Brief paper

Stochastic stability of Positive Markov Jump Linear Systems[☆]

Paolo Bolzern[a,1], Patrizio Colaneri[a], Giuseppe De Nicolao[b]

[a] Politecnico di Milano, Dipartimento di Elettronica, Informazione e Bioingegneria, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

[b] Università di Pavia, Dipartimento di Ingegneria Industriale e dell'Informazione, Via Ferrata 3, 27100 Pavia, Italy

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ABSTRACT

This paper investigates on the stability properties of Positive Markov Jump Linear Systems (PMJLS's), i.e. Markov Jump Linear Systems with nonnegative state variables. Specific features of these systems are highlighted. In particular, a new notion of stability (Exponential Mean stability) is introduced and is shown to be equivalent to the standard notion of 1-moment stability. Moreover, various sufficient conditions for Exponential Almost-Sure stability are worked out, with different levels of conservatism. The implications among the different stability notions are discussed. It is remarkable that, thanks to the positivity assumption, some conditions can be checked by solving Linear Programming feasibility problems.

1. Introduction

Markov Jump Linear Systems (MJLS's) are a popular class of stochastic systems that are well suited to describe dynamics characterized by random jumps between subsystems induced by external causes, such as random faults, unexpected events, and uncontrolled configuration changes (Boukas, 2005; Costa, Fragoso, & Marquez, 2005; Costa, Fragoso, & Todorov, 2013). Possible applications of jump systems include fault-tolerant systems (Aberkane, Ponsart, Rodrigues, & Sauter, 2008), networked control (Xiao, Xie, & Fu, 2010), communication networks for multi-agent systems (Meskin & Khorasani, 2009), macroeconomic models (do Val & Basar, 1999), pulp and paper industry (Khanbaghi, Malhamé, & Perrier, 2002), energy systems (Angeli & Kountouriotis, 2012) and epidemiology (Otero, Barmak, Dorso, Solari, & Natiello, 2011). The assumptions that the jumps are governed by an underlying Markov chain and the subsystems are linear make these systems amenable to a thorough theoretical analysis while preserving great flexibility.

It is remarkable that, even in the case of jumps between linear time-invariant subsystems, the fundamental property of state stability is much more involved than in the deterministic case and presents several interesting, sometimes intriguing, facets. Indeed, various notions of stochastic stability can be studied which are not equivalent. *Mean-square stability*, implying asymptotic convergence to zero of the expected squared norm of the state is a classical and widely investigated notion. Necessary and sufficient conditions for mean-square stability of MJLS's are available both in continuous and discrete-time (Boukas, 2005; Costa et al., 2005; Feng, Loparo, Ji, & Chizeck, 1992). Mean-square stability is a particular instance of δ*-moment stability*, where the squared norm is replaced by a generic positive power δ of the norm, see Fang, Loparo, and Feng (1994). However, as pointed out in Kozin (1969), these stability properties, dealing with the convergence of the moments of the state norm, would be better replaced by the direct study of the convergence to zero of almost all the sample paths of the state. As a matter of fact, this last notion, that goes under the name of *almost-sure stability*, is much closer to the practical concerns of the user. The connections between the different stability notions are now well understood, see e.g. Fang et al. (1994), Fang and Loparo (2002). It is known that δ-moment stability with a certain δ implies δ-moment stability with smaller values of δ. Furthermore, almost-sure stability is implied by δ-moment stability for any value of δ. Unfortunately, there do not exist direct and easy-to-check necessary and sufficient conditions for verifying almost-sure stability. The alternative is between relatively simple sufficient conditions, that may be however conservative (Fang, 1997; Fang et al., 1994; Tanelli, Picasso, Bolzern, & Colaneri, 2010), and a randomized criterion related to the average norm contractivity over an interval, that can be made arbitrarily close to necessity at the cost of increased computational burden (Bolzern, Colaneri, & De Nicolao, 2006). For

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E-mail addresses: bolzern@elet.polimi.it (P. Bolzern), colaneri@elet.polimi.it (P. Colaneri), giuseppe.denicolao@unipv.it (G. De Nicolao).

[1] Tel.: +39 0223993598

this reason, mean-square stability, though more restrictive, may appear more convenient, especially for design purposes, as it can benefit from the availability of necessary and sufficient conditions checkable by standard tools.

In this paper we consider the case in which all the subsystems belong to the class of *linear positive systems* (Farina & Rinaldi, 2000), whose state variables remain nonnegative whenever initialized in the positive orthant. Positive systems are frequently used to describe biological systems (e.g. compartmental models) or population dynamics. Other applications are found in chemical reactions, queue processes, traffic modeling, to mention but a few. The stability properties of deterministic positive systems present peculiar features which simplify the analysis. In particular, asymptotic stability can be assessed using linear Lyapunov functions, see e.g. Farina and Rinaldi (2000). It is therefore natural to investigate whether similar simplifications carry over to the stochastic case of *Positive Markov Jump Linear Systems* (PMJLS's), a study that, to the authors' knowledge, is not available in the literature.

With reference to the notions of mean-square, δ-moment and almost-sure stability, we provide a comprehensive picture of sufficient and/or necessary conditions highlighting the role of positivity both from the theoretical and the computational viewpoint. The key questions addressed in the paper regard the possibility of establishing equivalences between stability notions as well as the possible existence of a stability notion equipped with easy to handle necessary and sufficient conditions but less conservative than mean-square stability. As a further contribution, we work out a number of sufficient conditions for almost-sure stability, discussing their degree of conservatism versus usability.

