$ we denote by $[p]$ the set of all elements of the form $p_0 + Jp_1$ when J varies in $\mathbb{S}$.

Remark 2.2 The set $[p]$ is a 2-sphere which is reduced to the point p when $p \in \mathbb{R}$.

We now recall the definition of slice hyperholomorphic functions.

Definition 2.3 (Slice hyperholomorphic functions) Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on $\mathbb{C}_I$.

(1) We say that f is a left slice hyperholomorphic function (or hyperholomorphic for short) if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

(2) We say that f is right slice hyperholomorphic function (or right hyperholomorphic for short) if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I\right) = 0.$$

(3) In the sequel we will denote by $\mathcal{R}^L(U)$ (resp., $\mathcal{R}^R(U)$) the right (resp., left) $\mathbb{H}$-vector space of left (resp., right) hyperholomorphic functions on the open set U. When we do not distinguish between $\mathcal{R}^L(U)$ and $\mathcal{R}^R(U)$ we will use the symbol $\mathcal{R}(U)$.

The natural open sets on which slice hyperholomorphic functions are defined are axially symmetric, i.e., open sets that contain the 2-sphere $[p]$ whenever they contain p, which are also s-domains, i.e., they are domains which remain connected when intersected with any complex plane $\mathbb{C}_I$.

Given two left slice hyperholomorphic functions f, g, it is possible to introduce a binary operation called the $\star$-product, such that $f \star g$ is a slice hyperholomorphic function. Let $f, g : \Omega \subseteq \mathbb{H}$ be slice hyperholomorphic functions such that their restrictions to the complex plane $\mathbb{C}_I$ can be written as $f_I(z) = F(z) + G(z)J$, $g_I(z) = H(z) + L(z)J$, where $J \in \mathbb{S}$, $J \perp I$. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$ and they exist by the splitting lemma; see [28, p. 117]. We can now give the following definition.

Definition 2.4 Let f, g be slice hyperholomorphic functions defined on an axially symmetric open set $\Omega \subseteq \mathbb{H}$. The $\star$-product of f and g is defined as the unique left slice hyperholomorphic function on Ω whose restriction to the complex plane $\mathbb{C}_I$ is given by

$$\begin{aligned}&\big(F(z) + G(z)J\big) \star \big(H(z) + L(z)J\big)\\ &\quad := \big(F(z)H(z) - G(z)\overline{L(\bar{z})}\big) + \big(G(z)\overline{H(\bar{z})} + F(z)L(z)\big)J.\end{aligned} \tag{2.2}$$

When f, g are expressed by power series, i.e., $f(p) = \sum_{n=0}^{\infty} p^n a_n$, $g(p) = \sum_{n=0}^{\infty} p^n b_n$, then $(f \star g)(p) = \sum_{n=0}^{\infty} p^n c_n$, where $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ is obtained by convolution on the coefficients. This product extends the product of quaternionic polynomials with right coefficients, see [43], to series. Analogously, one can introduce a $\star$-product for right slice hyperholomorphic functions. For more details, we refer the reader to [28]. When considering both products in the same formula, or when confusion may arise, we will write $\star_l$ or $\star_r$ according to whether we are using the left or the right slice hyperholomorphic product. When there is no subscript, we will mean that we are considering the left $\star$-product.

Given a slice hyperholomorphic function f, we can define its slice hyperholomorphic reciprocal $f^{-\star}$; see [23, 28]. In this paper it will be sufficient to know the following definition.

Definition 2.5 Given $f(p) = \sum_{n=0}^{\infty} p^n a_n$, let us set

$$f^c(p) = \sum_{n=0}^{\infty} p^n \bar{a}_n, \qquad f^s(p) = \big(f^c \star f\big)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^{n} a_r \bar{a}_{n-r},$$

where the series converge. The left slice hyperholomorphic reciprocal of f is then defined as

$$f^{-\star} := \big(f^s\big)^{-1} f^c.$$

3 Formulations of the Quaternionic Functional Calculus

Here we briefly recall the possible formulations of the quaternionic functional calculus that we will use in the sequel. Let V be a two-sided quaternionic Banach space, and let $\mathcal{B}(V)$ be the two-sided vector space of all right linear bounded operators on V.

Definition 3.1 (The S-spectrum and the S-resolvent sets of quaternionic operators) Let $T \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \left\{ s \in \mathbb{H} : T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I} \text{ is not invertible} \right\}.$$

The S-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

The notion of the S-spectrum of a linear quaternionic operator T is suggested by the definition of the S-resolvent operator that is the analogue of the Riesz resolvent operator for the quaternionic functional calculus.

Definition 3.2 (The S-resolvent operator) Let V be a two-sided quaternionic Banach space, $T \in \mathcal{B}(V)$, and $s \in \rho_S(T)$. We define the left S-resolvent operator as

$$S_L^{-1}(s,T) := -\left(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}\right)^{-1}(T - \overline{s}\mathcal{I}), \tag{3.1}$$

and the right S-resolvent operator as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})\left(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}\right)^{-1}. \tag{3.2}$$

Theorem 3.3 *Let $T \in \mathcal{B}(V)$ and let $s \in \rho_S(T)$. Then, the left S-resolvent operator satisfies the equation*

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \tag{3.3}$$

while the right S-resolvent operator satisfies the equation

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{3.4}$$

Definition 3.4 Let V be a two-sided quaternionic Banach space, $T \in \mathcal{B}(V)$ and let $U \subset \mathbb{H}$ be an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$ and such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. We say that U is a T-admissible open set.

We can now introduce the class of functions for which we can define the two versions of the quaternionic functional calculus.

Definition 3.5 Let V be a two-sided quaternionic Banach space, $T \in \mathcal{B}(V)$ and let W be an open set in $\mathbb{H}$.

(1) A function $f \in \mathcal{R}^L(W)$ is said to be locally left hyperholomorphic on $\sigma_S(T)$ if there exists a T-admissible domain $U \subset \mathbb{H}$ such that $\overline{U} \subset W$ on which f is left hyperholomorphic. We will denote by $\mathcal{R}^L_{\sigma_S(T)}$ the set of locally left hyperholomorphic functions on $\sigma_S(T)$.
(2) A function $f \in \mathcal{R}^R(W)$ is said to be locally right hyperholomorphic on $\sigma_S(T)$ if there exists a T-admissible domain $U \subset \mathbb{H}$ such that $\overline{U} \subset W$ on which f is right hyperholomorphic. We will denote by $\mathcal{R}^R_{\sigma_S(T)}$ the set of locally right hyperholomorphic functions on $\sigma_S(T)$.

Using the left S-resolvent operator S_L^{-1}, we now give a result that motivates the functional calculus; analogous considerations can be done using S_R^{-1} with obvious modifications.

Definition 3.6 (The quaternionic functional calculus) Let V be a two-sided quaternionic Banach space and $T \in \mathcal{B}(V)$. Let $U \subset \mathbb{H}$ be a T-admissible domain and set $ds_I = -ds\,I$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\, f(s), \quad \text{for } f \in \mathcal{R}^L_{\sigma_S(T)}, \tag{3.5}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\, S_R^{-1}(s,T), \quad \text{for } f \in \mathcal{R}^R_{\sigma_S(T)}. \tag{3.6}$$

4 Projectors, Right and S-Spectrum

An important result that we will prove in this section is that the Riesz projector associated with a given quaternionic operator T commutes with T itself. We begin by recalling the definition of projectors and some of their basic properties that still hold in the quaternionic setting.

Definition 4.1 Let V be a quaternionic Banach space. We say that P is a projector if $P^2 = P$.

It is easy to show that the following properties hold:

(1) The range of P, denoted by $\mathrm{ran}(P)$ is closed.
(2) $v \in \mathrm{ran}(P)$ if and only if $Pv = v$.
(3) If P is a projector, then $I - P$ is also a projector and $\mathrm{ran}(I-P)$ is closed.
(4) $v \in \mathrm{ran}(I-P)$ if and only if $(I-P)v = v$, that is, if and only if $Pv = 0$; as a consequence, $\mathrm{ran}(I-P) = \ker(P)$.
(5) For every $v \in V$ we have $v = Pv + (I-P)v$; $Pv \in \mathrm{ran}(P)$, $(I-P)v \in \ker(P)$. So v can be written as $v' = Pv$ and $v'' = (I-P)v$. Since $\mathrm{ran}(P) \cap \ker(P) = \{0\}$ we have the decomposition $V = \mathrm{ran}(P) \oplus \ker(P)$.

