Families of moment matching based, low order approximations for linear systems[☆]

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1. Introduction

In the problem of model reduction moment matching techniques represent an efficient tool, see e.g. [1–3] for an overview for linear systems. In such techniques the (reduced order) model is obtained by constructing a lower degree rational function that approximates the original transfer function (assumed rational). The low degree rational function matches the original transfer function and its derivatives at various points in the complex plane. There are several possible (equivalent) notions of moments for a linear system. The first classical notion of moment has been given in [3], based on the series expansion of the transfer function of the linear system (see also [4–6]).

An alternative approach has been taken in [7] where the rational interpolation and tangential interpolation problems have been recast in terms of finding the projections by solving Sylvester equations. Recently in [8], a new framework for the solution to the realization problem has been proposed. The moment matching problem has been recast in terms of the Loewner matrix and solutions to Sylvester equations, with matrices constructed from tangential interpolation data. The result is a reduced order model that achieves moment matching and is minimal. More recently, in [9,10] new definitions of moments in a time-domain framework have been given. Hence another equivalent definition of moments is presented in the relation with the steady-state response (if it exists) of the system driven by a signal generator (a novel interpretation of the results in [7]). The reduced order model that achieves moment matching at ν points is a parametric model, the extra parameter being tuned such that certain properties are preserved. Based on the dual Sylvester equation, a new definition of moment dual to the previous one is obtained. The reduced order model that achieves moment matching at ν points is also a parametric one. Furthermore in [11] a connection between the different families of models is established.

In this paper we present the families of reduced order models based on the associated notions of moment. We analyze the controllability and the observability properties of the reduced order models. If the models are not minimal, we obtain systems of dimensions lower than the number of interpolation points, i.e., we consider the problem of pole–zero cancellations occurring in

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the reduced order models. We also state that, generically, the lowest dimension is half of the number of interpolation points, if the number of interpolation points is even and half plus one, if the number of interpolation points is odd, i.e., a number of poles and zeros less than, or equal to half of the number of interpolation points are canceled. In other words, we provide a selection of the free parameters that yield the solution to the pole–zero cancellation problem. To this end, we compute the parameters that help identify the model of minimal order that matches a prescribed number of moments. Furthermore, the problem of matching higher numbers of moments is studied in the time-domain setting. Thus, the series, parallel or feedback interconnection between the two reduced order models, obtained with the two latter definitions of moments, is proposed, yielding reduced order models of dimensions equal to the number of matched moments. Under some mild assumptions these models match the moments of the original systems at 2ν points. From a different perspective, this approach yields a way of splitting the moment matching problem into problems of lower dimensions, i.e., the interconnection between N models that match ν points, yields a reduced order model of dimension $N\nu$, that matches $N\nu$ moments of the original system.

The paper is organized as follows. In Section 2 we give an overview of the notion of moment for a transfer function and of the Krylov projection based reduced order models that match a prescribed number of moments, as well as a brief overview of the notion of moments in a time-domain framework. We also present the families of parameterized reduced order models that achieve moment matching. In Section 3 we analyze the controllability and observability properties and the pole–zero cancellation problem, for all the families of parameterized reduced order models that achieve moment matching i.e., find the (sets of) parameters such that pole–zero cancellations occur. The result consists of subclasses of models of orders lower than the number of matched moments (i.e., the number of chosen interpolation points). We prove that, generically, the largest number of cancellations is half of the number of matched moments. Performing all possible cancellations results in models that match a number of moments which are equal to twice their dimension. In Section 4 we compute reduced order models that match larger number of moments, by interconnecting models from different classes. The paper ends with some conclusions.

This paper is a preliminary step to develop a model reduction theory for nonlinear systems continuing the work in [10]. Preliminary results are found in [12].

Notation. $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. $\mathbb{C}^0$ is the set of complex numbers with zero real part and $\mathbb{C}^-$ denotes the set of complex numbers with negative real part. $A^* \in \mathbb{C}^{n\times m}$ denotes the transpose and complex conjugate of the matrix $A \in \mathbb{C}^{m\times n}$. If A is a real matrix, then $A^* = A^T$, where A^T is the transpose of A. $\sigma(A)$ denotes the set of eigenvalues of the matrix A and $\emptyset$ denotes the empty set.

2. Preliminaries

We consider a single-input, single-output[1] linear, time invariant system described by the equations

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$ and the associated transfer function $K : \mathbb{C} \to \mathbb{C}$,

$$K(s) = C(sI - A)^{-1}B. \tag{2}$$

When needed, we use the notation (A, B, C) to refer to a system described by (1). Throughout, we assume that (1) is controllable and observable, i.e., minimal.

2.1. The notion of moment and moment matching

In this section we recall the notion of moments of a linear system based on the associated transfer function.

Definition 1 ([3]). The 0-moment at $s_1 \in \mathbb{C}$, $s_1 \notin \sigma(A)$, of system (2) is the complex number $\eta_0(s_1) = C(s_1I - A)^{-1}B$. The k-moment at s_1 of system (2) is the complex number $\eta_k(s_1) = \frac{(-1)^k}{k!} \frac{d^k[C(sI-A)^{-1}B]}{ds^k}\Big|_{s=s_1}$, $k \geq 1$, integer. ∎

The point s_1 is called an interpolation point. The approximation problem can be formulated as follows: given system (1), the interpolation point s_1 and $k \geq 0$, find a system $(A_{red}, B_{red}, C_{red})$, where $A_{red} \in \mathbb{R}^{\nu\times\nu}$, $B_{red} \in \mathbb{R}^\nu$, $C_{red}^T \in \mathbb{R}^\nu$ with transfer function $K_{red}(s) = C_{red}(sI-A_{red})^{-1}B_{red}$, such that $\eta_k(s_1) = \hat{\eta}_k(s_1)$, for $k = 0, \ldots, \nu-1$, where $\hat{\eta}_k(s_1)$ are the moments of $K_{red}(s)$, $k = 0, \ldots, \nu - 1$. If $s_1 = \infty$, then the moments are the Markov parameters and the problem is known as partial realization. If the moments are chosen at $s_1 = 0$, then the problem is called Padé approximation, see e.g. [3] and the references therein. Alternatively one may consider moment matching at multiple interpolation points: given a system (A, B, C) and a set of interpolation points $s_1, s_2, \ldots, s_\nu \notin \sigma(A)$, find $(A_{red}, B_{red}, C_{red})$, where $A_{red} \in \mathbb{R}^{\nu\times\nu}$, $B_{red} \in \mathbb{R}^\nu$, $C_{red} \in \mathbb{R}^\nu$, with transfer function $K_{red}(s) = C_{red}(sI - A_{red})^{-1}B_{red}$, such that $K^{(j)}(s_k) = K^{(j)}_{red}(s_k)$, for $k = 1, \ldots, \nu$ and $j = 0, 1, \ldots, l$, where $K^{(j)} = \frac{d^jK(s)}{ds^j}$. Throughout the rest of the paper, without loss of generality we consider $j = 0$ and we assume that the interpolation points are not eigenvalues of A. We also assume that $\nu < n$.

2.2. Krylov projections

In this section we recall two different notions of moments based on Krylov projections. This definition allows for development of efficient numerical algorithms for the computation of reduced order models, i.e., the Arnoldi and Lanczos algorithms, see e.g. [1, 13–17] and references therein. These algorithms achieve moment matching through iterative procedures, without the computation of moments as in Definition 1.

Consider a linear system (1). Let $s_1, s_2, \ldots, s_\nu, s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu} \in \mathbb{C}$, $s_i \neq s_j$, $i \neq j$ and let $V \in \mathbb{C}^{n\times\nu}$ and $W \in \mathbb{C}^{n\times\nu}$ be, respectively

$$V = [(s_1I - A)^{-1}B \; (s_2I - A)^{-1}B \cdots (s_\nu I - A)^{-1}B], \tag{3a}$$

$$W = [(s_{\nu+1}I - A^*)^{-1}C^* \; (s_{\nu+2}I - A^*)^{-1}C^* \cdots (s_{2\nu}I - A^*)^{-1}C^*]. \tag{3b}$$

Definition 2. 1. Let $\theta \in \mathbb{C}^{1\times\nu}$, $\theta = [\theta_1 \; \theta_2 \cdots \theta_\nu]$ be such that $\theta = CV$. The moments of system (1) at $s_1, s_2, \ldots, s_\nu$ are the elements θ_i, $i = 1, \ldots, \nu$. We call V the right Krylov projection matrix.

2. Let $\vartheta = [\vartheta_1 \; \vartheta_2 \cdots \vartheta_\nu]^T \in \mathbb{C}^\nu$ be such that $\vartheta = W^*B$. The moments of system (1) at $s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu}$ are the elements ϑ_i, $i = \nu + 1, \ldots, 2\nu$. We call W the left Krylov projection. ∎

[1] The same arguments hold for multiple-input–multiple-output systems.

Using this definition, an interpolation problem is solved. The following result presents the solution of the interpolation problem as families of reduced order models (and their duals) that achieve moment matching at ν interpolation points.

Theorem 1 *([3]). The following statements hold.*

1. *Let θ be the matrix containing the 0-moments of (1) at $\{s_1, s_2, \ldots, s_\nu\}$. Let $\xi(t) \in \mathbb{R}^\nu$ and consider a linear model defined by the equations*

$$\Sigma_{\mathbf{W}} : \begin{cases} \dot{\xi} = \mathbf{W}^* AV\xi + \mathbf{W}^* Bu, \\ \eta = CV\xi, \end{cases} \tag{4}$$

where V is given by relation (3a) and $\mathbf{W} \in \mathbb{C}^{n\times\nu}$ is a matrix satisfying $\mathbf{W}^*V = I$. Let $\hat{\theta} \in \mathbb{C}^{1\times\nu}$ be the moments of (4) as in Definition 2. Then $\Sigma_{\mathbf{W}}$ as in (4) defines a class of reduced order models of (1), parameterized in $\mathbf{W}$, that achieve moment matching at $s_1, s_2, \ldots, s_\nu \in \mathbb{C}$, i.e., $\theta = \hat{\theta}$.