The paper is organized as follows. After introducing in Section 2 the adopted notation and some useful properties, the class of PMJLS's is presented in Section 3 and the various stability notions are reported in Section 4. The main results of the paper are discussed in Section 5, followed by some numerical examples in Section 6. The paper ends with some concluding remarks.

2. Notation and basic properties

In this paper we will conform with the standard convention of denoting scalar and vectors with small letters and matrices with capital letters. The ith entry of vector x will be indicated as x_i and the (i, j)th entry of matrix A as a_{ij}. Moreover, vectors are usually assumed as column vectors and the suffix $'$ corresponds to vector or matrix transposition. The symbol $\mathbf{1}_n$ denotes the n-dimensional vector with all entries equal to 1. The symbol I_n denotes the identity matrix of order n. In both cases, the suffix n will be omitted when the vector (or matrix) size is clear from the context. The symbol e_i stands for the ith column of the identity matrix (again the dimension will be clear from the context).

A (column or row) vector $x = [x_i] \in \Re^n$ is said to be *positive* if all its entries are strictly greater than 0. In that case, we will say that $x \gg 0$. A vector $x = [x_i] \in \Re^n$ is said to be *nonnegative* if all its entries are greater than or equal to 0. In that case, we will say that $x > 0$. Similar definitions and notation apply when x is either *negative* ($x \ll 0$) or *nonpositive* ($x < 0$). The expressions $x \gg y$, $x > y$, $x \ll y$, $x < y$ indicate that the difference $x - y$ is positive, nonnegative, negative, nonpositive, respectively. To indicate that a square matrix $P \in \Re^{n\times n}$ is *positive definite* (*positive semi-definite*), we will use the symbol $P \succ 0$ ($P \succeq 0$). The notation $P \prec 0$ ($P \preceq 0$) means that P is *negative definite* (*negative semi-definite*).

A square matrix $A = [a_{ij}] \in \Re^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are nonnegative, namely $a_{ij} \geq 0$ for $i \neq j$. For a Metzler matrix A, it is known that its eigenvalue λ with maximum real part is always real and is called the *Frobenius eigenvalue*. The corresponding eigenspace is generated by a positive eigenvector, called the *Frobenius eigenvector*, see e.g. Berman and Plemmons (1994), Farina and Rinaldi (2000). A dynamical linear system described by the differential equation $\dot{x}(t) = Ax(t)$, where A is a Metzler matrix, is called a *positive system* because it enjoys the property that any trajectory starting in the positive orthant remains indefinitely confined in it.

A square matrix is *Hurwitz* if all its eigenvalues lie in the open left half plane. A Metzler matrix is Hurwitz if and only if there exist a vector $c \gg 0$ such that $c'A \ll 0$, see e.g. (Farina & Rinaldi, 2000).

The symbols $\|x\|$ and $\|A\|$ will be used to denote a generic *norm* of vector $x \in \Re^n$ and the corresponding *induced norm* of matrix $A \in \Re^{n\times n}$. In particular $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$.

The *measure* of matrix $A \in \Re^{n\times n}$ is defined as $\mu(A) = \lim_{h\to 0} \frac{\|I+Ah\|-1}{h}$ and it depends on the adopted matrix norm. It is well known that $\|\exp(At)\| \leq \exp(\mu(A)t)$, $t \geq 0$. It can be shown that $\mu_1(A) = \max_j \left(a_{jj} + \sum_{i=1,\, i\neq j}^n |a_{ij}|\right)$. These results can be found in Desoer and Vidyasagar (1975).

Given a set of vectors $z_i \in \Re^n$, $i = 1, 2, \ldots, N$, the symbol $v_z = \mathrm{vec}[z_i]$ represents the vector obtained by stacking vectors $z_1, z_2, \ldots, z_N$, into a single nN-dimensional vector. For two matrices $A \in \Re^{n\times m}$, $B \in \Re^{p\times q}$, the expression $C = A \otimes B$ stands for the usual *Kronecker product*, obtained by orderly collecting the blocks $a_{ij}B$ into the matrix $C \in \Re^{np\times mq}$. For two square matrices $A \in \Re^{n\times n}$, $B \in \Re^{p\times p}$, the *Kronecker sum* is defined as $D = A \oplus B = A \otimes I_p + I_n \otimes B \in \Re^{np\times np}$. Properties of Kronecker operators can be found in Graham (1981).

The expectation of a stochastic variable v will be denoted as $E[v]$. The symbol $\Pr\{\cdot\}$ will be used for the probability of an event.

3. Positive Markov Jump Linear Systems

In this paper the attention will be focused on the class of continuous-time Markov jump linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \geq 0 \tag{1}$$

where $x(t) \in \Re^n$, $\sigma(t) \in \mathcal{S} = \{1, 2, \ldots, N\}$, and the matrices $A_i \in \Re^{n\times n}$, $i \in \mathcal{S}$ are Metzler matrices, i.e. real square matrices whose off-diagonal entries are nonnegative. The process $\sigma(t)$ is a time-homogeneous Markov stochastic process with infinitesimal generator $\Lambda \in \Re^{N\times N}$. Precisely, let

$$\Pr\{\sigma(t+h) = j|\sigma(t) = i\} = \lambda_{ij}h + o(h), \quad i \neq j \tag{2}$$

where $h > 0$, and $\lambda_{ij} \geq 0$ is the transition rate from mode i at time t to mode j at time $t + h$. The diagonal entries of Λ are defined as