Theorem 4.2 *Let $T \in \mathcal{B}(V)$ and let $\sigma_S(T) = \sigma_{1S} \cup \sigma_{2S}$, with* $\mathrm{dist}(\sigma_{1S}, \sigma_{2S}) > 0$. *Let U_1 and U_2 be two open sets such that $\sigma_{1S} \subset U_1$ and $\sigma_{2S} \subset U_2$, with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Set*

$$P_j := \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I, \quad j = 1,2, \tag{4.1}$$

$$T_j := \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\, s, \quad j = 1,2. \tag{4.2}$$

Then the following properties hold:

(1) *P_j are projectors and $TP_j = P_jT$ for $j = 1, 2$.*
(2) *For $\lambda \in \rho_S(T)$ we have*

$$P_j S_L^{-1}(\lambda, T)\lambda - T_j S_L^{-1}(\lambda, T) = P_j, \quad j = 1, 2, \tag{4.3}$$

$$\lambda S_R^{-1}(\lambda, T) P_j - S_R^{-1}(\lambda, T) T_j = P_j, \quad j = 1, 2. \tag{4.4}$$

Proof The fact that P_j are projectors is proved in [28]. Let us prove that $TP_j = P_jT$. Observe that the functions $f(s) = s^m$, for $m \in \mathbb{N}_0$, are both right and left slice hyperholomorphic. So the operator T can be written as

$$T = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, s = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} s\, ds_I\, S_R^{-1}(s, T);$$

analogously, for the projectors P_j we have

$$P_j = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s, T) ds_I = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} ds_I S_R^{-1}(s, T).$$

From the identity

$$T_j = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s, T) ds_I s = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} s\, ds_I S_R^{-1}(s, T)$$

we can compute TP_j as:

$$TP_j = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} T S_L^{-1}(s, T) ds_I$$

and using the resolvent equation (3.3) it follows that

$$\begin{aligned} TP_j &= \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} \big[S_L^{-1}(s, T)s - \mathcal{I}\big] ds_I = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s, T) s\, ds_I \\ &= \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s, T) ds_I s = T_j. \end{aligned}$$

Now consider

$$P_jT = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} ds_I S_R^{-1}(s, T) T$$

and using the resolvent equation (3.4) we obtain

$$P_jT = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} ds_I \big[s S_R^{-1}(s, T) - \mathcal{I}\big] = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} ds_I s S_R^{-1}(s, T) = T_j$$

so we have the equality $P_jT = TP_j$. To prove (4.3), for $\lambda \in \rho_S(T)$, consider and compute

$$P_j S_L^{-1}(\lambda, T)\lambda = \frac{1}{2\pi} \int_{\partial(U_j\cap\mathbb{C}_I)} ds_I S_R^{-1}(s, T) S_L^{-1}(\lambda, T)\lambda.$$

Using the S-resolvent equation (3.3) it follows that

$$P_j S_L^{-1}(\lambda, T)\lambda = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} ds_I S_R^{-1}(s, T)\big[T S_L^{-1}(\lambda, T) + I\big]$$
$$= \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} ds_I \big[S_R^{-1}(s, T)T\big] S_L^{-1}(\lambda, T) + P_j.$$

By the S-resolvent equation (3.4) we get

$$P_j S_L^{-1}(\lambda, T)\lambda = \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s ds_I S_R^{-1}(s, T) S_L^{-1}(\lambda, T) + P_j$$
$$= T_j S_L^{-1}(\lambda, T) + P_j$$

which is (4.3). Relation (4.4) can be proved in an analogous way. □

In analogy with the classical case, we will call the operator P_j a Riesz projector.

Our next result, of independent interest, is the validity of the decomposition of the S-spectrum which is based on the Riesz projectors. A simple but crucial result will be the following lemma.

Lemma 4.3 *Let $T \in \mathcal{B}(V)$ and let $\lambda \in \rho_S(T)$. Then the operator $(T^2 - 2\lambda_0 T + |\lambda|^2 \mathcal{I})^{-1}$ commutes with every operator A that commutes with T.*

Proof Since A commutes with T we have that

$$\big(T^2 - 2\lambda_0 T + |\lambda|^2 \mathcal{I}\big)A = A\big(T^2 - 2\lambda_0 T + |\lambda|^2 \mathcal{I}\big).$$

We get the statement by multiplying the above relation on both sides by $(T^2 - 2\lambda_0 T + |\lambda|^2 \mathcal{I})^{-1}$. □

Note that, unlike what happens in the classical case in which an operator A which commutes with T also commutes with the resolvent operator, here an operator A commuting with T just commutes with $(T^2 - 2\lambda_0 T + |\lambda|^2 \mathcal{I})^{-1}$. But this result is enough to prove the validity of the next theorem.

Theorem 4.4 *Let $T \in \mathcal{B}(V)$, suppose that P_1 is a projector in $\mathcal{B}(V)$ commuting with T, and let $P_2 = I - P_1$. Let $V_j = P_j(V)$, $j = 1, 2$, and define the operators $T_j = T P_j = P_j T$. Denote by $\widetilde{T}_j$ the restriction of T_j to V_j, $j = 1, 2$. Then*

$$\sigma_S(T) = \sigma_S(\widetilde{T}_1) \cup \sigma_S(\widetilde{T}_2).$$

Proof First of all, note that $T = T_1 + T_2$, $\quad T_1(V_2) = T_2(V_1) = \{0\}$, and that $T_j(V_j) \subseteq V_j$.

We have to show that $\rho_S(T) = \rho_S(\widetilde{T}_1) \cap \rho_S(\widetilde{T}_2)$. Let us assume that $\lambda \in \rho_S(T)$ and consider the identity

$$
\begin{aligned}
T^2-2\lambda_0 T+|\lambda|^2\mathcal{I} &= \big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big)(P_1+P_2)\\
&= \big(T_1^2-2\lambda_0 T_1+|\lambda|^2 P_1\big)+\big(T_2^2-2\lambda_0 T_2+|\lambda|^2 P_2\big). \quad (4.5)
\end{aligned}
$$

If we set

$$Q_\lambda(T):=\big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big)^{-1}$$

we have

$$Q_\lambda(T)=(P_1+P_2)Q_\lambda(T)(P_1+P_2)=P_1Q_\lambda(T)P_1+P_2Q_\lambda(T)P_2; \qquad (4.6)$$

in fact, by Lemma 4.3 and by the relation $P_1P_2=P_2P_1=0$, we deduce

$$P_1Q_\lambda(T)P_2=P_2Q_\lambda(T)P_1=0.$$

We now multiply the identity (4.5) by $Q_\lambda(T)$ on the left, and by (4.6) we obtain

$$\mathcal{I}=\big(P_1Q_\lambda(T)P_1+P_2Q_\lambda(T)P_2\big)\big[\big(T_1^2-2\lambda_0 T_1+|\lambda|^2 P_1\big)+\big(T_2^2-2\lambda_0 T_2+|\lambda|^2 P_2\big)\big].$$

Again using Lemma 4.3 and $P_1P_2=P_2P_1=0$ we obtain

$$\mathcal{I}=P_1Q_\lambda(T)P_1\big(T_1^2-2\lambda_0 T_1+|\lambda|^2 P_1\big)+P_2Q_\lambda(T)P_2\big(T_2^2-2\lambda_0 T_2+|\lambda|^2 P_2\big). \quad (4.7)$$

Let us set

$$Q_{\lambda,j}(T):=P_jQ_\lambda(T)P_j, \quad j=1,2.$$

We immediately observe that

$$Q_{\lambda,j}(T)(V_j)\subseteq V_j, \quad j=1,2,$$

and from (4.7) we deduce

$$Q_{\lambda,j}(T)\big(T_j^2-2\lambda_0 T_j+|\lambda|^2 P_j\big)=P_j, \quad j=1,2.$$

As a consequence, $Q_{\lambda,j}(T)$ restricted to V_j is the inverse of $(\widetilde{T}_j^2-2\lambda_0\widetilde{T}_j+|\lambda|^2P_j)$ and so we conclude that $\lambda\in\rho_S(\widetilde{T}_1)\cap\rho_S(\widetilde{T}_2)$.

