Analytical method for stagnation point calculation: theoretical developments and application to a hydraulic barrier design (Sicily, Italy)

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The present research paper describes a new method to locate stagnation points, which does not require complex calculations, nor the development of a numerical model. Instead using a simple trigonometric relationship, the areas of location for the stagnation point, can be found, with respect to the centre of the hydraulic barrier. The trigonometric equation is used to draw triangles whose vertices correspond to stagnation points, supposing that they belong to the line bisector connecting the two wells; the interpolation of the vertices is the groundwater divide.

The computed results are compared to a numerical real case model (an industrial site in Sicily, Italy) to assess the performance of the new method.

Stagnation points are defined as the points in an aquifer where the equilibrium between all the forces causes the stagnation of groundwater. Determining their location, together with the location of areas where groundwater velocity is very low, is important in many situations, as previous works have shown (e.g. Kasenow 1968; Winter 1976; Mary & Anderson 1981, 1992); in particular:

(a) By interpolating all the stagnation points, it is possible to draw the groundwater divide of a capture zone. This is an absolutely essential step in order to evaluate the effectiveness of any groundwater remediation action.

(b) Stagnation points often identify areas where the flow field cannot be easily modified, hence the need to know their position in order to design a drainage system (Winter 1976).

The study of stagnation points and the corresponding analysis of well problems and drainage systems has been addressed by many authors (Muskat 1946; Hantush 1965; Fanelli 1971; Bear 1979; Javandel 1986; Strack 1989; Christ *et al.* 1999; Christ & Goltz 2002; Intaraprasong & Zhan 2007). Shan (1999) has provided an analytical solution for the groundwater divide in a two wells co-linear system, showing the importance of well locations. Christ & Goltz (2002) have determined an analytical method to capture zones for a non-co-linear system, showing that moving a well down gradient from the co-linear wells position, the stagnation points of the capture curve moves in the same direction. Furthermore, the same authors have proposed a general formulation for multiple linear wells. Skvortsov & Suyucheva (2004) have presented a solution for an injection well in a straight-parallel natural stream. Lastly, Lu *et al.* (2009) showed an analysis of stagnation points for a pumping well inside the recharge areas, demonstrating that a zero velocity point is function of different extraction rates. In fact, for a low pumping rate, there are always three stagnation points as the rate of infiltration that is not withdrawn, produces the separation streamlines outside the recharge area. However for the well with a high pumping rate, there is just one sole stagnation point outside the recharge area.

Colombo *et al.* (2012) have computed the stagnation point's position as a function of different locations and extracted flow rates, observing that for particular situations, the central stagnation point of the barrier is very far downstream from the wells. To identify these particular zero-velocity zones, a groundwater level survey is needed. The search for these sectors then takes place through the reconstruction of the network flow, with the help of a piezometer which highlights the distribution of the piezometric levels on the vertical up to the depth of interest. The understanding of the phenomena governing the formation and presence of stagnation points in the situations described above, is greatly improved by the availability of a quick and efficient computational method.

This paper describes a simple analytical trigonometric relationship to calculate the position of the stagnation point in a two well extraction system. The simplified mathematical procedure is applied to a real case problem and successfully verified by comparison to a numerical model representing the real setting.

Previous research: analytical method for computation of stagnation points

Shan (1999) and Christ & Goltz (2004) model's considered both multiple wells and a location in any point of the complex plane (x, y), but has been developed in the Colombo *et al.* (2012) study. In that study a complex analytical model has been applied in order to identify the location of stagnation points for different hydraulic barrier geometries. A simplified method is described in this paper to locate stagnation points in a two-well geometry. The method provides a convenient way to analyze the performance of hydraulic barriers.