$$\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$$

so that Λ is a Metzler matrix satisfying $\Lambda\mathbf{1} = 0$. Let τ_k, $k = 0, 1, \ldots, \tau_0 = 0$, be the successive sojourn times between jumps. Then, assuming that after the kth jump the system stays in mode i, from (2) it follows that τ_k is exponentially distributed with parameter $-\lambda_{ii}$. Let $\pi_i(t) = \Pr\{\sigma(t) = i\}$ and $\pi(t) = [\pi_1(t) \ \ldots \ \pi_N(t)]'$. Given an initial probability distribution $\pi(0) = [\pi_{01} \ \ldots \ \pi_{0N}]'$, where $\pi_{0i} := \Pr\{\sigma(0) = i\}$, the time evolution of the probability distribution $\pi(t)$ obeys the differential equation

$$\dot{\pi}(t)' = \pi(t)'\Lambda.$$

Note that $\mathbf{1}'\pi(t) = 1$, $\forall t \geq 0$, i.e. $\pi(t)$ is a unit-sum vector. Moreover, if the Markov process is irreducible (see, e.g. Bremaud, 1998), then, for any $\pi(0)$, $\pi(t)$ converges as $t \to \infty$ to a stationary probability vector $\bar{\pi}$ which is the unique unit-sum Frobenius left eigenvector of Λ associated with the Frobenius–Perron null eigenvalue, Berman and Plemmons (1994). In the sequel, it is assumed that the Markov process is irreducible. Moreover, the symbol $E_{\bar{\pi}}[\cdot]$

will denote the expectation defined over stationary realizations of the Markov process $\sigma(t)$, namely those starting with $\pi(0) = \bar{\pi}$.

Thanks to the Metzler property of the dynamic matrices A_i, if the initial state $x(0)$ has nonnegative entries ($x(0) > 0$) the state $x(t)$ at any $t > 0$ has nonnegative entries as well. For this reason, this class of systems will be denoted as PMJLS's (Positive Markov Jump Linear Systems). Remarkably, positive linear systems enjoy the so-called monotonicity property. Precisely, consider the evolution of system (1) starting from two different initial states, say $x(0) = x_a$ and $x(0) = x_b$, with $x_a \gg x_b$. Then, for any given realization of $\sigma(t)$, it results that $x_a(t) \gg x_b(t)$, $\forall t > 0$.

4. Stability notions

For the PMJLS (1), various notions of stochastic stability can be introduced. Some of them refer to the convergence of suitable moments of the state norm (δ-moment stability). Another notion (almost-sure stability) deals with the convergence to zero of almost all the sample paths of the state. In addition, we will consider a stability definition peculiar to positive stochastic systems, namely the convergence to zero of the expectation of the state vector (mean stability). In the sequel, it will be assumed that convergence is always of the exponential type.

Definition 1. For any real scalar $\delta > 0$, the PMJLS (1) is said to be exponentially δ-moment stable if there exist positive real scalars α and β such that

$$E[\|x(t)\|^{\delta}] < \alpha e^{-\beta t}\|x(0)\|^{\delta}$$

for any initial condition $x(0) > 0$ and any initial probability distribution $\pi(0)$.

For $\delta = 2$, the above definition coincides with the well-known notion of Exponential Mean-Square stability (EMS stability).

Definition 2. The PMJLS (1) is said to be exponentially almost-sure stable (EAS-stable) if there exists a positive real scalar ρ such that, for any initial condition $x(0) > 0$ and any initial probability distribution $\pi(0)$, it results that

$$Pr\left\{\limsup_{t\to\infty}\frac{1}{t}\ln\|x(t)\| \le -\rho\right\} = 1.$$

Definition 3. The PMJLS (1) is said to be exponentially mean stable (EM stable) if there exist positive real scalars α and β such that

$$E[x(t)] < \alpha e^{-\beta t}\|x(0)\|\mathbf{1}$$

for any initial condition $x(0) > 0$ and any initial probability distribution $\pi(0)$.

In the three definitions above, convergence must be ascertained for all possible initial states $x(0)$. Thanks to the monotonicity property of PMJLS's, it is sufficient to check convergence just for a single strictly positive state $x(0) = x_a \gg 0$. Indeed, if $x(0) = x_b \neq x_a$, there exists a positive constant ϑ such that $\vartheta x_b \ll x_a$. Thus, for any realization of the Markov process $\sigma(t)$ the corresponding state trajectories satisfy $\vartheta x_b(t) \ll x_a(t)$, $\forall t > 0$. By linearity, it follows that fulfillment of the stability conditions for $x(0) = x_a$ entails their fulfillment also for $x(0) = x_b$. Notice also that, under the assumption of irreducibility of the Markov chain, the choice of the initial probability distribution $\pi(0)$ is immaterial.

It is well known that the above stability notions are not equivalent to each other. Concerning the relationships, it can be shown (Fang et al., 1994) that, if $\delta_2 > \delta_1 > 0$, then exponential δ-moment stability with $\delta = \delta_2$ implies exponential δ-moment stability with $\delta = \delta_1$. Moreover, exponential δ-moment stability implies EAS stability.

5. Stability analysis

A first result of this paper establishes the equivalence between exponential 1-moment stability and EM stability.