Conversely, assume that $\lambda\in\rho_S(\widetilde{T}_1)\cap\rho_S(\widetilde{T}_2)$. Let us set

$$\widetilde{Q}_{\lambda,j}(T):=\big(\widetilde{T}_j^2-2\lambda_0\widetilde{T}_j+|\lambda|^2P_j\big)^{-1}$$

and define

$$\widetilde{Q}=P_1\widetilde{Q}_{\lambda,1}(T)P_1+P_2\widetilde{Q}_{\lambda,2}(T)P_2.$$

We have

$$
\begin{aligned}
\widetilde{Q}\big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big) &= \big[P_1\widetilde{Q}_{\lambda,1}(T)P_1+P_2\widetilde{Q}_{\lambda,2}(T)P_2\big]\big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big)\\
&= P_1\big(\widetilde{T}_1^2-2\lambda_0\widetilde{T}_1+|\lambda|^2P_1\big)^{-1}P_1\big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big)\\
&\quad + P_2\big(\widetilde{T}_2^2-2\lambda_0\widetilde{T}_2+|\lambda|^2P_2\big)^{-1}P_2\big(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I}\big)\\
&= P_1+P_2=\mathcal{I}.
\end{aligned}
$$

Analogously $(T^2-2\lambda_0 T+|\lambda|^2\mathcal{I})\widetilde{Q}=\mathcal{I}$. So $\lambda\in\rho_S(T)$. □

In all our discussions on the functional calculus we have used the notion of S-spectrum. However, in the literature, other types of spectra are also used: the so-called left spectrum and the right spectrum. In order to discuss the notion of right spectrum it is not necessary to assume that V is a two-sided linear space, so we will consider quaternionic right linear spaces. We recall the following definition.

Definition 4.5 Let $T : V \to V$ be a right linear quaternionic operator on a right quaternionic Banach space V. We denote by $\sigma_R(T)$ the right spectrum of T, that is, $\sigma_R(T) = \{s \in \mathbb{H} : Tv = vs \text{ for } v \in V, v \neq 0\}$.

As has been widely discussed in the literature, one can also define the left spectrum, i.e., the set of $s \in \mathbb{H}$ such that $Tv = sv$. However, the notion of left spectrum is not very useful; see [1]. The S-spectrum and the left spectrum are not, in general, related; see [28]. The right spectrum is more useful and more studied. It has a structure similar to the one of the S-spectrum; indeed, whenever it contains an element s, it also contains the whole 2-sphere $[s]$. However, the operator $\mathcal{I}s - T$, where $(\mathcal{I}s)(v) := vs$, is not a right linear operator; thus the notion of right spectrum is not associated with a linear resolvent operator, and this represents a disadvantage since it prevents defining a functional calculus. The following result, see [21], states that the right spectrum coincides with the point S-spectrum and thus $\sigma_R(T)$ can now be related to the linear operator $T^2 - 2s_0T + |s|^2\mathcal{I}$.

Theorem 4.6 *Let T be a right linear quaternionic operator. Then its point S-spectrum coincides with the right spectrum.*

Theorem 4.6 is crucial since all known results on the right spectrum become valid also for the point S-spectrum.

Let us now consider the two eigenvalue problems:

$$Tv = vs, \quad v \neq 0,$$

and

$$\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)w = 0, \quad w \neq 0.$$

As is well known, the right eigenvectors do not form a right linear subspace of V, while the S-eigenvectors do, as one can immediately verify. We have the following proposition which will be useful in the sequel.

Proposition 4.7 *Let v be a right eigenvector associated with $s \in \sigma_R(T)$. Then we have*

$$\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)v = 0.$$

Proof Since $Tv = vs$, it follows that $T^2v = T(vs) = vs^2$. Thus we have

$$\big(T^2 - 2\, s_0T + |s|^2\mathcal{I}\big)v = vs^2 - 2s_0sv + |s|^2v = v\big(s^2 - 2s_0s + |s|^2\big) = 0,$$

where we have used the identity $s^2 - 2s_0s + |s|^2 = 0$, which holds for every $s \in \mathbb{H}$. □

5 A Result on Negative Squares

In this section we will consider power series of the form $K(p,q)=\sum_{n,m=0}^{\infty} p^n \times a_{n,m}\overline{q}^m$, where $a_{n,m}=a_{n,m}^* \in \mathbb{H}^{N\times N}$. It is immediate that $K(p,q)$ is a function slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$; moreover, the assumption on the coefficients $a_{n,m}$ implies that $K(p,q)$ is Hermitian.

Proposition 5.1 *Let $(a_{n,m})_{n,m\in\mathbb{N}_0}$ denote a sequence of $N\times N$ quaternionic matrices such that $a_{n,m}=a_{m,n}^*$, and assume that the power series*

$$K(p,q)=\sum_{n,m=0}^{\infty} p^n a_{n,m}\overline{q}^m$$

converges in a neighborhood V of the origin. Then the following are equivalent:

(1) *The function $K(p,q)$ has κ negative squares in V.*
(2) *All the finite matrices $A_\mu \overset{\text{def.}}{=} (a_{n,m})_{n,m=0,\ldots,\mu}$ have at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for at least one $\mu\in\mathbb{N}_0$.*

Proof Let $r>0$ be such that $B(0,r)\subset V$, and let I, J be two units in the unit sphere of purely imaginary quaternions $\mathbb{S}$ (see (2.1) for the latter). Then

$$a_{n,m}=\frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2} e^{-Int}K\big(re^{It},re^{Js}\big)e^{Jms}dtds.$$

This expression does not depend on the specific choice of I and J. Furthermore, we take $I=J$ and so:

$$A_\mu=\frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}\begin{pmatrix} I_N\\ e^{-Jt}I_N\\ \vdots\\ e^{-J\mu t}I_N\end{pmatrix} K\big(re^{Jt},re^{Js}\big)\big(I_N\ e^{Js}I_N\ \cdots\ e^{J\mu s}I_N\big)dtds.$$

Now write

$$K(p,q)=K_+(p,q)-F(p)F(q)^*,$$

where F is $\mathbb{H}^{N\times\kappa}$ valued. The function F is built from functions of the form $p\mapsto K(p,q)$ for a finite number of q's, and so is a continuous function of p, and so is $K_+(p,q)$. See [7, pp. 8–9]. Thus

$$A_\mu=A_{\mu,+}-A_{\mu,-}$$

where

$$A_{\mu,+}=\frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}\begin{pmatrix} I_N\\ e^{-Jt}I_N\\ \vdots\\ e^{-J\mu t}I_N\end{pmatrix} K_+\big(re^{Jt},re^{Js}\big)$$
$$\times\big(I_N\ e^{Js}I_N\ \cdots\ e^{J\mu s}I_N\big)dtds$$

$$A_{\mu,-} = \frac{1}{4r^{n+m}\pi^2} \iint_{[0,2\pi]^2} \begin{pmatrix} I_N \\ e^{-Jt} I_N \\ \vdots \\ e^{-J\mu t} I_N \end{pmatrix} F\big(re^{Jt}\big) F\big(re^{Js}\big)^*$$
$$\times \big(I_N \; e^{Js} I_N \; \cdots \; e^{J\mu s} I_N\big) dt ds.$$

These expressions show that A_μ has at most κ strictly negative eigenvalues.

Conversely, assume that all the matrices A_μ have at most κ strictly negative eigenvalues, and define

$$K_\mu(p,q) = \sum_{n,m=0}^{\mu} p^m a_{n,m} \overline{q}^m.$$

Then, K_μ has at most κ negative squares, as is seen by writing A_μ as a difference of two positive matrices, one of rank κ. Since, pointwise,

$$K(p,q) = \lim_{\mu \to \infty} K_\mu(p,q),$$

the function $K(p,q)$ has at most κ negative squares.

To conclude the proof, it remains to see that the number of negative squares of $K(p,q)$ and A_μ is the same. Assume that $K(p,q)$ has κ negative squares, but that the A_μ have at most $\kappa' < \kappa$ strictly negative eigenvalues. Then, the argument above shows that $K(p,q)$ would have at most κ' negative squares, which contradicts the hypothesis. The other direction is proved in the same way. $\square$

As consequences we have:

Proposition 5.2 *In the notation of the preceding proposition, the number of negative squares is independent of the neighborhood V.*

Proof This is because the coefficients $a_{n,m}$ do not depend on the given neighborhood. $\square$

Proposition 5.3 *Assume that $K(p,q)$ is $\mathbb{H}^{N\times N}$ valued and has κ negative squares in V and let $\alpha(p)$ be an $\mathbb{H}^{N\times N}$-valued slice hyperholomorphic function and such that $\alpha(0)$ is invertible. Then the function*

$$B(p,q) = \alpha(p) \star K(p,q) \star_r \alpha(q)^* \tag{5.1}$$

has κ negative squares in V.