2. *Let ϑ be the vector containing the 0-moments of (1) at $\{s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu}\}$ and. Let $\xi(t) \in \mathbb{R}^\nu$ and consider a linear model defined by the equations*

$$\Sigma_{\mathbf{V}} : \begin{cases} \dot{\xi} = W^* A\mathbf{V}\xi + W^* Bu, \\ \eta = C\mathbf{V}\xi, \end{cases} \tag{5}$$

where W is given by relation (3b) and $\mathbf{V} \in \mathbb{C}^n$ is a matrix satisfying $W^*\mathbf{V} = I$. Let $\hat{\vartheta} \in \mathbb{C}^\nu$ be the moments of (5) as in Definition 2. Then $\Sigma_{\mathbf{V}}$ as in (5) defines a class of reduced order models of (1), parameterized in $\mathbf{V}$, that achieve moment matching at $s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu} \in \mathbb{C}$, i.e., $\vartheta = \hat{\vartheta}$. ■

Note that not all reduced order models $\Sigma_{\mathbf{W}}$ (or $\Sigma_{\mathbf{V}}$, respectively) preserve properties such as stability, passivity, structure etc. New results show that the preservation of such properties depends on the choice of interpolation points $\{s_1, s_2, \ldots, s_\nu\}$ (or $\{s_{\nu+1}, s_{\nu+2}, \ldots, s_{2\nu}\}$, respectively), see [18–22].

The definition of the matrices V and W allows for the construction of projections matrices that, used for model reduction, lead to reduced order models that achieve matching at 2ν points. In other words, there exists a parameter $\mathbf{W} = \widetilde{W}$ such that from the class of models $\Sigma_{\mathbf{W}}$ there exists a model $\Sigma_{\widetilde{W}}$ of order ν, that matches 2ν moments. Dually, there exists a parameter $\mathbf{V} = \widetilde{V}$ such that from the class of models $\Sigma_{\mathbf{V}}$ there exists a model $\Sigma_{\widetilde{V}}$ of order ν, that matches 2ν moments. Let $s_1, s_2, \ldots, s_\nu, \ldots, s_{2\nu} \in \mathbb{C}$, $s_i \neq s_j$, $i \neq j$, with $i, j = 1, \ldots, 2\nu$ and assume V^*W and W^*V are invertible, respectively, with V as in (3a) and W as in (3b). Let $\widetilde{V} \in \mathbb{C}^{n\times\nu}$ and $\widetilde{W} \in \mathbb{C}^{n\times\nu}$ be, respectively

$$\widetilde{W} = W(V^*W)^{-1}, \tag{6a}$$

$$\widetilde{V} = V(W^*V)^{-1}. \tag{6b}$$

Note that $\widetilde{W}^*V = I$ and $W^*\widetilde{V} = I$, respectively.

Theorem 2 *([3]). The following statements hold.*

1. *Assume $\mathbf{W} = \widetilde{W}$ and let $\xi(t) \in \mathbb{R}^\nu$. If $\hat{\theta} \in \mathbb{C}^\nu$ are the 0-moments of $\Sigma_{\widetilde{W}}$ at $\{s_1, \ldots, s_\nu\}$ and $\hat{\vartheta} \in \mathbb{C}^\nu$ are the moments of $\Sigma_{\widetilde{W}}$ at $\{s_{\nu+1}, \ldots, s_{2\nu}\}$, then $\Sigma_{\widetilde{W}} \in \Sigma_{\mathbf{W}}$ is a reduced order model of (1) achieving moment matching at $\{s_1, \ldots, s_{2\nu}\}$, i.e., $\theta = \hat{\theta}$ and $\vartheta = \hat{\vartheta}$.*
2. *Assume $\mathbf{V} = \widetilde{V}$ and let $\xi(t) \in \mathbb{R}^\nu$. If $\hat{\theta} \in \mathbb{C}^\nu$ are the 0-moments of $\Sigma_{\widetilde{V}}$ at $\{s_1, \ldots, s_\nu\}$ and $\hat{\vartheta} \in \mathbb{C}^\nu$ are the moments of $\Sigma_{\widetilde{V}}$ at $\{s_{\nu+1}, \ldots, s_{2\nu}\}$, then $\Sigma_{\widetilde{V}} \in \Sigma_{\mathbf{V}}$ is a reduced order model of (1) achieving moment matching at $\{s_1, \ldots, s_{2\nu}\}$, i.e., $\theta = \hat{\theta}$ and $\vartheta = \hat{\vartheta}$.* ■

2.3. Time-domain moment matching

In this section we give a brief overview of a notion of moment in a time domain setting, see [10] for a more detailed analysis. Based on this notion families of parameterized reduced order models are developed. The free parameters do not depend on the choice of interpolation points and can be used for enforcing additional properties.

Consider the linear system (1) and let the matrices $S \in \mathbb{R}^{\nu\times\nu}$, $L \in \mathbb{R}^{1\times\nu}$ and $Q \in \mathbb{R}^{\nu\times\nu}$, $R \in \mathbb{R}^\nu$ be such that the pair (L, S) is observable and the pair (Q, R) is controllable, respectively. Consider the Sylvester equation

$$A\Pi + BL = \Pi S, \tag{7}$$

in the unknown $\Pi \in \mathbb{C}^{n\times\nu}$ and its dual

$$Q\Upsilon = \Upsilon A + RC, \tag{8}$$

in the unknown $\Upsilon \in \mathbb{C}^{\nu\times n}$. Assume that $\sigma(A) \cap \sigma(S) = \emptyset$. Since Σ is minimal, the Sylvester equation (7) has a unique solution Π and rank $\Pi = \nu$. Assuming $\sigma(A) \cap \sigma(Q) = \emptyset$, then Eq. (8) has a unique solution Υ and rank $\Upsilon = \nu$. (See e.g., [23]).

Definition 3. 1. Let $\phi = [\phi_1\ \phi_2 \cdots \phi_\nu] \in \mathbb{C}^{1\times\nu}$ be such that

$$\phi = C\Pi. \tag{9}$$

We call the moments of system (1) at $\sigma(S)$ the elements ϕ_i, $i = 1, \ldots, \nu$. The interpolation points are the eigenvalues of S, i.e., $\{s_1, s_2, \ldots, s_\nu\} = \sigma(S)$.

2. Let $\varphi = [\varphi_1\ \varphi_2 \cdots \varphi_\nu]^T \in \mathbb{C}^\nu$ be such that

$$\varphi = \Upsilon B. \tag{10}$$

We call the moments of system (1) at $\sigma(Q)$ the elements φ_i, $i = 1, \ldots, \nu$. The interpolation points are the eigenvalues of Q, i.e., $\{s_1, s_2, \ldots, s_\nu\} = \sigma(Q)$. ■

Based on Definition 3, we define a family of parameterized models of order ν that achieve moment matching at the interpolation points $\{s_1, \ldots, s_\nu\} = \sigma(S)$.

Theorem 3 *([10,11]).*

1. *Let the pair (L, S) be observable and assume $\sigma(A) \cap \sigma(S) = \emptyset$. Let $\xi(t) \in \mathbb{R}^\nu$ and consider the family of linear models*

$$\Sigma_G : \begin{cases} \dot{\xi} = (S - GL)\xi + Gu, \\ \eta = C\Pi\xi, \end{cases} \tag{11}$$

parameterized in $G \in \mathbb{C}^\nu$, where Π is the unique solution of (7). Assume $\sigma(S - GL) \cap \sigma(S) = \emptyset$. Let $\hat{\phi} \in \mathbb{C}^{1\times\nu}$ be the moments of (11) at $\sigma(S)$. Then (11) describes a family of reduced order models of (1), parameterized in G and achieving moment matching at $\sigma(S)$, i.e., $\phi = \hat{\phi}$.

2. *Let the pair (Q, R) be controllable and assume $\sigma(A) \cap \sigma(Q) = \emptyset$. Let $\xi(t) \in \mathbb{R}^\nu$ and consider the family of linear models*

$$\Sigma_H : \begin{cases} \dot{\xi} = (Q - RH)\xi + \Upsilon Bu, \\ \eta = H\xi, \end{cases} \tag{12}$$

parameterized in $H \in \mathbb{R}^{1\times\nu}$, where Υ is the unique solution of (8). Assume $\sigma(Q - RH) \cap \sigma(Q) = \emptyset$. Let $\hat{\varphi} \in \mathbb{C}^{1\times\nu}$ be the moments of (12) at $\sigma(Q)$. Then (12) describes a family of reduced order models of (1), parameterized in H and achieving moment matching at $\sigma(Q)$, i.e., $\varphi = \hat{\varphi}$. ■

Note that the moments as in Definition 1 are equivalent to the notions in Definition 3. Selecting (L, S) and (Q, R) in canonical forms, easy computations yield $[\eta(s_1) \cdots \eta(s_\nu)] = \phi = \varphi$.