Theorem 1. *System (1) is exponentially 1-moment stable if and only if it is EM stable.*

Proof. Suppose that system (1) is exponentially 1-moment stable and take α and β according to Definition 1 with $\delta = 1$. Then, letting $x_i(t)$ denote the ith entry of $x(t)$, it results that, for all $i = 1, 2, \dots, n$,

$$E[x_i(t)] \le E[\|x(t)\|] < \alpha\|x(0)\|e^{-\beta t}$$

so implying EM stability, as in Definition 3. Conversely, assume that the system is EM stable with α and β given by Definition 3. It is straightforward to see that

$$E\left[\sum_{i=1}^{n} x_i(t)\right] \le n\alpha\|x(0)\|e^{-\beta t}.$$

Since $x(t)$ is a nonnegative vector, it is immediate to see that (without any loss of generality, here we refer to the Euclidean norm of $x(t)$)

$$\|x(t)\| \le \sqrt{n}\sum_{i=1}^{n} x_i(t).$$

Therefore

$$E[\|x(t)\|] \le \sqrt{n}E\left[\sum_{i=1}^{n} x_i(t)\right] \le n\sqrt{n}\alpha\|x(0)\|e^{-\beta t}$$

so that the system is exponentially 1-moment stable. □

The next theorem provides necessary and sufficient conditions for EM stability.

Theorem 2. *The following three statements are equivalent:*

(i) *System (1) is EM stable.*
(ii) *There exist strictly positive vectors $c_i \in \Re^n$, $i = 1, 2, \dots, N$ such that the following inequalities are satisfied*

$$c_i' A_i + \sum_{j=1}^{N} \lambda_{ij} c_j' \ll 0. \tag{3}$$

(iii) *The $nN \times nN$ matrix*

$$\tilde{A} = \begin{bmatrix} A_1 + \lambda_{11}I_n & \lambda_{21}I_n & \cdots & \lambda_{N1}I_n \\ \lambda_{12}I_n & A_2 + \lambda_{22}I_n & \cdots & \lambda_{N2}I_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N}I_n & \lambda_{2N}I_n & \cdots & A_N + \lambda_{NN}I_n \end{bmatrix} \tag{4}$$

is Hurwitz.

Proof. (i) ⇔ (iii)

Let $E[x(t)] = q(t)$. It can be shown (see e.g. Bolzern, Colaneri, and De Nicolao (2010)) that $q(t) = \sum_{i=1}^{N} q_i(t)$, where

$$\dot{q}_i(t) = A_i q_i(t) + \sum_{j=1}^{N} \lambda_{ji} q_j(t).$$

Letting $v_q(t) = \mathrm{vec}[q_i(t)]$, it is immediate to verify that the previous equation can be rewritten as $\dot{v}_q(t) = \tilde{A}v_q(t)$, with $\tilde{A}$ defined in (4). Now notice that, thanks to positivity, $q(t)$ tends exponentially to zero if and only if $q_i(t)$, $i = 1, 2, \dots, N$ tend exponentially to zero. Then the conclusion follows.

(ii) ⇔ (iii)

Since $\tilde{A}$ is a Metzler matrix, it is Hurwitz stable if and only if there exists a strictly positive vector v_c such that $v_c'\tilde{A} \ll 0$. Letting $v_c = \text{vec}[c_i]$, the equivalence between (ii) and (iii) follows. □

While the positivity of the system allows the derivation of specific necessary and sufficient conditions for 1-moment stability, as shown in Theorems 1 and 2 above, apparently this does not happen for 2-moment (mean-square, or EMS) stability. Therefore, we just recall the well-known result below, see e.g. Fang and Loparo (2002).

Theorem 3. *The following three statements are equivalent:*

(i) *System (1) is EMS stable.*

(ii) *There exist positive definite matrices $P_i \in \Re^{n\times n}$, $i = 1, 2, \ldots, N$, such that the following inequalities are satisfied*

$$P_iA_i + A_i'P_i + \sum_{j=1}^{N} \lambda_{ij}P_j \prec 0. \tag{5}$$

(iii) *The $n^2N \times n^2N$ matrix*

$$\hat{A} = \begin{bmatrix} A_1 \oplus A_1 + \lambda_{11}I_{n^2} & \lambda_{21}I_{n^2} & \cdots & \lambda_{N1}I_{n^2} \\ \lambda_{12}I_{n^2} & A_2 \oplus A_2 + \lambda_{22}I_{n^2} & \cdots & \lambda_{N2}I_{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N}I_{n^2} & \lambda_{2N}I_{n^2} & \cdots & A_N \oplus A_N + \lambda_{NN}I_{n^2} \end{bmatrix} \tag{6}$$

is Hurwitz.

Let us turn now to EAS stability. It is known that such a notion of stability is related to the negativity of the expectation of the so-called top Lyapunov exponent, defined as

$$\lambda = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi(t, 0)\|$$

where $\Phi(t, 0)$ is the transition matrix associated with system (1). A necessary and sufficient condition having general validity for all MJLS's has been derived in Bolzern et al. (2006). It relates EAS stability with the average contractivity of the state norm over a sufficiently long time interval.

Theorem 4. *System (1) is EAS stable if and only if there exists a real scalar $T > 0$ such that*

$$E_{\bar{\pi}}\left[\log \|\Phi(T, 0)\|\right] < 0$$

where $\Phi(t, 0)$ is the transition matrix of the system over the interval $[0, t]$.

In general, checking the condition of Theorem 4 in closed form is not possible, so that randomized procedures are usually employed to assess stability with a prescribed confidence level, see Bolzern et al. (2006). This motivates the search of easier-to-check sufficient conditions.

