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ON THE EXISTENCE OF OPTIMAL CONTROLS FOR SPDES WITH BOUNDARY NOISE AND BOUNDARY CONTROL*

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Abstract. We consider a stochastic optimal control problem for a heat equation with boundary noise and boundary control. Under suitable assumptions on the coefficients, we prove existence of optimal controls in a strong sense by solving the related stochastic Hamiltonian system.

Key words. stochastic control, maximum principle, stochastic evolution equation, forward-backward stochastic differential system

AMS subject classifications. 93E20, 60H30, 60H15

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1. Introduction.

In this paper we are concerned with the existence of optimal controls for a stochastic optimal control problem related to the following stochastic heat equation, in which boundary noise and boundary control are allowed:

(1.1)
$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(t,\xi) + b(\xi)u^0(t,\xi) + g(\xi)\dot{W}(t,\xi), \qquad t \in [0,T],\ \xi \in (0,\pi), \\ y(0,\xi) = x(\xi), \\ \dfrac{\partial y}{\partial \xi}(t,0) = u_t^1 + \dot{\tilde{W}}_t, \quad \dfrac{\partial y}{\partial \xi}(t,\pi) = u_t^2. \end{cases}$$

In the above equation $\tilde{W}$ is a standard real Wiener process and $\dot{W}(\tau,\xi)$ is a space-time white noise on $[0,T]\times[0,\pi]$; $\tilde{W}$ and W are both defined on a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$ and are independent. By $\{\mathcal{F}_t,\ t\in[0,T]\}$ we will denote the natural filtration of $(\tilde{W},W)$, completed in the usual way; u^0 and (u^1,u^2) are $\mathcal{F}_t$-predictable square integrable processes and represent, respectively, the distributed and the boundary control.

The majority of existing articles on stochastic control deal only with distributed parameter controls (see, for instance, [12] and the references therein). Such a situation is not always realistic, since in practice such controls are not easy to implement or do not occur naturally. So in this paper we allow the control to act also on the boundary. Boundary control problems have been widely studied in the deterministic literature (see [24]) and have been addressed in the stochastic case as well (see [10], [16], [23], [28]). Notice that we are able to treat equations where the control acts through the entire boundary, while the noise affects only one point at the boundary; our treatment is different from [8], where the techniques used impose some restrictions in this sense.

The problem is considered in its *strong formulation*, i.e., without changing the reference probability space $(\Omega,\mathcal{F},\mathcal{P})$. The stochastic optimal control problem consists

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of minimizing the following cost functional over all admissible controls:

$$J(x,u^0,u^1,u^2)=\mathbb{E}\int_0^T\int_0^\pi(\bar{l}^0(s,\xi,y(s,\xi))+\bar{g}(u_s^0(\xi),u_s^1,u_s^2))\,d\xi\,ds + \mathbb{E}\int_0^\pi \bar{h}(\xi,y(T,\xi))\,d\xi, \tag{1.2}$$

where $\bar{g}$ and $\bar{h}$ satisfy suitable assumptions specified in section 2.2. Here we need to mention that $\bar{g}$ is allowed to have quadratic growth with respect to the control, and the control processes are not necessarily bounded. Equation (1.1) will be reformulated as a stochastic evolution equation in $H=L^2((0,\pi))$:

$$\begin{cases} dX_t = AX_t\,dt + [(\lambda-A)D+B]u_t\,dt + (\lambda-A)D_1\,d\tilde{W}_t + G\,dW_t, & t\in[0,T],\\ X_0 = x, \end{cases} \tag{1.3}$$

where B and G are, as usual, the multiplication operators related to b and g, respectively. The operators D and D_1 transform boundary data in elements of the domain of a suitable fractional power of $(\lambda-A)$ so that both $(\lambda-A)D$ and $(\lambda-A)D_1$ are unbounded operators. Notice that (1.3) can be considered as the model for a more general class of state equations; see section 2.2 for more details.

An approach to proving existence of optimal controls is the dynamic programming principle, and the solution, in a sufficiently regular sense, e.g., mild, of the related Hamilton–Jacobi–Bellman (HJB) equation. Because of the presence of the boundary noise, the transition semigroup related to (1.3) does not have sufficient smoothing properties, so the associated HJB equation cannot be solved in a mild sense by a fixed point argument, as done in [14] and, under weaker assumptions on the transition semigroup, in [29].

The HJB equation is solvable in the sense of viscosity solutions (see, e.g., [16]), and the presence of the noise as a forcing term is necessary in their approach. Moreover, since in (1.3) the control is not assumed to be in the image of G nor in the image of $(\lambda-A)D_1$, the HJB equations cannot be solved by means of backward stochastic differential equations (BSDEs in the following); see the pioneering paper [32] and the infinite dimensional extension in [12]. When HJB equations can be solved by means of BSDEs, boundary noise and boundary control problems for the heat equations are treated as in [8], in the case of Neumann boundary conditions, and the techniques have been extended to the case of Dirichlet boundary conditions in [30], by also using results in [11]. We also mention that in the dynamic programming approach, existence of optimal controls is proved in the weak sense, since once the HJB equation is solved, the synthesis of the optimal controls is subject to the solution of the so-called closed loop equation. In many cases the closed loop equation can be solved only in the weak sense because of the lack of the regularity of the feedback law.

Concerning other possible approaches to proving the existence of optimal controls, in [13], by extending finite dimensional techniques, existence of optimal controls in the case of Hilbert space valued controlled diffusions is proved in a relaxed sense. In [5] existence of *quasi*-optimal controls is proved for a control problem related to a controlled state equation with distributed control and noise via the Ekeland principle. Their setting is infinite dimensional as in the present paper, but they do not prove existence of optimal controls in a strong sense and, moreover, in the state equation no unbounded terms are allowed. On the other hand, they can bypass convexity assumptions either on the coefficients (still very regular) of the cost functional or of the control space U.

Another approach to proving existence of optimal controls is related to the stochastic maximum principle (see, e.g., [20]), which provides necessary conditions for optimality. When these conditions are also sufficient, existence of optimal controls can be proved by solving the related forward-backward stochastic Hamiltonian system; see, e.g., [21]. In both [20] and [21] the setting is finite dimensional. In this paper we generalize this approach to the infinite dimensional setting. The maximum principle (see [18]), where the boundary case is treated, provides as usual only necessary conditions for the optimal control to be verified. In the present paper we prove further that, under suitable assumptions—see section 2.2—these conditions are indeed sufficient. So the solution to the related Hamiltonian system fully characterizes the optimal control. The Hamiltonian system is a fully coupled forward-backward system, and in our case it turns out to be

$$
(1.4)\quad \begin{cases} d\bar{X}_t = A\bar{X}_t\,dt + [E+B]\gamma([E+B]^*\bar{Y}_t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G(t,\bar{X}_t)\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + l_x(t,\bar{X}_t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t\in[0,T], \\ \bar{X}_0 = x,\ \bar{Y}_T = -h_x(\bar{X}_T), \end{cases}
$$

where $\bar{X}$, $\bar{Y}$, and $(\bar{Z},\tilde{Z})$ are all infinite dimensional. $H(t,x,u,y) := -l(t,x,u) + \langle [E+B]^*y, u\rangle$, is the Hamiltonian function, and $\gamma : H \to U$ is such that $H(t,x,\gamma([E+B]^*y),y) = \inf_{u\in U} H(t,x,u,y)$. Because of the infinite dimensional setting and of the presence of unbounded operators, the results obtained in the solution of this infinite dimensional forward-backward system are of independent interest.

Indeed, the solution of fully coupled forward-backward systems is a difficult topic already in the finite dimensional case; see [1] and again [27] for examples of finite dimensional FBSDEs where there is no hope of proving existence of a solution.

Among the large literature in finite dimensions (see, e.g., the book [27]), we can distinguish two main approaches. The first approach, known as the *four-step scheme*, relies on the connections between SDEs with deterministic coefficients and nonlinear PDEs; see the pioneering paper [26]. Such an approach is not suitable for an infinite dimensional extension, since in this case less a priori estimates on the solution of the related PDE are known. A local existence for an infinite dimensional FBSDE is proved in [17] by fixed point: that approach is successful there also because there are no other unbounded operators except from the infinitesimal generators A and B, so that result is not directly applicable to the present case. Moreover, in [17] existence and uniqueness in an arbitrary length time interval is not achieved in general but only in the case when the forward equation fulfills the *structure condition* and the backward equation is one dimensional.

The second approach applies under monotonicity assumptions: different types of conditions have been investigated in this framework and we refer the reader to Hu and Peng [22], Peng and Wu [35], Yong [36], and Pardoux and Tang [33].

In the present paper, we solve FBSDE (1.4) by adapting the *bridge method* introduced in [22] to the infinite dimensional framework. New difficulties arise due to the presence of the infinitesimal generators and also because an unbounded operator is applied to the backward unknown Y in the forward equation so that one has to prove some extra regularity for Y in order to give meaning to the system in the space

H. The regularity of the adjoint unknown is a typical task when one wants to prove maximum principle in infinite dimension (see [20] and [18]); in this case new difficulties arise since the backward equation is coupled with the forward and the whole system has to be considered. In order to apply the *bridge method* we have to study the following auxiliary FBSDE:

$$\begin{cases} d\bar{X}_t = A\bar{X}_t\,dt - [E+B][E+B]^*\bar{Y}_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + \bar{X}_t\,dt + h_0(t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t \in [0,T], \\ \bar{X}_0 = x,\; -\bar{Y}_T = \bar{X}_T + g_0. \end{cases} \tag{1.5}$$

Unlike in [22], this linear auxiliary FBSDE is not immediately solvable. To solve FBSDE (1.5), we notice that such a system is the Hamiltonian system associated with an affine quadratic optimal control problem with state equation

$$\begin{cases} dX_t = AX_t\,dt + [E+B]u_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \qquad t \in [0,T], \\ X_0 = x \end{cases} \tag{1.6}$$

and cost functional

$$J(x,u) = \frac{1}{2}\mathbb{E}\int_0^T (|X_t + h_0(t)|^2 + |u_t|^2)\,dt + \frac{1}{2}\mathbb{E}|X_T + g_0|^2, \tag{1.7}$$

where b_0 and h_0 are suitable stochastic processes. Therefore we introduce the Riccati equation (deterministic) corresponding to the linear terms and a backward SDE to deal with the affine terms (see section 3.2) in order to get a solution to system (1.5). Again, because of the infinite dimensional setting and of the presence of unbounded operators, the solution and the regularity of this auxiliary backward SDE are of independent interest.

Once we prove that system (1.5) has a unique solution, for every suitable b_0 and h_0 we can start to "build" the *bridge* to get a solution to our original system (1.4) and then eventually solve our control problem.

The paper is organized as follows. In section 2 we state the notation and the problem; in section 3 we collect results on the stochastic maximum principle in the boundary case, prove sufficient conditions for optimality, and finally state our main result on the existence of optimal controls; and in section 4 we prove existence and uniqueness of a mild solution for the stochastic Hamiltonian system by applying the bridge method to this setting, and we conclude by proving the existence of optimal controls.

2. Preliminaries and statement of the problem.

2.1. Notation. Given a Banach space X, the norm of its elements x will be denoted by $|x|_X$, or even by $|x|$, when no confusion is possible. If V is another Banach space, $L(X,V)$ denotes the space of bounded linear operators from X to V, endowed with the usual operator norm. Finally, we say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X;V)$ if it is continuous, Gâteaux differentiable on X, and $\nabla F : X \to L(X,V)$ is strongly continuous. The letters Ξ, H, K, and U will always be used to denote Hilbert spaces. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, equipped with a subscript to specify the space, if necessary. All the Hilbert spaces are assumed to be real and separable; $L_2(\Xi, H)$ is the space of Hilbert–Schmidt operators from Ξ to H, respectively.

Given an arbitrary but fixed time horizon T, we consider all stochastic processes as defined on subsets of the time interval $[0,T]$. Let $Q \in L(K)$ be a symmetric nonnegative operator, not necessarily trace class, and let $\tilde{W} = (\tilde{W}_t)_{t\in[0,T]}$ be a Q-Wiener process with values in K, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $W = (W_t)_{t\in[0,T]}$ be a cylindrical Wiener process with values in Ξ, defined on the same probability space and independent of $\tilde{W}$. By $\{\mathcal{F}_t,\ t \in [0,T]\}$ we will denote the natural filtration of $(\tilde{W}, W)$, augmented with the family $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}$; see, for instance, [6] for its definition. Obviously, the filtration $(\mathcal{F}_t)$ satisfies the usual conditions of right-continuity and completeness. All the concepts of measurability for stochastic processes will refer to this filtration. By $\mathcal{P}$ we denote the predictable σ-algebra on $\Omega \times [0,T]$ and by $\mathcal{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ.

Next we define three classes of stochastic processes with values in a Hilbert space V.

- $L^2_{\mathcal{P}}(\Omega \times [0,T]; V)$ denotes the space of equivalence classes of processes $Y \in L^2(\Omega\times[0,T]; V)$ admitting a predictable version. It is endowed with the norm

$$|Y| = \left(\mathbb{E}\int_0^T |Y_s|^2\, ds\right)^{1/2}.$$

- $C_{\mathcal{P}}([t,T]; L^p(\Omega; V))$, $p \in [1, +\infty]$, $t \in [0,T]$, denotes the space of V-valued processes Y such that $Y : [t,T] \to L^p(\Omega, V)$ is continuous and Y has a predictable modification, endowed with the norm

$$|Y|_{C_{\mathcal{P}}([t,T];L^p(\Omega;V))} = \left(\sup_{s\in[t,T]} \mathbb{E}|Y_s|^p_V\right)^{1/p}.$$

  Elements of $C_{\mathcal{P}}([t,T]; L^p(\Omega; V))$ are identified up to modification.
- For a given $p \geq 2$, $L^p_{\mathcal{P}}(\Omega; C([0,T]; V))$ denotes the space of predictable processes Y with continuous paths in V such that the norm

$$\|Y\|_p = \left(\mathbb{E}\sup_{s\in[0,T]} |Y_s|^p\right)^{1/p}$$

  is finite. The elements of $L^p_{\mathcal{P}}(\Omega; C([0,T]; V))$ are identified up to indistinguishability.