The mathematical model presented in the previous work has undergone some simplifications, that is, the behavior of an homogeneous isotropic confined aquifer has been simulated, with uniform thickness B (m) and constant Darcy velocity U (m/s). A

(a)

(b)

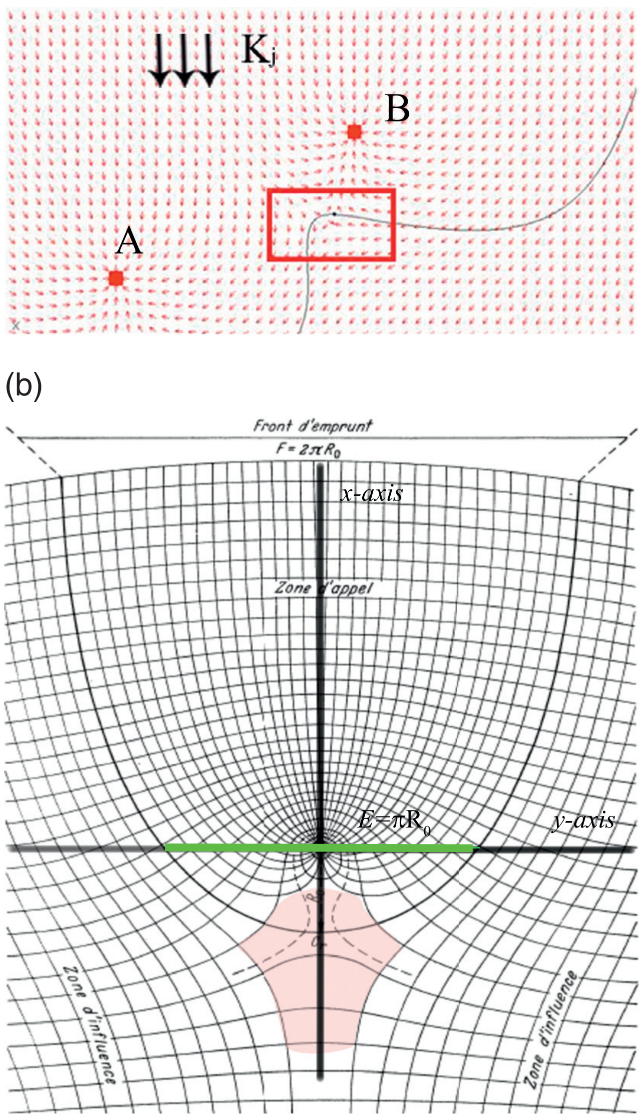

Fig. 1. (a) The red area (shaded in the print version) represents the lower velocity flow region (modified by Schoeller H. 1962). R_0 is the minimum distance between the well and the groundwater divide of the capture curve and the green line (thick horizontal line in the print version) represents the parameter E. **(b)** A typical formation of a stagnation point where the water is rather still. (the vectors have different direction. By increasing flow rate, a unique piezometric depression is formed as the sum of all wells superposition).

steady state groundwater flow is considered. The complex potential w (Javandel & Tsang 1984), due to the linearity of Laplace's formula, can be expressed as a superposition of the piezometric effects of pumping in several wells (both injecting or extracting) and of the uniform flow. Equation (1) shows that

$$w = \varphi + i\psi = -Jze^{-i\alpha} + \sum_{j=1}^{N} \frac{Q_j}{2\pi B} ln\left(z - z_j\right) + C \qquad (1)$$

where w is the complex potential of the overall system, J [m/s] is the Darcian velocity of a uniform regional flow, α is the angle between the regional flow direction and the x-axis, B [m] is the aquifer thickness, if $Q_j [m^3/s] \geq 0$, N is the number of wells, z ($z = x + iy$) is the coordinate in the complex plane where the potential w is evaluated, z_j ($z_j = a_j \pm ib_j$) the well coordinates j in the complex plane (x, y) where a [m] e b [m] are the coordinates of the wells in the real plane x, y, , $i = \sqrt{-1}$ and C is a constant of integration that depends on boundary condition.