Proof Write $K(p,q) = \sum_{n,m=0}^{\infty} p^n a_{n,m} \overline{q}^m$ and $\alpha(p) = \alpha_0 + p\alpha_1 + \cdots$. The $\mu \times \mu$ main block matrix B_μ corresponding to the power series (5.1) is equal to

$$B_\mu = L A_\mu L^*,$$

where

$$L = \begin{pmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \vdots & \vdots & & & \\ \alpha_\mu & \alpha_{\mu-1} & \cdots & \alpha_1 & \alpha_0 \end{pmatrix}$$

Since $\alpha_0 = \alpha(0)$ is assumed invertible, the signatures of A_μ and B_μ are the same for every $\mu \in \mathbb{N}_0$. By Proposition 5.1 it follows that the kernels A and B have the same number of negative squares. □

6 Operators in Quaternionic Pontryagin Spaces

This section contains some definitions and results on right quaternionic Pontryagin spaces. Some of the statements hold when we replace Pontryagin spaces by Krein spaces.

The following result, proved in the complex plane in [38, Theorem 2.4, p. 18], is very useful for studying convergence of sequences in Pontryagin spaces. It implies in particular that in a reproducing kernel Pontryagin space, convergence is equivalent to convergence of the self-inner product together with pointwise convergence. The proof of the quaternionic case appears in [6, Proposition 12.9, p. 471].

Proposition 6.1 *Let* $(\mathscr{P}, [\cdot,\cdot])$ *denote a quaternionic right Pontryagin space. The sequence* f_n *of elements in* $\mathscr{P}$ *tends to* $f \in \mathscr{P}$ *if and only if the following two conditions hold*:

$$\lim_{n\to\infty} [f_n, f_n] = [f, f],$$

and

$$\lim_{n\to\infty} [f_n, g] = [f, g] \quad \textit{for } g \textit{ in a dense subspace of } \mathscr{P}.$$

We endow $\mathbb{H}^N$ with the inner product

$$[u, v]_{\mathbb{H}^N} = v^* u.$$

Furthermore, a Hermitian form will be defined as having the following linearity condition:

$$[fa, gb] = \overline{b}[f, g]a. \tag{6.1}$$

Remark 6.2 When we consider two-sided Pontryagin vector spaces, we require an additional property on the inner product with respect to the left multiplication, i.e.,

$$[av, av] = |a|^2 [v, v].$$

This property is satisfied, for example, in $\mathbb{H}^N$ with the inner product described above.

Theorem 6.3 *Let T be a contraction in a two-sided quaternionic Pontryagin space such that T has no S-spectrum on the unit sphere and it satisfies*

$$\left[S_L^{-1}(\lambda,T)\lambda v, S_L^{-1}(\lambda,T)\lambda v\right] \leq \left[S_L^{-1}(\lambda,T)v, S_L^{-1}(\lambda,T)v\right], \quad \textit{for } |\lambda| = 1.$$

Then T has a maximal negative invariant subspace, and this subspace is unique.

Proof Let $|\lambda| = 1$ so that the operator $S_L^{-1}(\lambda,T)$ exists. The fact that T is a contraction implies the inequality

$$\left[TS_L^{-1}(\lambda,T)v, TS_L^{-1}(\lambda,T)v\right] < \left[S_L^{-1}(\lambda,T)v, S_L^{-1}(\lambda,T)v\right]$$

for $v \neq 0$. Using the S-resolvent equation, one deduces

$$\left[S_L^{-1}(\lambda,T)\lambda v + \mathcal{I}v, S_L^{-1}(\lambda,T)\lambda v + \mathcal{I}v\right] < \left[S_L^{-1}(\lambda,T)v, S_L^{-1}(\lambda,T)v\right]$$

from which one gets

$$\begin{aligned}&\left[S_L^{-1}(\lambda,T)\lambda v, S_L^{-1}(\lambda,T)\lambda v\right] + [v,v] + \left[S_L^{-1}(\lambda,T)\lambda v, v\right] + \left[v, S_L^{-1}(\lambda,T)\lambda v\right]\\ &\quad < \left[S_L^{-1}(\lambda,T)v, S_L^{-1}(\lambda,T)v\right].\end{aligned}$$

So, using the hypothesis, we finally get

$$[v,v] + \left[S_L^{-1}(\lambda,T)\lambda v, v\right] + \left[v, S_L^{-1}(\lambda,T)\lambda v\right] < 0.$$

In the above inequality we replace $S_L^{-1}(\lambda,T)\lambda$ by $S_L^{-1}(\lambda,T)\lambda d\lambda_I$, where $d\lambda_I = -Ie^{I\theta}d\theta$, and integrate over $[0, 2\pi]$. Recalling the definition of Riesz projector

$$P = -\frac{1}{2\pi}\int_{\partial(\mathbb{B}\cap\mathbb{C}_I)} S_L^{-1}(\lambda,T)\lambda d\lambda_I$$

we obtain

$$[v,v] < [Pv,v] + [v,Pv]$$

and so

$$[v,v] < 2\,\mathrm{Re}[Pv,v].$$

Theorem 4.2 implies that $PT = TP$, and the rest of the proof follows as in Theorem 11.1, p. 76 in [38]. □

For right quaternionic Pontryagin spaces, we have the following result.

Proposition 6.4 *A contraction T in a right quaternionic Pontryagin space $\mathcal{P}$ possessing an eigenvalue λ with $|\lambda| > 1$ has a maximal negative invariant subspace.*

Proof Let $v \neq 0$ be an eigenvector associated with the right eigenvalue λ. Then we have

$$[Tv, Tv] = [v\lambda, v\lambda] < [v,v],$$

from which we deduce

$$|\lambda|^2[v, v] < [v, v]$$

and so $[v, v] < 0$. Consider the right subspace $\mathcal{M}$ generated by v. Then any element in $\mathcal{M}$ is of the form va, $a \in \mathbb{H}$ and $[va, va] < 0$. The subspace $\mathcal{M}$ is invariant under the action of T; indeed, $T(va) = T(v)a = v\lambda a$. Thus $\mathcal{M}$ is a negative invariant subspace of $\mathcal{P}$. Then $\mathcal{M}$ is maximal or it is contained in another negative invariant subspace $\mathcal{M}_1$, and iterating this procedure we obtain a chain of inclusions $\mathcal{M} \subset \mathcal{M}_1 \subset \cdots$ which should end because $\mathcal{P}_-$ is finite dimensional. □

In view of Definition 6.6 below, it is useful to recall the following result (see [49, Corollary 6.2, p. 41]).

Proposition 6.5 *A Hermitian matrix H with entries in $\mathbb{H}$ is diagonalizable, and its eigenvalues are real. Furthermore, eigenvectors corresponding to different eigenvalues are orthogonal in $\mathbb{H}^N$. Let (t, r, s) denote the signature of H. There exists an invertible matrix $U \in \mathbb{H}^{N\times N}$ such that*

$$H = U \begin{pmatrix} \sigma_{tr} & 0 \\ 0 & 0_{s\times s} \end{pmatrix} U^*, \tag{6.2}$$

where $\sigma_{tr} = \left(\begin{smallmatrix} I_t & 0 \\ 0 & -I_r \end{smallmatrix}\right)$.

Definition 6.6 Let A be a continuous right linear operator from the quaternionic Pontryagin space $\mathscr{P}$ into itself. We say that A has κ negative squares if for every choice of $N \in \mathbb{N}$ and of $f_1, \ldots, f_N \in \mathscr{P}$, the Hermitian matrix $H \in \mathbb{H}^{N\times N}$ with jk entry equal to

$$[Af_k, f_j]_{\mathscr{P}} \tag{6.3}$$

has at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for some choice of $N, f_1, \ldots, f_N$.

Note that the above definition is coherent with the right linearity condition (6.1). If we replace the f_k by $f_k h_k$, where $h_k \in \mathbb{H}$, the new matrix has jk entry

$$[Af_kh_k, f_jh_j]_{\mathscr{P}} = \overline{h_j}[Af_k, f_j]_{\mathscr{P}}h_k,$$

and so

$$\big([Af_kh_k, f_jh_j]_{\mathscr{P}}\big)_{j,k=1,\ldots,N} = D^*\big([Af_k, f_j]_{\mathscr{P}}\big)_{j,k=1,\ldots,N}D,$$

with

$$D = \operatorname{diag}(h_1, h_2, \ldots, h_N).$$

In the case of left linear operators, (6.1) is then replaced by

$$[af, bg] = b[f, g]\overline{a},$$

and the roles of j and k have to be interchanged in (6.3). This problem does not appear in the commutative case.