Given an element Φ of $L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi, V))$ or of $L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(K, V))$, the Itô stochastic integrals $\int_0^t \Phi(s)\, dW(s)$ and $\int_0^t \Phi(s)\, d\tilde{W}(s)$, $t \in [0,T]$, are V-valued martingales belonging to $L^2_{\mathcal{P}}(\Omega; C([0,T]; V))$. The previous definitions have obvious extensions to processes defined on subintervals of $[0,T]$ or defined on the entire positive real line $\mathbb{R}^+$.

2.2. Optimal control problem and state equation. Let H be a separable real Hilbert space, and let U be a separable Hilbert space, called the space of controls. We set the space $L^2_{\mathcal{P}}(\Omega\times[0,T]; U)$ to be the space of admissible controls, and we denote it by $\mathcal{U}$.

We make the following assumptions that we denote by (A):

(A.1) $A : D(A) \subset H \to H$ is a linear, unbounded operator that generates a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ which is also analytic and such that $|e^{tA}|_{L(H,H)} \leq e^{\omega t}$, $t \geq 0$, for some $\omega \in \mathbb{R}$. This means in particular that every $\lambda > \omega$ belongs to the resolvent set of A.

(A.2) $B \in L(U;H)$ and $G \in L(\Xi, H)$, and there exist constants $\Delta > 0$ and $\gamma \in [0, 1/2[$ such that

$$|e^{sA}G|_{L_2(\Xi,H)} \leq \frac{\Delta}{(1 \wedge s)^\gamma}$$

for every $s \in \mathbb{R}^+$.

(A.3) D is a continuous linear operator $D : U \to D((\lambda - A)^\alpha)$ for some $\frac{1}{2} < \alpha < 1$ and $\lambda > \omega$; see, for instance, [25] or [34] for the definition of the fractional power of the operator A.

(A.4) D_1 is a linear operator $D_1 : K \to H$, and there is a constant $\frac{1}{2} < \beta < 1$ such that the following holds:

$$|e^{tA}(\lambda - A)D_1|_{L_2(K,H)} \leq \frac{C}{t^{1-\beta}}$$

for some $\lambda > 0$.

Remark 2.1. Notice that D_1 and D can have the same structure; indeed, if D_1 takes values in $D((\lambda - A)^\beta)$ and K is finite dimensional, then (A.4) holds. On the other hand, by the analyticity of A, also for D an estimate similar to the one for D_1 may follow.

We repeat for convenience the *state equation*, which is

$$\begin{cases} dX_t = AX_t\,dt + [(\lambda - A)D + B]u_t\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, & t \in [0, T], \\ X_0 = x; \end{cases} \tag{2.1}$$

from now on we will denote for simplicity $(\lambda - A)D := E$.

We will seek a *mild* solution to this equation, in the sense of [6], which is an $(\mathcal{F}_t)$-predictable process $X_t,\ t \in [0, T]$, with a continuous path in H such that $\mathcal{P}$-a.s.

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}[E + B]u_s\,ds + \int_0^t e^{(t-s)A}(\lambda - A)D_1\,d\tilde{W}_s + \int_0^t e^{(t-s)A}G\,dW_s, \qquad t \in [0, T]. \tag{2.2}$$

The *cost functional* to minimize, which depends on the initial state x and the control $u \in \mathcal{U}$, is

$$J(x, u) = \mathbb{E}\int_0^T l(t, X_t, u_t)\,dt + \mathbb{E}h(X_T), \tag{2.3}$$

where l and h verify (B).

(B.1) l is measurable; for all $t \in [0, T]$ and for all $u \in U$, $l(t, \cdot, u) \in \mathcal{G}^1(H; \mathbb{R})$; for all $t \in [0, T]$ and all $x \in H$, $l(t, x, \cdot) \in \mathcal{G}^1(U; \mathbb{R})$. Moreover, there exists a constant $\Delta > 0$ such that

$$|l_x(t, x, u)| + |l_u(t, x, u)| \leq \Delta(1 + |x|_H + |u|_U) \tag{2.4}$$

for all $t \in [0,T]$, $x \in H$, and $u \in U$.

(B.2) h is continuous and convex; moreover, $h(\cdot) \in \mathcal{G}^1(H;\mathbb{R})$ and there is a constant $\Delta > 0$ such that

$$|h_x(x)| \leq \Delta(1 + |x|_H) \tag{2.5}$$

for all $x \in H$. Moreover, there exists a constant $c_1 > 0$ such that

$$\langle h_x(x_1) - h_x(x_2), x_1 - x_2\rangle_H \leq -c_1|x_1 - x_2|^2 \text{ for any } x_1, x_2 \in H. \tag{2.6}$$

(B.3) The map l can be decomposed as $l(t,x,u) = l^0(t,x) + g(u)$, where l^0 and g are two convex functions. Moreover, there exists a constant $c_1 > 0$ such that

(2.7)

$$\langle l^0_x(t,x_1) - l^0_x(t,x_2), x_1 - x_2\rangle_H \geq c_1|x_1 - x_2|^2 \text{ for any } x_1, x_2 \in H, t \in [0,T].$$

(B.4) For any $t \in [0,T], x \in H, y \in D(E^*)$, we define

$$H(t,x,u,y) := -l(t,x,u) + \langle [E+B]^*y, u\rangle,$$

and we assume that there exists a function $\gamma : H \to U$ such that

$$H(t,x,\gamma([E+B]^*y),y) = \inf_{u\in U} H(t,x,u,y). \tag{2.8}$$

We assume, moreover, that there exist positive constants c_1 and Δ such that

$$\langle \gamma(y_1) - \gamma(y_2), y_1 - y_2\rangle_H \leq -c_1|y_1 - y_2|^2 \text{ for any } y_1, y_2 \in H, \tag{2.9}$$

$$|\gamma(y_1) - \gamma(y_2)|_H \leq \Delta|y_1 - y_2| \text{ for any } y_1, y_2 \in H. \tag{2.10}$$

Remark 2.2. Requiring that $l = l^0 + g$ is standard in the stochastic optimal control literature, even when considering the case of bounded controls; see, e.g., [15]; see also [12] where the case of $l = l^0 + g$ is considered as a case where all their hypotheses are verified.

Moreover, in the quadratic case considered here such a decomposition, or more generally, conditions on l similar to (B.3) and (B.4), can be found in both [15] and [31]. Besides their clear restrictions, they allow us to treat cost functionals that are not purely quadratic in x or in u, so that the techniques used in the LQ (linear quadratic) framework (such as the exponential transformation (see, e.g., [14], where it is briefly introduced) and the Riccati theory) cannot be exploited.

2.3. Heat equation with Neumann boundary conditions. In this section we present a concrete stochastic control problem that we will be able to treat, and we show how this model fits the "abstract" setting of section 2.2. As already stated in the introduction, one can refer to the book [24] for a review of the models that can be treated by this class of equations. We consider a heat equation on the interval $(0,\pi)$ with boundary noise and boundary control, and we focus our attention on the case where the control affects the entire boundary, and the noise affects only one point at the boundary. Namely, we consider the following equation:

(2.11)

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(t,\xi) + b(\xi)u^0(t,\xi) + g(\xi)\dot{W}(t,\xi), \qquad t \in [0,T],\ \xi \in (0,\pi), \\ y(0,\xi) = x(\xi), \\ \dfrac{\partial y}{\partial \xi}(t,0) = u^1_t + \dot{\tilde{W}}_t, \quad \dfrac{\partial y}{\partial \xi}(t,\pi) = u^2_t. \end{cases}$$

In the above equation $\tilde{W}$ is a standard real Wiener process and $\dot{W}(\tau,\xi)$ is a space-time white noise on $[0,T]\times[0,\pi]$; $\tilde{W}$ and W are independent. We will give sense to the notion of solution in the following.

We reformulate (2.11) as a stochastic evolution equation in $H = L^2(0,\pi)$. The operator A stands for the Laplace operator with homogeneous Neumann boundary conditions, which is the generator of an analytic semigroup in H:

$$\mathcal{D}(A) = \left\{ y \in H^2(0,\pi) : \frac{\partial y}{\partial \xi}(0) = \frac{\partial y}{\partial \xi}(\pi) = 0 \right\}, \qquad Ay = \frac{\partial^2 y}{\partial \xi^2} \text{ for } y \in \mathcal{D}(A).$$

The control process $u \in L^2_{\mathcal{P}}(\Omega \times [0,T], U)$, where $U = L^2(0,\pi)\times\mathbb{R}^2$ and $u = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \end{pmatrix}$. Fix $\lambda > 0$ and define

$$b^1(\xi) = -\frac{\cosh(\sqrt{\lambda}(\pi-\xi))}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)}, \qquad b^2(\xi) = \frac{\cosh(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)},$$

and note that they solve the Neumann problems

$$\begin{cases} \dfrac{\partial^2 b^i}{\partial \xi^2}(\xi) = \lambda b^i(\xi), \qquad \xi \in (0,\pi),\ i = 1,2, \\ \dfrac{\partial b^1}{\partial \xi}(0) = 1, \quad \dfrac{\partial b^1}{\partial \xi}(\pi) = 0, \\ \dfrac{\partial b^2}{\partial \xi}(0) = 0, \quad \dfrac{\partial b^1}{\partial \xi}(\pi) = 1. \end{cases}$$

So $b^i \in \mathcal{D}(\lambda - A)^\alpha = H^{2\alpha}$ for $1/2 < \alpha < 3/4$.

Equation (2.11) can now be reformulated as

(2.12)
$$\begin{cases} dX_t = AX_t\,dt + [(\lambda - A)D + B]u_t\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \qquad t \in [0,T], \\ X_0 = x, \end{cases}$$

where, for $u \in U$ and $h \in H$, $Du = (0, b^1(\cdot)u^1(\cdot), b^2(\cdot)u^2(\cdot))$, $D_1 = (0, b^1(\cdot), 0)$, $B = (b(\cdot), 0, 0)$, $Gh = g(\cdot)h(\cdot)$. With the notation of section 2.2, $K = \mathbb{R}$ and $\Xi = H$.

Equation (2.12) is still formal, since $(\lambda - A)D$ and $(\lambda - A)D_1$ do not take their values in H, and hence the precise meaning of (2.12) is given by its mild formulation.

An H-valued predictable process X is called a mild solution to (2.12) on $[0,T]$ if

$$\mathbb{E}\int_0^T |X_r|^2 dr < +\infty$$

and, for every $0 < t < T$, X satisfies the integral equation

$$X_t = e^{tA}x + \int_0^t e^{(t-r)A}[(\lambda - A)D + B]u_r dr + \int_0^t e^{(t-r)A}(\lambda - A)D_1\,d\tilde{W}_r + \int_0^t e^{(t-r)A} G dW_r.$$

Since $b^i \in \mathcal{D}(\lambda - A)^\alpha = H^{2\alpha}$, for $1/2 < \alpha < 3/4$, and by the analyticity of the semigroup e^{tA}, $t \geq 0$, the integral $\int_0^t e^{(t-r)A}(\lambda - A)Du_r dr$ and the stochastic integral $\int_0^t e^{(t-r)A}(\lambda - A)D_1\,d\tilde{W}_r$ are well defined; see also [8].

Notice that (2.12) does not satisfy any structure condition suitable to treat the related stochastic optimal control problem using backward stochastic differential equations, as in [8] and [30], where the case of a heat equation with Dirichlet boundary control and boundary noise is considered. Notice that in the present example, different from [8] and [30], the control affects the system in 0 and π and the noise acts only on 0, so that $\mathrm{Im}(D) \nsubseteq \mathrm{Im}(D_1)$.

The optimal control problem we wish to treat in this paper consists of minimizing the finite horizon cost

(2.13)
$$J(x, u^0, u^1, u^2) = \mathbb{E}\int_0^T \int_0^\pi \bar{l}(s, \xi, y(s,\xi), u^0_s(\xi), u^1_s, u^2_s)\, d\xi\, ds + \mathbb{E}\int_0^\pi \bar{h}(\xi, y(T,\xi))\, d\xi$$

over all admissible controls. The cost functional (2.13) can be written in an abstract way as in (2.3) by setting, for $s \in [0,T],\ x \in H,\ u \in U$,

$$l(s,x,u) = \int_0^\pi l(s, \xi, x(\xi), u^0_s(\xi), u^1_s, u^2_s), \quad h(x) = \int_0^\pi \bar{h}(\xi, x(\xi)).$$

We consider costs of the form $\bar{l}(s,\xi,y,u^0,u^1,u^2) = \bar{l}^0(s,\xi,y) + \bar{g}(\xi,u^0,u^1,u^2)$ so that l can be decomposed as in (B.3). From $\bar{l}^0$ and $\bar{g}$ we define l^0 and g as we have defined l:

$$l^0(s,x) = \int_0^\pi \bar{l}^0(s,\xi,x(\xi))d\xi, \quad g(u) = \int_0^\pi \bar{g}(\xi, u^0(\xi), u^1, u^2)d\xi.$$

We make suitable assumptions on $\bar{l}^0$, $\bar{g}$, $\bar{h}$ such that l^0, g, and h satisfy assumptions (B1)–(B3).

Hypothesis 2.3. *We assume the following:*

(1) *The map* $\bar{h} : [0,\pi]\times\mathbb{R} \to \mathbb{R}$ *is measurable, for a.a.* $\xi \in [0,\pi]$ $\bar{h}(\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ *it is continuous, convex, and differentiable, and there exist* $\Lambda \in L^\infty([0,\pi])$ *such that*
$$|h_x(\xi,x)| \le \Lambda(\xi)(1+|x|).$$
Moreover, we assume that for a.a. $\xi \in [0,\pi]$ $\bar{h}(\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ *it is dissipative; namely, for every* $x_1, x_2 \in \mathbb{R}$,
$$(\bar{h}(\xi,x_1) - \bar{h}(\xi,x_2))(x_1 - x_2) \le -c_1(x_1-x_2)^2$$
for some positive constant c_1.