The MIMO case. Consider a MIMO system of the form (1), with input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, i.e., $B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{p\times n}$

and the transfer function $K(s) \in \mathbb{C}^{p\times m}$. Let $S \in \mathbb{C}^{\nu\times\nu}$ and $L = [l_1\ l_2 \cdots l_\nu] \in \mathbb{C}^{m\times\nu}, l_i \in \mathbb{C}^m, i = 1, \ldots, \nu$, be such that the pair (L, S) is observable. Let $\Pi \in \mathbb{C}^{n\times\nu}$ be the unique solution of the Sylvester equation (7). Simple computations yield that the moments $\eta(s_i) = K(s_i)l_i, \eta(s_i) \in \mathbb{C}^p, i = 1, \ldots, \nu$ of system (1) at $\{s_1, \ldots, s_\nu\} = \sigma(S)$ are in one-to-one relation with $C\Pi$. Consider the following system

$$\begin{aligned} \dot{\xi} &= F\xi + Gu, \\ \psi &= H\xi, \end{aligned} \tag{13}$$

with $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}^p$, $G \in \mathbb{C}^{\nu\times m}$ and $H \in \mathbb{C}^{p\times\nu}$. The model reduction problem for MIMO systems boils down to finding a ν-th order model described by Eqs. (13) which satisfies the conditions

$$K(s_i)l_i = \widehat{K}(s_i)l_i, \quad i = 1, \ldots, \nu, \tag{14}$$

where $\widehat{K}(s) = H(sI - F)^{-1}G$ is the transfer function of (13). The relations (14) are called the right tangential interpolation conditions, see [24]. It immediately follows that the solution to this problem is provided by a direct application of Theorem 3, i.e., a class of reduced order MIMO models that achieve moment matching in the sense of satisfying the tangential interpolation conditions (14) is given by $\Sigma_G = (S - GL, G, C\Pi)$ as in (11).

Similarly, we may define the left tangential interpolation problem and its solution. To this end, let $\mathcal{Q} \in \mathbb{C}^{\nu\times\nu}$ and $\mathcal{R} = [r_1^* \cdots r_\nu^*]^* \in \mathbb{C}^{\nu\times p}, r_i \in \mathbb{C}^{1\times p}, i = 1, \ldots, \nu$, be such that the pair $(\mathcal{Q}, \mathcal{R})$ is controllable. Let $\Upsilon \in \mathbb{C}^{\nu\times n}$ be the unique solution of (8). Hence the moments $\eta(s_i) = r_iK(s_i), \eta(s_i) \in \mathbb{C}^{1\times m}, i = 1, \ldots, \nu$, of system (1) at $\{s_1, \ldots, s_\nu\} = \sigma(\mathcal{Q})$ are in one-to-one relation with ΥB. The model reduction problem boils down to finding a ν-th order model described by Eqs. (13) which satisfy the conditions

$$r_iK(s_i) = r_i\widehat{K}(s_i), \quad i = 1, \ldots, \nu. \tag{15}$$

The relations (15) are called the left tangential interpolation conditions, see [24] and the solution to this problem is provided by a direct application of Theorem 3, i.e., a class of reduced order MIMO models that achieve moment matching in the sense of satisfying the tangential interpolation conditions (15) is given by $\Sigma_H = (\mathcal{Q} - \mathcal{R}H, \Upsilon B, H)$ as in (12).

Note finally that also in the MIMO case the models are parameterized in L and $\mathcal{R}$, respectively. Their choice is important in establishing appropriate directions for interpolation. Throughout the rest of the paper we discuss the SISO case, i.e., $m = p = 1$, the results being easily extended to tangential interpolation for MIMO systems. However, when necessary, we make specific remarks about the latter case.

2.4. On the equivalence of various families of reduced order models

The *equivalence* between the families of reduced order models described by Eqs. (11), (12), (4) and (5) is now established, see [11]. In other words, there exist parameters G and H, respectively, which provide a subclass of models from the classes $\Sigma_{\mathbf{W}}$ or $\Sigma_{\mathbf{V}}$.

First we establish relations between the projections V and W and the solutions of the Sylvester equations Π and Υ, respectively.

Lemma 1 (*[11]*). *The following statements hold.*

1. *Consider the matrix Π, solution of the Sylvester equation (7) and the projector V defined by Eq. (3a). There exists a square, non-singular, matrix $T \in \mathbb{C}^{\nu\times\nu}$ such that $\Pi = VT$.*
2. *Consider the matrix Υ, solution of the Sylvester equation (8) and the projector W defined by Eq. (3b). There exists a square, non-singular, matrix $T \in \mathbb{C}^{\nu\times\nu}$ such that $\Upsilon = TW$.* ■

$$\begin{tikzcd}[row sep=large, column sep=large]
\Sigma_G \arrow[r, shift left, "\exists"] \arrow[d, leftrightarrow, "\exists\,!"'] & \Sigma_{\mathbf{W}} \arrow[l, shift left, "\exists\,!"] \\
\Sigma_H \arrow[r, shift left, "\exists"] & \Sigma_{\mathbf{V}} \arrow[l, shift left, "\exists\,!"]
\end{tikzcd}$$

Fig. 1. Graphical illustration of Theorem 4.

Theorem 4 (*[11]*). *Consider the families of reduced order models $\Sigma_{\mathbf{W}}$, $\Sigma_{\mathbf{V}}$, Σ_G and Σ_H described by Eqs. (4), (5), (11) and (12), respectively. Then the following statements hold.*

(a) *For any $\mathbf{W}$ there exists a (unique) G such that $\Sigma_G = \Sigma_{\mathbf{W}}$ and $\sigma(S) \cap \sigma(S - GL) = \emptyset$.*
(b) *For any G such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$ there exists a $\mathbf{W}$ such that $\Sigma_G = \Sigma_{\mathbf{W}}$.*
(c) *For any $\mathbf{V}$ there exists a (unique) H such that $\Sigma_{\mathbf{V}} = \Sigma_H$ and $\sigma(Q) \cap \sigma(Q - RH) = \emptyset$.*
(d) *For any H such that $\sigma(Q) \cap \sigma(Q - RH) = \emptyset$ there exists a $\mathbf{V}$ such that $\Sigma_V = \Sigma_H$.*
(e) *Assume $Q = S$. For any H such that $\sigma(Q) \cap \sigma(Q - RH) = \emptyset$ there exists a unique G such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$ and $\Sigma_G = \Sigma_H$ and vice versa.* ■

An illustration of the results expressed by Theorem 4 is given in Fig. 1. Therein, the symbol $\exists$ ($\exists\,!$, respectively) denotes that for a given model in the family from where the arrow originates there exists (exists and is unique, respectively) a model in the family where the arrow terminates.

The parameters G and H can be selected, respectively, such that certain properties of the approximating model, e.g., stability, passivity, prescribed relative degree, port-Hamiltonian structure, etc., are preserved/enforced, see e.g. [10,25,26]. The selections of G and H are independent of the interpolation points used for moment matching, i.e., the choices of G and H do not depend on the definition of L and S or Q and R, respectively.

3. Minimality analysis of moment matching models

The selections of the free parameters help to identify (sub-classes of) reduced order models that achieve moment matching and satisfy desired properties. Following these arguments, we analyze the controllability and observability properties of the families of models. As a result we determine the subclasses of models of order less than the number of matched moments. Furthermore, we determine the number of maximal pole–zero cancellations possible, i.e., we compute the model of the lowest order that achieves a prescribed number of moments. Preliminary results are found in [12].

Throughout the rest of the paper we make the following standing assumption.

Assumption 1. The pair (L, S) is observable and the pair (Q, R) is controllable. $\sigma(S) \cap \sigma(A) = \emptyset$ and $\sigma(Q) \cap \sigma(A) = \emptyset$. Furthermore, G is such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$ and H is such that $\sigma(Q) \cap \sigma(Q - RH) = \emptyset$.

Assumption 1 is not restrictive, i.e., the freedom of choosing the interpolation points allows for the controllability and observability assumptions to be made. Furthermore, the assumptions on the spectra of the aforementioned matrices only ensure that the interpolation points are not among the poles of the given linear system and the reduced order models, respectively, i.e., ensure the moments are well defined.

We seek G and H that provide models from the classes Σ_G and Σ_H, respectively, of orders lower than ν, i.e., the number of matched moments. Furthermore, we compute G and H that yield the (unique) lowest order models, i.e., half the number of matched moments/interpolation points, from the classes Σ_G and Σ_H. The results of this subsection solve the problem of matching a number of moments, equal to double the order of the reduced model, from a pole–zero cancellation point of view.

Since we assume that $\sigma(S)\cap\sigma(S-GL)=\emptyset$ the pair $(S-GL, G)$ is controllable. However, the pair $(C\Pi, S-GL)$ is not necessarily observable, i.e., Σ_G is not necessarily minimal. However, if G is such that Σ_G has relative degree ν, then Σ_G is observable. A similar argument follows for the Σ_H case, i.e., the pair $(H, Q-RH)$ is observable, yet the pair $(Q-RH, \Upsilon B)$ is not necessarily controllable, i.e. Σ_H is not necessarily minimal. If the models Σ_G or Σ_H, are not minimal, respectively, then a number of poles and zeros can be canceled, yielding subclasses of parameterized reduced order models of orders less than ν.

Theorem 5. *Let Σ_G and Σ_H be reduced order models that match ν moments of the system (1), respectively. Let $K_G(s)=C\Pi(sI-S+GL)^{-1}G$ be the transfer function of Σ_G and let $K_H(s)=H(sI-Q+RH)^{-1}\Upsilon B$ be the transfer function of Σ_H. Assume Σ_G and Σ_H are not minimal. Let $k\in\mathbb{N}$ and let $z=[z_1\cdots z_k]^T\in\mathbb{C}^k, z_i\neq z_j, i=1,\dots,k, j=1,\dots,k$ be, such that $z_i\notin\sigma(S)$ and $z_i\notin\sigma(Q)$. Then the following statements hold.*

(G1) *Assume $\nu=2k$. Let*

$$M(z)=\begin{bmatrix}L(z_1I-S)^{-1}\\ L(z_2I-S)^{-1}\\ \vdots\\ L(z_kI-S)^{-1}\end{bmatrix}\in\mathbb{C}^{k\times 2k},$$
$$N(z)=\begin{bmatrix}C\Pi(z_1I-S)^{-1}\\ C\Pi(z_2I-S)^{-1}\\ \vdots\\ C\Pi(z_kI-S)^{-1}\end{bmatrix}\in\mathbb{C}^{k\times 2k},\tag{16}$$
$$T(z)=\begin{bmatrix}N(z)\\ M(z)\end{bmatrix}\in\mathbb{C}^{2k\times 2k}.$$

Then there exists a unique $G=T^{-1}(z)\begin{bmatrix}-\P\\0\end{bmatrix}$, with $\P=[1\cdots 1]^T$, such that the numbers z_i are both zeros and poles of $K_G(s)$, i.e., k pole–zero cancellations occur in $K_G(s)$. Furthermore, setting $[\kappa_1\ \kappa_2]=C\Pi T^{-1}(z)$, the cancellations yield a model of minimal order k given by

$$\hat{K}(s)=-\frac{\sum_{i=1}^{k}\kappa_{1i}\prod_{j\neq i}(s-z_j)}{\prod_{j=1}^{k}(s-z_i)+\sum_{i=1}^{k}\kappa_{2i}\prod_{j\neq i}^{k}(s-z_j)},\tag{17}$$

where $\kappa_i=[\kappa_{i1}\ \kappa_{i2}\cdots\kappa_{ik}]$, $i=1,2$.