The complex potential w may be separated into two parts: the real part Φ represents the constant head lines

$$\varphi = -J\left(xcos\alpha + ysin\alpha\right) + \sum_{j=1}^{N} \frac{Q}{4\pi B} ln\left[\left(x - a_j\right)^2 + \left(y - b_j\right)^2\right] \qquad (2)$$

and the imaginary part Ψ represents the stream function,

$$\Psi = J\left(xsin\alpha - ycos\alpha\right) + \sum_{j=1}^{N} \frac{Q_j}{2\pi B} tan^{-1}\left(\frac{y - b_j}{x - a_j}\right) \qquad (3)$$

In order to calculate the location of the groundwater divide generated by the hydraulic barrier, the stream function at a stagnation point, where velocity is zero, must be evaluated first. The stagnation point can be calculated (Christ & Goltz 2002) deriving *w* as a function of *z* and setting it equal to zero.

$$\frac{dw}{dz} = -Je^{-i\alpha} + \sum_{j=1}^{N} \frac{Q_j}{2\pi B\left(z - z_j\right)} \qquad (4)$$

For example for a 5 wells barrier a zero flow equation can be obtained as:

$$isin\alpha z^5 + \beta' z^4 - \gamma i z^3 - \delta z^2 + \varepsilon i z + \epsilon = 0 \qquad (5)$$

where the complex coefficients are:

$$\beta' = 2bsin\alpha + 5R_0, \gamma = b^2 sin\alpha + 5a^2 sin\alpha + 8R_0 b,$$
$$\delta = 15a^2 R_0 + 3b^2 R_0 + 8a^2 bsin\alpha,$$
$$\varepsilon = 16a^2 bR_0 + 4a^4 sin\alpha + 4a^2 b^2 sin\alpha,$$
$$\epsilon = 4a^4 R_0 + 4a^2 b^2 R_0$$

These values depend on the spatial distribution of the wells, on the groundwater flow direction and on radius R_0 *[m]* (in Fig. 1a R_0 is the minimum distance measured along the x-axis between the well and the groundwater divide).

A trigonometric computation of stagnation points for a hydraulic barrier

For a single well, the formation of stagnation points is a consequence of the perfect balance between the opposing forces in the fluid field; in particular it can be remarked that a sort of equilibrium can occur between the natural hydraulic gradient J and the hydraulic gradient of the piezometric depression induced by the pumping well. Where the natural hydraulic gradient (and the vector representing the natural velocity KJ of groundwater flow) has the same magnitude and a direction opposite to that of the hydraulic gradient caused by the well discharge, their application

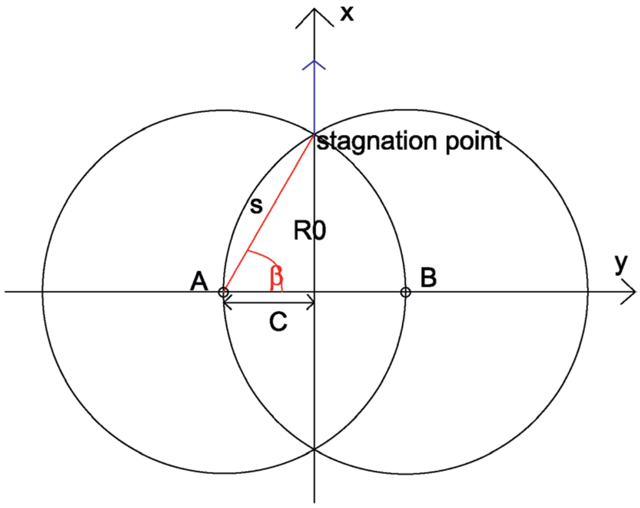

Fig. 2. Arrangement of two co-linear wells. It could be observed the formation of a stagnation point. The distance 2C between the two wells lies on the y axis. PS is the stagnation point as an intersection between the two groundwater divide with the same pumping rate.

point becomes a stagnation point. Its location matches the distance R_0 [m] from the well, which can be obtained from Bear (1979)'s relation. R_0, is commonly defined 'fictitious radius' of the well. This relation is valid only for a single well because it does not take into account the superposition between the cones of depression produced by the simultaneous operation of two or more wells.