(2) *The map* $\bar{l}^0 : [0,T]\times[0,\pi]\times\mathbb{R} \to \mathbb{R}$ *is measurable and for a.a.* $t \in [0,T]$ *and* $\xi \in [0,\pi]$, $\bar{l}^0(t,\xi,\cdot)$ *is continuous, convex, and differentiable, and there exists* $\Lambda \in L^\infty([0,\pi])$ *such that for all* $\xi \in [0,\pi]$ *and for all* $x \in \mathbb{R}$,
$$|\bar{l}^0_x(t,\xi,x)| \le \Lambda(\xi)(1+|x|).$$
Moreover, we assume that for a.a. $t \in [0,\pi]$ *and* $\xi \in [0,\pi]$ $\bar{l}^0(t,\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ *it is dissipative; namely, for every* $x_1, x_2 \in \mathbb{R}$,
$$(\bar{l}^0(t,\xi,x_1) - \bar{l}^0(t,\xi,x_2))(x_1-x_2) \ge -c_1(x_1-x_2)^2$$
for some positive constant c_1.

(3) *The map* $\bar g : [0,\pi]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is measurable and for a.a.* $\xi\in[0,\pi]$, $\bar g(\xi,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}$ *is continuous, convex, and differentiable, and there exist* $\Lambda\in L^\infty([0,\pi])$ *such that*

$$|\bar g_{u^0}(\xi,u^0,u^1,u^2)|\le\Lambda(\xi)(1+|u^0|),$$

and a constant $c>0$ *such that*

$$|\bar g_{u^i}(\xi,u^0,u^1,u^2)|\le c(1+|u^i|),\qquad i=1,2.$$

3. Main results.

In this section we come back to the abstract formulation of the problem, introducing the scheme we follow to find the optimal control: in section 3.1 we state the necessary condition for the existence of optimal controls, so we collect some results already present in the literature about the maximum principle; then in section 3.2 we prove that the condition is also sufficient under convexity assumptions on the coefficients which are standard in these topics (see the monograph [37]); and in sections 3.3 and 3.4 we focus our attention on the existence of optimal controls (see Theorem 3.5), which will pass through the solution of the Hamiltonian system (see Theorem 3.4).

3.1. Maximum principle.

Let us assume that there exists an optimal control $\bar u\in\mathcal{U}$: under hypotheses stated previously we have that there exists a unique mild solution $\bar X$ to (2.1) corresponding to $\bar u$; see, for instance, [6]. So $(\bar u,\bar X)$ is an *optimal pair* for the control problem described by (2.1) and (2.3). We introduce the following *forward-backward* system, composed by the state equation corresponding to the optimal control $\bar u$ and its adjoint equation:

$$\left\{\begin{array}{l} d\bar X_t = A\bar X_t\,dt+[E+B]\bar u_t\,dt+(\lambda-A)D_1\,d\tilde W_t+G\,dW_t,\\[2mm] -d\bar Y_t = A^*\bar Y_t\,dt+l^0_x(t,\bar X_t)\,dt-\bar Z_t\,dW_t-\tilde Z_t\,d\tilde W_t,\qquad t\in[0,T],\\[2mm] \bar X_0=x,\ \bar Y_T=-h_x(\bar X_T).\end{array}\right. \tag{3.1}$$

Once the forward equation is solved, the adjoint equation is a backward equation depending on the parameter $\bar X$. The existence and uniqueness of a *mild* solution $(\bar Y,(\bar Z,\tilde Z))\in L^2_{\mathcal{P}}(\Omega;C([0,T];H))\times L^2_{\mathcal{P}}([0,T]\times\Omega;L_2(\Xi\times K,H))$ for such an equation was first proved in [19]. Notice that (3.1) is not a fully coupled FBSDE: once the forward equation is solved, the backward equation can be solved following [19]. We collect the mentioned results in the following proposition.

Proposition 3.1. *Assume* (A) *and* (B). *Then the forward-backward system* (3.1) *has a unique mild solution* $(\bar X,\bar Y,(\bar Z,\tilde Z))$. *Moreover,*

$$\sup_{t\in[0,T[}\mathbb{E}(T-t)^{2(1-\alpha)}\|\bar Y_t\|^2_{D(E^*)}<+\infty. \tag{3.2}$$

Proof. The regularity result can be proved as in Proposition 3.1 of [18]. ∎

Next we recall the maximum principle, giving the necessary condition for the existence of optimal controls, in the case of stochastic boundary problems, following [18].

Theorem 3.2. *Assume* (A) *and* (B). *Let* $(\bar u,\bar X)$ *be an optimal pair for the problem given by* (2.1) *and* (2.3). *Then there exists a unique pair* $(\bar Y,(\bar Z,\tilde Z))\in L^2_{\mathcal{P}}(\Omega;C([0,T];H))\times L^2_{\mathcal{P}}([0,T]\times\Omega;L_2(\Xi\times K,H))$ *solution of* (3.1) *such that*

$$\langle H_u(t,\bar X_t,\bar u_t,Y_t),v-\bar u_t\rangle\le 0\qquad\forall v\in U,\ a.e.\ t\in[0,T],\ \mathbb{P}\text{-}a.s., \tag{3.3}$$

where

$$H(t,x,u,p) := \langle (E+B)^* p, u\rangle_H - l(t,x,u), \quad (t,x,u,p) \in [0,T]\times H\times U\times D(E^*),\ \lambda > \omega.$$

Proof. The result follows from Theorem 4.6 of [18] taking F_x and G_x equal to zero; the presence of the bounded operator B does not introduce any new difficulty. The proof follows exactly in the same way. ∎

3.2. Sufficient condition for optimality. Now we present the following sufficient condition for optimality. Let us consider the forward-backward system (3.1): for any admissible control $\bar v \in \mathcal{U}$ there exists a solution $(\bar X, \bar Y, (\bar Z, \tilde Z))$; we say then that $(\bar v, \bar X, \bar Y, (\bar Z, \tilde Z))$ is an *admissible* 4*-tuple.*

THEOREM 3.3. *Assume* (A) *and* (B). *Let* $(\bar u, \bar X, \bar Y, (\bar Z, \tilde Z))$ *be an admissible* 4*-tuple. If*

$$\langle H_u(t,\bar X_t, \bar u_t, \bar Y_t), v - \bar u_t\rangle \le 0 \qquad \forall v \in U,\ a.e.\ t \in [0,T],\ \mathbb{P}\text{-}a.s., \tag{3.4}$$

then $(\bar u, \bar X)$ *is optimal for the problem given by* (2.1) *and* (2.3).

Proof. Let $\bar v \in \mathcal{U}$; hence $\bar u + \lambda(\bar v - \bar u) \in \mathcal{U}$ for all $\lambda \in [0,1]$. With the state equation being affine, we have that $\bar X^{\bar u + \lambda(\bar v - \bar u)} = \bar X + \lambda \tilde X^{\bar v - \bar u}$, where $\tilde X^{\bar v - \bar u}$ solves the equation

$$\begin{cases} d\tilde X_t^{\bar v - \bar u} = A\tilde X_t^{\bar v - \bar u}\, dt + (E+B)(\bar v_t - \bar u_t)\, dt, \\ \tilde X_0^{\bar v - \bar u} = 0, \end{cases}$$

which is, in mild form,

$$\tilde X_t^{v-u} = \int_0^t e^{(t-s)A}(E+B)(\bar v_s - \bar u_s)\, ds. \tag{3.5}$$

Therefore, by the convexity assumption of l^0, g, h we end up with

$$\begin{aligned} &J(x,\bar u) - J(x, \bar u + \lambda(\bar v - \bar u)) \\ &\quad = \mathbb{E}\int_0^T [l(t,\bar X_t, \bar u_t) - l(t, \bar X_t + \lambda \tilde X_t^{\bar v - \bar u}, \bar u_t + \lambda(\bar v_t - \bar u_t))]\, dt \\ &\qquad + \mathbb{E}[h(\bar X_T) - h(\bar X_t + \lambda \tilde X_T^{\bar v - \bar u})] \\ &\quad \le -\mathbb{E}\int_0^T \lambda\langle l_x^0(t,\bar X_t), \tilde X_t^{\bar v - \bar u}\rangle\, dt - \mathbb{E}\int_0^T \lambda \langle g_u(\bar u_t), \bar v_t - \bar u_t\rangle\, dt \\ &\qquad - \mathbb{E}\lambda\langle h_x(\bar X_T), \tilde X_T^{\bar v - \bar u}\rangle. \end{aligned}$$

Now following the usual approximation strategy, we multiply both equations for $\tilde X^{\bar v - \bar u}$ and $\bar Y$ by $n(n-A)^{-1} = nR(n,A)$ for $n > \lambda$, so that the two processes $\tilde X^{\bar v - \bar u, n} := nR(n,A)\tilde X^{\bar v - \bar u}$ and $\bar Y^n := nR(n,A)\bar Y$ both admit an Itô differential. Since $D(E^*) \equiv D((\lambda - A^*)^{1-\alpha})$, we can let n tend to ∞, and we get that

$$-\mathbb{E}\langle h_x(\bar X_T), \tilde X_T^{\bar v - \bar u}\rangle - \mathbb{E}\int_0^T \langle l_x^0(t,\bar X_t), \tilde X_t^{\bar v - \bar u}\rangle\, dt = \mathbb{E}\int_0^T \langle (v_t - u_t), [E+B]^* \bar Y_t\rangle\, dt.$$

Notice that

$$\mathbb{E}\int_0^T \langle nR(n,A)[E+B](v_t - u_t), \bar Y_t^n\rangle\, dt$$

makes sense since $\bar{Y}_t^n = nR(n,A)\bar{Y}_t$ and $\bar{Y}_t \in D(E^*) = D((\lambda - A^*)^{1-\alpha})$; therefore $\bar{Y}_t^n \in D(E^*)$ and $nR(n,A^*)\bar{Y}_t^n \in D(E^*)$. So

$$\mathbb{E}\int_0^T \langle nR(n,A)[E+B](v_t - u_t), \bar{Y}_t^n\rangle\, dt = \mathbb{E}\int_0^T \langle (v_t - u_t), [E+B]^* nR(n,A^*)\bar{Y}_t^n\rangle\, dt$$

is well defined. Now, thanks to (3.3), taking $\lambda = 1$ we get

$$J(x,\bar{u}) \le J(x,\bar{v}) \qquad \forall\, \bar{v} \in \mathcal{U}. \qquad \square$$

3.3. Hamiltonian system. Let us introduce the Hamiltonian system associated with our control problem:

(3.6)
$$\begin{cases} d\bar{X}_t = A\bar{X}_t\, dt + [E+B]\gamma([E+B]^*\bar{Y}_t)\, dt + (\lambda - A)D_1\, d\tilde{W}_t + G(t,\bar{X}_t)\, dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\, dt + l_x^0(t,\bar{X}_t)\, dt - \bar{Z}_t\, dW_t - \tilde{Z}_t\, d\tilde{W}_t, \qquad t \in [0,T], \\ \bar{X}_0 = x,\ \bar{Y}_T = -h_x(\bar{X}_T), \end{cases}$$

where γ is defined as in (2.8). Section 4 is devoted to the proof of the following result.

THEOREM 3.4. *Assume* (A) *and* (B). *Then there exists a unique solution* $(\bar{X}, \bar{Y}, (\bar{Z},\tilde{Z})) \in L^2_{\mathcal{P}}((0,T)\times\Omega; H) \times L^2_{\mathcal{P}}((0,T)\times\Omega; D(E)) \times L^2_{\mathcal{P}}((0,T)\times\Omega; L_2(\Xi\times K; H))$ *of the forward-backward system* (3.6). *Moreover, we have that*

$$\sup_{t\in[0,T[} (T-t)^{1-\alpha}\|\bar{Y}_t\|_{D(E^*)} < +\infty. \tag{3.7}$$

By the definition of the map γ we deduce that $(\gamma([E+B]^*\bar{Y}), \bar{X}, \bar{Y}, (\bar{Z},\tilde{Z}))$ is an admissible 4-tuple.

3.4. Main result. We can now state the main result of the paper.

THEOREM 3.5. *Assume* (A) *and* (B). *There exists a unique optimal pair given by the solution of system* (3.6) *for the control problem given by* (2.1) *and* (2.3).

Proof. Thanks to Theorem 3.4 we have an admissible 4-tuple $(\gamma([E+B]^T\bar{Y}), \bar{X}, \bar{Y}, (\bar{Z},\tilde{Z}))$ that, by definition of γ, verifies condition (3.4). So from Theorem 3.3 we deduce the thesis. $\square$

4. Proof of Theorem 3.4. System (3.6) is an infinite dimensional fully coupled forward-backward system. Besides the difficulties typical of the finite dimensional FBSDEs (see [27]), there are some additional ones due to the presence of unbounded operators. In particular we need to introduce the graph norm of E and prove some crucial estimates with respect to this stronger norm. Thanks to dissipativity hypotheses (2.6), (2.7), and (2.9), the more suitable method for getting a solution is the bridge method used in [22] whose infinite dimensional extension will be described in the next section.

4.1. The *bridge method* applied to an infinite dimensional system. This section is devoted to the *bridge method* for solving the Hamiltonian system (3.6), which is an FBSDE in an infinite dimensional Hilbert space H. This method is introduced in [22] in order to solve a nonlinear fully coupled FBSDE. First, a linear auxiliary FBSDE is studied, and then, making a sort of convex combination between the affine term in this linear FBSDE and the nonlinear terms in the original FBSDE, one gets the solution of the original FBSDE.

The main difference between the present paper and [22] is that in [22] the finite dimensional case is treated, and so the linear auxiliary FBSDE has a very special structure and is solvable by hand; in the present paper, since Y also takes its values in H, the auxiliary linear affine FBSDE has a different structure, and it takes some effort to be solved; see section 4.2. Namely, let $b_0, h_0 \in L^2_{\mathcal{P}}([0,T] \times \Omega; H)$ and $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$, and consider the following linear FBSDE:

$$\begin{cases} d\bar{X}_t = A\bar{X}_t\,dt - [E+B][E+B]^*\bar{Y}_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + \bar{X}_t\,dt + h_0(t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t \in [0,T], \\ \bar{X}_0 = x,\ -\bar{Y}_T = \bar{X}_T + g_0. \end{cases} \tag{4.1}$$

In the next section we prove the following proposition, according to which (4.1) admits a unique solution. The difficulties in solving this FBSDE are due to the fact that the forward SDE contains Y itself, unlike in [22], and to the presence of the unbounded term $[E+B][E+B]^*\bar{Y}$.