(H1) *Assume $\nu=2k$. Let*

$$\begin{aligned}M(z)&=\left[(z_1I-Q)^{-1}R\ (z_2I-Q)^{-1}R\right.\\ &\quad\left.\cdots(z_kI-Q)^{-1}R\right]\in\mathbb{C}^{2k\times k},\\ N(z)&=\left[(z_1I-Q)^{-1}\Upsilon B\ (z_2I-Q)^{-1}\Upsilon B\right.\\ &\quad\left.\cdots(z_kI-Q)^{-1}\Upsilon B\right]\in\mathbb{C}^{2k\times k},\\ T(z)&=[N(z)\ M(z)]\in\mathbb{C}^{2k\times 2k}.\end{aligned}\tag{18}$$

Then there exists a unique $H=[-\P\ \ 0]T^{-1}(z)$, with $\P=[1\cdots 1]$, such that the numbers z_i are both zeros and poles of $K_H(s)$, i.e., k pole–zero cancellations occur in $K_H(s)$. Furthermore, letting $\begin{bmatrix}\kappa_1\\ \kappa_2\end{bmatrix}=T^{-1}(z)\Upsilon B$, the cancellations yield a model of minimal order k given by $\hat{K}(s)$, as in (17).

(G2) *Assume $\nu=2k+1$. Then there exists a parameterized matrix $G(\alpha)\in\mathbb{C}^{2k+1}$, $\alpha\in\mathbb{C}$, such that the numbers z_i are both zeros and poles of $K_G(s)$, i.e., k pole–zero cancellations occur in $K_G(s)$, yielding a subclass of models $\Sigma_{G(\alpha)}$ of minimal order $k+1$, described by $K_{G(\alpha)}(s)$ as in (17), with $[\kappa_1(\alpha)\ \kappa_2(\alpha)]=C\Pi(\alpha)T^{-1}(z,\alpha)$, $T(z,\alpha)\in\mathbb{C}^{(2k+2)\times(2k+2)}$.*

(H2) *Assume $\nu=2k+1$. Then there exists a parameterized matrix $H(\alpha)=\left[H'(\alpha)\ \alpha\right]$, $\alpha\in\mathbb{C}$, $H'(\alpha)\in\mathbb{C}^{1\times 2k}$, such that the numbers z_i are both zeros and poles of $K_H(s)$, i.e., k pole–zero cancellations occur in $K_H(s)$, yielding a subclass of models $\Sigma_{H(\alpha)}$ of minimal order $k+1$, described by $K_{H(\alpha)}(s)$ as in (17), with $\begin{bmatrix}\kappa_1(\alpha)\\ \kappa_2(\alpha)\end{bmatrix}=T^{-1}(z,\alpha)\Upsilon B$, where $T(z,\alpha)\in\mathbb{C}^{(2k+2)\times(2k+2)}$.* ■

Proof of G1. Let Σ_G be a model of order $\nu=2k$, non-minimal. Then, according to Lemma A.1 we write

$$K_G(s)=C\Pi(sI-S+GL)^{-1}G=\frac{C\Pi(sI-S)^{-1}G}{1+L(sI-S)^{-1}G}.$$

To this end, G should be such that z_i, $i=1,\dots,k$ are zeros and poles of $K_G(s)$, i.e.,

$$L(z_iI-S)^{-1}G=-1,\quad i=1,\dots,k,$$
$$C\Pi(z_iI-S)^{-1}G=0,\quad i=1,\dots,k.$$

Hence by (16), $T(z)G=\begin{bmatrix}-\P\\0\end{bmatrix}$. By Lemma A.3 we have that $T(z)$ is invertible, hence $G=T^{-1}(z)\begin{bmatrix}-\P\\0\end{bmatrix}$. This is the unique G that yields k pole–zero cancellations. Applying the coordinate transformation $\zeta=[\zeta_1^T\ \zeta_2^T]^T=T(z)\xi$ and using the relations (A.2), Σ_G is described by the normal form

$$\dot{\zeta}_1=Z\zeta_1-\P u,$$
$$\dot{\zeta}_2=Z\zeta_2-\P\eta,$$
$$\eta=\kappa_1\zeta_1+\kappa_2\zeta_2,$$

where $Z=\mathrm{diag}\{z_1,\dots,z_k\}$ and $[\kappa_1\ \kappa_2]=C\Pi T^{-1}(z)$. Note that z_i are invariant zeros for (19). By further simple computations, $K_G(s)$ becomes $\hat{K}(s)$ of degree k. Note that z_i are invariant zeros of Σ_G. Furthermore, by the construction of $T(z)$ and the Kronecker–Capelli theorem, no further cancellations are possible.

Proof of G2. Let $\nu=2k+1$. Consider the matrix $T(z)\in\mathbb{C}^{2k\times(2k+1)}$ as in (16). Let $T'(z)\in\mathbb{C}^{2k\times 2k}$ and $t(z)\in\mathbb{C}^{2k}$ be such that $T(z)=[T'(z)\ t(z)]$. By construction, $T(z)G=\begin{bmatrix}-\P\\0\end{bmatrix}$. Denoting $G=[G'^T\ \alpha]^T$ we have $T'(z)G'+t(z)\alpha=\begin{bmatrix}-\P\\0\end{bmatrix}$. Noting that by Remark A.1, we assume, without loss of generality, that $T'(z)$ is invertible. This yields $G(\alpha)=\begin{bmatrix}T'^{-1}(z)\left(\begin{bmatrix}-\P\\0\end{bmatrix}-t(z)\alpha\right)\\ \alpha\end{bmatrix}$.

In order to determine $K_{G(\alpha)}(s)$ of order $k+1$ we consider the augmented problem of matching at $2k+2$ points. Let $\beta\in\mathbb{C}$ be determined by the parameterization of $G(\alpha)$ and $\bar{S}(\beta)=\mathrm{diag}\{S,\beta\}\in\mathbb{C}^{(2k+2)\times(2k+2)}$, $\bar{L}=[L\ 1]\in\mathbb{C}^{1\times(2k+2)}$. Note that since the pair (L,S) is observable, the pair $(\bar{L},\bar{S})$ is observable, too. Further, let $\bar{\Pi}(\beta)=[\Pi\ p(\beta)]\in\mathbb{C}^{n\times(2k+2)}$ satisfy the Sylvester equation $A\bar{\Pi}(\beta)+B\bar{L}=\bar{\Pi}(\beta)\bar{S}$. Then the moments of the given system at $\sigma(\bar{S})=\sigma(S)\cup\{\beta\}$ are $C\bar{\Pi}(\beta)=[C\Pi\ \eta_\beta]$, $\eta_\beta=Cp(\beta)=C(\beta I-A)^{-1}B$. Let $\bar{G}=[G^T\ 0]^T$. The class of reduced order models that match $2k+2$ moments at $\sigma(\bar{S})$ are

$$\bar{S}-\bar{G}\bar{L}=\mathrm{diag}\{S-GL,\beta\},\qquad \bar{G},\ \bar{H}=[C\Pi\ \eta_\beta].\tag{19}$$

Note that $\bar{K}(s) = \bar{H}(sI - \bar{S} - \bar{G}\bar{L})^{-1}\bar{G} = K_G(s)$. Let z_{k+1} be the additional, "dummy" pole/zero to be canceled, i.e., $\bar{K}(s) = \hat{K}(s)\frac{s-z_{k+1}}{s-z_{k+1}}$. Construct $\bar{T}(z, z_{k+1}, \beta)$ as in (16). Hence $\bar{G} = T^{-1}(z, z_{k+1}, \beta) \begin{bmatrix} -\mathbb{1}_{k+1} \\ 0_{k+1} \end{bmatrix}$. It follows from the even case that $\bar{T}(z, z_{k+1}, \beta)$ can be used as a coordinate transformation to compute the normal form of (19) with the invariant zeros z and z_{k+1}. Hence, performing $k+1$ cancellations (actually k, since z_{k+1} is canceled by default) yields $\hat{K}(s)$ described by (17), with κ_1 and κ_2 replaced by $[\bar{\kappa}_1(\beta), \ \bar{\kappa}_2(\beta)] = C\bar{\Pi}(\beta)T^{-1}(z, z_{k+1}, \beta)$. To complete the proof of the claim, note that from the matching condition $\eta_\beta = C\Pi(\beta I - S)^{-1}G(\alpha)$, β depends on the free parameter α, i.e., $\beta = \beta(\alpha)$ and so $\kappa_i(\beta) \to \kappa_i(\alpha)$, $i = 1, 2$ and $T(z, z_{k+1}, \beta) \to T(z, \alpha)$.