We point out the following notation. Let T be a bounded linear operator from the quaternionic right Pontryagin space $(\mathscr{P}_1, [\cdot,\cdot]_{\mathscr{P}_1})$ into the quaternionic right Pontryagin space $(\mathscr{P}_2, [\cdot,\cdot]_{\mathscr{P}_2})$, and let σ_1 and σ_2 denote two signature operators such that $(\mathscr{P}_1, \langle\cdot,\cdot\rangle_{\mathscr{P}_1})$ and $(\mathscr{P}_2, \langle\cdot,\cdot\rangle_{\mathscr{P}_2})$ are right quaternionic Pontryagin spaces, where

$$\langle x, y\rangle_{\mathscr{P}_j} = [x, \sigma_j y]_{\mathscr{P}_j}, \quad j = 1, 2.$$

We denote by $T^{[*]}$ the adjoint of T with respect to the Pontryagin structure and by T^* its adjoint with respect to the Hilbert space structure. Thus,

$$\begin{aligned}
[Tx, y]_{\mathscr{P}_2} &= \langle Tx, \sigma_2 y\rangle_{\mathscr{P}_2} \\
&= \left\langle x, T^*\sigma_2 y\right\rangle_{\mathscr{P}_1} \\
&= \left[x, \sigma_1 T^*\sigma_2 y\right]_{\mathscr{P}_1},
\end{aligned}$$

and so, as is well known in the complex case,

$$T^{[*]} = \sigma_1 T^*\sigma_2 \quad \text{and} \quad T^* = \sigma_1 T^{[*]}\sigma_2.$$

We will denote $\nu_-(A) = \kappa$. When $\kappa = 0$ the operator is called positive.

Theorem 6.7 *Let A be a bounded right linear self-adjoint operator from the quaternionic Pontryagin space $\mathscr{P}$ into itself, which has a finite number of negative squares. Then, there exists a quaternionic Pontryagin space $\mathscr{P}_1$ with* $\text{ind}_{\mathscr{P}_1} = \nu_-(A)$, *and a bounded right linear operator T from $\mathscr{P}$ into $\mathscr{P}_1$ such that*

$$A = T^{[*]}T.$$

Proof The proof follows that of [11, Theorem 3.4, p. 456], slightly adapted to the present non-commutative setting. Since A is Hermitian, the formula

$$[Af, Ag]_A = [Af, g]_{\mathscr{P}}$$

defines a Hermitian form on the range of A. Since $\nu_-(A) = \kappa$, there exists $N \in \mathbb{N}$ and $f_1, \ldots, f_N \in \mathscr{P}$ such that the Hermitian matrix M with ℓj entry $[Af_j, f_\ell]_{\mathscr{P}}$ has exactly κ strictly negative eigenvalues. Let $v_1, \ldots, v_\kappa$ be the corresponding eigenvectors, with strictly negative eigenvalues $\lambda_1, \ldots, \lambda_\kappa$. As recalled in Proposition 6.5, v_j and v_k are orthogonal when $\lambda_j \neq \lambda_k$. We can, and will, assume that vectors corresponding to a given eigenvalue are orthogonal. Then,

$$v_s^* M v_t = \lambda_t \delta_{ts}, \quad t, s = 1, \ldots, N. \tag{6.4}$$

In view of (6.1), and with

$$v_t = \begin{pmatrix} v_{t1} \\ v_{t2} \\ \vdots \\ v_{tN} \end{pmatrix}, \quad t = 1, \ldots, N,$$

we see that (6.4) can be rewritten as

$$[F_s, F_t]_A = \lambda_t \delta_{ts}, \quad \text{with } F_s = \sum_{k=1}^{N} A f_k v_{sk}, \ t, s = 1, \ldots, N.$$

The space $\mathscr{M}$ spanned by $F_1, \ldots, F_N$ is strictly negative, and it has an orthocomplement in $(\operatorname{Ran} A, [\cdot, \cdot]_A)$, say $\mathscr{M}^{[\perp]}$, which is a right quaternionic pre-Hilbert space. The space $\operatorname{Ran} A$ endowed with the quadratic form

$$\langle m + h, m + h \rangle_A = -[m, m]_A + [h, h]_A, \quad m \in \mathscr{M}, h \in \mathscr{M}^{[\perp]}$$

is a pre-Hilbert space, and we denote by $\mathscr{P}_1$ its completion. We note that $\mathscr{P}_1$ is defined only up to an isomorphism of Hilbert space. We denote by ι the injection from $\operatorname{Ran} A$ into $\mathscr{P}_1$ such that

$$\langle f, f \rangle_A = \left\langle \iota(f), \iota(f) \right\rangle_{\mathscr{P}_1}.$$

We consider the decomposition $\mathscr{P}_1 = \iota(M) \oplus \iota(M)^{\perp}$, and endow $\mathscr{P}_1$ with the indefinite inner product

$$\left[\iota(m) + h, \iota(m) + h\right]_{\mathscr{P}_!} = [m, m]_A + \langle h, h \rangle_{\mathscr{P}_1}.$$

See [38, Theorem 2.5, p. 20] for the similar argument in the complex case. Still following [11], we define

$$Tf = \iota(Af), \quad f \in \mathscr{P}.$$

We now prove that T is a bounded right linear operator from $\mathscr{P}$ into $\iota(\operatorname{Ran} A) \subset \mathscr{P}_1$. Indeed, let $(f_n)_{n \in \mathbb{N}}$ denote a sequence of elements in $\mathscr{P}$ converging (in the topology of $\mathscr{P}$) to $f \in \mathscr{P}$. Since $\operatorname{Ran} A$ is dense in $\mathscr{P}_1$, using Proposition 6.1 it is therefore enough to prove that:

$$\lim_{n \to} [T f_n, T f_n]_{\mathscr{P}_1} = [Tf, Tf]_{\mathscr{P}_1},$$

and

$$\lim_{n \to \infty} [T f_n, T g]_{\mathscr{P}_1} = [Tf, Tg]_{\mathscr{P}_1}, \quad \forall g \in \mathscr{P}.$$

By the definition of the inner product, the first equality amounts to

$$\lim_{n \to} [A f_n, f_n]_{\mathscr{P}} = [Af, f]_{\mathscr{P}},$$

which is true since A is continuous, and similarly for the second claim. Therefore, T has an adjoint operator, which is also continuous. The equalities (with $f, g \in \mathscr{P}$)

$$\begin{aligned}\left[f, T^{[*]}Tg\right]_{\mathscr{P}} &= [Tf, Tg]_{\mathscr{P}_1} \\ &= \left[Tf, \iota(Ag)\right]_{\mathscr{P}_1} \\ &= \left[\iota(Af), \iota(Ag)\right]_{\mathscr{P}_1} \\ &= [Af, Ag]_A \\ &= [f, Ag]_{\mathscr{P}}\end{aligned}$$

show that $T^{[*]}T = A$. □

We note the following. As is well known, the completion of a pre-Hilbert space is unique up to an isomorphism of Hilbert spaces, and the completion need not be in general a subspace of the original pre-Hilbert space. Some identification is needed. In [38] (see [38, 2.4, p. 19] and also in [11]) the operator ι is not used, and the space $\mathscr{P}_1$ is written directly as a direct sum of $\mathscr{M}$ and of the completion of the orthogonal of $\mathscr{M}$. This amounts to identifying the orthogonal of $\mathscr{M}$ as being a subspace of its abstract completion.

7 The Structure Theorem

We first give some background to provide motivation for the results presented in this section. Denote by R_0 the backward-shift operator:

$$R_0 f(z) = \frac{f(z) - f(0)}{z}.$$

Beurling's theorem can be seen as the characterization of R_0-invariant subspaces of the Hardy space $\mathbf{H}_2(\mathbb{D})$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. These are the spaces $\mathbf{H}_2(\mathbb{D}) \ominus j\mathbf{H}_2(\mathbb{D})$, where j is an inner function. Equivalently, these are the reproducing kernel Hilbert spaces with reproducing kernel $k_j(z, w) = \frac{1-j(z)\overline{j(w)}}{1-z\overline{w}}$ with j inner. When replacing j inner by s analytic and contractive in the open unit disk, it is more difficult to characterize reproducing kernel Hilbert spaces $\mathscr{H}(s)$ with reproducing kernel $k_s(z, w)$. Allowing for s not necessarily scalar valued, de Branges gave a characterization of $\mathscr{H}(s)$ spaces in [30, Theorem 11, p. 171]. This result was extended in [7, Theorem 3.1.2, p. 85] to the case of Pontryagin spaces. The theorem below is the analog of de Branges's result in the slice hyperholomorphic setting, in which the backward-shift operator R_0 is now defined as

$$R_0 f(p) = p^{-1}\big(f(p) - f(0)\big) = \big(f(p) - f(0)\big) \star_\ell p^{-1}.$$