Proposition 4.1. *Let $b_0, h_0 \in L^2_{\mathcal{P}}([0,T] \times \Omega; H)$ and $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$, and assume that A, E, B, D_1, and G satisfy assumptions* (A)*; then the linear FBSDE* (4.1) *admits a unique mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$ satisfying, moreover,*

$$\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}_t\|^2_{D((\lambda - A^*)^{1-\alpha})} < +\infty.$$

The proof of this proposition is given in the next section.

The aim of the present section is to prove the following result on the *bridge method*, in which the solution of the FBSDE (4.1) is in some sense related to the solution of the starting FBSDE (3.6).

Namely, let us define, for $x \in H$, $y \in H \cap D(E)$, and $\alpha \in [0,1]$,

$$\begin{aligned} b^\alpha(y) &= \alpha[E+B]\gamma([E+B]^*y) + (1-\alpha)[E+B][E+B]^*(-y), \\ h^\alpha(t,x) &= \alpha l^0_x(t,x) + (1-\alpha)(x), \\ g^\alpha(x) &= \alpha h_x(x) - (1-\alpha)(x). \end{aligned} \tag{4.2}$$

Consider the following FBSDE:

$$\begin{cases} d\bar{X}_t = A\bar{X}_t\,dt + b^\alpha(\bar{Y}_t)\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + h^\alpha(\bar{X}_t)\,dt + h_0(t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t \in [0,T], \\ \bar{X}_0 = x,\ -\bar{Y}_T = g^\alpha(\bar{X}_t) + g_0. \end{cases} \tag{4.3}$$

This is, with α varying in $[0,1]$, the system that links the linear FBSDE (4.1) to the original FBSDE (3.6).

Notice that the linear FBSDE (4.1) is equal to the FBSDE (4.3) with $\alpha = 0$, and it admits an adapted solution satisfying, moreover, $\mathbb{E}\sup_{t\in[0,T]}(T-t)^{2(1-\alpha)}\|(E+$

$B)\bar{Y}_t\|^2 < +\infty$, as stated in Proposition 4.1. In the next lemma we prove that if (4.3) admits a solution for some α_0, then it admits a solution for any $\alpha \in (\alpha_0, \alpha_0 + \delta)$, with a suitable $\delta > 0$.

LEMMA 4.2. *Let A, E, B, D_1, and G satisfy assumptions* (A), *and let γ, l, and h satisfy assumptions* (B). *Assume that for some $\alpha = \alpha_0$ and for any $b_0, h_0 \in L^2_{\mathcal{P}}((0,T)\times\Omega; H)$ and any $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$, (4.3) admits a mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$ satisfying, moreover,*

$$\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}_t\|^2_{D((\lambda-A^*)^{1-\alpha})} < +\infty.$$

Then there exists $\delta_0 \in (0,1)$ depending only on constants appearing in (A) *and* (B) *such that for all $\delta \in [0,\delta_0]$ and all $\alpha \in [\alpha_0, \alpha_0+\delta]$ and for any $b_0, h_0 \in L^2_{\mathcal{P}}([0,T]\times\Omega; H)$ and any $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$, FBSDE (4.3) admits a mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$ satisfying, moreover,*

$$\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}_t\|^2_{D((\lambda-A^*)^{1-\alpha})} < +\infty.$$

Proof. We notice that for $\alpha = \alpha_0 + \delta$ coefficients in (4.2) can be rewritten as

$$\begin{aligned}
&b^{\alpha_0+\delta}(y) = b^{\alpha_0}(y) + \delta[E+B]\gamma([E+B]^*y) + \delta[E+B][E+B]^*y,\\
&h^{\alpha_0+\delta}(t,x) = h^{\alpha_0}(t,x) + \delta l^0_x(t,x) - \delta x,\\
&g^{\alpha_0+\delta}(x) = g^{\alpha_0}(x) + \delta h_x(x) + \delta x.
\end{aligned}$$

Notice that by our assumptions it follows that for $\alpha = \alpha_0$ the FBSDE (4.3) admits a mild solution.

We set $(\bar{X}^0, \bar{Y}^0, (\bar{Z}^0, \tilde{Z}^0)) = (0, 0, (0, 0))$. For $j \geq 0$ we solve iteratively the following FBSDEs, as stated in Proposition 4.1:

(4.4)

$$\begin{cases}
d\bar{X}^{j+1}_t = A\bar{X}^{j+1}_t\,dt + \Big(\alpha_0[E+B]\gamma([E+B]^*\bar{Y}^{j+1}_t) \\
\qquad\qquad -(1-\alpha_0)[E+B][E+B]^*\bar{Y}^{j+1}_t\Big)\,dt \\
\qquad + \Big(\delta[E+B]\gamma([E+B]^*\bar{Y}^j_t) + \delta[E+B][E+B]^*\bar{Y}^j_t\Big)\,dt \\
\qquad + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\
-d\bar{Y}^{j+1}_t = A^*\bar{Y}^{j+1}_t\,dt + h_0(t)\,dt + \alpha_0 l^0_x(t, \bar{X}^{j+1}_t)\,dt + (1-\alpha_0)\bar{X}^{j+1}_t\,dt \\
\qquad + \delta(l^0_x(t,\bar{X}^j_t) - \bar{X}^j_t)\,dt - \bar{Z}^{j+1}_t\,dW_t - \tilde{Z}^{j+1}_t\,d\tilde{W}_t, \qquad t \in [0,T], \\
\bar{X}^{j+1}_0 = x, \quad -\bar{Y}^{j+1}_T = \alpha_0 h_x(\bar{X}^{j+1}_T) - (1-\alpha_0)\bar{X}^{j+1}_T + \delta h_x(\bar{X}^j_T) + \delta\bar{X}^j_T + g_0.
\end{cases}$$

By extending some statements to the mild solution (see the lines below), we will prove that such an FBSDE admits a mild solution $(\bar{X}^{j+1}, \bar{Y}^{j+1}, (\bar{Z}^{j+1}, \tilde{Z}^{j+1})) \in L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$ satisfying, moreover,

$$\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}^{j+1}_t\|^2_{D((\lambda-A^*)^{1-\alpha})} < \infty.$$

Indeed, for $j = 1$, FBSDE (4.4) is equal to FBSDE (4.3). So by hypothesis the solution, with the required regularity, exists. By induction, assume that for j a solution, with the required regularity, exists, and we show that also for $j+1$ a solution exists too. By setting $\tilde b_0(t)=\delta[E+B]\gamma([E+B]^*\bar Y_t^j)+\delta[E+B][E+B]^*\bar Y_t^j+b_0(t)$, $\tilde h_0(t)=\delta(l_x^0(t,\bar X_t^j)-\bar X_t^j)+h_0(t)$, $\tilde g_0=\delta h_x(\bar X_T^j)+\delta\bar X_T^j+g_0$, FBSDE (4.4) is equal to FBSDE (4.3) with $\tilde b_0$, $\tilde h_0$, and $\tilde g_0$ in place of b_0, h_0, and g_0, respectively. This time $\tilde b_0\notin L^2_{\mathcal P}(\Omega\times[0,T];H)$; indeed, $\tilde b_0$ is not well defined as an element of H. Nevertheless, in the mild formulation of X_t, $\tilde b_0$ appears in integral form and is affected by the regularizing properties of the semigroup: the integral $\int_t^T e^{(s-t)A}\tilde b_0(s)ds$ is well defined and bounded in $L^2_{\mathcal P}(\Omega\times[0,T];H)\cap L^2_{\mathcal P}(\Omega,C([0,T];H))$.

So by our assumptions, for all $j\geq 0$, there exists a mild solution $(\bar X^{j+1},\bar Y^{j+1},(\bar Z^{j+1},\tilde Z^{j+1}))\in L^2_{\mathcal P}(\Omega;C([0,T];H))\times L^2_{\mathcal P}(\Omega;C([0,T];H))\times L^2_{\mathcal P}(\Omega\times[0,T];L_2(\Xi\times K;H))$ satisfying, moreover,

$$\mathbb E\sup_{t\in[0,T]}(T-t)^{2(1-\alpha)}\|\bar Y_t^{j+1}\|^2_{D((\lambda-A)^{1-\alpha})}<\infty.$$

Next we define, for every $t\in[0,T]$,

$$\hat X_t^{j+1}=\bar X_t^{j+1}-\bar X_t^j;\quad \hat Y_t^{j+1}=\bar Y_t^{j+1}-\bar Y_t^j;$$

$$\hat Z_t^{j+1}=\bar Z_t^{j+1}-\bar Z_t^j;\quad \hat{\tilde Z}_t^{j+1}=\tilde Z_t^{j+1}-\tilde Z_t^j.$$

We note that $(\hat X^{j+1},\hat Y^{j+1},(\hat Z^{j+1},\hat{\tilde Z}^{j+1}))$ solves

$$\begin{cases} d\hat X_t^{j+1}=A\hat X_t^{j+1}\,dt+\alpha_0[E+B](\gamma([E+B]^*\bar Y_t^{j+1})-\gamma([E+B]^*\bar Y_t^j))\,dt\\ \qquad\qquad -(1-\alpha_0)[E+B][E+B]^*\hat Y_t^{j+1}\,dt+\delta[E+B][E+B]^*\hat Y_t^j\,dt\\ \qquad\qquad +\delta[E+B](\gamma([E+B]^*\bar Y_t^j)-\gamma([E+B]^*\bar Y_t^{j-1}))\,dt,\\ -d\hat Y_t^{j+1}=A^*\hat Y_t^{j+1}\,dt+\alpha_0(l_x^0(t,\bar X_t^{j+1})-l_x^0(t,\bar X_t^j))\,dt+(1-\alpha_0)\hat X_t^{j+1}\,dt\\ \qquad\qquad +\delta(l_x^0(t,\bar X_t^j)-l_x^0(t,\bar X_t^{j-1}))\,dt-\delta\hat X_t^j\,dt-\hat Z_t^{j+1}\,dW_t\\ \qquad\qquad -\hat{\tilde Z}_t^{j+1}\,d\tilde W_t,\qquad t\in[0,T],\\ \hat X_0^{j+1}=0,\\ -\hat Y_T^{j+1}=\alpha_0(h_x(\bar X_T^{j+1})-h_x(\bar X_T^j))-(1-\alpha_0)\hat X_T^{j+1}\\ \qquad\qquad +\delta(h_x(\bar X_T^j)-h_x(\bar X_T^{j-1}))+\delta\hat X_T^j.\end{cases}$$

Moreover, by our assumptions for every j, $\mathbb E\sup_{t\in[0,T]}|\hat X_t^j|^2<+\infty$.

Next we have to apply the Itô formula: in order to do this we have to approximate X and Y with elements of the domain of A. Namely, for $n>\lambda$, we denote as usual $R(n,A):=(n-A)^{-1}$. We set $(\hat X^{n,j+1},\hat Y^{n,j+1},(\hat Z^{n,j+1},\hat{\tilde Z}^{n,j+1}))=(nR(n,A)\hat X^{j+1},nR(n,A)\hat Y^{j+1},(nR(n,A)\hat Z^{j+1},nR(n,A)\hat{\tilde Z}^{j+1}))$. We also denote $E_n+B_n:=nR(n,A)(E+B)$, and we note that $(\hat X^{n,j+1},\hat Y^{n,j+1},(\hat Z^{n,j+1},\hat{\tilde Z}^{n,j+1}))$ solves the following FBSDE:

$$\begin{cases} d\hat{X}_t^{n,j+1} = A\hat{X}_t^{n,j+1}\,dt + \alpha_0[E_n+B_n](\gamma([E+B]^*\bar{Y}_t^{j+1}) - \gamma([E+B]^*\bar{Y}_t^{j}))\,dt \\ \qquad - (1-\alpha_0)[E_n+B_n][E+B]^*\hat{Y}_t^{j+1}\,dt + \delta[E_n+B_n][E+B]^*\hat{Y}_t^{j}\,dt \\ \qquad + \delta[E_n+B_n](\gamma([E+B]^*\bar{Y}_t^{j}) - \gamma([E+B]^*\bar{Y}_t^{j-1}))\,dt, \\ -d\hat{Y}_t^{n,j+1} = A^*\hat{Y}_t^{n,j+1}\,dt + \alpha_0 nR(n,A)(l_x^0(t,\bar{X}_t^{j+1}) - l_x^0(t,\bar{X}_t^{j}))\,dt \\ \qquad + (1-\alpha_0)\hat{X}_t^{n,j+1}\,dt + \delta nR(n,A)(l_x^0(t,\bar{X}_t^{j}) - l_x^0(t,\bar{X}_t^{j-1}))\,dt \\ \qquad - \delta\hat{X}_t^{n,j}\,dt - \hat{Z}_t^{n,j+1}\,dW_t - \hat{\tilde{Z}}_t^{n,j+1}\,d\tilde{W}_t, \qquad t\in[0,T], \\ \hat{X}_0^{n,j+1} = 0, \\ -\hat{Y}_T^{n,j+1} = \alpha_0 nR(n,A)(h_x(\bar{X}_T^{j+1}) - h_x(\bar{X}_T^{j})) - (1-\alpha_0)\hat{X}_T^{n,j+1} \\ \qquad + \delta nR(n,A)(h_x(\bar{X}_T^{j}) - h_x(\bar{X}_T^{j-1})) + \delta\hat{X}_T^{n,j}. \end{cases}$$

By applying the Itô formula to $\langle \hat{X}_t^{n,j+1}, \hat{Y}_t^{n,j+1}\rangle$, and then integrating over $[0,T]$ and taking expectation, we get