The proofs of statements (H1) and (H2) follow the same arguments, hence they are omitted. □

According to [10, Proposition 1], system Σ_G parameterizes all ν-th order models that match the moments of (1) at $\sigma(S)$. Hence, there exists a unique G with at most $\nu/2$ degrees of freedom allowing $\nu/2$ cancellations of poles and zeros. Above this number, information about the ν moments is lost, resulting in loss of matching. If ν is even, after $\nu/2$ cancellations, we get a unique model of minimal order $\nu/2$, from the family Σ_G that matches ν moments. If $\nu = 2k+1$, after k cancellations we obtain a family of parameterized models of order $k+1$ with one degree of freedom, that match $\nu = 2k+1$ moments. A similar argument follows for the Σ_H model. The result in Theorem 5 for $s_i = 0, i = 1, \dots, 2\nu$ is in accordance with the results from the Padé approximation theory, see e.g., [27].

Remark 1. From a practical point of view it is desired to have transfer functions with real coefficients. Assume $K_G(s)$ has real coefficients. If $z_i \in \mathbb{R}$, $i = 1, \dots, k$, employing Theorem 5, means that we cancel k zeros and poles, hence $\hat{K}(s)$ as in (17) has real coefficients, too. If some $z_i \in \mathbb{C} - \mathbb{R}$, since $K_G(s)$ has real coefficients, the complex z_i's are in complex conjugate pairs. Hence, by (17), $\hat{K}(s)$ has real coefficients, too. Similar arguments hold for Σ_H. ■

Remark 2. Consider the case in which $\nu \geq n$. Assume $\nu = n + \mu$. By Theorem 3 there exists a class of parameterized models $\Sigma_G = (S - GL, G, C\Pi)$ that match the $n+\mu$ moments $C\Pi$ at $\sigma(S)$. Finding the set of matrices G that allow for μ pole–zero cancellations yields that the system Σ belongs to a subclass of models of order n that match $n+\mu$ moments. Furthermore, if $\mu = \nu$, employing Theorem 5, yields that the system (1) is the unique n-th order model (from the class of models that achieve moment matching) that matches $2n$ moments given by $C\Pi$.

If $\nu = n$, then, by Assumption 1, the unique solution $\Pi \in \mathbb{C}^{n \times n}$ of (7) is invertible. Hence (7) becomes $\Pi^{-1}A\Pi = S - \Pi^{-1}BL$. Denoting by K_G the transfer function of any model from the class Σ_G, yields that for $G = \Pi^{-1}B \Rightarrow K(s) = K_{\Pi^{-1}B}(s)$, i.e., system (1) is a model from the class Σ_G for a specific choice of G. Similar arguments follows for the Σ_H class of models. ■

Remark 3. Unfortunately, the results of Theorem 5 are not generally applicable to the multiple-input, multiple-output case. However, for very conservative cases, such as $k \geq 2p$, where p is the number of outputs, there is the possibility of finding a G as in Theorem 5, provided some rank constraints are met. However, for instance, for the case $p > 2k$ the problem does not even have a solution. In this case, in order to find G such that the reduced order model is minimal, one should follow a classic observable decomposition of the system and retain the controllable and observable part of the realization.

Example 1. Consider the reduced order model from [10, Example 1], i.e.,

$$C\Pi = [\eta_0 \ \eta_1 \ \eta_2], \qquad L = [1\ 0\ 0], \qquad S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

with $\eta_0^2 + \eta_1^2 + \eta_2^2 \neq 0$, where η_0, η_1 and η_2 are the given zero, first and second order moments at zero, respectively. All parameters G such that the model has relative degree one, are given by

$$G = \gamma \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} + \delta \begin{bmatrix} \eta_1 \\ \eta_2 - \eta_0 \\ -\eta_1 \end{bmatrix},$$

with $\gamma \neq 0$ and $\delta \in \mathbb{R}$. 0 is not a pole of the model if and only if $\gamma \notin \left\{ -\frac{\delta\eta_1}{\eta_0}, \ \frac{\delta\eta_1}{\eta_2}, \ \frac{\delta(\eta_0 - \eta_2)}{\eta_1} \right\}$. The transfer function of the reduced order model is

$$K_{\gamma\delta}(s) = \frac{(\eta_0^2 + \eta_1^2 + \eta_2^2)s^2 + (\eta_0\eta_1 + \eta_1\eta_2)\gamma - (\eta_0^2 + \eta_1^2\delta)s - \eta_0\eta_1\gamma + \eta_0\eta_2\gamma}{s^3 + (\gamma\eta_0 + \delta\eta_1)s + \gamma\eta_2 - \delta\eta_1}.$$

If $\delta = -\frac{a_2\gamma^2 + a_1\gamma + a_0}{b_1\gamma + b_0}$, with

$$a_0 = \eta_0^3,$$
$$a_1 = -\eta_1^3\eta_2 + 3\eta_0\eta_1\eta_2^2 + \eta_0^2\eta_1\eta_2 + \eta_0^2\eta_1\eta_2 + \eta_0\eta_1^3 + 2\eta_1\eta_0^3,$$
$$a_2 = \eta_1^4\eta_0 + \eta_0^2\eta_1^2\eta_2 + \eta_2^5 + 2\eta_1^2\eta_2^3 + \eta_0^2\eta_2^3 + \eta_1^4\eta_2 + \eta_0^3\eta_1^2 + \eta_0\eta_1^2\eta_2^2,$$
$$b_0 = \eta_1^4 + \eta_0^2\eta_2^2 + \eta_0^2\eta_1^2 - \eta_0^3\eta_2 - 2\eta_0\eta_1^2\eta_2,$$
$$b_1 = \eta_1^5 - \eta_2^3\eta_1\eta_0 - \eta_0^3\eta_1\eta_2 + \eta_1^3\eta_2^2 + \eta_0^2\eta_1^3,$$

then a pole and a zero are canceled, and a second order model is obtained, with the transfer function given in Box I. Since 0 is not a pole of the model, then $K_\gamma(s)$ characterizes the family of lowest order models that match the moments $C\Pi$. Further canceling is possible, i.e., for $\gamma = \frac{-1}{\eta_0^2+\eta_1^2+\eta_2^2}\frac{\eta_0^2}{\eta_1}$, we get $\hat{K}(s) = \frac{-\eta_0^2}{\eta_1 s - \eta_0}$, that has no information about η_2, hence matching is not possible. For this value of γ the condition that the interpolation points are not poles of the reduced order models is not satisfied. ■

Example 2. The result of Theorem 5 is generically true, however there are *other* cases in which there exist systems of order less than k that match, say $\nu = 2k$ moments.

Consider the second order transfer function

$$K(s) = \frac{as + b}{s^2 + cs + d}.$$

We want to find the parameters $a, b, c, d \in \mathbb{R}$, such that $K(0) = \eta_0$ and $K(1) = \eta_0$. Indeed $K(s)$ matches the moments η_0 at 0 and 1, if and only if $a = \eta_0(1+c)$, $b = \eta_0 d$, $d \neq 0$ and $c + d \neq -1$, i.e.,

$$K(s) = \eta_0 \frac{(1+c)s + d}{s^2 + cs + d}. \tag{21}$$

However, there exists a constant function (i.e., a transfer function of order 0), $\hat{K}(s) = \eta_0$, that matches the moments η_0 at $s = 0$ and $s = 1$ of $K(s)$ as in (21) (see also Fig. 2).

In general, there exists a function of order 0, i.e., a constant that matches l moments of a rational function $\frac{r(s)}{p(s)}$, where $r(s)$ and $p(s)$ are polynomials in s, i.e., the interpolation points are the zeros of the error function. This is proven by noting that the equation

$$\frac{r(s_j)}{p(s_j)} = \eta_0, \quad j = 1, \dots, l,$$

has l solutions s_j, which are the zeros of the polynomial equation $r(s) - \eta_0 p(s) = 0$. Hence, the constant function η_0 matches l moments of the rational function r/p.

$$K_\gamma(s) = \frac{-\gamma(\eta_1^2(\eta_1^2+\eta_2^2-\eta_0\eta_2)+\eta_0^2\eta_1^2-\eta_0\eta_2^3-\eta_0^3\eta_2)s+\eta_0\gamma(\eta_1\eta_2^2+\eta_1^3+\eta_0^2\eta_1)+\eta_0^3}{(-\eta_1^2+\eta_0\eta_2)s^2+(\eta_1\eta_0+\gamma\eta_2(\eta_0^2+\eta_1^2+\eta_2^2))s-\eta_0^2-\gamma\eta_1(\eta_0^2+\eta_1^2+\eta_2^2)}$$

Box I.

Furthermore, let $r(s) = a_{l-1}s^{l-1} + a_{l-2}s^{l-2} + \cdots + a_0$, $a_i \in \mathbb{C}$, $i = 0, \ldots, l$ and $p(s) = s^l + b_{l-1}s^{l-1} + b_{l-2}s^{l-2} + \cdots + b_0$, $b_i \in \mathbb{C}$, $i = 0, \ldots, l$, with $2l < k < 2k = \nu$. The moment matching problem is to find the coefficients a_i and b_i such that

$$\frac{r(s_j)}{p(s_j)} = \eta_j, \quad j = 1, \ldots, \nu,$$

i.e., the rational function r/p that matches ν moments. In the matrix form, the problem is rewritten as

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \alpha = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \tag{22}$$

where $\alpha = [a_{l-1} \cdots a_0 \; b_{l-1} \cdots b_0]^T \in \mathbb{C}^{2l}$, $\gamma_1 = [\eta_1 s_1^l \cdots \eta_l s_l^l]^T \in \mathbb{C}^{2l}$ and $\gamma_2 = [\eta_{l+1} s_{l+1}^l \cdots \eta_\nu s_\nu^l]^T \in \mathbb{C}^{\nu-2l}$. $\Gamma_1 \in \mathbb{C}^{2l\times 2l}$ and $\Gamma_2 \in \mathbb{C}^{(\nu-2l)\times 2l}$ are matrices with a Vandermonde-like structure with elements depending on s_j and η_j. Assuming Γ_1 is invertible, (22) yields $\alpha = \Gamma_1^{-1}\gamma_1$ which further yields a (restrictive) condition on the moments η_j and s_j, i.e.,

$$\Gamma_2\Gamma_1^{-1}\gamma_1 = \gamma_2 \Leftrightarrow \gamma_1 \in \operatorname{Im}\Gamma_2\Gamma_1^{-1}. \tag{23}$$

In other words, there exists a rational function r/p of order $l < k$ that matches $\nu = 2k$ moments only if the moments and the interpolation points satisfy condition (23). ■

3.1. On the minimality of the Krylov models

Based on the minimal order results from Theorem 5 and the equivalence results from Theorem 4 we determine the (subclasses of) Krylov based reduced order models, of minimal order, subsets of $\Sigma_{\mathbf{W}}$ or $\Sigma_{\mathbf{V}}$, respectively. Hence we present a method which allows for efficient computations of (minimal order) models, that match a prescribed number of moments, larger than the order of the models.