In order to prove the result, we will need a fact which is a direct consequence of Lemma 3.6 in [13]: If f, g are two left slice hyperholomorphic functions then

$$(f \star_l g)^* = g^* \star_r f^*.$$

Theorem 7.1 *Let $\sigma \in \mathbb{H}^{N\times N}$ be a signature matrix, and let $\mathscr{M}$ be a Pontryagin space of $\mathbb{H}^N$-valued functions slice hyperholomorphic in a spherical neighborhood V of the origin, and invariant under the operator R_0. Assume, moreover, that*

$$[R_0 f, R_0 f]_{\mathscr{M}} \leq [f, f]_{\mathscr{M}} - f(0)^* \sigma f(0). \tag{7.1}$$

Then there exists a Pontryagin space $\mathscr{P}$ such that $\operatorname{ind}_- \mathscr{P} = \nu_-(\sigma)$ *and an* $\mathbf{L}(\mathscr{P}, \mathbb{H}^N)$*-valued slice hyperholomorphic function S such that the elements of $\mathscr{M}$ are the restrictions to V of the elements of $\mathscr{P}(S)$.*

Proof We follow the proof in [7, Theorem 3.1.2, p. 85]. Let $\mathscr{P}_2 = \mathscr{M} \oplus \mathbb{H}_\sigma$, and denote by C the point evaluation at the origin. We divide the proof into a number of steps.

STEP 1: *Let $p \in V$ and $f \in \mathscr{M}$. Then,*

$$f(p) = C \star (I - pR_0)^{-\star} f. \tag{7.2}$$

STEP 2: *The reproducing kernel of $\mathscr{M}$ is given by*

$$K(p, q) = C \star (I - pR_0)^{-\star} \big(C \star (I - qR_0)^{-\star}\big)^*.$$

STEP 3: *Let E denote the operator*

$$E = \begin{pmatrix} R_0 \\ C \end{pmatrix} : \quad \mathscr{M} \longrightarrow \mathscr{P}_2.$$

There exists a quaternionic Pontryagin space $\mathscr{P}_1$ with $\operatorname{ind}_{\mathscr{P}_1} = \nu_-(J)$, *and a bounded right linear operator T from $\mathscr{M}$ into $\mathscr{P}_1$ such that*

$$I_{\mathscr{M}} - EE^{[*]} = T^{[*]} T. \tag{7.3}$$

Write (see [7, (1.3.14), p. 26])

$$\begin{pmatrix} I_{\mathscr{M}} & 0 \\ E & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{M}} & 0 \\ 0 & I_{\mathscr{P}_2} - EE^{[*]} \end{pmatrix} \begin{pmatrix} I_{\mathscr{M}} & E^{[*]} \\ 0 & I_{\mathscr{P}_2} \end{pmatrix}$$
$$= \begin{pmatrix} I_{\mathscr{M}} & E^{[*]} \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{M}} - E^{[*]}E & 0 \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{M}} & 0 \\ E & I_{\mathscr{P}_2} \end{pmatrix}.$$

Thus,

$$\nu_-\big(I_{\mathscr{P}_2} - EE^{[*]}\big) + \nu_-(\mathscr{M}) = \nu_-\big(I_{\mathscr{M}} - E^{[*]}E\big) + \nu_-(\mathscr{P}_2), \tag{7.4}$$

and noting that $\nu_-(\mathscr{P}_2) = \nu_-(\mathscr{M}) + \nu_-(\sigma)$, we have (see also [7, Theorem 1.3.4(1), p. 25])

$$\nu_-\big(I_{\mathscr{P}_2} - EE^{[*]}\big) + \nu_-(\mathscr{M}) = \nu_-\big(I_{\mathscr{M}} - E^{[*]}E\big) + \nu_-(\mathscr{M}) + \nu_-(\sigma).$$

Equation (7.1) can be rewritten as $I - E^{[*]}E \geq 0$, and in particular $\nu_-(I - E^{[*]}E) = 0$. Thus,

$$\nu_-\big(I_{\mathscr{P}_2} - EE^{[*]}\big) = \nu_-(\sigma).$$

Applying Theorem 6.7, we obtain the factorization (7.3).

We set

$$T^{[*]} = \begin{pmatrix} B \\ D \end{pmatrix} : \mathscr{P}_1 \longrightarrow \mathscr{M} \oplus \mathbb{H}_\sigma,$$

and

$$V = \begin{pmatrix} R_0 & B \\ C & D \end{pmatrix}.$$

Let

$$S(p) = D + pC \star (I_{\mathscr{M}} - pA)^{-\star} B.$$

STEP 4: *We have that*

$$\sigma - S(p)\sigma S(q)^* = C \star (I - pR_0)^{-\star} \star (I - p\overline{q})\sigma_{\mathscr{M}}\big((I - qA)^{-\star}\big)^* \star_r C^*,$$

where $\sigma_{\mathscr{M}}$ is a fundamental symmetry for $\mathscr{M}$.

The computation is as in our previous paper [13]. □

We note that a corollary of (7.4) is:

Theorem 7.2 *Let T be a contraction between right quaternionic Pontryagin spaces of the same index. Then, its adjoint is a contraction.*

Proof Indeed, when $\nu_-(\mathscr{M}) = \nu_-(\mathscr{P}_2)$ we have

$$\nu_-\big(I_{\mathscr{P}_2} - EE^{[*]}\big) = \nu_-\big(I_{\mathscr{M}} - E^{[*]}E\big).$$ □

8 Blaschke Products

As is well known and easy to check, a rational function r is analytic in the open unit disk and takes unitary values on the unit circle if and only if it is a finite Blaschke product. If one allows poles inside the unit disk, then r is a quotient of finite Blaschke products. This is a very special case of a result of Krein and Langer discussed in Sect. 9 below. In particular, such a function cannot have a pole (or a zero) on the unit circle. The case of matrix-valued rational functions which take unitary values (with respect to a possibly indefinite inner product space) plays an important role in the theory of linear systems. When the metric is indefinite, poles can occur on the unit circle. See, for instance, [5, 14, 32, 46].

Slice hyperholomorphic functions have zeros that are either isolated points or isolated 2-spheres. If a slice hyperholomorphic function f has zeros at $Z = \{a_1, a_2, \ldots, [c_1], [c_2], \ldots\}$ then its reciprocal $f^{-\star}$ has poles at the set $\{[a_1], [a_2], \ldots,$

$[c_1], [c_2], \ldots\}$, $a_i, c_j \in \mathbb{H}$. So the sphere associated with a zero of f is a pole of $f^{-\star}$. In other words, the poles are always 2-spheres, as one may clearly see from the definition of $f^{-\star} = (f^s)^{-1} f^c$; see also [47].

We now recall the definitions of Blaschke factors, see [13], and then discuss the counterpart of rational unitary functions in the present setting. For the Blaschke factors, it is necessary to give two different definitions, depending on whether the zero of a Blaschke factor is a point, see Definition 8.1, or a sphere, see Definition 8.3.

Definition 8.1 Let $a \in \mathbb{H}$, $|a| < 1$. The function

$$B_a(p) = (1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} \tag{8.1}$$

is called Blaschke factor at a.

Remark 8.2 Let $a \in \mathbb{H}$, $|a| < 1$. Then, see Theorem 5.5 in [13], the Blaschke factor $B_a(q)$ takes the unit ball $\mathbb{B}$ to itself and the boundary of the unit ball to itself. Moreover, it has a unique zero for $p = a$.

The Blaschke factor having zeros at a sphere is defined as follows.

Definition 8.3 Let $a \in \mathbb{H}$, $|a| < 1$. The function

$$B_{[a]}(p) = \big(1 - 2\,\mathrm{Re}(a)p + p^2|a|^2\big)^{-1}\big(|a|^2 - 2\,\mathrm{Re}(a)p + p^2\big) \tag{8.2}$$

is called Blaschke factor at the sphere $[a]$.

Remark 8.4 The definition of $B_{[a]}(p)$ does not depend on the choice of the point a that identifies the 2-sphere. In fact, all the elements in the sphere $[a]$ have the same real part and module. It is immediate that the Blaschke factor $B_{[a]}(p)$ vanishes on the sphere $[a]$.

The following result has been proven in [13], Theorem 5.16.