(4.5)

$$\begin{aligned} &-\mathbb{E}(\langle \hat{X}_T^{n,j+1}, \alpha_0 nR(n,A)(h_x(\bar{X}_T^{j+1}) - h_x(\bar{X}_T^{j})) - (1-\alpha_0)\hat{X}_T^{n,j+1}\rangle \\ &+ \delta\langle nR(n,A)(h_x(\bar{X}_T^{j}) - h_x(\bar{X}_T^{j-1})) + \hat{X}_T^{n,j}, \hat{X}_T^{n,j+1}\rangle) \\ &= \mathbb{E}\int_0^T \Big[\langle \alpha_0[E_n+B_n](\gamma([E+B]^*\bar{Y}_t^{j+1}) - \gamma([E+B]^*\bar{Y}_t^{j})), \hat{Y}_t^{n,j+1}\rangle \\ &+ (1-\alpha_0)\langle [E_n+B_n][E+B]\hat{Y}_t^{j+1}, \hat{Y}_t^{n,j+1}\rangle\Big]\,dt \\ &- \mathbb{E}\int_0^T \Big[\alpha_0\langle nR(n,A)(l_x^0(t,\bar{X}_t^{j+1}) - l_x^0(t,\bar{X}_t^{j})), \hat{X}_t^{n,j+1}\rangle + (1-\alpha_0)|\hat{X}_t^{n,j+1}|^2\Big]\,dt \\ &+ \delta\mathbb{E}\int_0^T \langle(\gamma([E+B]^*\bar{Y}_t^{j}) - \gamma([E+B]^*\bar{Y}_t^{j-1})) + [E+B]^*\hat{Y}_t^{j}, [E_n+B_n]^*\hat{Y}_t^{n,j+1}\rangle\,dt \\ &- \delta\mathbb{E}\int_0^T \langle nR(n,A)(l_x^0(t,\bar{X}_t^{j}) - l_x^0(t,\bar{X}_t^{j-1})) + \hat{X}_t^{n,j}, \hat{X}_t^{n,j+1}\rangle\,dt. \end{aligned}$$

Next we want to let $n\to+\infty$ in (4.5): in order to do this we need to recover at first the $L^2_{\mathcal{P}}(\Omega\times[0,T];H)$-convergence of $\hat{X}^{n,j}$, $\hat{Y}^{n,j}$ to $\hat{X}^j$, and $\hat{Y}^j$, respectively. For what concerns $\hat{X}^{n,j}$, notice that $e^{(t-s)A}$ and $(nR(n,A)-I)$ commute, so in a mild form we get

$$
\begin{aligned}
&\hat{X}_t^{n,j+1} - \hat{X}_t^{j+1} \\
&\quad = \int_0^t (nR(n,A) - I)e^{(t-s)A}\alpha_0[E+B](\gamma([E+B]^*\bar{Y}_s^{j+1}) - \gamma([E+B]^*\bar{Y}_s^j))\, ds \\
&\qquad - \int_0^t (nR(n,A) - I)e^{(t-s)A}(1-\alpha_0)[E+B][E+B]^*\hat{Y}_s^{j+1}\, ds \\
&\qquad + \delta \int_0^t (nR(n,A) - I)e^{(t-s)A}[E+B][E+B]^*\hat{Y}_s^j\, ds \\
&\qquad + \delta \int_0^t (nR(n,A) - I)e^{(t-s)A}[E+B](\gamma([E+B]^*\bar{Y}_s^j) - \gamma([E+B]^*\bar{Y}_s^{j-1}))\, ds.
\end{aligned}
$$

Let us consider the first integral; for the others the same conclusion follows in a similar way. For a.e. $s \in [0,t]$ and for all $0 < t \le T$, and for a.e. $\omega \in \Omega$, $\alpha_0 e^{(t-s)A}[E + B](\gamma([E+B]^*\bar{Y}_s^{j+1}) - \gamma([E+B]^*\bar{Y}_s^j))$ is an element in H and

$$
\begin{aligned}
&|\alpha_0 e^{(t-s)A}[E+B](\gamma([E+B]^*\bar{Y}_s^{j+1}) - \gamma([E+B]^*\bar{Y}_s^j))| \\
&\qquad \le C\Delta\alpha_0(t-s)^{-(1-\alpha)}|[E+B]^*\hat{Y}_s^{j+1}|,
\end{aligned}
$$

so that, by dominated convergence, as $n \to \infty$,

$$
\mathbb{E} \sup_{t\in[0,T]} |\hat{X}_t^{n,j+1} - \hat{X}_t^{j+1}|^2 \to 0.
$$

In a similar way we can get that $\hat{Y}_t^{n,j+1} \to \hat{Y}_t^{j+1}$ in $L^2_{\mathcal{P}}(\Omega \times [0,T]; H)$ and, moreover, as $n \to \infty$,

$$
\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|(\hat{Y}_t^{n,j+1} - \hat{Y}_t^{j+1})\|^2_{D((\lambda - A^*)^{1-\alpha})} \to 0.
$$

So we can let $n \to \infty$ in (4.5); by assumptions (B) we get

$$
\begin{aligned}
&\min\{c_1, 1\}\mathbb{E}|\hat{X}_T^{j+1}|^2 \\
&\quad \le \delta\Delta\mathbb{E}|\hat{X}_T^{j+1}|\,|\hat{X}_T^j| - \alpha_0 c_1 \mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^{j+1}|^2\, dt \\
&\qquad - (1-\alpha_0)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^{j+1}|^2\, dt \\
&\qquad + \delta(\Delta+1)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^j||\hat{Y}_t^{j+1}|\, dt - (\alpha_0 c_1 + 1 - \alpha_0)\mathbb{E}\int_0^T |\hat{X}_t^{j+1}|^2\, dt \\
&\qquad + \delta\Delta\mathbb{E}\int_0^T |\hat{X}_t^j|\,|\hat{X}_t^{j+1}|\, dt.
\end{aligned}
$$

By applying Young inequalities several times we finally get

$$
\begin{aligned}
&\mathbb{E}|\hat{X}_T^{j+1}|^2 + \left[\mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^{j+1}|^2 dt + \mathbb{E}\int_0^T |\hat{X}_t^{j+1}|^2 dt\right] \\
&\qquad \le c'(\delta,\Delta,c_1)\mathbb{E}|\hat{X}_T^j|^2 + c'(\delta,c_1)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^j|^2 dt \\
&\qquad\quad + c'(\delta,\Delta,c_1)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}_t^j|^2 dt,
\end{aligned}
$$

where $c'(\delta,\Delta,c_1)$ and $c'(\delta,c_1)$ are constants depending, respectively, only on $\delta\,,\Delta\,,c_1$ and $\delta\,,c_1$, respectively. Now notice that

$$\hat{X}^j_T = \alpha_0 \int_0^T e^{(T-t)A}[E+B](\gamma([E+B]^*\bar{Y}^j_t) - \gamma([E+B]^*\bar{Y}^{j-1}_t))\,dt$$
$$+ (1-\alpha_0)\int_0^T e^{(T-t)A}[E+B][E+B]^*\hat{Y}^j_t\,dt + \delta \int_0^T e^{(T-t)A}[E+B][E+B]^*\hat{Y}^{j-1}_t\,dt$$
$$+ \delta \int_0^T e^{(T-t)A}[E+B](\gamma([E+B]^*\bar{Y}^{j-1}_t) - \gamma([E+B]^*\bar{Y}^{j-2}_t))\,dt.$$

So, by assumptions (A) and the Hölder inequality,

$$\mathbb{E}|\hat{X}^j_T|^2 \le [c\Delta^2\alpha_0^2 + c(1-\alpha_0)^2]\left(\int_0^T (T-t)^{-2(1-\alpha)}\,dt\right)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}^j_t|^2 dt$$
$$+ \delta^2 c'\left(\int_0^T (T-t)^{-2(1-\alpha)}\,dt\right)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}^{j-1}_t|^2\,dt.$$

Now, arguing as in [22, proof of Lemma 3.2], we get that there exists $\delta_0 \in (0,1)$ depending only on c_1,Δ,T such that for every $\delta \in (0,\delta_0]$, we get

$$\begin{aligned}
&\mathbb{E}\int_0^T |[E+B]^*\hat{Y}^{j+1}_t|^2 dt + \mathbb{E}\int_0^T |\hat{X}^{j+1}_t|^2 dt \\
&\le \frac{1}{4}\left[\mathbb{E}\int_0^T |\hat{X}^j_t|^2 dt + \mathbb{E}\int_0^T |[E+B]^*\hat{Y}^j_t|^2 dt\right] \\
&+ \frac{1}{8}\left[\mathbb{E}\int_0^T |\hat{X}^{j-1}_t|^2 dt + \mathbb{E}\int_0^T |[E+B]^*\hat{Y}^{j-1}_t|^2 dt\right].
\end{aligned}$$

From this we deduce that $(\bar{X}^j_t,\bar{Y}^j_t)_{j\ge 1}$ is a Cauchy sequence in $L^2_{\mathcal{P}}(\Omega\times[0,T],H)\times L^2_{\mathcal{P}}(\Omega\times[0,T],H)$, and we denote by $(\bar{X}_t,\bar{Y}_t)$ its limit.

In order to prove that $(\bar{X}^j_t,\bar{Y}^j_t)_{j\ge 1}$ converge to $(\bar{X}_t,\bar{Y}_t)$ also in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))\times L^2_{\mathcal{P}}(\Omega, C([0,T],H))$ we go to the mild formulation of the equations solved by $\bar{X}^j_t$ and $\bar{Y}^j_t$, and thus, similarly to the previous calculations,

$$\mathbb{E}\sup_{t\in[0,T]}|\hat{X}^j_t|^2 \le [c\Delta^2\alpha_0^2 + c(1-\alpha_0)^2]\left(\int_0^T (T-s)^{-2(1-\alpha)}\,ds\right)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}^j_s|^2 ds$$
$$+ \left(\int_0^t (T-s)^{-2(1-\alpha)}\,ds\right)\mathbb{E}\int_0^T |[E+B]^*\hat{Y}^{j-1}_s|^2\,ds.$$

We notice that by the previous choice of δ we get that $(\bar{X}^j)_{j\ge 1}$ is a Cauchy sequence in $L^2_{\mathcal{P}}(\Omega, C([0,T],H))$ so that $\bar{X}^j \to \bar{X}$ in $L^2_{\mathcal{P}}(\Omega, C([0,T],H))$. For what concerns the convergence of $\bar{Y}^j$ in $L^2_{\mathcal{P}}(\Omega, C([0,T],H))$ we have to first recover the convergence of $(\bar{Z}^j,\tilde{Z}^j)$ in $L^2_{\mathcal{P}}(\Omega\times[0,T];L_2(\Xi\times K;H))$. In the mild formulation, $\bar{Y}^{j+1}$ solves the following BSDE:

$$\bar{Y}^{j+1}_t = -e^{(T-t)A^*}\left[\alpha_0 h_x(\bar{X}^{j+1}_T) - (1-\alpha_0)\bar{X}^{j+1}_T + \delta h_x(\bar{X}^j_T) + \delta\bar{X}^j_T + g_0\right]$$
$$+ \alpha_0\int_t^T e^{(s-t)A^*} l^0_x(s,\bar{X}^{j+1}_s)\,ds + (1-\alpha_0)\int_t^T e^{(s-t)A^*}\bar{X}^{j+1}_s\,ds \tag{4.6}$$

$$+ \delta \int_t^T e^{(s-t)A^*} l_x^0(s, \bar{X}_s^j)\, ds - \delta \int_t^T e^{(s-t)A^*} \bar{X}_s^j\, ds$$
$$- \int_t^T e^{(s-t)A^*} \bar{Z}_s^{j+1}\, dW_s - \int_t^T e^{(s-t)A^*} \tilde{Z}_s^{j+1}\, d\tilde{W}_s + \int_t^T e^{(s-t)A^*} h_0(s)\, ds.$$

Arguing as in [19], by the extended martingale representation theorem (see also [20] and [38]), for every $s \in [0,T]$ there exists $(K^{j+1}(s,\cdot), \tilde{K}^{j+1}(s,\cdot)) \in L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(\Xi, H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(K,H))$ and $(L^{j+1}, \tilde{L}^{j+1}) \in L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(\Xi, H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(K,H))$ such that the following representation for $(\bar{Z}_s^j, \tilde{Z}_s^j)$ holds:

$$\bar{Z}_s^j = \int_s^T e^{(\alpha-s)A^*} K^j(\alpha, s)\, d\alpha + e^{(T-s)A^*} L_s^{j+1},$$
$$\tilde{Z}_s^j = \int_s^T e^{(\alpha-s)A^*} \tilde{K}^j(\alpha, s)\, d\alpha + e^{(T-s)A^*} \tilde{L}_s^{j+1}.$$

Note that for all $\theta \geq s$, $K^j(s,\theta) = 0$ a.e. and $\tilde{K}^j(s,\theta) = 0$ a.e. The following estimates hold:

$$\mathbb{E} \int_0^T \int_0^s \left[|K^{j+1}(s,\theta)|^2 + |\tilde{K}^{j+1}(s,\theta)|^2 \right] d\theta\, ds \leq 4\mathbb{E} \int_0^T |f_s^{j+1}|^2\, ds, \tag{4.7}$$

where

$$f_s^{j+1} := \alpha_0 l_x^0(s, \bar{X}_s^{j+1}) + (1-\alpha_0) e^{(s-t)A^*} \bar{X}_s^{j+1} + \delta l_x^0(s, \bar{X}_s^j)\, ds - \delta \bar{X}_s + h_0(s),$$

and

$$\begin{aligned} &\mathbb{E} \int_0^T [|L_s^{j+1}|^2 + |\tilde{L}_s^{j+1}|^2]\, ds \\ &\leq 4\mathbb{E} |\alpha_0 [h_x(\bar{X}_T^{j+1}) - h_x(\bar{X}_T^j)] + (\alpha_0 - 1)\hat{X}_T^{j+1} + \delta [h_x(\bar{X}_T^{j+1}) - h_x(\bar{X}_T^j)] + \delta \hat{X}_T^j|^2. \end{aligned} \tag{4.8}$$

By the definitions of $(K^j, \tilde{K}^j)$ and $(L^j, \tilde{L}^j)$, by estimates (4.7) and (4.8), and by previous estimates on the L^2-norms of $\hat{X}^j$, it is possible to prove that $(\bar{Z}_s^j, \tilde{Z}_s^j)$ is a Cauchy sequence in $L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$, and we denote by $(\bar{Z}_s, \tilde{Z}_s)$ its limit.