The calculation of stagnation points as a function of aquifer parameters and flow rate can in fact be applied only in the case of two wells extracting the same flow rate in a homogeneous aquifer (as shown in Fig. 1b). Each extraction well will have an associated stagnation point (Javandel 1986; Shan 1999; Christ & Goltz 2002; Christ & Goltz 2004). However, under particular conditions, such as in the presence of infiltration (Lu *et al.* 2009) with an increasing withdrawal (Colombo *et al.* 2012), the stagnation points converge to a single location.

Under those assumptions, the stagnation point is found at the conjunction between the limits of the superposing cones of depression, as seen in Figure 1b which shows the output of a numerical model. The radius of influence of each of the wells, theoretically infinite, is in fact limited in real case by infiltration or recharge factors.

For a single well, the dimension E of the groundwater divide of the well shown in Figure 1a, is of great importance: it can be seen that only the water particles before the groundwater divide and at a distance not superior to the R_0 (Reduced Radius of influence, Schoeller (1962) or fictitious radius of influence) are attracted by the well.

Schoeller (1962) has demonstrated that the length of the water supply front F is proportional both to the radius R_0 and to the dimension E (Equation 6). In this case, using the Bear's equation (1979) relating the geometry of the cone of depression in an inclined water table to the flow rate and the aquifer transmissivity.

$$R_0 = \frac{Q}{2\pi TJ} = \frac{E}{\pi} = \frac{F}{2\pi} \tag{6}$$

where Q (m^3/s) is the well flow rate, T (m^2/s) is the aquifer transmissivity, J (-) is the groundwater natural gradient, E (m) is the width of the cone of depression. In Figure 1a, R_0 is the minimum distance, measured along the x-axis, between the well and the limit of the cone of depression. In any case, when the flow rates are different for two different configuration (i.e Q_1,Q_2=$2Q_1$), the stagnation point is still found at the crossing between the limits of the cones of depression, and it will be closer to the well pumping at the lower rate configuration (Javandel & Tsang 1984; Strack 1989; Christ & Goltz 2002; Intaraprasong & Zhan 2007).

Figure 2 shows the geometry for the problem of two wells separated by the distance 2C (Q being the flow rate extracted by each of the wells A and B). The angle β is formed between the line joining well A, the stagnation point and the horizontal line joining the wells. The intersection between the cones of depression of the two wells (circumferences) is the stagnation point which lies on the x-axis.

The β angle can be determined by applying Dupuit (1863)'s equation with the superposition of the groundwater direction J as follows

$$y - h = \frac{Q \ln \frac{s}{r}}{2\pi T} - \mathrm{J} s \sin\beta \tag{7}$$

where *s [m]* is the distance of stagnation point as shown in Figure 2 from the i-well and *r [m]* is the well radius, *y-h [m]* is the drawdown, *J [-]* is the groundwater natural gradient, *T [m$^{2/}$s]* is the aquifer transmissivity and *Q [m^3/s]* is the flow rate of each well. The intersection between two circumferences coincides with the maximum or the minimum of the function and is obtained deriving the Equation (7) with respect to x and equaling it to zero.

$$\frac{dy}{ds} = \frac{Q}{2\pi Ts} - \mathrm{J} \sin\beta = 0 \tag{8}$$

Dividing Equation (8) by J, the first term is the expression of the fictitious radius *R_0 [m]* Equation (6). Equation (8) can therefore be rewritten as follows:

$$s \sin\beta = R_0 \tag{8b}$$

Equation (8b) is verified with $\beta = 0$, where R_0 equals zero.
Considering the Figure 2, the following equation:

$$\tan\beta = \frac{R_0}{C} \tag{9}$$

combined with Equation (6) gives the final form Equation (10) that predicts the precise location of the central stagnation point for two wells with the same flow rate where C is the half distance between the two wells.

$$\tan\beta = \frac{Q}{2\pi CTJ} \tag{10}$$

In fact, the β angle calculcated using Equation (10) allows one to identify the downstream point of the piezometric depression, which is located near the symmetric axis of the co-linear distance between wells. The methodology of triangle construction has been applied to a real case (Sicily, Italy) in order to confirm its usefulness in detecting down-gradient influence of a hydraulic barrier in a polluted site.