Theorem 5. *Any of the following conditions is sufficient for EAS stability of system (1):*

(i) $E_{\bar{\pi}}[\mu(A_\sigma)] = \sum_{i=1}^{N} \bar{\pi}_i \mu(A_i) < 0.$

(ii) $\sum_{i=1}^{N} \bar{\pi}_i(-\lambda_{ii} \log \alpha_i - \beta_i) < 0$, *where α_i and β_i are real positive scalars such that $\|\exp(A_it)\| \le \alpha_i \exp(-\beta_it)$, $\forall t \ge 0$.*

(iii) *There exists a strictly positive vector $c \in \Re^n$ such that*

$$E_{\bar{\pi}}\left[\sup_{d\gg 0} \frac{c'A_\sigma d}{c'd}\right] = \sum_{i=1}^{N} \bar{\pi}_i \sup_{d\gg 0} \frac{c'A_id}{c'd} < 0.$$

(iv) *There exist real strictly positive vectors $c_i \in \Re^n$ and real scalars η_i, $i = 1, 2, \ldots, N$, such that the following inequalities are satisfied:*

$$c_i'A_i + \sum_{j=1}^{N} \lambda_{ij}c_j' - \eta_ic_i' \ll 0, \quad \forall i \tag{7}$$

with $\sum_{i=1}^{N} \bar{\pi}_i\eta_i \le 0.$

Proof. Conditions (i) and (ii) are not specific to PMJLS's and their proofs can be found in Fang and Loparo (2002), and Tanelli et al. (2010), respectively.

Condition (iii) can be seen as a particularization to positive systems of Infante's condition (Kozin, 1969). Precisely, define $V(x) = c'x$. Then, along the trajectories of system (1), it results that

$$\dot{V}(x(t)) = \left(\frac{c'A_{\sigma(t)}x(t)}{c'x(t)}\right) V(x(t)) \le q(t)V(x(t))$$

with

$$q(t) = \sup_{d\gg 0} \frac{c'A_{\sigma(t)}d}{c'd}.$$

Hence $V(x(t)) \le V(x(0))e^{tr(t)}$, where

$$r(t) = \frac{1}{t}\int_0^t q(\tau)d\tau.$$

Thanks to ergodicity, $r(t)$ converges almost surely to $\bar{r} = E_{\bar{\pi}}[r(t)] < 0$. In conclusion, $V(x(t))$ (and hence $x(t)$) converges exponentially to zero almost surely, for any initial state.

As for condition (iv), notice that inequalities (7) coincide with inequalities (3) when A_i is replaced by $\bar{A}_i = A_i - \eta_iI_n$. Then, in view of Theorem 2, the PMJLS

$$\dot{\bar{x}}(t) = \bar{A}_{\sigma(t)}\bar{x}(t) \tag{8}$$

endowed with the same infinitesimal generator Λ is 1-moment stable, and therefore EAS stable as well. This means that the expected value of the top Lyapunov exponent $\bar{\lambda}$ associated with system (8) is negative. The transition matrix of this system, namely $\bar{\Phi}(t, 0)$, is related to $\Phi(t, 0)$ by the expression $\bar{\Phi}(t, 0) = \Phi(t, 0)\exp(-\int_0^t \eta_{\sigma(\tau)}d\tau)$. Hence

$$\log \|\Phi(t, 0)\| = \log \|\bar{\Phi}(t, 0)\| + ts(t)$$

where $s(t) = \frac{1}{t}\int_0^t \eta_{\sigma(\tau)}d\tau$. Thanks to ergodicity, $\lim_{t\to\infty} s(t) = \bar{s}$, with

$$E_{\bar{\pi}}[\bar{s}] = \sum_{i=1}^{N} \bar{\pi}_i\eta_i \le 0.$$

As a consequence, the expected value of the top Lyapunov exponent λ of system (1) satisfies $E_{\bar{\pi}}[\lambda] = E_{\bar{\pi}}[\bar{\lambda}] + E_{\bar{\pi}}[\bar{s}] < 0$, so concluding the proof. □

The next theorem shows some interesting implications among the sufficient conditions of Theorem 5.

Theorem 6. *With reference to the statement of Theorem 5, condition* (i) *implies* (ii)*. Moreover,* (i) *implies both* (iii) *and* (iv) *provided that $\mu(\cdot)$ denotes the matrix measure $\mu_1(\cdot)$ induced by the 1-norm. Finally, condition* (iii) *implies* (iv).

Proof. The fact that (i) implies (ii) easily follows by letting $\alpha_i = 1$, $\forall i$ and $\beta_i = -\mu(A_i)$, $\forall i$. Now, suppose that (i) holds with $\mu(\cdot)$

replaced with $\mu_1(\cdot)$. Notice that, for a Metzler matrix A, it holds that $\mu_1(A) = \max_k \mathbf{1}' A e_k$. Therefore, letting $c = \mathbf{1}$ it turns out that

$$\begin{aligned} E_{\bar{\pi}}\left[\sup_{d \gg 0} \mathbf{1}' A_\sigma d\right] &= E_{\bar{\pi}}\left[\sup_{d \gg 0} \mathbf{1}' A_\sigma \sum_{k=1}^{n} e_k d_k\right] \\ &= E_{\bar{\pi}}\left[\sum_{k=1}^{n} \sup_{d \gg 0} \mathbf{1}' A_\sigma e_k d_k\right] \\ &\le E_{\bar{\pi}}\left[\sum_{k=1}^{n} \sup_{d \gg 0} \max_k \mathbf{1}' A_\sigma e_k d_k\right] \\ &= E_{\bar{\pi}}\left[\sum_{k=1}^{n} \sup_{d \gg 0} \mu_1(A_\sigma) d_k\right] = n E_{\bar{\pi}}\left[\mu_1(A_\sigma)\right] < 0. \end{aligned}$$

Hence, condition (iii) is satisfied. Now, address the implication (i) $\Rightarrow$ (iv). If condition (i) is satisfied, then

$$\sum_{i=1}^{N} \bar{\pi}_i \mu_1(A_i) < 0.$$

Then there exists a sufficiently small constant ϵ such that

$$\sum_{i=1}^{N} \bar{\pi}_i \mu_1(A_i) + \epsilon < 0.$$

Now let $\eta_i = \mu_1(A_i) + \epsilon$ and $c_i = \mathbf{1}$, $\forall i$. Recalling that $\sum_{j=1}^{N} \lambda_{ij} = 0$ and $\mu_1(A) = \max_k \mathbf{1}' A e_k$, it is straightforward to verify that all inequalities of condition (iv) are satisfied.