Theorem 8.5 *A Blaschke product having zeros at the set*

$$Z = \big\{(a_1, \mu_1), (a_2, \mu_2), \ldots, \big([c_1], \nu_1\big), \big([c_2], \nu_2\big), \ldots\big\},$$

where $a_j \in \mathbb{B}$, a_j have respective multiplicities $\mu_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \ldots$, $[a_i] \neq [a_j]$ if $i \neq j$, $c_i \in \mathbb{B}$, the spheres $[c_j]$ have respective multiplicities $\nu_j \geq 1$, $j = 1, 2, \ldots$, $[c_i] \neq [c_j]$ if $i \neq j$ and

$$\sum_{i,j \geq 1} \big(\mu_i\big(1 - |a_i|\big) + \nu_j\big(1 - |c_j|\big)\big) < \infty$$

is given by

$$\prod_{i \geq 1} \big(B_{[c_i]}(p)\big)^{\nu_i} \prod_{j \geq 1}^{\star} \big(B_{a'_j}(p)\big)^{\star \mu_j},$$

where $a_1' = a_1$ and $a_j' \in [a_j]$, for $j = 2, 3, \ldots$, are suitably chosen elements.

Remark 8.6 It is not difficult to compute the slice hyperholomorphic inverses of the Blaschke factors using Definition 2.5. The slice hyperholomorphic reciprocal of $B_a(p)$ and $B_{[a]}(p)$ are, respectively,

$$B_a(p)^{-\star} = \frac{a}{|a|}(a-p)^{-\star} \star (1 - p\bar{a}),$$

$$B_{[a]}(p)^{-\star} = \left(|a|^2 - 2\,\mathrm{Re}(a)p + p^2\right)^{-1}\left(1 - 2\,\mathrm{Re}(a)p + p^2|a|^2\right).$$

The reciprocal of a Blaschke product is constructed by taking the reciprocal of the factors, in the reverse order.

Remark 8.7 The zeroes of $B_{[a]}(p)$ are poles of $B_{[a]}(p)^{-\star}$ and vice versa. The Blaschke factor $B_a(p)$ has a zero at $p = a$ and a pole at the 2-sphere $[1/\bar{a}]$, while $B_a(p)^{-\star}$ has a zero at $p = 1/\bar{a}$ and a pole at the 2-sphere $[a]$.

Proposition 8.8 *Let $\sigma \in \mathbb{H}^{N\times N}$ denote a signature matrix (that is, $\sigma = \sigma^* = \sigma^{-1}$) and let $(C, A) \in \mathbb{H}^{N\times M} \times \mathbb{H}^{M\times M}$ be such that $\cap_{n=0}^{\infty} \ker CA^n = \{0\}$. Let P be an invertible and Hermitian solution of the Stein equation*

$$P - A^*PA = C^*\sigma C. \tag{8.3}$$

Then, there exist matrices $(B, D) \in \mathbb{H}^{M\times N} \times \mathbb{H}^{\times N\times N}$ such that the function

$$S(p) = D + pC \star (I_M - pA)^{-\star}B \tag{8.4}$$

satisfies

$$\sigma - S(p)\sigma S(q)^* = C\star(I_M - pA)^{-\star}\left(P^{-1} - pP^{-1}\overline{q}\right)\star_r \left(I_M - A^*\overline{q}\right)^{-\star_r} \star_r C^*. \tag{8.5}$$

Before the proof, we mention the following. The vectors $f_1, f_2, \ldots$ in the quaternionic Pontryagin space $(\mathcal{P}, [\cdot,\cdot]_{\mathcal{P}})$ are said to be orthonormal if

$$[f_j, f_\ell]_{\mathcal{P}} = \begin{cases} 0, & \text{if } j \neq \ell, \\ \pm 1, & \text{if } j = \ell. \end{cases}$$

The set $f_1, f_2, \ldots$ is called an orthonormal basis if the closed linear span of the f_j is all of $\mathcal{P}$. In the proof we used the fact that in a finite-dimensional quaternionic Pontryagin space an orthonormal family can be extended to an orthonormal basis. This is true because any non-degenerate closed space in a quaternionic Pontryagin space admits an orthogonal complement. See [6, Proposition 10.3, p. 464].

Proof of Proposition 8.8 Following our previous paper [13], the statement is equivalent to finding matrices $(B, D) \in \mathbb{H}^{M\times N} \times \mathbb{H}^{\times N\times N}$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} P^{-1} & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} P^{-1} & 0 \\ 0 & \sigma \end{pmatrix}, \tag{8.6}$$

or equivalently,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} P & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & \sigma \end{pmatrix}. \tag{8.7}$$

By Proposition 6.5, there exists a matrix $V \in \mathbb{H}^{M\times M}$ and $t_1, s_1 \in \mathbb{N}_0$ such that

$$P = V\sigma_{t_1 s_1} V^*.$$

Equation (8.3) can be then rewritten as

$$V^{-1}A^*V\sigma_{t_1,s_1}V^*AV^{-*} + V^{-1}C^*\sigma CV^{-*} = \sigma_{t_1,s_1},$$

and expresses that the columns of the $\mathbb{H}^{(M+N)\times M}$ matrix

$$\begin{pmatrix} V^*AV^{-*} \\ CV^{-*} \end{pmatrix} = \begin{pmatrix} V^* & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} V^{-*}$$

are orthogonal in $\mathbb{H}^{M+N}$, endowed with the inner product

$$[u, v] = u_1^*\sigma_{t_1,s_1}u_1 + u_2^*\sigma u_2, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{8.8}$$

the first t_1 columns having self-inner product equal to 1 and the next s_1 columns having self-inner product equal to -1. We can complete these columns to form an orthonormal basis of $\mathbb{H}^{(M+N)\times(M+N)}$ endowed with the inner product (8.8), that is, we find a matrix $X \in \mathbb{H}^{(M+N)\times N}$

$$\begin{pmatrix} \begin{pmatrix} V^*AV^{-*} \\ CV^{-*} \end{pmatrix} & X \end{pmatrix} \in \mathbb{H}^{(M+N)\times(M+N)}$$

unitary with respect to (8.8). From

$$\begin{pmatrix} \begin{pmatrix} V^*AV^{-*} \\ CV^{-*} \end{pmatrix} & X \end{pmatrix}^* \begin{pmatrix} \sigma_{t_1 s_1} & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \begin{pmatrix} V^*AV^{-*} \\ CV^{-*} \end{pmatrix} & X \end{pmatrix} = \begin{pmatrix} \sigma_{t_1 s_1} & 0 \\ 0 & \sigma \end{pmatrix},$$

we obtain (8.6) with

$$\begin{pmatrix} B \\ D \end{pmatrix} = X \begin{pmatrix} V^* & 0 \\ 0 & I_N \end{pmatrix}. \tag{8.9}$$

□

When the signature matrix σ is taken to be equal to I_N we can get another more explicit formula for S.

Proposition 8.9 *In the notation and hypotheses of the previous theorem, assume $\sigma = I_N$. Then, $(I_M - A)$ is invertible and the function*

$$S(p) = I_N - (1-p)C \star (I_M - pA)^{-\star}P^{-1}(I_M - A)^{-*}C^* \tag{8.10}$$

satisfies

$$I_N - S(p)S(q)^* = C \star (I_M - pA)^{-\star}\left(P^{-1} - pP^{-1}\overline{q}\right) \star_r \left(I_M - A^*\overline{q}\right)^{-\star_r} \star_r C^*. \tag{8.11}$$

Note that formula (8.10) is not a realization of the form (8.4). It can be brought to the form (8.4) by writing

$$\begin{aligned} S(p) &= S(0) + S(p) - S(0) \\ &= I_N - CP^{-1}(I_M - A)^{-*}C^* \\ &\quad + pC \star (I_M - pA)^{-\star}(I_M - A)P^{-1}(I_M - A)^{-*}C^*. \end{aligned}$$

Proof of Proposition 8.9 We write for p, q, where the various expressions make sense,

$$\begin{aligned} S(p)I_N S(q)^* - I_N &= \left(I_N - (1-p)C \star (I_M - pA)^{-\star}P^{-1}(I_M - A)^{-*}C^*\right) \\ &\qquad \times \left(I_N - (1-q)C \star (I_M - qA)^{-\star}P^{-1}(I_M - A)^{-*}C^*\right)^* - I_N \\ &= C \star (I_M - pA)^{-\star} \star \Delta \star_r \left(I_M - A^*\overline{q}\right)^{-\star_r} \star_r C^*, \end{aligned}$$

where

$$\begin{aligned} \Delta &= -(1-p)P^{-1} \star (I_M - A)^{-*}\left(I_M - A^*\overline{q}\right)^{-\star_r} \\ &\quad - (I_M - pA) \star (I - \overline{q}(I_M - A)^{-1}P^{-1} \\ &\quad + P^{-1}(I_M - A)^{-*}C^*C \star (1-p) \star_r (1 - \overline{q}) \star_r (I_M - A)^{-1}P^{-1}. \end{aligned}$$

Taking into account the Stein equation (8.3) with $\sigma = I_M$, we have

$$\Delta = P^{-1}(I_M - A)^{-*} \star \Delta_1 \star_r (I_M - A)^{-1}P^{-1},$$

where, after some computations,

$$\begin{aligned} \Delta_1 &= \left\{-(1-p) \star \left(I_M - \overline{q}A^*\right) \star_r P(I_M - A)\right. \\ &\quad \left. - (I_M - A)^*P \star (I_M - pA) \star_r (1 - \overline{q}) + (1-p) \star \left(P - A^*PA\right) \star_r (1 - \overline{q})\right\} \\ &= \left(I_M - A^*\right)P(I_M - A). \end{aligned}$$

□

We note that a space $\mathscr{P}(S)$ can be finite dimensional without S being square. For instance,

$$S(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & b_a(p) \end{pmatrix}.$$

On the other hand, finite-dimensional $\mathscr{P}(S)$ spaces for square S correspond to the σ-unitary functions studied in linear system theory. The factorization theory of these functions (that is, the slice hyperholomorphic counterpart of [3–5]), will be considered in a future publication.