An application of the stagnation point theory to a hydraulic barrier

A hydraulic barrier has been designed to prevent the contaminated groundwater from flowing towards the Mediterranean sea,

(a)

(b)

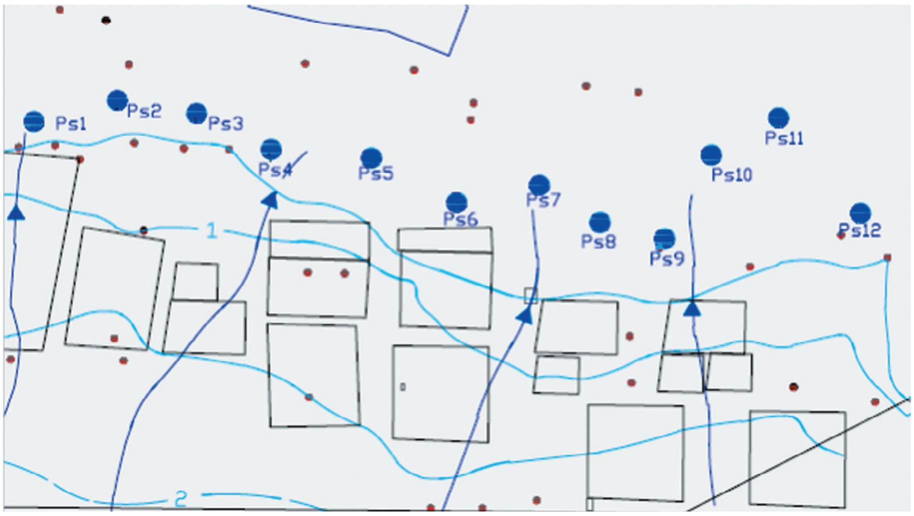

Fig. 3. (a) The studied area in Sicily. **(b)** The studied site: 12 wells hydraulic barrier located for defending Mediterranean sea by the pollution losses from reservoir located upstream the barrier.

at an industrial site along the Sicilian coast (Fig. 3a). After collecting the necessary field data, a barrier of 12 wells, all pumping the same flow rate, has been designed in order to avoid the advance of polluted particles in the sea (Fig. 3b). The effectiveness of the solution has been assessed through the application of Equation (10).

Equation (10) can estimate the unknown vertex of the triangles formed for each couple of wells following the groundwater gradient direction, expressed by blue lines in Figure 3. As aforementioned, the vertex corresponds to that area where the aquifer velocity is nearly zero. In this area, the water polluted particles are almost static.

The parameters used for the computation are in Table 1 where *J [-]* is the mean groundwater natural gradient, *Q [m³/h]* is the flow rate barrier variable from 6 to 7 m^3/h as a function of the location and the distance between wells; the flow rate Q is always constant for the two wells operating at the same time. *T [m²/s]* is the aquifer transmissivity in the studied area.

Equation (10) has been applied using the parameters shown in Table 1. The results are shown in Table 2.

The obtained β values allow the location of the unknown triangle vertex to be calculated, with respect to the line linking wells A and B. These vertices are located downstream from the groundwater divide and hence it is possible to assess the effectiveness of the barrier (it's capturing all the up-gradient water). The line connecting the stagnation points represents the groundwater divide drawn by the cones of depression of the barrier wells. Indeed the flow lines starting downstream from the line drawn by the stagnation points escape the wells capture. Furthermore, through the points located upstream of this line, the flow lines converge toward the wells.

Discussion of results

Equation (10) can be used to calculate the position of stagnation points, has been validated through comparison with the results of a model built with the numerical code Modflow (McDonald & Harbaugh 1984). For this purpose, a model has been designed using the same hydrogeological parameters listed in Table 3, which have been used in the analytical computation. The single-layer model domain has been discretized by a grid of 204 rows and 363 columns with dimensions varying between 100 m and 1 m, (the latter being the dimension in the well barrier area).