We are ready to prove that $\bar{Y}^j \to \bar{Y}$ in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))$. Noting that $\mathbb{E}^{\mathcal{F}_t} \bar{Y}_t^{j+1} = \bar{Y}_t^{j+1}$, by standard estimates we obtain

$$\begin{aligned} \mathbb{E} \sup_{t \in [0,T]} |\hat{Y}_t^{j+1}|^2 &\leq c(T,A,\Delta) \left[\mathbb{E} |\hat{X}_T^{j+1}|^2 + \delta^2 (1+\Delta^2) \mathbb{E} |\hat{X}_T^j|^2 \right] \\ &\quad + \alpha_0^2 c(T,A,\Delta)^2 \mathbb{E} \int_0^T |\hat{X}_s^{j+1}|^2\, ds + \delta^2 c(T,A,\Delta) \mathbb{E} \int_0^T |\hat{X}_s^j|^2\, ds. \end{aligned}$$

From this, again by using the previous estimates on the L^2-norm of $\hat{X}^j$ and of $\hat{Y}^j$, it is possible to prove that $\hat{Y}^j$ is a Cauchy sequence in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))$, and the claim follows.

Finally, we have to prove that $\mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}_t\|^2_{D((\lambda - A)^{1-\alpha})}$ is bounded.

Let $\alpha \in [\alpha_0, \alpha_0 + \delta]$. In its mild formulation, $\bar{Y}$ solves the following BSDE:

$$\begin{aligned}\bar{Y}_t &= e^{(T-t)A^*}\left[-\alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0\right] \\ &\quad + \alpha \int_t^T e^{(s-t)A^*}\left(\alpha l_x^0(s,\bar{X}_s) + (1-\alpha)\bar{X}_s\right) ds + \int_t^T e^{(s-t)A^*} h_0(s)\, ds \\ &\quad - \int_t^T e^{(s-t)A^*} \bar{Z}_s \, dW_s - \int_t^T e^{(s-t)A^*} \tilde{Z}_s \, d\tilde{W}_s.\end{aligned}$$

Notice also that

$$\begin{aligned}\bar{Y}_t &= \mathbb{E}^{\mathcal{F}_t}\bar{Y}_t = \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*}\left[-\alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0\right] \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{(s-t)A^*}\left(\alpha l_x^0(s,\bar{X}_s) + (1-\alpha)\bar{X}_s\right) ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{(s-t)A^*} h_0(s)\, ds.\end{aligned} \tag{4.9}$$

The semigroup $(e^{tA})_{t\geq 0}$ is analytic, and for every $t > 0$, e^{tA} maps H into $\mathcal{D}(A)$, so since $E = (\lambda - A)D$, we get that $\bar{Y}_t \in \mathcal{D}(E)$. Moreover, by the mild equality (4.9) satisfied by $\bar{Y}_t$ we get

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,T]} |(T-t)^{2(1-\alpha)}(E+B)^*\bar{Y}_t|^2 \\ &\leq c\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}|(E+B)^* e^{(T-t)A^*}\left[-\alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0\right]|^2 \\ &+ c\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}\left|(E+B)^*\int_t^T e^{(s-t)A^*}\left(\alpha l_x^0(s,\bar{X}_s) + (1-\alpha)\bar{X}_s\right) ds\right|^2 \\ &+ c\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}\left|(E+B)^*\int_t^T e^{(s-t)A^*} h_0(s)\, ds\right|^2 = I + II + III.\end{aligned}$$

Moreover, remember again that $E = (\lambda - A)D$: since D takes its values in $\mathcal{D}(\lambda - A)^\alpha$ and also by the analyticity of A, we get, for every $t > 0$ and every $f \in H$,

$$|(E+B)^* e^{tA^*} f| \leq c t^{-(1-\alpha)}|f|.$$

So

$$\begin{aligned}I &\leq c\mathbb{E} \sup_{t\in[0,T]} (1 + |\bar{X}_T|)^2 < +\infty, \\ II &\leq c\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}\int_t^T |(s-t)^{-(1-\alpha)}\left(1 + |\bar{X}_s|\right) ds|^2 \\ &\leq \mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}\int_t^T (s-t)^{-2(1-\alpha)} ds \int_0^T \left(1 + |\bar{X}_s|^2\right) ds < +\infty, \\ III &\leq c\mathbb{E} \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)}\left|\int_t^T (s-t)^{-(1-\alpha)} h_0(s)\, ds\right|^2 < +\infty.\end{aligned}$$

So we have finally proved that there exists $\delta_0 \in (0,1)$ such that for all $\delta \in [0, \delta_0]$, for all $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, and for all $b_0, h_0 \in L^2_{\mathcal{P}}([0,T] \times \Omega; H)$ and $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$ the FBSDE (4.3) admits a unique mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega; C([0,T]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi \times K; H))$ satisfying, moreover,

$$E \sup_{t\in[0,T]} (T-t)^{2(1-\alpha)} \|\bar{Y}_t\|^2_{D((\lambda - A^*)^{1-\alpha})} < \infty. \qquad \square$$

Remark 4.3. As noticed in the proof, Lemma 4.2 holds even for any b_0 such that the term $\int_t^T e^{(s-t)A}\tilde{b}_0(s)ds$ is well defined and bounded in $L^2_{\mathcal{P}}(\Omega\times[0,T];H)\cap L^2_{\mathcal{P}}(\Omega, C([0,T];H))$.

Remark 4.4. We notice that the presence of a diffuse control is not required in our methods. Indeed, if $B=0$, then we can consider as an auxiliary linear FBSDE

$$\begin{cases} d\bar{X}_t = A\bar{X}_t\,dt - [E+I][E+I]^*\bar{Y}_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + \bar{X}_t\,dt + h_0(t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t\in[0,T], \\ \bar{X}_0 = x,\ -\bar{Y}_T = \bar{X}_T + g_0, \end{cases}$$

and we can apply the bridge method, linking this FBSDE to the FBSDE

$$\begin{cases} d\bar{X}_t = A\bar{X}_t\,dt + E\gamma(E^*\bar{Y}_t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G(t,\bar{X}_t)\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + l^0_x(t,\bar{X}_t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t\in[0,T], \\ \bar{X}_0 = x,\ \bar{Y}_T = -h_x(\bar{X}_T). \end{cases}$$

4.2. An auxiliary LQ control problem. This section is devoted to the solution of the affine FBSDE. Let $b_0, h_0 \in L^2_{\mathcal{P}}((0,T)\times\Omega;H)$ and $g_0\in L^2(\Omega,\mathcal{F}_T;H)$, and consider

$$\begin{cases} d\bar{X}_t = A\bar{X}_t dt - [E+B][E+B]^*\bar{Y}_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \\ -d\bar{Y}_t = A^*\bar{Y}_t\,dt + \bar{X}_t\,dt + h_0(t)\,dt - \bar{Z}_t\,dW_t - \tilde{Z}_t\,d\tilde{W}_t, \qquad t\in[0,T], \\ \bar{X}_0 = x,\ -\bar{Y}_T = \bar{X}_T + g_0. \end{cases} \tag{4.10}$$

This system is the Hamiltonian system corresponding to the control problem with state equation

$$\begin{cases} dX_t = AX_t\,dt + [E+B]u_t\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde{W}_t + G\,dW_t, \qquad t\in[0,T], \\ X_0 = x, \end{cases} \tag{4.11}$$

and cost functional

$$J(x,u) = \frac{1}{2}\mathbb{E}\int_0^T (|X_t + h_0(t)|^2 + |u_t|^2)\,dt + \frac{1}{2}\mathbb{E}|X_T + g_0|^2 \tag{4.12}$$

to minimize over all $u\in\mathcal{U}$. We will exploit this interpretation through the control problem in order to solve (4.10). For this purpose we introduce the following Riccati equation:

$$\begin{cases} -\dfrac{dP_t}{dt} = A^*P_t + P_tA - P_t(E+B)(E+B)^*P_t + I, \qquad t\in[0,T], \\ P_T = I \end{cases} \tag{4.13}$$

and the following backward equation, to cope with the affine terms:

$$
\begin{cases} -dr_t = A^* r_t\, dt - P_t(E+B)(E+B)^* r_t\, dt + P_t b_0(t)\, dt \\ \qquad -h_0(t)\, dt - q_t\, W_t - \tilde{q}_t\, d\tilde{W}_t, \qquad t \in [0,T], \\ r_T = g_0. \end{cases} \tag{4.14}
$$

As in [3], we denote by $\Sigma(H)$ the space of self adjoint linear operators in H and by $C_s([0,T];\Sigma(H))$ the space of all strongly continuous mappings from $[0,T]$ to $\Sigma(H)$, which is $P:[0,T]\to\Sigma(H)$, and which for every $h\in H$ $t\mapsto P_t h$ is continuous.

In the book [3, part IV, Chapter 2, Theorem 2.1], it is proved that the first equation of (4.13) has a solution in the space $C_{s,\alpha}([0,T];\Sigma(H))$, that is, the set of all $P\in C_s([0,T];\Sigma(H))$ such that

(i) $P(t)x \in D((-A^*)^{1-\alpha})$ for all $x\in H$, $t\in[0,T[$,

(ii) $(-A^*)^{1-\alpha}P \in C([0,T[;L(H))$,

(iii) $\lim_{t\to T}(T-t)^{1-\alpha}(-A^*)^{1-\alpha}P_t x = 0$ for all $x\in H$.

Moreover, define

$$
\|P\|_1 = \sup_{t\in[0,T[} (T-t)^{1-\alpha}\|(-A^*)^{1-\alpha}P(t)\|. \tag{4.15}
$$

The space $C_{s,\alpha}([0,T];\Sigma(H))$, endowed with the norm

$$
\|P\|_\alpha = \|P\| + \|P\|_1 \tag{4.16}
$$

is a Banach space.

For simplicity we will denote the couple $(q,\tilde{q})$ as $\hat{q}$ along with the comprehensive Wiener process $\hat{W}_t := (W_t, \tilde{W}_t)$.

We can now prove existence and uniqueness of a solution to (4.14).

THEOREM 4.5. *Assume* (A) *and* (B). *Then* (4.14) *has a unique mild solution* $(r_t,\hat{q}) \in L^2_{\mathcal{P}}(\Omega; C([0,T];H)) \times L^2_{\mathcal{P}}(\Omega\times[0,T]; L_2(\Xi\times K;H))$*; moreover,*

$$
\mathbb{E}\sup_{t\in[0,T[}(T-t)^{2(1-\alpha)}|r_t|^2 < \infty. \tag{4.17}
$$

Proof. We will prove existence and uniqueness by a fixed point technique. Let us define a map $\Gamma:\mathcal{Y}\to\mathcal{Y}$, where

$$
\mathcal{Y} := \left\{ \begin{array}{l} (r,\hat{q}) \in L^2_{\mathcal{P}}(\Omega; C([0,T];H)) \times L^2_{\mathcal{P}}(\Omega\times[0,T]; \\ L_2(\Xi\times K;H)) : \mathbb{E}\sup_{t\in[0,T[}(T-t)^{2(1-\alpha)}|r_t|^2 < \infty \end{array} \right\}
$$

such that $\Gamma((r',\hat{q}')) = (r,\hat{q})$ is the *mild* solution to

$$
\begin{cases} r_t = e^{A^*(T-t)}g_0 - \displaystyle\int_t^T e^{A^*(s-t)}P_s(E+B)(E+B)^* r'_s\, ds \\ \qquad + \displaystyle\int_t^T e^{A^*(s-t)}P_s b_0(s)\, ds \\ \qquad - \displaystyle\int_t^T e^{A^*(s-t)}h_0(s)\, ds - \int_t^T e^{A^*(s-t)}\hat{q}_s\, d\hat{W}_s, \qquad t\in[0,T]. \end{cases} \tag{4.18}
$$

We will prove the following:

(1) $\Gamma((r',\hat{q}')) \in \mathcal{Y}$;

(2) for any $\alpha < 1$ there exists $\delta \in [0,T[$ that depends only on α, on constants appearing in (A) and (B), and on T such that

$$\|(r^1,\hat{q}^1)-(r^2,\hat{q}^2)\|_{\mathcal{Y}_\delta} \le \alpha \|(r'^1,\hat{q}'^1)-(r'^2,\hat{q}'^2)\|_{\mathcal{Y}_\delta} \tag{4.19}$$

for some $\delta > 0$, where we set

(4.20)

$$\mathcal{Y}_\delta := \Big\{ (r,\hat{q}) \in L^2_{\mathcal{P}}(\Omega; C([T-\delta,T];H)) \times L^2_{\mathcal{P}}(\Omega\times(T-\delta,T); L_2(\Xi\times K;H)) : \mathbb{E} \sup_{t\in[T-\delta,T[} (T-t)^{2(1-\alpha)} |(\lambda-A^*)^{1-\alpha} r_t|^2 < \infty \Big\}.$$

The space $\mathcal{Y}_\delta$ endowed with the norm

$$\|(r,\hat{q})\|^2_{\mathcal{Y}_\delta} := \mathbb{E}\sup_{t\in[T-\delta,T]} |r_t|^2 + \mathbb{E}\sup_{t\in[T-\delta,T[} (T-t)^{2(1-\alpha)} |(\lambda-A^*)^{1-\alpha} r_t|^2 + \mathbb{E}\int_{T-\delta}^T |\hat{q}_t|^2\,dt$$

is a Banach space.

Proof of statement (1). We introduce the approximating problems for $k > \lambda$:

$$\begin{cases} -\dfrac{dP^k_t}{dt} = A^* P^k_t + P^k_t A - P^k_t (E^k+B)(E^k+B)^* P^k_t + I, & t\in[0,T],\\ P^k_T = I, \end{cases} \tag{4.21}$$

where $E^k := (\lambda-A)^{1-\alpha} kR(k,A)(\lambda-A)^\alpha D$, with, as usual, $R(k,A) := (k-A)^{-1}$.