9 Krein–Langer Factorization

In the classical case, functions S for which the kernel (1.1) has a finite number of negative squares have a special structure: They can be written as the quotient of a Schur function and of a finite Blaschke product. This is a result of Krein and Langer. See, for instance, [39]. In this section, we present some related results.

Proposition 9.1 *Let S be an $\mathbb{H}^{N\times M}$-valued slice hyperholomorphic function in $\mathbb{B}$ and of the form*

$$S(p) = B(p)^{-\star} \star S_0(p), \quad p \in \mathbb{B}, \tag{9.1}$$

where B is an $\mathbb{H}^{N\times N}$-valued Blaschke product and S_0 is an $\mathbb{H}^{N\times M}$-valued Schur multiplier. Then, S is a generalized Schur function.

Proof We follow the argument in [3, §6.2]. We have for $n \in \mathbb{N}_0$ and $p, q \in \mathbb{B}$

$$\begin{aligned} p^n\big(I_N - S(p)S(q)^*\big)\overline{q}^n &= p^n B(p)^{-\star} \star \big(B(p)B(q)^* \\ &\quad - S_0(p)S_0(q)^*\big) \star_r \big(B(q)^*\big)^{-\star_r}\overline{q}^n \\ &= p^n B(p)^{-\star} \star \big(B(p)B(q)^* - I_N \\ &\quad + I_N - S_0(p)S_0(q)^*\big) \star_r \big(B(q)^*\big)^{-\star_r}\overline{q}^n. \end{aligned}$$

Thus

$$K_S(p,q) = B(p)^{-\star} \star \big(K_{S_0}(p,q) - K_B(p,q)\big)\big(B(q)^*\big)^{-\star_r}, \tag{9.2}$$

where K_{S_0} and K_B are defined as in (1.1). Using Proposition 5.3 with $\kappa = 0$, we see that formula (9.2) expresses the kernel K_S as a difference of two positive definite kernels, one being finite dimensional. It follows that K_S has a finite number of negative squares in $\mathbb{B}$. $\square$

Theorem 9.2 *Let S be an $\mathbb{H}^{N\times M}$-valued slice hyperholomorphic function in $\mathbb{B}$, and such that the associated space $\mathcal{P}(S)$ is finite dimensional. Then, S admits a representation of the form* (9.1).

Proof Since the coefficients spaces are quaternionic Hilbert spaces, R_0 is a contraction in $\mathcal{P}(S)$. We proceed along the lines of [7, §4.2 p. 141] and divide the proof into a number of steps.

STEP 1: *The operator R_0 has no eigenvalues of modulus* 1.

Indeed, let $f \in \mathcal{P}(S)$ and $\lambda \in \mathbb{H}$ be such that $R_0 f = f\lambda$. Assume $|\lambda| = 1$. From

$$[R_0 f, R_0 f]_{\mathcal{P}(S)} \le [f, f]_{\mathcal{P}(S)} - f(0)^* f(0), \quad f \in \mathcal{P}(S) \tag{9.3}$$

we get

$$[f\lambda, f\lambda] \le [f, f] - f(0)^* f(0)$$

and so $f(0)=0$. Reiterating (9.3) with $R_0 f$ instead of f we get $(R_0 f)(0)=0$, and in a similar way, $(R_0^n f)(0)=0$ for $n=2,3,\ldots$. But the $(R_0^n f)(0)$ are the coefficients of the power series of f, and so $f=0$.

STEP 2: *Let κ be the number of negative squares of K_S. Then, R_0 has a κ-dimensional negative invariant subspace.*

We write in matrix form $A=R_0$ and C the point evaluation at the origin, and denote by H the matrix corresponding to the inner product in $\mathcal{P}(S)$. Thus

$$A^*HA \le H.$$

Without loss of generality we assume that A is in Jordan form (see [48, 49]), and we denote by $\mathcal{L}_+$ (resp., $\mathcal{L}_-$) the linear span of the generalized eigenvectors corresponding to eigenvalues in $\mathbb{B}$ (resp., outside the closure of $\mathbb{B}$). Since there are no eigenvalues on $\partial\mathbb{B}$, $\mathbb{H}^N$ (where $N=\dim\mathcal{P}(S)$) is spanned by $\mathcal{L}_+$ and $\mathcal{L}_-$. As in [36, Theorem 4.6.1, p. 57], one shows that

$$\dim\mathcal{L}_+ \le i_+(H) \quad \text{and} \quad \dim\mathcal{L}_- \le i_-(H),$$

where $i_+(H)$ is the number of positive eigenvalues of H (and similarly for $i_-(H)$), and by dimension argument equality holds there. Thus $\mathcal{L}_-$ is a κ-dimensional invariant subspace of A.

Let G denote the solution of the matrix equation

$$G - A^*GA = C^*C.$$

STEP 3: *Let $\mathcal{M}$ be the space corresponding to $\mathcal{L}_-$ in $\mathcal{P}(S)$, endowed with the metric defined by G. Then $\mathcal{M}$ is contractively included in $\mathcal{P}(S)$.*

Let M denote the Gram matrix of $\mathcal{M}$ in the $\mathcal{P}(S)$ inner product. We show that $M \ge P$. Indeed, in view of (9.3), the matrix M satisfies

$$A^*MA \le M - C^*C.$$

In view of (8.3), the matrix $M-P$ satisfies $A^*(M-P)A \le M-P$, or equivalently (since A is invertible)

$$M - P \le A^{-*}(M-P)A^{-1}$$

and so, for every $n \in \mathbb{N}$,

$$M - P \le \left(A^{-*}\right)^n (M-P)A^{-n}. \tag{9.4}$$

Since the S-spectrum of A is outside the closed unit ball, we have by the S-spectral radius theorem (see [15, Theorem 3.10, p. 616], [28, Theorem 4.12.6, p. 155]

$$\lim_{n\to\infty} \left\| A^{-n} \right\|^{1/n} = 0,$$

and so $\lim_{n\to\infty} \|(A^{-*})^n(P-M)A^{-n}\| = 0$. Thus entrywise

$$\lim_{n\to\infty} \left(A^{-*}\right)^n (P-M)A^{-n} = 0$$

and it follows from (9.4) that $M - P \leq 0$.

By Proposition 8.8,

$$\mathcal{M} = \mathcal{P}(B).$$

when $\mathcal{M}$ is endowed with the P metric. Furthermore,

STEP 4: *The kernel* $K_S(p,q) - K_B(p,q)$ *is positive.*

Let $k_{\mathcal{M}}(p,q)$ denote the reproducing kernel of $\mathcal{M}$ when endowed with the $\mathcal{P}(S)$ metric. Then

$$k_{\mathcal{M}}(p,q) - K_B(p,q) \geq 0$$

and

$$K_S(p,q) - k_{\mathcal{M}}(p,q) \geq 0.$$

On the other hand,

$$K_S(p,q) - K_B(p,q) = K_S(p,q) - k_{\mathcal{M}}(p,q) + k_{\mathcal{M}}(p,q) - K_B(p,q)$$

and so is positive definite.

To conclude, we apply Proposition 5.1 to

$$K_S(p,q) - K_B(p,q) = B(p) \star \left(I_N - S_0(p)S_0(q)^*\right) \star_r B(q)^*,$$

where $S(p) = B(p)^{-\star} \star S_0(p)$, to check that S_0 is a Schur function. $\square$

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