Constant head conditions have been applied at the Southern and Northern boundaries (122 m. a.s.l. at the Southern and 112.2 m a.s.l.

(a)

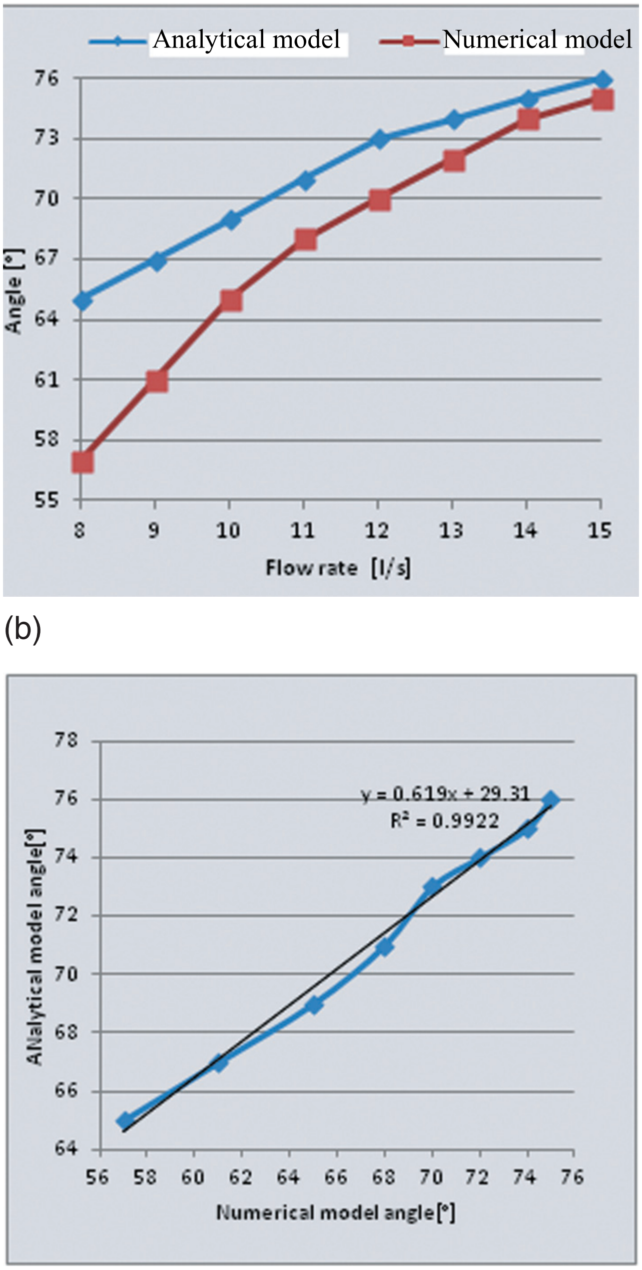

Fig. 4. (**a**) Behavior of angle in function to the extraction flow rate from the two wells (the blue line (online version only) is the analytical solution and the red (online version only) is the numerical one). (**b**) Correlation coefficient between analytical angle and numerical one.

at the Northern) in order to obtain a hydraulic gradient of 0.001. Each of the two wells has been simulated by applying the same time-variant flux condition. Table 4 lists the results obtained by the application of the analytical computation and the numerical model.

The comparison between the β angle values resulting in different settings highlights small differences in the range of about 1 to 8 degrees. The difference is minimal at the highest flow rates (from 8 to 15 l/s), as shown in Figure 4. This behaviour is due to the fact that at the lower flow rates, the barrier is ineffective as the cones of depression of the two wells become separated and two stagnation points are created by each well.