From [3] we know that (4.21) has a unique mild solution $P^k \in C_{s,\alpha}([0,T];\Sigma(H))$ for every k and, moreover, the following holds (see [3, part IV, Chapter 2, Lemma 2.1]):

(4.22)

$$\begin{cases} \lim_{k\to\infty} P^k(\cdot)x = P(\cdot)x \quad \text{in } C([0,T];H),\\ \lim_{k\to\infty} (T-\cdot)^{1-\alpha}(-A^*)^{1-\alpha} P^k(\cdot)x = (T-\cdot)^{1-\alpha}(-A^*)^{1-\alpha} P(\cdot)x \quad \text{in } C([0,T];H). \end{cases}$$

Given P^k we introduce also

$$\begin{cases} -dr^k_t = A^* r^k_t\,dt - P^k_t(E^k+B)(E^k+B)^* r'_t\,dt \\ \qquad + P^k_t b_0(t)\,dt - h_0(t)\,dt - \hat{q}^k_t\,d\hat{W}_t, \qquad t\in[0,T],\\ r^k_T = g_0. \end{cases} \tag{4.23}$$

Existence and uniqueness of a mild solution for (4.23) in $L^2_{\mathcal{P}}(\Omega; C([0,T];H)) \times L^2_{\mathcal{P}}(\Omega\times(0,T); L_2(\Xi\times K;H))$ can be deduced by [19, Prop. 2.1]. Now we can prove that

(4.24)

$$\mathbb{E}\int_0^T |\hat{q}^k_t|^2\,dt \le C\Big[\mathbb{E}|g_0|^2 + \mathbb{E}\Big(\int_0^T |P^k_s(E^k+B)(E^k+B)^* r'_s|\,ds\Big)^2 + \mathbb{E}\int_0^T |P^k_t b_0(s)|^2\,ds + \mathbb{E}\int_0^T |h_0(s)|^2\,ds\Big].$$

The former estimate can be achieved by evaluating $d_t|r_t|^2$ and exploiting the fact that, with A^* being the generator of a contraction semigroup, $\langle A^*y,y\rangle \le \omega|y|^2$ for any $y\in D(A^*)$. Since r^k does not belong to $D(A^*)$, we multiply r^k by $nR(n,A)$

for $n > \omega$ in order to perform the Itô formula. Let us set $r_t^{n,k} = nR(n,A)r_t^k$ and $\hat{q}^{n,k} = nR(n,A)\hat{q}_t^k$; hence,

$$\text{(4.25)}\quad \begin{cases} -dr_t^{n,k} = A^* r_t^{n,k}\,dt - nR(n,A)P_t^k(E^k+B)(E^k+B)^* r_t'\,dt + nR(n,A)P_t^k b_0(t)\,dt \\ \qquad\qquad - nR(n,A)h_0(t)\,dt - \hat{q}_t^{n,k}d\hat{W}_t, \qquad t\in[0,T], \\ r_T^{n,k} = nR(n,A)g_0. \end{cases}$$

Now we can evaluate $d_t|r_t^{n,k}|^2$:

$$\text{(4.26)}\quad -d_t|r_t^{n,k}|^2 = 2\langle A^* r_t^{n,k}, r_t^{n,k}\rangle\,dt - 2\langle f_t^{n,k}, r_t^{n,k}\rangle\,dt - 2\langle \hat{q}_t^{n,k}, r_t^{n,k}\rangle\,d\hat{W}_t - |\hat{q}_t^{n,k}|^2\,dt,$$

where

$$f_t^{n,k} = nR(n,A)[P_t^k(B+E^k)(B+E^k)^* {r'}_t + P_t^k b_0(t) + h_0(t)].$$

Now, similarly to [17, Prop. 3.4] (see also [4, Lemma 3.1]), we get

$$\text{(4.27)}\quad \mathbb{E}\int_0^T |\hat{q}_t^{n,k}|^2\,dt \le C\left[\mathbb{E}\sup_{t\in[0,T]}|r_t^{n,k}|^2 + \mathbb{E}\left(\int_0^T |f_t^{n,k}|\,dt\right)^2\right],$$

where the constant C depends on constants appearing in (A) and (B) and T. Letting n tend to ∞ we obtain estimate (4.24). Now, bearing in mind that $\sup_{t\in[0,T]}|P_t^k| \le M$ with M independent of k, thanks to (4.22) and to the Banach–Steinhaus theorem, we obtain that

$$\text{(4.28)}\quad \begin{aligned} \mathbb{E}\int_0^T |\hat{q}_t^k|^2\,dt \le C\Bigg[\mathbb{E}\Bigg(&\sup_{s\in[0,T]}(T-s)^{(2-2\alpha)}[|(\lambda-A^*)^{1-\alpha}r_s'|^2 \\ &+ |r_s'|^2]\,ds\int_0^T s^{\alpha-1}(T-s)^{2\alpha-2}\,ds\Bigg)^2 \\ &+ \mathbb{E}|g_0|^2 + \mathbb{E}\int_0^T |b_0(s)|^2\,ds + \mathbb{E}\int_0^T |h_0(s)|^2\,ds\Bigg]. \end{aligned}$$

Let us consider $k, m > \omega$:

$$\begin{aligned} r_t^k - r_t^m = &-\int_0^T e^{A^*(s-t)}[P_s^k(B+E^k)(B+E^k)^* - P_s^m(B+E^m)(B+E^m)^*]{r'}_s\,ds \\ &- \int_0^T e^{A^*(s-t)}(\hat{q}_s^k - \hat{q}_s^m)\,d\hat{W}_s. \end{aligned}$$

We have that

$$\begin{aligned} r_t^k - r_t^m &= \mathbb{E}^{\mathcal{F}_t}(r_t^k - r_t^m) \\ &= -\mathbb{E}^{\mathcal{F}_t}\int_0^T e^{A^*(s-t)}[P_s^k(B+E^k)(B+E^k)^* - P_s^m(B+E^m)(B+E^m)^*]{r'}_s\,ds, \end{aligned}$$

and, since $|(\lambda - A^*)^{1-\alpha} e^{sA^*}|_{L(H)} \le c\, s^{1-\alpha}$,

$$\begin{aligned}
&\sup_{t\in[0,T[} |(\lambda - A^*)^{1-\alpha}(r_t^k - r_t^m)|^2 \\
&\le c\,\mathbb{E}^{\mathcal{F}_t} \int_0^T s^{\alpha-1} |[P_s^k(B+E^k)(B+E^k)^* - P(B+E)(B+E)^*] r_s'|\, ds \\
&\quad + c\,\mathbb{E}^{\mathcal{F}_t} \int_0^T s^{\alpha-1} |[P_s^m(B+E^m)(B+E^m)^* - P(B+E)(B+E)^*] r_s'|\, ds.
\end{aligned}$$

Hence, taking into account that

$$\mathbb{E} \sup_{s\in[0,T[} (T-s)^{2(1-\alpha)} |(\lambda - A^*)^{1-\alpha} r_s'|^2 < \infty, \tag{4.29}$$

by dominated convergence we end up with

$$\lim_{k,m\to+\infty} \mathbb{E} \sup_{t\in[0,T[} (T-t)^{2(1-\alpha)} |(\lambda - A^*)^{1-\alpha}(r_t^k - r_t^m)|^2 = 0. \tag{4.30}$$

Similarly we have that

$$\lim_{k,m\to+\infty} \mathbb{E}|(r_t^k - r_t^m)|^2 = 0. \tag{4.31}$$

Moreover, from former calculations we have that

$$\begin{aligned}
&\mathbb{E}\int_0^T |\hat{q}_t^k - \hat{q}_t^m|^2\, dt \\
&\le C\Bigg[\mathbb{E}\bigg(\int_0^T |[P_s^k(B+E^k)(B+E^k)^* - P(B+E)(B+E)^*] r_s'|\, ds\bigg)^2 \\
&\quad + \mathbb{E}\bigg(\int_0^T |[P_s^m(B+E^m)(B+E^m)^* - P(B+E)(B+E)^*] r_s'|\, ds\bigg)^2\Bigg].
\end{aligned} \tag{4.32}$$

Thus, the limit processes r and $\hat{q}$ solve (4.18), and we have the desired regularity. Now we have to prove (4.19). The equation for $(r^1 - r^2, q^1 - q^2)$ is

$$\begin{aligned}
r_t^1 - r_t^2 = &- \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{A^*(s-t)} P_s(B+E)(B+E)^*(r_s^1 - r_s^2)\, ds \\
&- \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{A^*(s-t)} P_s(B+E)(B+E)^*(q_s^1 - q_s^2)\, ds.
\end{aligned}$$

Hence, following previous procedures, we have

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[T-\delta,T[} (T-t)^{2(1-\alpha)} |(\lambda - A^*)^{1-\alpha}(r_t^1 - r_t^2)|^2 \\
&\le M\delta^{2(2\alpha-1)} \mathbb{E} \sup_{t\in[T-\delta,T[} (T-t)^{2(1-\alpha)} |(\lambda - A^*)^{1-\alpha}(r_t'^1 - r_t'^2)|^2,
\end{aligned}$$

$$\mathbb{E} \sup_{t\in[T-\delta,T]} |r_t^1 - r_t^2|^2 \le M\delta^{2(2\alpha-1)} \mathbb{E} \sup_{t\in[T-\delta,T[} |(\lambda - A^*)^{1-\alpha}(r_t'^1 - r_t'^2)|^2,$$

and

$$\mathbb{E}\int_{T-\delta}^{T}|\hat{q}_t^1-\hat{q}_t^2|^2\,dt \le M\left[\mathbb{E}\bigg(\int_{T-\delta}^{T}|P_t(B+E)(B+E)^*(r_t'^1-r_t'^2)|\,dt\bigg)^2\right]$$
$$\le M\delta^{2(2\alpha-1)}\mathbb{E}\sup_{t\in[T-\delta,T[}|(\lambda-A^*)^{1-\alpha}(r_t'^1-r_t'^2)|^2,$$

where the constant M depends on α and on constants appearing in hypotheses (A) and (B), and it is such that $M\delta^{2\alpha-1}<1$ if δ is sufficiently small. Therefore one can repeat the procedure in $[T-2\delta, T-\delta]$ and so on in order to cover, in a finite number of steps, the whole interval $[0,T]$. ☐

It remains to show that if we define $\bar{Y}_t = P_t\bar{X}_t + r_t$, then there is one solution to FBSDE (4.10).

PROPOSITION 4.6. *Let assumptions* (A) *hold and let* $b_0, h_0 \in L^2_{\mathcal{P}}(\Omega\times[0,T];H)$, $g_0 \in L^2(\Omega;H)$. *Then the FBSDE* (4.10) *admits a unique mild solution* $(\bar{X},\bar{Y},(\bar{Z},\tilde{Z}))\in L^2_{\mathcal{P}}(\Omega;C([0,T];H))\times L^2_{\mathcal{P}}(\Omega;C([0,T];H))\times L^2_{\mathcal{P}}(\Omega\times[0,T];L_2(\Xi\times K;H))$ *satisfying, moreover,*

$$\mathbb{E}\sup_{t\in[0,T]}(T-t)^{2(1-\alpha)}\|(E+B)\bar{Y}_t\|^2<+\infty.$$

Proof. Let us denote by P^k the solution of the Riccati equation (4.21) and, for $j>\omega$, by $A_j := jR(j,A)$ the Yosida approximants of A. We denote by $P^{j,k}$ the solution of the Riccati equation (4.21) with A_j in place of A:

(4.33)
$$\begin{cases} -\dfrac{dP_t^{j,k}}{dt} = A_j^*P_t^{j,k}+P_t^{j,k}A_j - P_t^{j,k}(E^k+B)(E^k+B)^*P_t^{j,k}+I, & t\in[0,T],\\ P_T^{j,k}=I. \end{cases}$$

By $(r^k,(q^k,\tilde{q}^k))$ and $(r^{n,k},(q^{n,k},\tilde{q}^{n,k}))$ we denote, respectively, the solution of the BSDEs (4.23) and (4.25). Moreover, we denote by $\bar{X}$ and $\bar{X}^k$, respectively, the solution of

$$\begin{cases} d\bar{X}_t = A\bar{X}_t dt - [E+B][E+B]^*(P_t\bar{X}_t+r_t)\,dt + b_0(t)\,dt\\ \qquad + (\lambda-A)D_1\,d\tilde{W}_t + G\,dW_t,\\ \bar{X}_0 = x \end{cases} \tag{4.34}$$

and

$$\begin{cases} d\bar{X}_t^k = A\bar{X}_t^k\,dt - [E^k+B][E^k+B]^*(P_t^k\bar{X}_t^k+r_t^k)\,dt\\ \qquad + b_0(t)\,dt + (\lambda-A)D_1\,d\tilde{W}_t + G\,dW_t,\\ \bar{X}_0^k = x. \end{cases} \tag{4.35}$$

We also set $\bar{X}^{n,k} = nR(n,A)\bar{X}^k$, which is the solution of

(4.36)

$$\begin{cases} d\bar{X}_t^{n,k} = A\bar{X}_t^{n,k}\,dt - nR(n,A)[E^k+B][E^k+B]^*(P_t^k\bar{X}_t^k + r_t^k)\,dt \\ \qquad\qquad + nR(n,A)b_0(t)\,dt + nR(n,A)(\lambda - A)D_1\,d\tilde{W}_t + nR(n,A)G(t,\bar{X}_t)\,dW_t, \\ \bar{X}_0^{n,k} = nR(n,A)x. \end{cases}$$

By applying the Itô formula to $P_t^{j,k}\bar{X}_t^{n,k} + r_t^{n,k}$, we obtain

$$\begin{aligned} P_t^{j,k}\bar{X}_t^{n,k} + r_t^{n,k} &= e^{(T-t)A_j^*}[\bar{X}_T^{n,k} + nR(n,A)g_0] \\ &+ \int_t^T e^{(s-t)A_j^*}\left(A^* r_s^{n,k} - A_j^* r_s^{n,k}\right)ds \\ &+ \int_t^T e^{(s-t)A_j^*}\left(nR(n,A)P_s^k b_0(s) - P_s^{j,k} nR(n,A)b_0(s)\right)ds \\ &+ \int_t^T e^{(s-t)A_j^*}P_s^{j,k}\left(A_j\bar{X}_s^{n,k} - A\bar{X}_s^{n,k}\right)ds \\ &- \int_t^T e^{(s-t)A_j^*}P_s^{j,k}[E^k+B][E^k+B]^*P_s^{j,k}\bar{X}_s^{n,k}\,ds \\ &- \int_t^T e^{(s-t)A_j^*}nR(n,A)P_s^k[E^k+B][E^k+B]^* r_s^{n,k}\,ds \\ &+ \int_t^T e^{(s-t)A_j^*}P_s^{j,k}nR(n,A)[E^k+B][E^k+B]^*(P_s^k\bar{X}_s^k + r_s^k)\,ds \\ &- \int_t^T e^{(s-t)A_j^*}(\bar{X}_s^{n,k} - nR(n,A)h_0(s))\,ds \\ &+ \int_t^T e^{(s-t)A_j^*}\left(P_s^{j,k}nR(n,A)(\lambda - A)D_1 - \tilde{q}_s^{n,k}\right)d\tilde{W}_t \\ &+ \int_t^T e^{(s-t)A_j^*}\left(P_s^{j,k}nR(n,A)G - q_t^{n,k}\right)dW_t. \end{aligned}$$

We start by letting $j \to \infty$. It follows by assumption (A.1) that $\|e^{tA_j}\| \le e^{\omega t}$. Keeping this in mind, and since $r^{n,k}, \bar{X}^{n,k} \in \mathcal{D}(A)$, and, moreover, since $P^{j,k}$ is uniformly bounded in j, we get that the integrals $\int_t^T e^{(s-t)A_j^*}\left(A^* r_s^{n,k} - A_j^* r_s^{n,k}\right)ds$ and $\int_t^T e^{(s-t)A_j^*}P_s^{j,k}\left(A_j\bar{X}_s^{n,k} - A\bar{X}_s^{n,k}\right)ds$ converge to 0 as $j \to \infty$.