Finally, we show that (iii) $\Rightarrow$ (iv). Assume that there exists $c \gg 0$ satisfying condition (iii) and define

$$\eta_i = \sup_{d \gg 0} \frac{c' A_i d}{c' d} + \epsilon > \frac{c' A_i v}{c' v}, \quad \forall v \gg 0. \tag{9}$$

By choosing ϵ small enough, it is possible to guarantee that $E_{\bar{\pi}}[\eta_\sigma] = \sum_{j=1}^{N} \bar{\pi}_i \eta_i < 0$. What is left to show is that there exist vectors $c_i \gg 0$ satisfying the inequalities (7). To this purpose take $c_i = c$, $\forall i$. Since the summation appearing in (7) vanishes, it suffices to show that $c' A_i - \eta_i c' \ll 0$. In view of (9), for any arbitrary vector $v \gg 0$ it results that,

$$\left(c' A_i - \eta_i c'\right) v < \left(c' A_i - \frac{c' A_i v}{c' v} c'\right) v = 0.$$

Hence $c' A_i - \eta_i c' \ll 0$ and the proof is completed. □

Remark 7. It is interesting to remark that, when none of the matrices A_i is Hurwitz, conditions (i)–(iii) of Theorem 5 can never be met with. As a matter of fact, for a non-Hurwitz (unstable) matrix the measure $\mu(A_i)$ and the parameter β_i appearing in the bound on the norm of its transition matrix are always nonnegative. If all matrices A_i are unstable, then conditions (i) and (ii) can never be satisfied. As for condition (iii), when A_i is Metzler and non-Hurwitz, for any vector $c \gg 0$, the maximum element of the row vector $c' A_i$ is greater than or equal to 0. In turn, this implies that there exists $d \gg 0$ such that $c' A_i d \ge 0$. Then, $\sup_{d \gg 0} \frac{c' A_i d}{c' d} \ge 0$ and the expectation of condition (iii) can never be negative.

Remark 8. Note that, for a scalar system ($n = 1$), conditions (i)–(iv) provided in Theorem 5 are also necessary for EAS stability. Indeed, if the PMJLS (1) is scalar, necessity of condition (i) was proven in Fang et al. (1994). In view of Theorem 6, necessity of (ii)–(iv) directly follows.

In summary, the relationships among the different stability notions for PMJLS's can be depicted as in Fig. 1. While the chain

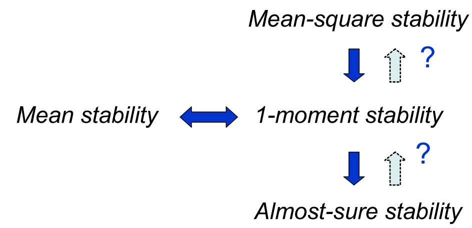


Fig. 1. Picture of true and conjectured implications between the different stability notions.

of implications going from EMS stability to EAS stability is valid for any generic MJLS, the equivalence between EM stability and exponential 1-moment stability is specific to PMJLS's. One may wonder whether positivity may imply that these stability notions are all equivalent. As a matter of fact, the conjectured implications associated with the question marks in the figure, are falsified through the counterexample provided in Example 1 of the next section.

Remark 9. From a computational viewpoint, testing the EM stability condition (ii) of Theorem 2 reduces to a feasibility problem for the Linear Programming (LP) inequalities (3). This is a problem that standard LP-solvers can efficiently deal with, see e.g. Luenberger and Ye (2008). As for condition (iii) of Theorem 2, note that the matrix $\tilde{A}$ is Metzler, so that the check on its stability is again a linear feasibility problem, of greater dimension, consisting in finding a vector $\tilde{c} \gg 0$ such that $\tilde{c}' \tilde{A} \ll 0$.

Turning now to the criteria of EMS stability of Theorem 3, checking condition (ii) requires the solution of a feasibility problem for the Linear Matrix Inequalities (5), for which efficient tools of Convex Programming are available, see e.g. Boyd, El Ghaoui, Feron, and Balakrishnan (1994). Alternatively, observing that $\hat{A}$ is Metzler, EMS stability can be tested through condition (iii) looking for a vector $\hat{c} \gg 0$ such that $\hat{c}' \hat{A} \ll 0$.

EAS stability can be assessed by means of the sufficient conditions of Theorem 3. Note that condition (iv) involves the study of feasibility of a Bilinear Programming problem, as it requires the determination of both vectors c_i and scalars η_i, that appear combined in a product in the inequalities (7). General Nonlinear Programming methods or specific Bilinear Programming methods can be used to solve this problem, see e.g. Luenberger and Ye (2008).

6. Numerical examples

In this Section two simple examples of PMJLS's are presented. The first one demonstrates that the two implications conjectured in Fig. 1 do not hold true in general.