By increasing the well discharge, the two stagnation points start to converge into a unique point moving downstream, thus the

Table 1. *Hydrogeological parameters used to compute stagnation point with Equation (10)*

	PARAMETERS
Natural gradient j (-)	0.001
Flow rate Q (m^3/s)	From 0.0018 to 0.0021 for each couple of wells
Transmissivity T (m^2/s)	1.83E-03

Table 2. *R_o (m) and β values computed for each couple of wells of the hydraulic barrier*

Wells	Q(m^3/s)	β(RAD)	R_o(m)	β(°)
PS39-PS50	0.0020	0.98	173.70	56.237151
PS50-PS51	0.0018	1.03	151.99	59.197976
PS51-PS52	0.0018	0.99	151.99	56.698709
PS52-PS53	0.0021	0.96	182.39	55.273983
PS53-PS54	0.0021	0.98	182.39	56.394239
PS54-PS55	0.0021	1.07	182.63	61.392909
PS55-PS40	0.0018	1.04	152.23	59.672126
PS40-PS41	0.0018	1.08	151.99	61.854897
PS41-PS42	0.0021	0.98	182.39	56.019716
PS42-PS43	0.0021	1.13	182.39	64.757617

Table 3. *Hydrogeological parameters of the studied area*

k_J (-)	0.001
K (m/s)	0.001
B (m)	40

Table 4. *Comparison between analytical model and numerical one*

Flow rate [l/s]	β (analytical)	β (numerical)	Numerical-Analytical differences
8	65	57	−8
9	67	62	−6
10	69	66	−4
11	71	69	−3
12	73	71	−3
13	74	73	−2
14	75	74	−1
15	76	75	−1

effectiveness of the barrier is assured by the fact that the flow (or up gradient water) is completely captured by the wells. This behaviour is shown in Figure 5.

The location of stagnation points is independent from water density. The results of the computations (Fig. 5) highlight that the flow rate must be high enough to cause a sufficient superposition of the cones of depression (which can be measured in terms of the value of E as shown in Equation (6). The interpolation of the vertices of all the triangles shows a complete water capture.

According to the model, the presence of a very low seepage velocity areas - immediately downstream of the groundwater divide - is sharp, as shown in Figure 6 for the case of a withdrawal of 15 l/s per well. The yellow area is broadly extended downstream from the groundwater divide involving a portion of aquifer outside of the wells capture area. As computed in Table 4, in Figure 6 only two wells have been represented with the same discharge rate (PS4 and PS5); if all the well barriers discharge simultaneously, the superposition could be located more downstream the zero-velocity area.

Figure 6 shows the low velocity areas corresponding to the triangle vertex calculated using Equation (10). The area can also be

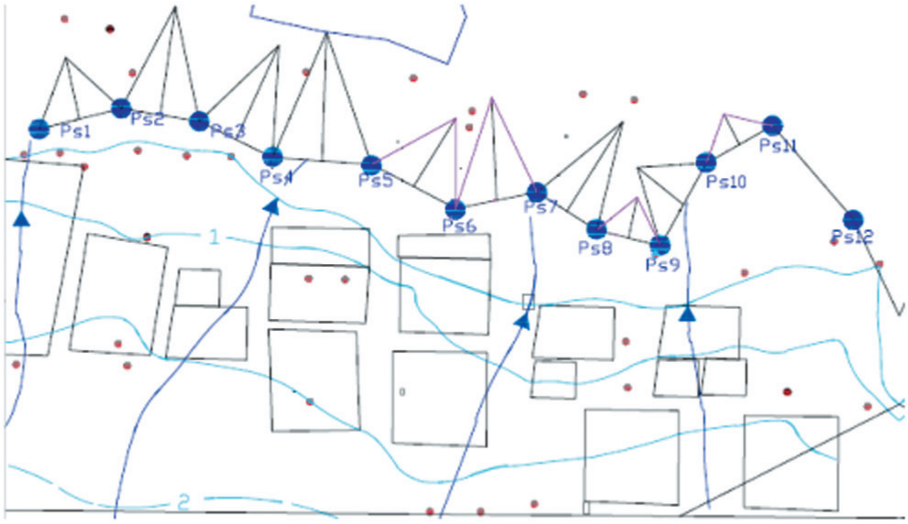

Fig. 5. Computation of triangle in the industrial site. The figure suggests that the flow rate is at least large enough to ensure water capture by superposition of groundwater divide.