With similar considerations, by adding and subtracting $e^{(s-t)A_j^*}P_s^{j,k}[E^k+B][E^k+B]^*P_s^k\bar{X}_s^{n,k}$ and $e^{(s-t)A_j^*}P_s^k[E^k+B][E^k+B]^*P_s^k\bar{X}_s^{n,k}$ we get that the integral $\int_t^T e^{(s-t)A_j^*}P_s^{j,k}[E^k+B][E^k+B]^*P_s^{j,k}\bar{X}_s^{n,k}\,ds$ converges to $\int_t^T e^{(s-t)A^*}P_s^k[E^k+B][E^k+B]^*P_s^k\bar{X}_s^{n,k}\,ds$ as $j \to \infty$.

In an analogous and simpler way we also get that $\int_t^T e^{(s-t)A_j^*}P_s^{j,k}nR(n,A)[E^k+B][E^k+B]^*(P_s^k\bar{X}_s^k + r_s^k)\,ds$ converges to $\int_t^T e^{(s-t)A^*}P_s^k nR(n,A)[E^k+B][E^k+B]^*(P_s^k\bar{X}_s^k + r_s^k)\,ds$.

By adding and subtracting $P_s^k nR(n,A)b_0(s)$ it is possible to see that

$$\int_t^T e^{(s-t)A_j^*}\left(nR(n,A)P_s b_0(s) - P_s^{j,k}nR(n,A)b_0(s)\right)ds$$

converges to $\int_t^T e^{(s-t)A^*}\left(nR(n,A)P_s^k b_0(s) - P_s^k nR(n,A)b_0(s)\right) ds$ as $j \to \infty$, and it is immediate that $\int_t^T e^{(s-t)A_j^*} nR(n,A)h_0(s)\, ds \to \int_t^T e^{(s-t)A^*} nR(n,A)h_0(s)\, ds$ as $j \to \infty$.

For what concerns the stochastic integrals, we notice that the integrands are square integrable with respect to s, uniformly with respect to j.

Now, first let $j \to \infty$, then take the conditional expectation, and eventually let $n \to \infty$ (recall also that $\|P_k\|_1$ is bounded uniformly with respect to k):

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_t} P_t^k \bar{X}_t^k + r_t^k &= \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*}[\bar{X}_T^k + g_0] + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*}\left(P_s^k b_0(s) - P_s^k b_0(s)\right) ds \\
&\quad - \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*}\left[P_s^k[E^k + B][E^k + B]^*(P_s^k \bar{X}_s^k + r_s^k)\right] ds \\
&\quad + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*}(\bar{X}_s^k - h_0(s))\, ds \\
&\quad + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} P_s^k[E^k + B][E^k + B]^*(P_s^k \bar{X}_s^k + r_s^k)\, ds \\
&= \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*}[\bar{X}_T^k + g_0] + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*}(\bar{X}_s^k - h_0(s))\, ds.
\end{aligned}
$$

Now notice that

$$
\begin{aligned}
\bar{X}_t^k - \bar{X}_t = \int_0^t e^{(t-s)A}\big([E^k + B][E^k + B]^*(P_s^k \bar{X}_s^k + r_S^k) \\
- [E + B][E + B]^*(P_s \bar{X}_s + r_s)\big)\, ds.
\end{aligned}
$$

By the convergence of P^k to P (see [3, Chapter IV, section 2, Lemma 2.1 and Theorem 2.1], by adding and subtracting suitable terms and by virtue of the Gronwall lemma, we get that $\bar{X}^k \to \bar{X}$ in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))$. By adding and subtracting $P_t^k \bar{X}_t$ we also get that $P_t^k \bar{X}_t^k \to P_t \bar{X}_t$ in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))$, since $\sup_{t\in[0,T]} |P_t^k| \le M$ independent of k, thanks to (4.22) and the Banach–Steinhaus theorem. With similar arguments we also get that $(T-t)^\alpha(\lambda - A^*)^{1-\alpha} P_t^k \bar{X}_t^k$ converges to $(T-t)^\alpha(\lambda - A^*)^{1-\alpha} P_t \bar{X}_t$ in $L^2_{\mathcal{P}}(\Omega, C([0,T], H))$. With similar and simpler arguments we finally get

$$
P_t \bar{X}_t + r_t = \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*}[\bar{X}_T + g_0] - \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \bar{X}_s\, ds - \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} h_0(s)\, ds.
$$

Again by the extended martingale representation theorem (see also [20] and [38]), we deduce that for almost all $s \in [0,T]$,

$$
\begin{aligned}
Z_s &= \int_s^T e^{(\alpha-s)A^*} K(\alpha, s)\, d\alpha + e^{(T-s)A^*} L_s, \\
\tilde{Z}_s &= \int_s^T e^{(\alpha-s)A^*} \tilde{K}(\alpha, s)\, d\alpha + e^{(T-s)A^*} \tilde{L}_s,
\end{aligned}
$$

where $(K(s,\cdot), \tilde{K}(s,\cdot)) \in L^2_{\mathcal{P}}(\Omega\times[0,T], L_2(\Xi, H)) \times L^2_{\mathcal{P}}(\Omega\times[0,T], L_2(K,H))$ for every for all $\theta \ge s$ $K(s,\theta) = \tilde{K}(s,\theta) = 0$ a.e. and $(L, \tilde{L}) \in L^2_{\mathcal{P}}(\Omega\times[0,T], L_2(\Xi, H)) \times L^2_{\mathcal{P}}(\Omega\times[0,T], L_2(K,H))$ such that

$$\mathbb{E}\int_0^T\int_0^s \left[|K(s,\theta)|^2+|\tilde K(s,\theta)|^2\right]\,d\theta\,ds \le 4\mathbb{E}\int_0^T |(\bar X_s - h_0(s))|^2\,ds, \tag{4.37}$$

$$E\int_0^T (|L_\theta|^2+|\tilde L_\theta|^2)\,d\theta \le 4\mathbb{E}[|\bar X_T|^2+|g_0|^2]. \tag{4.38}$$

Therefore, by the definitions of $(K,\tilde K)$ and $(L,\tilde L)$ and by estimates (4.37)–(4.38), it follows that $(\bar Z_s,\tilde Z_s)\in L^2_{\mathcal P}(\Omega\times[0,T];L_2(\Xi\times K;H))$ and $(X,Y,(Z,\tilde Z))$ is a solution to FBSDE (4.10). □

4.3. Existence and uniqueness of the mild solution of the FBSDE. In this section we prove Theorem 3.4 by using results proved in section 4.

Proof of Theorem 3.4. *Existence*. We follow the proof of Theorem 3.1, existence part, in [22], with suitable changes due to the different framework. For $\alpha\in[0,1]$ consider the FBSDE

(4.39)
$$\begin{cases} d\bar X_t = A\bar X_t,dt + b^\alpha(\bar Y_t)\,dt + b_0(t)\,dt + (\lambda - A)D_1\,d\tilde W_t + G\,dW_t,\\ -d\bar Y_t = A^*\bar Y_t\,dt + h^\alpha(\bar X_t)\,dt + h_0(t)\,dt - \bar Z_t\,dW_t - \tilde Z_t\,d\tilde W_t, \qquad t\in[0,T],\\ \bar X_0 = x,\ -\bar Y_T = g^\alpha(\bar X_t)+g_0. \end{cases}$$

For $\alpha=0$ the FBSDE (4.39) admits a mild solution: by section 4.2 we know that FBSDE (4.11) admits a mild solution, and for $\alpha=0$, FBSDE (4.39) coincides with FBSDE (4.11). By Lemma 4.2 there exists δ_0 such that for all $\alpha\in[0,\delta_0]$ the FBSDE (4.39) admits a mild solution with the required regularity. Then, by the arbitrary choice of b_0, h_0, and g_0 (see also Remark 4.3), we can solve (4.39) for $\alpha\in[\delta_0,2\delta_0]$, $[2\delta_0,3\delta_0],\ldots$: notice that δ_0 does not depend on α. We arrive at solving (4.39) for $\alpha=1$, and since we can choose b_0, h_0, and g_0 arbitrarily, we arrive at proving the existence of an adapted solution $(\bar X,\bar Y,(\bar Z,\tilde Z))$ of (3.6) with the required regularity.

Uniqueness. In order to prove uniqueness we follow [22, Theorem 3.1, uniqueness part] and the proof of Lemma 4.2 in the present paper. For $i=1,2$, let $(\bar X^i,\bar Y^i,(\bar Z^i,\tilde Z^i))$ be two solutions of (3.6). In order to apply the Itô formula, we have to approximate these solutions with elements in the domain of A; namely, we set $(\bar X^{n,i},\bar Y^{n,i},(\bar Z^{n,i},\tilde Z^{n,i})) = (nR(n,A)\bar X^i, nR(n,A)\bar Y^i,(nR(n,A)\bar Z^i, nR(n,A)\tilde Z^i))$, $i=1,2$, and as in Lemma 4.2, we also denote $E_n+B_n := nR(n,A)(E+B)$. By applying the Itô formula to $\langle \bar X^{n,1}_t-\bar X^{n,2}_t,\bar Y^{n,1}_t-\bar Y^{n,2}_t\rangle$, and then integrating over $[0,T]$ and taking expectation, we get

$$\begin{aligned} &-\mathbb{E}\langle \bar X^{n,1}_T-\bar X^{n,2}_T, nR(n,A)(h_x(\bar X^1_T)-h_x(\bar X^2_T))\rangle\\ &\quad= \mathbb{E}\int_0^T \langle [E_n+B_n](\gamma([E+B]^*\bar Y^1_t)-\gamma([E+B]^*\bar Y^2_t)),\bar Y^{n,1}_t-\bar Y^{n,2}_t\rangle\,dt\\ &\qquad -\mathbb{E}\int_0^T \langle nR(n,A)(l^0_x(t,\bar X^1_t)-l^0_x(t,\bar X^2_t)),\bar X^{n,1}_t-\bar X^{n,2}_t\rangle\,dt. \end{aligned} \tag{4.40}$$

Next we want to let $n\to+\infty$ in (4.40): arguing as in Lemma 4.2, we deduce that $\bar X^{n,i}\to\bar X^i$ in $L^2_{\mathcal P}(\Omega;C([0,T],H))$ for $i=1,2$, and that $\bar Y^{n,i}\to\bar Y^i$ in $L^2_{\mathcal P}(\Omega;$

$C([0,T],H))$ for $i=1,2$, and, moreover, $\mathbb{E}\sup_{t\in[0,T]}(T-t)^{2(1-\alpha)}|[E+B]^*(\bar{Y}^{n,i}-\bar{Y}^i)|^2\to 0$ for $i=1,2$. We also have $\mathbb{E}\sup_{t\in[0,T]}(T-t)^{2(1-\alpha)}|[E_n+B_n]^*\bar{Y}^i-[E+B]^*\bar{Y}^i|^2\to 0$, for $i=1,2$. So, letting $n\to\infty$ in (4.40), we get

$$\begin{aligned}
&-\mathbb{E}\langle \bar{X}^1_T-\bar{X}^2_T, h_x(\bar{X}^1_T)-h_x(\bar{X}^2_T)\rangle\\
&\quad=\mathbb{E}\int_0^T\langle [E+B](\gamma([E+B]^*\bar{Y}^1_t)-\gamma([E+B]^*\bar{Y}^2_t)),\bar{Y}^1_t-\bar{Y}^2_t\rangle\,dt\\
&\qquad-\mathbb{E}\int_0^T\langle l^0_x(t,\bar{X}^1_t)-l^0_x(t,\bar{X}^2_t),\bar{X}^1_t-\bar{X}^1_t\rangle\,dt.
\end{aligned}$$

So, by assumptions (B), we get

$$\mathbb{E}|\bar{X}^1_T-\bar{X}^2_T|^2+\mathbb{E}\int_0^T|[E+B]^*(\bar{Y}^1_t-\bar{Y}^2_t)|^2\,dt+\mathbb{E}\int_0^T|\bar{Y}^1_t-\bar{Y}^2_t|^2\,dt\le 0,$$

and so the uniqueness follows. ∎

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