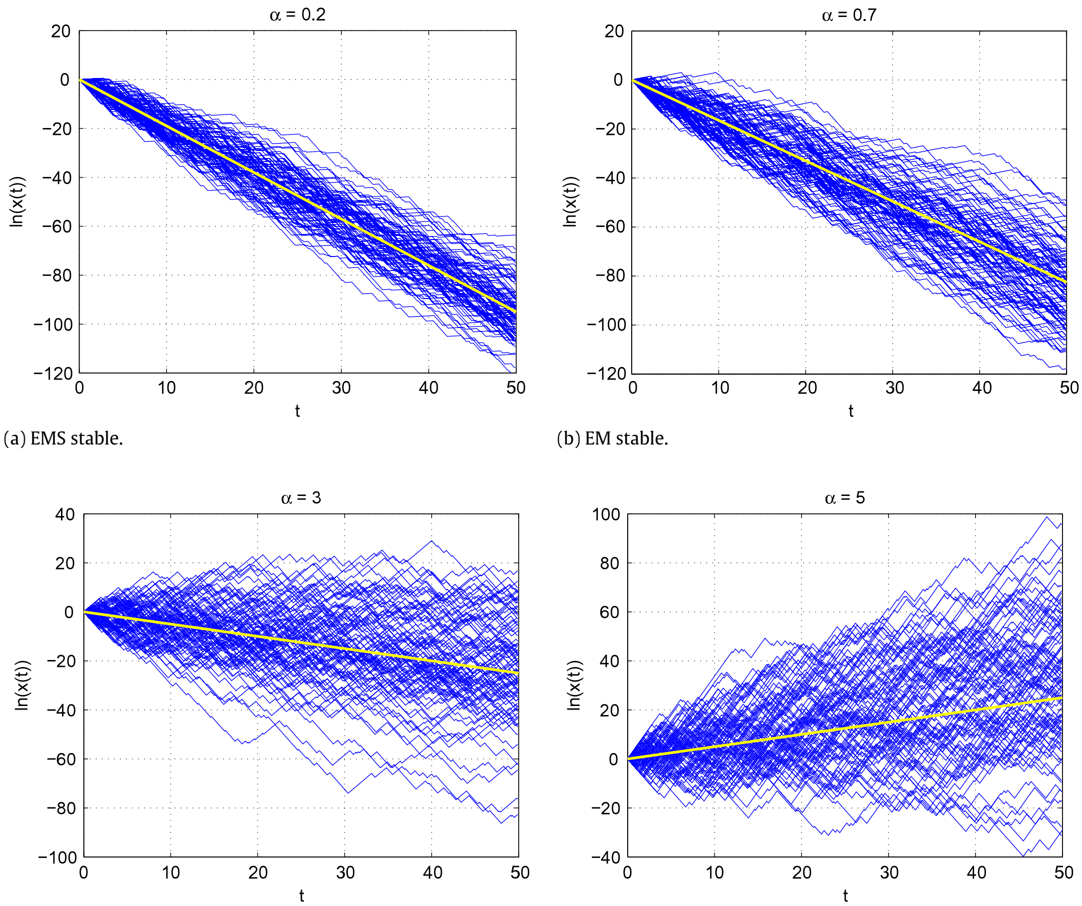
Example 1. Let $n = 1$, $N = 2$, $A_1 = \alpha$, $A_2 = -4$ and

$$\Lambda = \begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix}.$$

The associated stationary probability distribution is given by $\bar{\pi} = [0.5 0.5]'$. Since the system is scalar, condition (i) of Theorem 5 is necessary for EAS stability. Then, observing that $E_{\bar{\pi}}[\mu(A_\sigma)] = \bar{\pi}_1 A_1 + \bar{\pi}_2 A_2 = 0.5(\alpha - 4)$, one can conclude that the system is EAS stable if and only if $\alpha < 4$. To assess EM stability, use condition (iii) of Theorem 2. Since

$$\tilde{A} = \begin{bmatrix} \alpha - 1.5 & 1.5 \\ 1.5 & -5.5 \end{bmatrix}$$

it turns out that the system is EM stable if and only if both eigenvalues of $\tilde{A}$ are negative, namely for $\alpha < 12/11$. Repeating the same



(a) EMS stable.

(b) EM stable.

(c) EAS stable.

(d) Unstable.

Fig. 2. One hundred realizations of $\ln(x(t))$ in Example 1 for different values of the parameter α.

analysis with

$$\hat{A} = \begin{bmatrix} 2\alpha - 1.5 & 1.5 \\ 1.5 & -9.5 \end{bmatrix}$$

and checking condition (iii) of Theorem 3, it results that the system is EMS stable if and only if $\alpha < 12/19$. Evidently, the implications conjectured in Fig. 1 do not hold.

In order to appreciate the different behavior of the system for different values of the parameter α, 100 realizations of the stochastic process $x(t)$, starting from $x(0) = 1$, were simulated on a finite time-interval with $\alpha = 0.2$ (the PMJLS is EMS stable), $\alpha = 0.7$ (the PMJLS is EM stable), $\alpha = 3$ (the PMJLS is EAS stable) and $\alpha = 5$ (the PMJLS is unstable), respectively. The results are plotted in Fig. 2 in the four cases. The logarithmic scale on the vertical axis has been chosen for better readability. Note that EAS stability implies that almost all the realizations converge to $-\infty$. It is apparent that EM stability and EMS stability are more stringent. The straight line in each subplot represents the line $\bar{\lambda}t$ where $\bar{\lambda}$ is the expected value of the top Lyapunov exponent, which, in this scalar example, can be easily computed as $\bar{\lambda} = \bar{\pi}_1 A_1 + \bar{\pi}_2 A_2 = 0.5(\alpha - 4)$. The PMJLS is EAS stable if and only if the slope of this line is negative.

In the second Example, it is shown that a PMJLS composed of two unstable subsystems may still be stable.