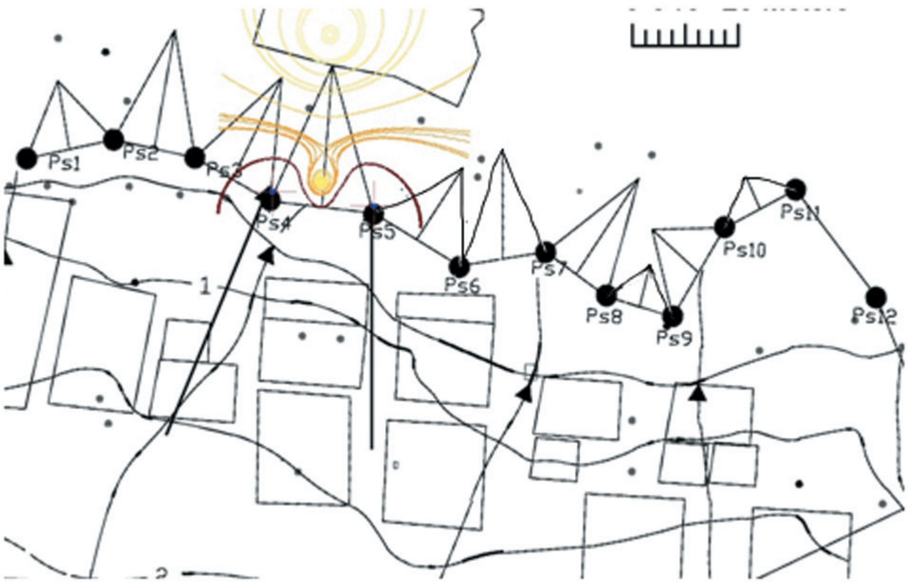

Fig. 6. Iso-velocity curves and stagnation point of the barrier. The low area velocity can be also more downstream than stagnation point itself (the grey area).

extended downstream from the watershed up to a distance of about 15 m, which is half the distance between the wells.

Conclusions

The identification and study of the formation of stagnation point is an important field of research with the application in designing optimal distance of well barriers.

The aim of this paper has been to provide a simple analytical relationship in order to identify the position of stagnation points for a two wells setting. In fact, Equation (10) allows the secondary stagnation point to be calculated, appearing along an equidistant line from two wells extracting the same flow rate in a homogeneous and isotropic aquifer. The stagnation point is the locus of equilibrium between the flow component due to well withdrawal and the natural flow.

Stagnation points can only develop along the equidistant lines between two wells and at a distance equal to the fictitious radius of influence. As secondary stagnation points belong to the limit of the capture area, the calcualtions presented in this study are useful in order to assess the effectiveness of a hydraulic barrier. Analyzing the velocity distribution, an area of stagnation (rather than a unique stagnation point) can be found in certain cases. The dimensions of the area are usually less than the half distance between the wells and their locations are generally downstream from the limit of the capture area.

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Appendix

Dividing Equation (4) by the Darcy velocity and introducing the fictitious radius R_0 (m), the potential function of geometry co-linear wells w can be obtained as follow:

$$w = ze^{-i\alpha} + R_0 \begin{bmatrix} ln(z_A) + ln(z_B - a - ib) + \\ ln(z_C - 2a) + ln(z_D + a - ib) + \\ ln(z_E + 2a) \end{bmatrix}$$

where a, b are the well coordinate j in the complex domain; the derivative of above equation becomes

$$\frac{dw}{dz} = e^{-i\alpha} + R_0 \begin{bmatrix} \left(\frac{1}{z_A}\right) + \frac{1}{(z_B - a - ib)} + \frac{1}{(z_C - 2a)} + \\ \frac{1}{(z_D + a - ib)} + \frac{1}{(z_E + 2a)} \end{bmatrix} = 0$$

By calculating the sum of the fraction in the previous equation, and rearranging the governing equation, Equation (5) can be obtained.

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