Corollary 1. *The following statements hold.*

1. *Consider the family of $\nu = 2k$ order models $\Sigma_{\mathbf{W}}$, as in (4), $k \in \mathbb{N}$. There exists a set of matrices $\bar{W}$ such that the subclass of models $\Sigma_{\bar{W}} \subseteq \Sigma_{\mathbf{W}}$ that match $2k$ moments, have minimal realizations of order k. Furthermore, all models $\Sigma_{\bar{W}}$ have the same (unique) transfer function of degree k.*
2. *Consider the family of $\nu = 2k$ order models $\Sigma_{\mathbf{V}}$, as in (5), $k \in \mathbb{N}$. There exists a set of matrices $\bar{V}$ such that the subclass of models $\Sigma_{\bar{V}} \subseteq \Sigma_{\mathbf{V}}$ that match $2k$ moments, have minimal realizations of order k. Furthermore, all models $\Sigma_{\bar{V}}$ have the same (unique) transfer function of degree k.* ■

Proof. Let $\Sigma_{\bar{G}}$ be the k-th order model that matches $2k$ prescribed moments, where $\bar{G} = T^{-1}(\cdot)\begin{bmatrix} -\P \\ 0 \end{bmatrix}$, with $\P = [1 \cdots 1]^T$, with T as in (16). By Theorem 4, there exists $\bar{W}$ such that $\Sigma_{\bar{W}} = \Sigma_{\bar{G}}$, i.e., by the controllability of (1), there exists $\bar{W}$ satisfying $\bar{W}^*B = G$, $\bar{W}^*V = I$, with $V = \Pi$, where Π is the unique solution of (7). Furthermore, noting that $\bar{W}^*AV = S - \bar{G}L$ and $C\Pi = CV$, which means that $K_{\bar{W}}(s) = K_{\bar{G}}(s) = \hat{K}(s)$, with $K(s)$ as in (17), completes the proof. The proof of the second statement is identical, hence omitted. □

Based on this result, we present a procedure to compute the approximant which matches a number of moments equal to twice

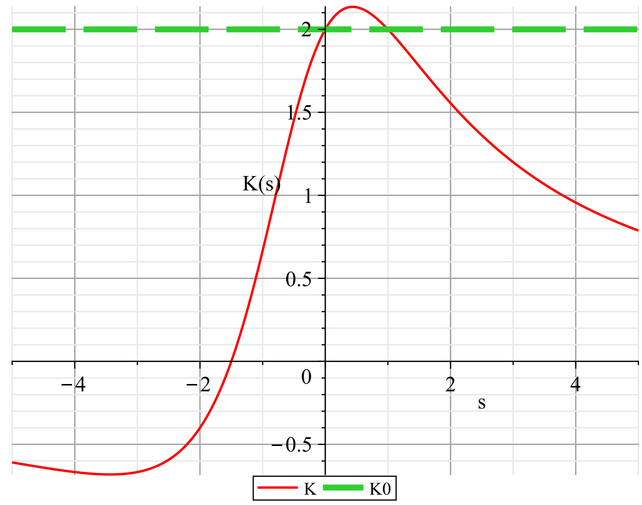

Fig. 2. $\hat{K}(s) = 2$ matches the 0-moment 2 at $s = 0$ and the 0-moment 2 at $s = 1$ of $K(s) = \frac{4s+6}{s^2+s+3}$.

the order of the approximant. Given a linear, minimal system (1) of order n and a set of 2ν interpolation points, $\nu < n$, find a model of order ν that matches 2ν moments at the given interpolation points.

Algorithm 1. Computation of a model of order ν that matches 2ν moments.

- Using any efficient numerical method, compute the class of reduced order models $\Sigma_{\mathbf{W}}$, and implicitly CV.
- Solve the linear algebraic system $T(\cdot)\bar{G} = \begin{bmatrix} -\P \\ 0 \end{bmatrix}$, with $\P = [1 \cdots 1]^T$ and T as in (16).
- Construct a projection matrix $\bar{W}$ which satisfies $\bar{W}^*B = \bar{G}$.
- The reduced order model is $\Sigma_{\bar{W}}$ as in (4) with the transfer function $\hat{K}(s)$ as in (17). ■

With a little modification, Algorithm 1 can be used for the computation of sub-families of models of order less than ν. Furthermore, the difference to matching 2ν points is, that here, we do not need to build double sided projections, one is sufficient for matching the first ν moments with the rest of additional ν moments matched through the particular instance of the parameter G.

The results hold for $\nu = 2k + 1$, too.

Remark 4. Consider the ν-th order model $\Sigma_{\widetilde{W}}$ as in Theorem 2, which matches 2ν points. Applying Corollary 1, there exists $\widetilde{G} \in \mathbb{C}^\nu$ such that $\Sigma_{\widetilde{G}} \in \Sigma_G$ matches 2ν moments, i.e., $\widetilde{G} = U^{-1}\widetilde{W}^*B$, U invertible. Now, consider the ν order model $\Sigma_{\bar{G}}$ with the transfer function (17). Then, there exists U invertible, such that $\begin{bmatrix} \widetilde{G} \\ 0 \end{bmatrix} = U\bar{G}$. By uniqueness, the transfer functions associated to the models, satisfy $K_{\widetilde{W}} = K_{\widetilde{G}} = K_{\bar{G}} = \hat{K}(s)$, with $\hat{K}(s)$ as in (17). Furthermore, let $(F, \bar{G}, H)$ be a non-minimal 2ν order realization of $\Sigma_{\bar{G}}$, with the transfer function $\hat{K}(s)$. $\Sigma_{\bar{G}}$ matches the 2ν prescribed moments if there exists an invertible matrix $P \in \mathbb{C}^{2\nu\times 2\nu}$ such that $HP = C\Pi$ and P is the unique solution of $FP + \bar{G}L = PS$ (see [10]). Since P is invertible, we have that $(F, \bar{G})$ is controllable (see also Section 3). However, since the pair (H, F) is not observable, pick P to be

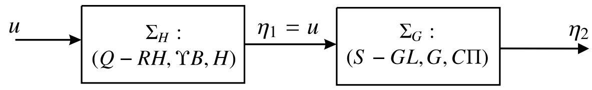

Fig. 3. The reduced order model matching moments at 2ν interpolation points (series connection).

the coordinate transformation that yields the following observable decomposition

$$F = \begin{bmatrix} F_1 & 0 \\ F_2 & F_3 \end{bmatrix}, \qquad P^{-1}\bar{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \qquad H = [H_1\ 0],$$

with (F_1, G_1, H_1) minimal. The matching conditions yield $F_1 = S_1 - G_1 L_1$ and H_1 satisfying $[H_1\ 0]P = C\Pi$, where $S = \text{diag}\{S_1, S_2\}$, $L = [L_1\ L_2]$. The application of Theorem 4 establishes a relation between $\Sigma_{\widetilde{W}}$ and $\Sigma_{\bar{G}}$ with the minimal realization $(S_1 - G_1 L_1, G_1, H_1)$, i.e., there exists an invertible matrix U, such that $G_1 = U^{-1}\widetilde{W}B$ and $H_1 = CVU$, with V as in (3a). ■

4. Interconnections of moment matching models

The families of models (11) (parameterized in G) and (12) (parameterized in H) approximate system (1) achieving moment matching at ν interpolation points, i.e., say $\{s_1, \ldots, s_\nu\} = \sigma(S)$ and say $\{s_{\nu+1}, \ldots, s_{2\nu}\} = \sigma(Q)$, respectively. We assume that $s_k \neq s_j$, for all $k = 1, \ldots, \nu$ and $j = \nu + 1, \ldots, 2\nu$. In the sequel we propose reduced order models (parameterized in G and H) that approximate (1) and match its moments at 2ν points $\{s_1, \ldots, s_{2\nu}\}$.

Assumption 2. $\sigma(S) \cap \sigma(Q) \cap \sigma(S - GL) \cap \sigma(Q - RH) = \emptyset$.

Let $\epsilon \in \mathbb{C}^{1\times 2\nu}$ be the moments of (1) at $\sigma(S)$ and $\sigma(Q)$.