Example 2. Let $n = 2$, $N = 2$ and

$$A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0.1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

Note that both subsystems are unstable. However, by letting $\eta_1 = 0.1$, $\eta_2 = -0.2$, $c_1 = [0.54530.3759]'$, $c_2 = [0.65070.3714]'$, it turns out that all conditions (iv) of Theorem 5 are satisfied, so that one can conclude that the PMJLS is EAS stable. By checking that both matrices $\tilde{A}$ of (4) and $\hat{A}$ of (6) are Hurwitz, it can be easily shown that the system is EM stable and EMS stable as well.

7. Concluding remarks

In this paper we have provided a thorough picture of stochastic stability of Positive Markov Jump Linear Systems. We have shown that these systems enjoy a number of specific properties induced by positivity of the state. In particular, it holds that 1-moment stability is equivalent to the asymptotic convergence of the expectation of the state (mean stability) and can be assessed through a necessary and sufficient condition corresponding to a Linear Programming feasibility problem. Since mean stability is less demanding than mean-square stability, such a result offers a viable alternative to ascertain almost-sure stability, which is recognized as being closer to the practical concerns of the user. Exploiting positivity, new sufficient conditions of almost-sure stability have also been derived, with different levels of conservatism and usability. In perspective, the results discussed in the paper will prove useful in tackling control and filtering problems of PMJLS's.

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Paolo Bolzern was born in Milano, Italy in 1955. He received his Doctor's degree (Laurea) in Electronic Engineering from the Politecnico di Milano in 1978. From 1979 to 1981 he worked on modeling and identification of environmental systems at IIASA, Laxenburg, Austria, and at the Politecnico di Milano. From 1983 to 1987, he was a research fellow with the Centro di Teoria dei Sistemi of the National Research Council (CNR) in Milan. In July 1987, he has joined the Politecnico di Milano, where he is currently a Full Professor of Automatic Control in the Dipartimento di Elettronica, Informazione e Bioingegneria. In the period 2000–2002 he served as vice-chairman of the Department and, from 2003 to 2011, he was the Coordinator of the Systems and Control area. He is a member of IEEE, an Associate Editor in the Conference Editorial Board of IEEE-CSS and in the EUCA (European Control Association) Conference Editorial Board, a representative of the General Assembly of EUCA, a member of the IFAC Technical Committee on Optimal Control and a member of the IFAC Publication Committee. He was recognized as an outstanding reviewer of the IEEE Transactions on Automatic Control (2001) and Automatica (2011–2012). His current research interests include robust filtering and control, coordination of multi-agent systems and control of deterministic and stochastic switched systems. On these and other related topics he has authored or co-authored some 80 research papers published in international journals and books. He is a co-author with R. Scattolini and N. Schiavoni of the textbook (in Italian, also translated into Spanish) *Fondamenti di Controlli Automatici* (McGraw-Hill, 2008).

Patrizio Colaneri was born in Palmoli, Italy, in 1956. He received the Laurea degree in Electrical Engineering in 1981 and the Ph.D. degree (Dottorato di Ricerca) in Automatic Control in 1988. After a few years in industry and at the National Research Council of Italy, he joined the Politecnico di Milano where he is full professor of Automatica and served as head of the Ph.D. school on ICT (2007–2009). He spent a semester at the Systems Research Center of the University of Maryland (1989) and at the Hamilton Institute of the National University of Ireland (2009). He also collaborates with the Johannes Kepler University in Linz since 2000. Dr. Colaneri was a YAP (Young Author Prize) finalist at the 1990 IFAC World Congress, Tallinn, USSR. He is the chair the IFAC Coordinating Committee on Design Methods, a member of the editorial board of Int. J. Applied and Computational Mathematics, a subject editor of the International Journal of Robust and Nonlinear Control and a Senior Editor of the IEEE Transaction on Automatic Control. He was a member of the Council of EUCA (European Union Control Association), and has been serving for six years as Associate Editor of Automatica (certificate of outstanding service). In 2010 he was elevated to the degree of IEEE Fellow for contributions on periodic and switching control. Since 2011 he is also a Fellow of IFAC (International Federation of Automatic Control). His main interests are in the area of periodic systems and control, robust filtering and control, and switching control. He has authored/co-authored more than 200 papers and five books, including "Control Theory and Design: an RH_2 and RH_∞ viewpoint", published by Academic Press in 1997, "Periodic Systems: Filtering and Control", Springer Verlag, 2009.

Giuseppe De Nicolao received the cum laude degree in electronic engineering from the Politecnico di Milano in 1986. He has been a visiting scientist at the Biomathematics and Biostatistics Unit of the Institute of Pharmacological Researches "Mario Negri" (1987–88) and the Department of Systems Engineering of the Australian National University, Canberra (1991). After being a researcher (1988–1992) at the Center of System Theory of the Italian National Research Council, in 1992 he joined the University of Pavia as an associate professor. Since 2000, he is full professor of Model Identification and Data Analysis in the Department of Industrial and Information Engineering of the University of Pavia. He has been a keynote speaker at the 1998 IFAC workshop on "Nonlinear model predictive control: Assessment and future directions for research". He has served as associate editor of the IEEE Transactions on Automatic Control, IEEE Transactions on Control Systems Technology, and Automatica. His research interests include Bayesian learning, neural networks, model predictive control, optimal and robust filtering and control, deconvolution techniques, modeling, identification model-based drug development, artificial pancreas, advanced process control and fault diagnosis for semiconductor manufacturing. On these subjects he has authored or co-authored more than 110 papers in peer-reviewed journals and is co-inventor of four patents.