Theorem 6. *Let $[\xi_1^T(t)\ \xi_2^T(t)]^T \in \mathbb{R}^{2\nu}$. Consider the linear model*

$$\Sigma_{HG} : \begin{cases} \dot{\xi}_1 = (Q - RH)\xi_1 + \Upsilon Bu, \\ \dot{\xi}_2 = (S - GL)\xi_2 + GH\xi_1, \\ \eta_2 = C\Pi\xi_2, \end{cases} \tag{24}$$

parameterized in $H \in \mathbb{R}^{1\times\nu}$ and $G \in \mathbb{R}^\nu$ (see also Fig. 3). Assume that the pair $(Q, R - \Upsilon B)$ is controllable and the pair $(L - C\Pi, S)$ is observable. Furthermore, assume that the interpolation points are not zeros of (1). Let $\hat{\epsilon} \in \mathbb{C}^{1\times 2\nu}$ be the moments of (24) at $\sigma(S)$ and $\sigma(Q)$. Let $2\nu \leq n$, then (24) is a reduced order model of (1), that achieves moment matching at $\sigma(S)$ and $\sigma(Q)$, i.e., $\epsilon = \hat{\epsilon}$ if and only if G and H are such that $\sigma(S) = \sigma(Q - RH + \Upsilon BH)$ and $\sigma(Q) = \sigma(S - GL + GC\Pi)$. ■

Proof. To start with, the controllability and observability assumptions on the pairs $(Q, R - \Upsilon B)$ and $(L - C\Pi, S)$, respectively, imply the existence of matrices H and G such that the eigenvalue conditions are satisfied. Let $K_G(s)$ be the transfer function of (11) and $K_H(s)$ be the transfer function of (12). Hence, the transfer function of (24) is $\hat{K}(s) = K_G(s)K_H(s)$. The model (24) achieves moment matching at $s_i \in \{s_1, \ldots, s_{2\nu}\}$ if the error function $E(s) = K(s) - \hat{K}(s)$ satisfies $E(s_i) = 0$, where $K(s)$ is the transfer function of (1). Let $s_i \in \sigma(Q)$. Then, by Theorem 3, we have that $K_H(s_i) = K(s_i)$. Then $E(s_i) = K(s_i) - K_G(s_i)K_H(s_i) = K(s_i) - K(s_i)K_G(s_i) = K(s_i)(1 - K_G(s_i))$. Thus, $E(s_i) = 0 \Leftrightarrow -K_G(s_i) + 1 = 0$, i.e. s_i is a zero of the system $1 - \hat{K}(s_i)$, which is equivalent to $\exists\, [v_1^T\ v_2]^T \in \mathbb{R}^{\nu+1}$ such that $\begin{bmatrix} S - GL & G \\ -C\Pi & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s_i v_1 \\ 0 \end{bmatrix}$. This is equivalent to $s_i \in \sigma(S - GL + GC\Pi)$. The other eigenvalue condition is proven in a similar way. □

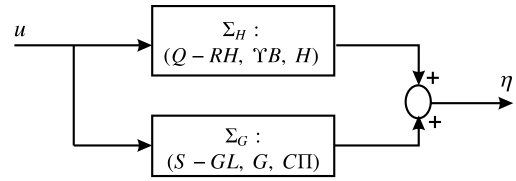

Fig. 4. The reduced order model matching moments at 2ν interpolation points (parallel connection).

Remark 5. The assumptions that the pair $(Q, R - \Upsilon B)$ is controllable and the pair $(L - C\Pi, S)$ is observable do not affect the generality of the result, since S, L, Q and R are related to the choice of interpolation points. Furthermore, since the pair (L, S) is chosen observable, and since observability is a generic property, there exist $v \in \mathbb{C}^{1\times\nu}$ such that $(L - v, S)$ is observable. A similar statement holds for the controllability of the pair $(Q, R - \Upsilon B)$. ■

Theorem 7. *Let $[\xi_1^T(t)\ \xi_2^T(t)]^T \in \mathbb{R}^{2\nu}$. Consider the linear model*

$$\Sigma_{H+G} : \begin{cases} \dot{\xi}_1 = (Q - RH)\xi_1 + \Upsilon Bu, \\ \dot{\xi}_2 = (S - GL)\xi_2 + Gu, \\ \eta = C\Pi\xi_2 + H\xi_1, \end{cases} \tag{25}$$

parameterized in $H \in \mathbb{R}^{1\times\nu}$ and $G \in \mathbb{R}^\nu$ (see also Fig. 4). Let $\hat{\epsilon} \in \mathbb{C}^{1\times 2\nu}$ be a vector containing the moments of (25) at $\sigma(S)$ and $\sigma(Q)$. Let $2\nu \leq n$, then (25) is a reduced order model of (1), that achieves moment matching at $\sigma(S)$ and $\sigma(Q)$, i.e., $\epsilon = \hat{\epsilon}$, if and only if there exist matrices $G \neq 0$ and $H \neq 0$ such that s_i, $i = 1, \ldots, \nu$ are among the finite zeros of Σ_H and s_j, $j = \nu + 1, \ldots, 2\nu$ are among the finite zeros of Σ_G. ■

Proof. Let $K_G(s)$ be the transfer function of (11) and $K_H(s)$ be the transfer function of (12). Hence, the transfer function of (24) is $\hat{K}(s) = K_H(s) + K_G(s)$. The model (24) achieves moment matching at $s_i \in \{s_1, \ldots, s_{2\nu}\}$ if the error function $E(s) = K(s) - \hat{K}(s)$ satisfies $E(s_i) = 0$, where $K(s)$ is the transfer function of (1). Let $s_i \in \sigma(Q)$. Then, by Theorem 3, we have that $K_H(s_i) = K(s_i)$. Then $E(s_i) = K(s_i) - K_H(s_i) - K_G(s_i) = K(s_i) - H(s_i) - K_G(s_i) = -K_G(s_i)$. Thus, $E(s_i) = 0 \Leftrightarrow -K_G(s_i) = 0$, i.e. s_i is a zero of $K_G(s_i)$. The other zero condition is proven in a similar way. □

Remark 6. There exist matrices G and H such that the zeros are assigned according to the conditions in Theorem 7, see [10]. Since G assigns ν zeros, there are ν constraints on the elements of G, hence G is unique. Similarly, H is unique and so, there is only one model (25) that matches 2ν moments at $\sigma(S)$ and $\sigma(Q)$. ■

Remark 7. Both Theorems 6 and 7 can be extended to the case of matching $N\nu$ points, resulting in models of order at most $(N - 1)\nu$. However the conditions are more restrictive, e.g. in the parallel connection $(N - 1)\nu$ interpolation points must be zeros of the $(N - 1)\nu$ interconnected systems.

Based on Theorem 6 the problem of finding a reduced order model that matches ν interpolation points, can be solved as follows. First, compute two (classes of) reduced order models Σ_G and Σ_H that match $\nu/2$ moments, respectively and then apply Theorem 6 to find G and H such that the series interconnection of Σ_G and Σ_H matches ν moments. Further splitting the problem, one may solve ν first order moment matching problems, resulting in say $K_{g_1}(s), \ldots, K_{g_\nu}(s)$ first order models, parameterized in $g_1, \ldots, g_\nu$. Applying Theorem 6, results in a system of ν equations in the unknowns $g_1, \ldots, g_\nu$, i.e., $\prod_{i\neq j}^{\nu} K_{g_i}(s_j) = 1$. The solution $g_1, \ldots, g_\nu$ is such that the (unique) model $K_{g_1,\ldots,g_\nu}(s) = K_{g_1}(s)K_{g_2}(s)\cdots K_{g_\nu}(s)$ matches all the prescribed ν moments. Future work will include a thorough theoretical and numerical investigation of such arguments. ■

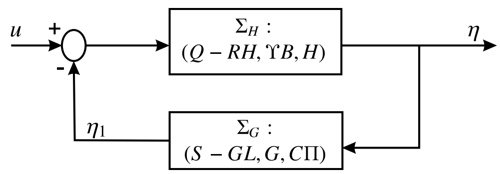

Fig. 5. The reduced order model matching moments at 2ν interpolation points (feedback connection).

Example 3. In this example we compute a third order model that matches three chosen moments, using the arguments from Remark 7. To this end, let $K_{g_1}(s) = \frac{g_1}{s+g_1}$ be such that $\eta(0) = K_{g_1}(0) = 1$, $K_{g_2}(s) = \frac{g_2}{s-1+g_2}$ be such that $\eta(1) = K_{g_2}(1) = -1$ and $K_{g_3}(s) = \frac{1}{2}\frac{g_3}{s+1+g_3}$ such that $\eta(-1) = K_{g_3}(-1) = 1/2$. Applying Theorem 6 yields $g_1 = 1.656$, $g_2 = 0.567$ and $g_3 = -2.906$. Hence the unique third order model that matches the moments $\eta(0)$, $\eta(1)$ and $\eta(-1)$ is defined by the transfer function $K_{g_1,g_2,g_3}(s) = K_{g_1}(s)K_{g_2}(s)K_{g_3}(s) = \frac{1.365}{s^3-0.683s^2-3.048s+1.365}$. ∎

Theorem 8. *Let $[\xi_1^T(t)\ \xi_2^T(t)]^T \in \mathbb{R}^{2\nu}$. Consider the linear model*

$$\Sigma_{HrG}: \begin{cases} \dot{\xi}_1 = (Q - RH)\xi_1 + \Upsilon BC\Pi\xi_2 + \Upsilon Bu, \\ \dot{\xi}_2 = (S - GL)\xi_2 + GH\xi_1, \\ \eta = H\xi_1, \end{cases} \tag{26}$$

parameterized in $H \in \mathbb{R}^{1\times\nu}$ and $G \in \mathbb{R}^{\nu}$ (see also Fig. 5). Assume that $\sigma(Q)$ does not contain any zero of the original model. Let $\hat{\epsilon} \in \mathbb{C}^{1\times 2\nu}$ be the moments of (26) at $\sigma(S)$ and $\sigma(Q)$. Let $2\nu \le n$, then (26) is a reduced order model of (1), parameterized in G and H and achieving moment matching at $\sigma(S)$ and $\sigma(Q)$, i.e., $\epsilon = \hat{\epsilon}$, if and only if, there exist G and H, such that $s_j, j = \nu+1, \ldots, 2\nu$ are among the finite zeros of Σ_G and the moments of Σ_H at $s_i, i = 1, \ldots, \nu$ match the moments of the (positive) feedback closed-loop interconnection of Σ_G with itself. ∎

Proof. Let $K_G(s)$ be the transfer function of (11) and $K_H(s)$ be the transfer function of (12). Hence, the transfer function of (24) is $\hat{K}(s) = \frac{K_H(s)}{1-K_G(s)K_H(s)}$. The model (24) achieves moment matching at $s_i \in \{s_1, \ldots, s_{2\nu}\}$ if the error function $E(s) = K(s) - \hat{K}(s)$ satisfies $E(s_i) = 0$, where $K(s)$ is the transfer function of (1). Let $s_i \in \sigma(Q)$. Then, by Theorem 3, we have that $K_H(s_i) = K(s_i)$. Then, $E(s_i) = 0 \Leftrightarrow K(s_i) - \frac{K_H(s_i)}{1-K_G(s_i)K_H(s_i)} = 0$. Let $s_i \in \sigma(Q)$, then by assumption, $K(s_i) = K_G(s_i)$ and we have that $K_G(s_i)K^2(s_i) = 0 \Leftrightarrow K_G(s_i) = 0$. Let $s_i \in \sigma(S)$, then $E(s_i) = 0 \Leftrightarrow K(s_i) - (K^2(s_i)+1)K_H(s_i) = 0 \Leftrightarrow K_H(s_i) = \frac{K_G(s_i)}{1+K_G^2(s_i)}$, which proves the last statement. □

Remark 8. If $\sigma(Q)$ coincides with the zeros of the original transfer function then the parameter G is free and can be used for other purposes, such as cancellation, resulting in lower order models. Hence we obtain families of reduced order models of order less than or equal to 2ν that match 2ν moments. ∎

5. Conclusions

In this paper we have presented several equivalent notions of moment for linear systems. Based on these notions, we have presented classes of parameterized reduced order models that achieve moment matching. We have analyzed the relations between models from different classes. We have analyzed the controllability and observability properties of the models belonging to each of the classes of reduced order models. We have obtained the subclasses of models of orders lower than the number of matched moments, based on computing the sets of parameters that allow for pole–zero cancellations to occur in the reduced order transfer function. Furthermore, we have computed the (subclass of) minimal order model(s), that is half of the number of matched moments, proving that, generically, the largest number of pole–zero cancellations is of order half the number of matched moments. Finally, we have presented reduced order models that match larger numbers of moments, based on series, parallel and feedback interconnections between models from different classes, paving the way for splitting the moment-matching problem into problems of smaller dimensions.

Appendix A. Preliminaries for the proof of Theorem 5

Lemma A.1. *Let $K_G(s) = C\Pi(sI - S + GL)^{-1}G$ be the transfer function of the reduced order model (11). Then*

$$K_G(s) = \frac{C\Pi(sI - S)^{-1}G}{1 + L(sI - S)^{-1}G}. \quad ∎ \tag{A.1}$$

Proof. The following sequence of equalities hold

$$\begin{aligned} &K_G(s)(1 + L(sI - S)^{-1}G) \\ &\quad = C\Pi(sI - S + GL)^{-1}G(1 + L(sI - S)^{-1}G) \\ &\quad = C\Pi(sI - S + GL)^{-1}(1 + L(sI - S)^{-1})G. \end{aligned}$$

Factoring $(sI - S)^{-1}$ yields

$$\begin{aligned} &K_G(s)(1 + L(sI - S)^{-1}G) \\ &\quad = C\Pi(sI - S + GL)^{-1}(sI - S + GL)(sI - S)^{-1}G \\ &\quad = C\Pi(sI - S)^{-1}G, \end{aligned}$$

which proves the claim. □

Lemma A.2. *Let $z = [z_1 \cdots z_k]^T \in \mathbb{R}^k$. Then*

$$M(z)G = 0, \tag{A.2a}$$
$$N(z)G = -\P, \tag{A.2b}$$
$$N(z)(S - GL) = ZN(z), \tag{A.2c}$$
$$M(z)(S - GL) = ZM(z) - \P C\Pi, \tag{A.2d}$$

where $Z = \mathrm{diag}\{z_1, \ldots, z_k\}$ and $M(z)$ and $N(z)$ are given in (16), with $\P = [1 \cdots 1]^T \in \mathbb{R}^k$. ∎

Proof. The relations (A.2b) and (A.2a) hold by construction. Noting that $C\Pi(z_iI - S)^{-1}(S - GL) = -C\Pi(z_iI - S)^{-1}(z_iI - S + GL - z_iI) = -C\Pi + z_iC\Pi(z_iI - S)^{-1} - \underbrace{C\Pi(z_iI - S)^{-1}G}_{0}L$ proves (A.2d). Furthermore, note that $L(z_iI - S)^{-1}(S - GL) = -L(z_iI - S)^{-1}(z_iI - S + GL - z_iI) = -L - L(z_iI - S)^{-1}GL + L(z_iI - S)^{-1}z_i = \underbrace{(-1 - L(z_iI - S)^{-1}G)}_{0}L + L(z_iI - S)^{-1}z_i$. Hence relation (A.2c) follows. □

Lemma A.3. *Consider a non-minimal reduced order model Σ_G that achieves moment matching. Then, the matrix $T(z) \in \mathbb{R}^{2k\times 2k}$ in (16) is invertible.* ∎

Proof. Assume $T(z)$ is not invertible. Hence, further assume that rank $T(z) = 2k - 1$. Then the row $L(z_iI - S)^{-1}, i = 1, \ldots, k$ is a linear combination of the other $2k - 1$ rows of $T(z)$, i.e., there exist $a_j \in \mathbb{C}, j = 1, \ldots, k-1$ and $b_i \in \mathbb{C}, j = 1, \ldots, k$ such that $L(z_iI - S)^{-1} = \sum_{j\ne i}^{k-1} a_jL(z_jI - S)^{-1} + \sum_{j=1}^{k} b_jC\Pi(z_jI - S)^{-1}$. Note now that

$$L = \sum_{j\ne i}^{k-1} a_jL(z_jI - S)^{-1}(z_iI - S) + \sum_{j=1}^{k} b_jC\Pi(z_jI - S)^{-1}(z_iI - S).$$

Using (A.2c) and (A.2d) yields

$$\begin{aligned} & L(z_jI-S)^{-1}(z_iI-S) \\ & \quad = L(z_jI-S)^{-1}z_j - L(z_jI-S)^{-1}S \\ & \quad = L(z_jI-S)^{-1}(S-GL) - L(z_jI-S)^{-1}S \\ & \quad = -\underbrace{L(z_jI-S)^{-1}G}_{-1}L = L, \end{aligned}$$

$$\begin{aligned} & C\Pi(z_jI-S)^{-1}(z_iI-S) \\ & \quad = C\Pi(z_jI-S)^{-1}z_j - C\Pi(z_jI-S)^{-1}S \\ & \quad = C\Pi(z_jI-S)^{-1}(S-GL) + C\Pi - C\Pi(z_jI-S)^{-1}S \\ & \quad = \underbrace{-C\Pi(z_jI-S)^{-1}G}_{0}L + C\Pi = C\Pi. \end{aligned}$$

Hence $L = \sum_{j\neq i}^{k-1} a_j L + \sum_{j=1}^{k} b_j C\Pi$, which further yields $\left(1 - \sum_{j\neq i}^{k-1} a_j\right)L = C\Pi \sum_{j=1}^{k} b_j$.

Since the pair $(C\Pi, S-GL)$ is not observable, there exists $v \neq 0$ such that

$$\begin{bmatrix} \lambda I - S + GL \\ C\Pi \end{bmatrix} v = 0, \quad \lambda \in \mathbb{C}.$$

Hence $(\lambda I - S + GL)v = 0 \Leftrightarrow \lambda \in \sigma(S-GL)$ and $Lv = 0$, which yields $(\lambda I - S)v = 0 \Leftrightarrow \lambda \in \sigma(S)$. This is a contradiction since $\sigma(S-GL) \cap \sigma(S) = \emptyset$. This means that $T(z)$ is invertible. □

Remark A.1. If $\nu = 2k+1$, then the matrix $T(z) \in \mathbb{C}^{(2k\times(2k+1))}$ has a full row rank, i.e., rank $T(z) = 2k$. The proof is similar to the proof of Lemma A.3. ■

Remark A.2. Dual results of Lemmas A.1–A.3, respectively, follow considering the class of models Σ_H as in Eqs. (12) and $M(z)$, $N(z)$ and $T(z)$ described by (18). ■

Appendix B. On the matrices $B - \Pi G$ and $C - H\Upsilon$

Lemma B.1. *The following statements hold.*

1. *Consider the controllable pair (A, B), $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^n$. Let (L, S), $S \in \mathbb{C}^{\nu\times\nu}$, $L \in \mathbb{C}^{1\times\nu}$ be an observable pair and let Π be the solution of the Sylvester equation (7). Assuming that $\nu < n$, then $B - \Pi G \neq 0$, for all $G \in \mathbb{C}^\nu$.*
2. *Consider the observable pair (C, A), $A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{1\times n}$. Let (Q, R), $Q \in \mathbb{C}^{\nu\times\nu}$, $R \in \mathbb{C}^\nu$ be a controllable pair and let Υ be the solution of the Sylvester equation (8). Assuming that $\nu < n$, then $C - H\Upsilon \neq 0$, for all $H \in \mathbb{C}^{1\times\nu}$.* ■

Proof. Assume that $\exists G \in \mathbb{C}^\nu$ such that $B - \Pi G = 0 \Leftrightarrow B = \Pi G$. Then the controllability matrix of the pair (A, B) is $\mathbf{R} = [B\ AB \cdots A^{n-1}B] = [\Pi G\ A\Pi G \cdots A^{n-1}\Pi G]$. Since Π satisfies (7), then $A\Pi = \Pi S - BL$ which yields $\mathbf{R} = \Pi[G\ (S-GL)G \cdots (S-GL)^{n-1}G]$. Hence rank $\mathbf{R} \leq \min\{\nu,\ \text{rank}\ [G\ (S-GL)G \cdots (S-GL)^{n-1}G]\}$, which yields rank $\mathbf{R} \leq \nu < n$. This contradicts the assumption that the pair (A, B) is controllable, hence $B - \Pi G \neq 0,\ \forall G$. The proof of the second statement follows similar arguments. □

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